

Barlevy, Gadi; Nagaraja, H. N.

**Working Paper**

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Working Paper, No. 2013-05

**Provided in Cooperation with:**  
Federal Reserve Bank of Chicago

*Suggested Citation:* Barlevy, Gadi; Nagaraja, H. N. (2013) : Properties of the vacancy statistic in the discrete circle covering problem, Working Paper, No. 2013-05, Federal Reserve Bank of Chicago, Chicago, IL

This Version is available at:  
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Federal Reserve Bank of Chicago

**Properties of the Vacancy Statistic in  
the Discrete Circle Covering Problem**

*Gadi Barlevy and H. N. Nagaraja*

July 2013

WP 2013-05

# Properties of the Vacancy Statistic in the Discrete Circle Covering Problem

Gadi Barlevy\* and H. N. Nagaraja†

July 3, 2013

## Abstract

Holst (1985) introduced a discrete spacings model that is related to the Bose-Einstein distribution and obtained the distribution of the number of vacant slots in an associated circle covering problem. We correct his expression for its probability mass function, obtain the first two moments, and describe their limiting properties. We also discuss an application of our results to a study of contagion in banking networks.

**Key Words:** *spacings; sampling without replacement; Bose-Einstein distribution; Hypergeometric probability.*

**2010 Mathematics Subject Classification:** Primary: 60C05; Secondary: 62E15.

## 1 Introduction

Consider  $r$  ( $\geq 2$ ) boxes arranged in a ring. Starting with one box chosen at random, number the boxes  $0, 1, \dots, r - 1$ . Draw  $m - 1$  boxes by simple random sampling without replacement from the boxes numbered  $1, 2, \dots, r - 1$ , where  $2 \leq m \leq r$ . Let  $1 \leq R_1 < \dots < R_{m-1} \leq r - 1$  be the drawn numbers, and set  $R_0 = 0$  and  $R_m = r$ . Define

$$S_k = R_k - R_{k-1}$$

for  $k = 1, 2, \dots, m$ , i.e.  $S_k$  are spacings. Next, for an integer  $b$  where  $1 \leq b \leq r$ , define

$$V = \sum_{k=1}^m (S_k - b)_+ \tag{1}$$

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\*gbarlevy@frbchi.org; Economic Research Department, Federal Reserve Bank of Chicago, 230 South LaSalle, Chicago, IL 60604, USA.

†hnagaraja@cph.osu.edu; College of Public Health, Division of Biostatistics, The Ohio State University, 1841 Neil Avenue, Columbus, OH 43210-1351, USA.

where  $(x)_+ = \max\{x, 0\}$ . Thus  $V$  represents the number of vacant slots in a discrete circle covering problem with arcs of length  $b$ .

Our interest is in basic characteristics of  $V$ , specifically its distribution and some of its moments. Holst (1985) has derived the marginal and joint distributions of  $S_k$  and showed that they are exchangeable. He also explored the connections between these random variables (rv's) and the Bose-Einstein distribution. Feller (1968, Sec II.5(a)) provides a nice introduction to the Bose-Einstein urn model.

Note that in the limit, as  $r \rightarrow \infty$  while  $b = ar$  for some constant  $a < 1$ , this problem converges to the well known circle covering problem in which  $m$  points are chosen uniformly from the unit circle, and each of the  $m$  points forms the end point of an arc of length  $a$  that is drawn. The limit of  $V/r$  corresponds to the fraction of the circle that is uncovered. This problem was originally analyzed by Stevens (1939) and has been extensively analyzed; see for example, Siegel (1978). However, the finite version of the problem has been less studied, even though as we discuss below this version can arise in practical applications.

We derive in Section 2 an explicit expression for the probability mass function (pmf) of  $V$  including an exploration of the range of its values; in this process we correct an error in the expression for the pmf given in Holst (1985). We give explicit expressions for the first two moments of  $V$  in Section 3 using several properties of the joint distribution of  $S_k$  derived by Holst (1985). We establish limiting properties of  $V$  in Section 4 and link our results to those of Siegel (1978). In Section 5, we discuss an application exploring contagion among interconnected financial intermediaries in which the discrete version of the circle covering problem arises under certain conditions. The application reveals a new generalization of the circle covering problem that has not been explored in previous work.

## 2 Exact Distribution of $V$

### 2.1 The Range

The value of  $V$  must be at least  $r - mb$  and it has to be non-negative. Further, the largest possible value of  $V$  occurs when the chosen boxes are consecutive and  $V$  takes on the value  $r - m - b + 1$ . Thus, the support of  $V$  is the set  $\{(r - mb)_+, \dots, (r - m - b + 1)_+\}$ , and so  $V$  is degenerate at 0 whenever  $r < m + b$ . Further when  $r - mb \geq 0$ , the total number of points in the support of  $V$  is  $(m - 1)(b - 1) + 1$  independently of  $r$ . Hence when  $b = 1$  and  $m < r$ , we have a single support point is at  $r - m > 0$ . We now examine the form of the pmf  $P(V = x)$  for various  $x$  values when  $r \geq m + b$  and  $b > 1$ .

### 2.2 Probability Mass Function of $V$

Holst (1985; Theorem 2.2) argues that  $P(V = x)$  is given by

$$\sum_{y=1}^m \binom{m}{y} \sum_{t=0}^{m-y} (-1)^t \binom{m-y}{t} \binom{x-1}{y-1} \binom{r-(y+t)b-x-1}{m-y-1} / \binom{r-1}{m-1}. \quad (2)$$

Note that expression (2) may include improper binomial coefficients  $\binom{n}{k}$  where either  $n < 0$  or  $k \notin \{0, \dots, n\}$ . Such terms are traditionally set to 0. We now argue that this convention may yield an incorrect expression for  $P(V = x)$  for  $x = 0$  and for  $x = r - mb$ , and offer correct expressions for  $P(V = x)$  for these cases.

Observe first that for  $x = 0$ , the right-hand side of (2) is always equal to 0, at odds with the fact that  $P(V = 0) = 1$  whenever  $r < m + b$ .

To obtain a proper expression for  $P(V = 0)$ , we use the observation noted by Holst that

$$P(V = 0) = P\left(\sum_{j=1}^m I(S_j > b) = 0\right), \quad (3)$$

where  $I(C) = 1$  if condition  $C$  is true and 0 else. Holst (1985) derives an expression for the right hand side of (3) in part (a) of his Theorem 2.2. Using his result, we can deduce that for  $x = 0$ , (2) must be replaced by

$$P(V = 0) = \left\{ \sum_{j=0}^m (-1)^j \binom{m}{j} \binom{r-jb-1}{m-1} \right\} / \binom{r-1}{m-1}. \quad (4)$$

We next turn to the case where  $r \geq m + b$  and  $x > 0$ , and examine the range of values for  $y$  and  $t$  for which the associated terms on the right side of (2) are all positive, i.e. when  $0 \leq t \leq \min\{m-y, [r-m-x-(b-1)y]/b\}$ , and  $1 \leq y \leq \min\{m-1, x, (r-m-x)/(b-1)\}$  for  $b > 1$  and  $1 \leq y \leq \min\{m-1, x\}$  for  $b = 1$ .

We shall now argue that for this range, (2) holds except when  $r - mb \geq m$  and  $x = r - mb$ . To see this, we begin with the observation by Holst in his proof of Theorem 2.2 that if  $I_k = I(S_k > b)$ , then  $P(V = x)$  must equal

$$\sum_{y=1}^m \binom{m}{y} \sum_{t=0}^{m-y} (-1)^t \binom{m-y}{t} P\left(\sum_{k=1}^y (S_k - b)_+ = x, I_1 = \dots = I_{y+t} = 1\right). \quad (5)$$

Expression (5) can in turn be rewritten as

$$\begin{aligned} & \sum_{y=1}^{m-1} \binom{m}{y} \sum_{t=0}^{m-y-1} (-1)^t \binom{m-y}{t} P\left(\sum_{k=1}^y (S_k - b)_+ = x, I_1 = \dots = I_{y+t} = 1\right) \\ & + \sum_{y=1}^m \binom{m}{y} (-1)^{m-y} P\left(\sum_{k=1}^y (S_k - b)_+ = x, I_1 = \dots = I_m = 1\right). \end{aligned} \quad (6)$$

Holst then computes the following probabilities based respectively on parts (E) and (D) of his Theorem 2.1:

$$P(I_1 = \cdots = I_{y+t} = 1) = \frac{\binom{r-(y+t)b-1}{m-1}}{\binom{r-1}{m-1}}$$

and

$$P\left(\sum_{k=1}^y (S_k - b)_+ = x \mid I_1 = \cdots = I_{y+t} = 1\right) = \frac{\binom{x-1}{y-1} \binom{r-(y+t)b-x-1}{m-y-1}}{\binom{r-(y+t)b-1}{m-1}} \quad (7)$$

and thus concludes that

$$P\left(\sum_{k=1}^y (S_k - b)_+ = x, I_1 = \cdots = I_{y+t} = 1\right) = \frac{\binom{x-1}{y-1} \binom{r-(y+t)b-x-1}{m-y-1}}{\binom{r-1}{m-1}}. \quad (8)$$

The expression for the conditional probability in (7) is valid and nonzero whenever  $y \leq m-1$ , and  $y+t < m$ .

Next we consider the last sum on the right in (6). Since the event  $\{I_1 = \cdots = I_m = 1\}$  implies  $S_i > b$  for  $i = 1, \dots, m$ , the sum  $\sum_{k=1}^y (S_k - b)_+$  is strictly increasing in  $y$  for  $y \leq m$  and  $\sum_{k=1}^y (S_k - b)_+ < \sum_{k=1}^m (S_k - b)_+ \equiv r - mb$ . This means

$$\sum_{y=1}^m \binom{m}{y} (-1)^{m-y} P\left(\sum_{k=1}^y (S_k - b)_+ = x, I_1 = \cdots = I_m = 1\right)$$

is 0 for any  $x > r - mb$  and when  $x = r - mb$  only the last term in the sum, namely  $P\left(\sum_{k=1}^m (S_k - b)_+ = r - mb, I_1 = \cdots = I_m = 1\right)$  is potentially positive. Since each term in  $\sum_{k=1}^m (S_k - b)_+$  is at least one, the sum should be at least  $m$ . In other words, the only nonzero term in the last sum on the right side of (6) is

$$\begin{aligned} & P\left(\sum_{k=1}^m (S_k - b)_+ = r - mb, I_1 = \cdots = I_m = 1\right) \\ &= P(I_1 = \cdots = I_m = 1) = P(S_1 > b, \dots, S_m > b) = P(S_1 > mb) \\ &= \binom{r-mb-1}{m-1} / \binom{r-1}{m-1} \end{aligned} \quad (9)$$

provided  $r - mb \geq m$ . Upon collecting all of our findings (in (4), (6), (8), and (9)), we have the following modification of Theorem 2.2 of Holst (1985).

**Theorem 1.** *The support of the rv  $V$  representing the length of the vacant region is given by  $\{(r - mb)_+, \dots, (r - m - b + 1)_+\}$ . When  $r < m + b$ ,  $V$  is degenerate at 0. When  $r > m$  and  $b = 1$ ,  $V$  is degenerate at  $(r - m)$ . When  $r \geq m + b$  and  $r - mb \leq 0$ ,  $P(V = 0)$  is given by (4). In all other cases,  $P(V = x)$  is given by*

$$\begin{aligned} & \left\{ \sum_{y=1}^{m-1} \binom{m}{y} \sum_{t=0}^{m-1-y} (-1)^t \binom{m-y}{t} \binom{x-1}{y-1} \binom{r-(y+t)b-x-1}{m-y-1} \right. \\ & \left. + I(x = r - mb \geq m) \binom{r-mb-1}{m-1} \right\} / \binom{r-1}{m-1}. \end{aligned} \quad (10)$$

**Remark 1:** The actual range of values for  $y$  and  $t$  for which the associated terms are positive are more restricted than given by the limits in the double sum in (10) in a way that depends on  $x$ . For example, when  $x = r - mb$ , the lowest value  $V$  can assume, the terms are positive for all  $1 \leq y \leq m - 1$  and  $0 \leq t \leq \min\{m - y - 1, m - y - (m - y)/b\}$ . In contrast, when  $x = r - m - b + 1$ , the highest value  $V$  can assume,  $y = 1$ ,  $t = 0$  is the only combination that produces a positive term. In that case, (10) yields

$$P(V = r - m - b + 1) = m \binom{r - 1}{m - 1},$$

a quantity free of  $b$ .

Table 1 below provides the pmf of  $V$  for  $r = 10$  and  $m = 5$ . It shows how the probabilities shift towards 0 as  $b$  increases.

Table 1: **The pmf  $P(V = x)$  for  $r = 10, m = 5$  for various  $b$  values**

$b$	$x$					
	0	1	2	3	4	5
1	0	0	0	0	0	1
2	0.008	0.159	0.476	0.317	0.040	
3	0.405	0.397	0.159	0.040		
4	0.802	0.159	0.040			
5	0.960	0.040				
6	1					

### 3 Moments of $V$

Instead of using the pmf for  $V$  to compute the first two moments of  $V$ , we take advantage of an exchangeability argument to derive them from those of  $S_k$ . We will need the following expressions for the first two moments of nonnegative integer valued rv's  $X$  and  $Y$ .

$$E(X) = \sum_{i=0}^{\infty} P(X > i); \tag{11}$$

$$E(X^2) = 2 \sum_{i=0}^{\infty} iP(X > i) + E(X); \tag{12}$$

$$E(XY) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(X > i, Y > j). \tag{13}$$

The first two are well-known. Equations (11) and (12) are given in, for example, David and Nagaraja (2003), p. 43, and go back to Feller's classical work.

Expression (13) is similar to known results for the continuous case; see, for example, the expression for the covariance in Barlow and Proschan (1981, p. 31), and the idea goes back to Hoeffding (1940); see Wellner (1994).

Here we give a short proof of (13) when  $X$  and  $Y$  are nonnegative integer valued rv's.

$$\begin{aligned} E(XY) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} ijP(X=i, Y=j) \\ &= \sum_{i=0}^{\infty} i \sum_{j=0}^{\infty} P(X=i, Y>j) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(X>i, Y>j) \end{aligned}$$

upon using the idea of the form on the right side of (11) twice.

The moments expressions simplify further by the use of the following well-known identity: For positive integers  $c \leq a$ ,

$$\sum_{k=c}^a \binom{k-1}{c-1} = \binom{a}{c}. \quad (14)$$

Harris, Hirst, and Mossinghoff (2008) derive this identity using an induction argument (see equation (2.10) in their book). We give a simple probabilistic proof.

*Proof.* Multiply both sides by  $(1/2)^a$ . Then the right side,  $\binom{a}{c}(1/2)^a$  represents the probability that in  $a$  tosses of a fair coin there are exactly  $c$  heads. Now if we have  $c$  heads, this event can be written as the union of disjoint events  $E_c, \dots, E_a$  where  $E_k$  is the event that we have exactly  $c$  heads and the  $c$ th head appears at the  $k$ th toss. By the negative binomial type argument we know that this probability is

$$\binom{k-1}{c-1}(1/2)^c(1/2)^{a-c} = \binom{k-1}{c-1}(1/2)^a.$$

Now sum this over  $k$  from  $c$  to  $a$ . □

**Theorem 2.** Let  $W_i = (S_i - b)_+$ , for  $i = 1, 2$ . Then the  $W_i$  are exchangeable and for  $r \geq m + b$

$$E(W_1) = \binom{r-b}{m} / \binom{r-1}{m-1}, \quad (15)$$

and

$$E(W_1^2) = (2(r-b) + 1)E(W_1) - 2m \frac{\binom{r-b+1}{m+1}}{\binom{r-1}{m-1}}, \quad (16)$$



For  $r \geq m + 2b$ ,

$$E(W_1 W_2) = \binom{r - 2b + 1}{m + 1} \bigg/ \binom{r - 1}{m - 1}, \quad (17)$$

and  $E(W_1 W_2) = 0$  if  $r < m + 2b$ .

*Proof.* Exchangeability follows from Theorem 2.1 of Holst (1985). Now

$$\begin{aligned} E(W_1) &= \sum_{i=0}^{r-m-b} P(W_1 > i) \quad [\text{from (11)}] \\ &= \sum_{j=b}^{r-m} P(S_1 > j) \\ &= \sum_{j=b}^{r-m} \binom{r-j-1}{m-1} \bigg/ \binom{r-1}{m-1} \quad [\text{from Th 2.1(B), Holst}]. \quad (18) \end{aligned}$$

From (14), the numerator on the right side of (18) reduces to  $\binom{r-b}{m}$ .

To establish (16), we use the expression for the second moment in (12). Consider

$$\begin{aligned} \sum_{i=0}^{\infty} iP(W_1 > i) &= \sum_{j=b}^{r-m} (j-b)P(S_1 > j) \\ &= \sum_{j=b}^{r-m} \{r - (r-j)\}P(S_1 > j) - bE(W_1) \\ &= r \sum_{j=b}^{r-m} P(S_1 > j) - \sum_{j=b}^{r-m} (r-j)P(S_1 > j) - bE(W_1) \\ &= (r-b)E(W_1) - \sum_{j=b}^{r-m} (r-j) \binom{r-j-1}{m-1} \bigg/ \binom{r-1}{m-1} \quad (19) \end{aligned}$$

from the expression for  $P(S_1 > i)$  in Theorem 2.1, Part (B) of Holst (1986). The numerator in the second term in (19) above can be expressed as

$$m \sum_{i=b}^{r-m} \binom{r-i}{m} = m \sum_{j=m+1}^{r-b+1} \binom{j-1}{m} = m \binom{r-b+1}{m+1}, \quad (20)$$

where the last equality follows from (14). Upon using (12) with  $W_1 = X$  and applying (19) and (20) we obtain (16).

Using (13) with  $W_1 = X$  and  $W_2 = Y$ , and applying Theorem 2.1 Part (E),

and Part (B) of Holst (1985) in succession, we obtain

$$\begin{aligned}
E(W_1W_2) &= \sum_{i \geq b} \sum_{j \geq b} P(S_1 > i, S_2 > j) \\
&= \sum_{i=b}^{r-m-b} \sum_{j=b}^{r-m-i} P(S_1 > i, S_2 > j) \\
&= \sum_{i=b}^{r-m-b} \sum_{j=b}^{r-m-i} P(S_1 > i+j) \\
&= \sum_{i=b}^{r-m-b} \sum_{j=b}^{r-m-i} \binom{r-i-j-1}{m-1} / \binom{r-1}{m-1}.
\end{aligned}$$

Now, with  $k = r - i - j$ ,

$$\sum_{j=b}^{r-m-i} \binom{r-i-j-1}{m-1} = \sum_{k=m}^{r-b-i} \binom{k-1}{m-1} = \binom{r-b-i}{m}$$

from (14). Hence

$$\sum_{i=b}^{r-m-b} \sum_{j=b}^{r-m-i} \binom{r-i-j-1}{m-1} = \sum_{i=b}^{r-m-b} \binom{r-b-i}{m}.$$

With  $k = r - b - i + 1$ , the above sum can be expressed as

$$\sum_{k=m+1}^{r-2b+1} \binom{k-1}{m} = \binom{r-2b+1}{m+1}.$$

Hence the claim in (17) holds. Clearly, when  $r < m + 2b$ , we cannot have both  $W_1, W_2$  be positive simultaneously and hence  $E(W_1W_2) = 0$ .  $\square$

From (1) and the exchangeability of the  $W_i$  we see that

$$\begin{aligned}
E(V) &= mE(W_1) \\
\text{Var}(V) &= m\text{Var}(W_1) + m(m-1)\text{Cov}(W_1, W_2) \\
&= m[E(W_1^2) - \{E(W_1)\}^2] + m(m-1)[E(W_1W_2) - \{E(W_1)\}^2] \\
&= mE(W_1^2) + m(m-1)E(W_1W_2) - m^2\{E(W_1)\}^2 \tag{21}
\end{aligned}$$

where the expectations on the right side of (21) are given by Theorem 2. Thus we have the following result.

**Theorem 3.** *If  $r \geq m + b$ , the first two moments of the rv  $V$  representing the*

number of vacant slots are given by

$$\begin{aligned}
E(V) &= m \frac{\binom{r-b}{m}}{\binom{r-1}{m-1}}, & (22) \\
\text{Var}(V) &= \frac{m(2(r-b)+1)\binom{r-b}{m} - 2m^2\binom{r-b+1}{m+1} + m(m-1)\binom{r-2b+1}{m+1}}{\binom{r-1}{m-1}} \\
&\quad - m^2 \left\{ \frac{\binom{r-b}{m}}{\binom{r-1}{m-1}} \right\}^2 & (23)
\end{aligned}$$

where the coefficient of  $m(m-1)$  in (23) is taken to be 0 whenever  $r < m + 2b$ .

**Notes:**

1. After deriving the expression for  $E(V)$ , we discovered it was previously reported in Ivchenko (1994). However, he does not derive a formula for the variance.
2. For  $r \geq m + b$ ,

$$E(V) = m \binom{r-b}{m} / \binom{r-1}{m-1} = \frac{(r-b)!(r-m)!}{(r-b-m)!(r-1)!}. \quad (24)$$

As seen from the second version in (24),  $E(V)$  is symmetric in  $b$  and  $m$ , even though the distribution for  $V$  is not symmetric in these parameters. The symmetry only holds for the first moment, but not the second moment.

3. As mentioned in Theorem 1, if  $r < m + b$ ,  $P(V = 0) = 1$ . Thus if  $b \geq r - m + 1$  or  $m + b > r$ , all the  $S_i$  are  $b$  or less. Thus,  $E(V) = \text{Var}(V) = 0$  whenever  $r < m + b$ . When  $b = 1$  and  $r > m$ ,  $V$  is degenerate at  $r - m$  and in that case  $E(V) = r - m$  and  $\text{Var}(V) = 0$ .

## 4 Limiting Properties of $V$

### 4.1 Limiting Distributions

Holst (1985; Theorem 3.2) shows that as  $r, b \rightarrow \infty$  such that  $b/r \rightarrow a$  for some  $0 < a < 1$ , then  $V/r \xrightarrow{d} V_a$  where  $V_a$  has the same distribution as the length of non-covered segments when  $m$  arcs of length  $a$  are dropped at random on a circle with unit circumference. Siegel (1978; Theorem 3) has shown that the distribution of  $V_a$  can be expressed as the mixture of a degenerate and a continuous rv. Specifically, he shows that  $P(V_a(m) = (1 - ma)_+) = p_a(m)$  where

$$p_a(m) = \sum_{i=0}^{m-1} (-1)^i \binom{m}{i} (1 - ia)_+^{m-1}, \quad ma > 1 \quad (25)$$

$$= (1 - ma)^{m-1}, \quad ma \leq 1, \quad (26)$$

and with probability  $1 - p_a(m)$ ,  $V_a(m)$  behaves like a continuous rv  $W_a(m)$  having the pdf  $f(w; a, m)$  given by

$$f(w; a, m) = \frac{m}{1 - p_a(m)} \sum_{i=1}^m \sum_{j=1}^{m-1} (-1)^{i+j} \binom{m-1}{i-1} \binom{m-1}{j} \binom{i-1}{j-1} w^{j-1} (1 - ia - w)_+^{m-j-1},$$

$$(1 - ma)_+ < w < 1 - a. \quad (27)$$

with the convention that  $(1 - ia - w)_+^0$  is interpreted as 1 if  $1 - ia - w \geq 0$ , and as 0, otherwise. We now show the following.

**Lemma 1.** *If  $r, b \rightarrow \infty$  such that  $b/r \rightarrow a, 0 < a < 1$ , then*

$$P\{V = (r - mb)_+\} \rightarrow p_a(m) \equiv P\{V_a = (1 - ma)_+\},$$

given by (25) when  $ma > 1$ , and by (26) when  $ma < 1$ . When  $ma = 1$ , both (25) and (26) reduce to 0.

*Proof.* When  $ma > 1$ ,  $r - mb$  is eventually negative, our interest then is in the limiting form of  $P(V = 0)$  given in (4). Consider the  $j$ th term there, excluding the factor  $(-1)^m \binom{m}{j}$ :

$$\frac{\binom{r-jb-1}{m-1}}{\binom{r-1}{m-1}} = \frac{(r - jb - 1) \cdots (r - jb - m + 1)}{(r - 1) \cdots (r - m + 1)},$$

if  $r - jb = r(1 - j(b/r)) > m - 1$ ; and it is 0 if  $r(1 - j(b/r)) \leq m - 1$ . So if  $b/r \rightarrow a$  with  $1 - ma < 0$ , the above ratio converges to  $(1 - ja)_+^{m-1}$ . Thus the limit is given by (25).

Whenever  $ma < 1$ , since  $r - mb = r(1 - m(b/r))$ ,  $r - mb$  eventually exceeds any fixed  $m$ . In that case the term (9) converges to  $(1 - ma)^{m-1}$ . The remaining finite number of terms in the numerator on the right in (10) are finite and each is of  $o(r^{m-1})$  whereas the denominator is  $O(r^{m-1})$ . Thus, the only nonzero term in the limit is that of (9) and it coincides with (26).

If  $ma = 1$ , (26) is obviously 0 and now we show that (25) also converges to 0 as  $ma \rightarrow 1^+$ . For this we consider the continuous uniform spacing problem where one chooses at random  $m - 1$  points  $U_1, \dots, U_{m-1}$  from the interval  $(0, 1)$ . With spacings defined as  $Y_i = U_{i:m-1} - U_{i-1:m-1}$ ,  $i = 1, \dots, m$  where  $U_{0:m-1} = 0$  and  $U_{m:m-1} = 1$ , it is known that the survival function of the continuous rv  $Y_{(m)}$  representing the maximal spacing is given by (see, e.g., David and Nagaraja, 2003, p. 135)

$$P(Y_{(m)} > a) = \sum_{i=1}^m (-1)^{i-1} \binom{m}{i} (1 - ia)_+^{m-1}, \quad (28)$$

for all  $a$  in  $(0, 1)$ . Since it corresponds to the maximal spacing,  $Y_{(m)} \geq 1/m$ , and so this probability is 1 whenever  $a \leq 1/m$  or  $ma \leq 1$ . The right side sum

in (25) is nothing but  $1 - P(Y_{(m)} > a)$  where  $P(Y_{(m)} > a)$  is given by (28) for all  $a$  in  $(0, 1)$ . Hence as  $ma \rightarrow 1^+$  the expression in (25) converges to 0.  $\square$

**Notes:**

4. When  $b = ar$ , with  $ma < 1$ , we have seen that the term in (9) converges to  $(1 - ma)^{m-1}$ , while the other terms contributing to  $P(V = r - mb)$  converge to 0, indicating the dominant nature of this term missing in Holst's Theorem 2.2 (1985).
5. Holst's Theorem 3.2 gives expressions for  $P(V_a = 0)$  and the pdf of the continuous part. Lemma 1 reveals that his expressions are imprecise and fail to properly account for the range of  $V$ .
6. Siegel's (1978) version of (25) [his expression (3.23)] has the summation that includes an additional term with  $i = m$ . In view of the assumption that  $ma > 1$ , the corresponding term is 0, and hence they coincide. Further, in view of the above lemma, we can conclude that when  $ma = 1$  both (25) and (26) hold.

For  $m = 5$  and selected  $r$  values, Figure 1 and Figure 2 respectively provide the normalized conditional pmf of  $V$  given the event  $\{V > (r - mb)_+\}$ , for  $a = 0.1$  and 0.25. The case of  $r = \infty$  corresponds to the conditional pdf  $f(w; a, m)$  of the continuous case, given in (27). Both these figures suggest that by the time  $r$  reaches 500, we are close to the limiting result, indicating that when the sampling fraction is under 1%,  $f(w; a, m)$  provides a close approximation to the conditional pmf of  $V$ .

Holst (1985, Theorem 4.2) has another limit result for a normalized  $V$ . Assuming  $b$  is held fixed and  $r, m \rightarrow \infty$  such that  $m/r = p = 1 - q$  and  $p \rightarrow p_0 = 1 - q_0, 0 < p_0 < 1$ , he shows that

$$(V - rq^b)/\sqrt{m} \xrightarrow{d} N(0, \sigma_0^2), \tag{29}$$

where  $\sigma_0^2 = (1 + q_0 - q_0^b)q_0^b - (q_0 + bp_0)^2 q_0^{2b-1}$ .

## 4.2 Limiting Moments

Since  $V/r$  is uniformly bounded, the convergence in distribution implies that  $E(V/r)^k \rightarrow E(V_a^k)$  when  $r \rightarrow \infty$  and  $b/r \rightarrow a$  and  $ma \neq 1$ . Siegel has shown in his Theorem 2 that,

$$E(V_a^k) = \binom{k+m-1}{m}^{-1} \sum_{i=1}^k \binom{k}{i} \binom{m-1}{i-1} (1 - ia)_+^{m+k-1}, k \geq 1. \tag{30}$$

Hence we can obtain approximations to any moment of  $V$  when  $m$  is small and  $r$  and  $b$  are large using the moments of  $V_a$ .

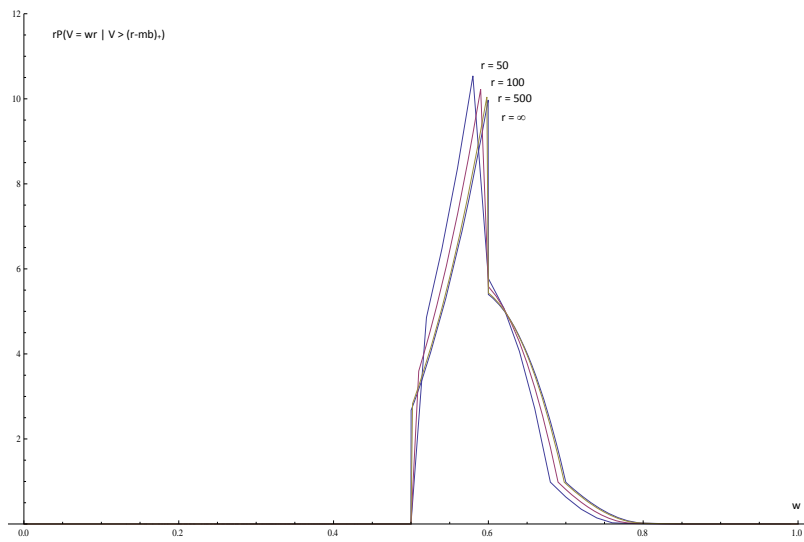


Figure 1:  $rP\{V = rw \mid V > (r - bm)_+\}$  for  $m = 5$ ,  $a = 0.1$ ,  $b = ar$ , for selected  $r$ ;  $f(w; a, m)$ , given in (27), corresponds to  $r = \infty$ .

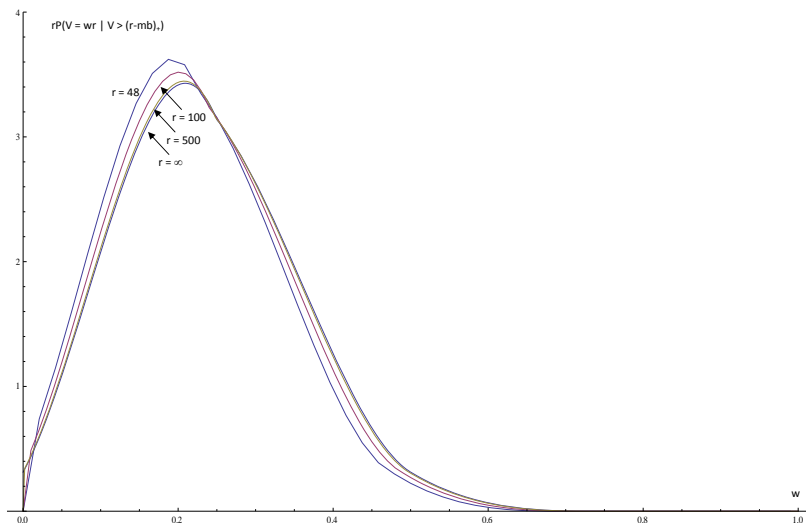


Figure 2:  $rP\{V = rw | V > (r - bm)_+\}$  for  $m = 5$ ,  $a = 0.25$ ,  $b = ar$ , for selected  $r$ ;  $f(w; a, m)$ , given in (27), corresponds to  $r = \infty$ .

Table 2 provides some key facts about the features of the distributions of  $V$  and the limiting rv  $V_a$  for  $m = 5$ ,  $r = 20, 50$  and  $b$  values up to 20 corresponding to a good range of  $a$  values. It shows that as  $a$  increases  $p_a(m)$  decreases for  $a \leq 1/m$ , it is 0 when  $ma = 1$  and then  $p_a(m)$  increases. Note that whenever  $a$  reaches  $1/m$  from below, the lower limit of the support of  $V_a$  moves towards 0 and whenever  $ma > 1$ , the lower limit remains at 0. This limiting pattern is closely followed by  $V$  when  $r = 50$ , but not that closely when  $r = 20$ . The moments converge fairly quickly to the limiting values. The mean is better approximated by the limit for small  $b (=ar)$  whereas for the standard deviation, large  $b$  values tend to be slightly more efficient.

**Remark 2:** When  $b$  is held fixed and  $r, m \rightarrow \infty$  such that  $m/r \rightarrow p_0 = 1 - q_0, 0 < p_0 < 1$ , convergence of moments is not evident from the limiting normal distribution. By using Stirling's approximation in the expression for  $E(V)$  computed earlier in Section 3, we can show that  $E(V - rq^b)/\sqrt{m} \rightarrow 0$  only when

$$\frac{m}{r} = p_0 + o\left(\frac{1}{\sqrt{m}}\right).$$

Table 2: **Properties of  $V$  and  $V_a$  for  $m = 5$ ,  $r = 20, 50$ , and selected  $b$  values;  $a = b/r$ .**

$b$	$P(V = (r - mb)_+)$	$p_a(m)$	$E(V)$	$rE(V_a)$	$SD(V)$	$rSD(V_a)$
$r = 20$						
1	1	0.3164	15	15.48	0	0.49
2	0.008	0.0625	11.05	11.14	0.80	1.14
3	0.405	0.0040	7.98	8.87	1.42	1.71
4	0.802	0	5.63	6.55	1.84	2.11
5	0.960	0.0040	3.87	4.75	2.04	2.31
$r = 50$						
1	1	0.6561	45	45.196	0	0.341
2	0.6407	0.4096	40.408	40.769	0.586	0.899
3	0.3882	0.2401	36.199	36.695	1.213	1.536
4	0.2189	0.1296	32.348	32.954	1.875	2.196
5	0.1121	0.0625	28.832	29.525	2.532	2.842
6	0.0502	0.0256	25.628	26.387	3.156	3.452
7	0.0183	0.0081	22.716	23.521	3.726	4.009
8	0.0047	0.0016	20.075	20.911	4.229	4.503
9	0.0006	0.0001	17.685	18.537	4.658	4.925
10	0.0000	0	15.528	16.384	5.007	5.270
11	0.0006	0.0001	13.587	14.436	5.273	5.538
12	0.0047	0.0016	11.845	12.678	5.457	5.728
13	0.0179	0.0081	10.287	11.095	5.560	5.842
14	0.0452	0.0246	8.897	9.675	5.586	5.881
15	0.0885	0.0545	7.661	8.404	5.541	5.852
16	0.1467	0.0989	6.566	7.27	5.430	5.759
17	0.2158	0.1561	5.601	6.262	5.261	5.609
18	0.2912	0.2226	4.752	5.369	5.041	5.408
19	0.3689	0.2944	4.010	4.581	4.780	5.164
20	0.4455	0.3680	3.363	3.888	4.486	4.886

## 5 Application to Banking Networks

Alvarez and Barlevy (2013), building on earlier work by Caballero and Simsek (2010), analyze a model of contagion among interconnected banks that encompasses the discrete circle covering problem analyzed here. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013) present a closely related model of financial contagion where the circle covering problem is a special case.

The Alvarez and Barlevy model assumes each bank owes a fixed amount  $\lambda$  to exactly one other bank, and that it is in turn owed  $\lambda$  by exactly one other bank. Hence, each bank will be connected to a set of other banks through a cyclical network in which banks are linked if they have a debt obligation between them. Let  $r$  denote the number of banks in a given cycle, and index banks in that cycle by  $i \in \{0, \dots, r - 1\}$  so that bank  $i$  owes  $\lambda$  to bank  $i + 1$  modulo



$r$ . We drop the reference to modulo  $r$  in what follows. Their model makes the following assumptions: (1) Each bank owns  $\mu$  worth of assets that it can sell to repay its outstanding obligations if it needs to, (2)  $m$  banks among the  $r$  banks, drawn uniformly from among the  $\binom{r}{m}$  possible selections, are “bad”, meaning they incur a loss of constant size  $\phi$ , and (3)  $\mu < \phi < \frac{r}{m}\mu$ . The last assumption implies bad banks incur losses that exceed what they can afford to pay by liquidating their assets, but total losses across all  $m$  bad banks are still less than the combined value of all assets held among all  $r$  banks in the network. Banks are required to pay their full obligation  $\lambda$  if possible, and must sell their asset holdings if they fall short.

Let  $x_i$  denote the amount bank  $i$  pays bank  $i + 1$  and assume that the bad banks are labeled  $R_i$ ,  $i = 0, \dots, m - 1$ , and these are the ones who have incurred a direct external loss of  $\phi$  and the others have no external losses. Given that banks must pay their obligations if they can, the payments  $\{x_i\}_{i=0}^{r-1}$  satisfy the following system of equations

$$\begin{aligned} x_i &= \min\{(x_{i-1} + \mu - \phi)_+, \lambda\}, & i \in \{R_0, \dots, R_{m-1}\} \\ &= \min\{x_{i-1} + \mu, \lambda\}, & i \notin \{R_0, \dots, R_{m-1}\}, \end{aligned}$$

with  $x_{-1} \equiv x_{r-1}$ . For  $\phi < \frac{r}{m}\mu$ , there exists a unique solution  $\{x_i\}_{i=0}^{r-1}$  to this system. Bank  $i$  is said to be insolvent if  $x_i < \lambda$ , i.e. if it cannot meet its obligation, and solvent otherwise. Each of the  $m$  bad banks are insolvent, since even if they received the full amount  $\lambda$  from the bank that is obligated to them, the fact that  $\mu < \phi$  implies  $x_{i-1} + \mu - \phi < \lambda$  and so they would be unable to pay in full even after liquidating their assets. Beyond these  $m$  bad banks, banks that do not directly suffer losses may still end up insolvent because they are exposed to bad banks either directly – meaning the bank that owes them  $\lambda$  is bad – or indirectly – meaning the bank that owes them  $\lambda$  is good but is exposed to a bad bank. One question of interest is to determine the number of banks that are insolvent, i.e. to gauge the extent of contagion.

The parameter  $\lambda$  plays an important role in determining the distribution of the number of insolvent banks. Now suppose  $\lambda \leq \phi - \mu$ , and  $\lambda = b\mu$ , for some integer  $b$ . Then  $(b + 1)\mu \leq \phi$ , and  $x_{R_k} = 0$ , and  $x_{R_k+j} = j\mu$ ,  $j = 1, \dots, \min\{b - 1, S_{k+1} - 1\}$ , for  $k = 0, \dots, m - 1$ . Hence, the number of insolvent banks starting from each bad bank is a fixed number  $b$  unless there is another bad bank that is closer. In this case, the number of solvent banks in the network corresponds to the number of vacant slots  $V$  we analyze in this paper with  $b$  equal to  $\frac{\lambda}{\mu}$ . Thus our results provide the small as well as large sample properties of the number of solvent banks in the Alvarez and Barlevy model with  $\lambda \leq \phi - \mu$ .

The situation where  $\lambda > \phi - \mu$  provides a new generalization of the discrete circle covering problem. Then  $b_k$ , the number of insolvent banks induced by the  $k$ th bad bank, becomes a rv that depends on the location of the other bad banks. However, unlike in Siegel and Holst (1982) who discuss the continuous case of the circle covering problem assuming the length of the arc  $b$  starting at any point is an i.i.d. rv, here the number of insolvent banks starting at each  $R_k$  will depend on the distribution of the spacings between the bad banks. To

elaborate,  $x_{R_k} = (x_{R_{k-1}} - (\phi - \mu))_+$  is no longer identically 0, and this will affect the number of associated insolvent banks. That is, the number of successive boxes that must be filled starting at the  $k$ th bad bank is a rv that depends on the location of that bad bank and the associated  $b_k$  here are all dependent on the entire collection of spacings  $\{S_k\}_{k=1}^m$ . Even then, for  $\lambda > m(\phi - \mu)$  Alvarez and Barlevy (2013) have shown that the number of solvent banks  $V$  (vacant slots) is degenerate and equals  $r - mb$  where  $b = \lambda/\mu$ . In the intermediate case where  $\phi - \mu < \lambda < m(\phi - \mu)$  the distribution of  $V$  is non-degenerate and unknown. We leave the investigation of its distribution for future work.

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