

Milchtaich, Igal

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Comparative Statics of Altruism and Spite

Igal Milchtaich

Department of Economics, Bar-Ilan University, Ramat Gan 52900, Israel
Email: milchti@mail.biu.ac.il Home Page: <http://faculty.biu.ac.il/~milchti>

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The equilibrium outcome of a strategic interaction between two or more people may depend on the weight they place on each other's payoff. A positive, negative or zero weight represents altruism, spite or complete selfishness, respectively. Paradoxically, the real, material payoff in equilibrium for a group of altruists may be lower than for selfish or spiteful groups. However, this can only be so if the equilibria involved are unstable. If they are stable, the total (equivalently, average) payoff can only increase or remain unchanged with an increasing degree of altruism. *JEL Classification*: C62, C72, D64.

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1 Introduction

Altruism and spite represent deviations in opposite directions from complete selfishness, or total indifference to the welfare of others. A person is altruistic or spiteful towards another if he is willing to take costly actions in order to benefit or harm the other person, respectively. This paper considers the question of the welfare consequences of such preferences. Is the average payoff in a group in which everyone is equally altruistic or spiteful towards the others higher or lower than in a group in which everyone is only concerned with his own good? Unlike much of the related literature (e.g., Frank, 1988; Ridley, 1997) this question only involves the consequences of deviations from complete selfishness, not their origin or evolution. Correspondingly, it only concerns the effect on (material) social welfare, rather than on the individual payoffs. (However, this distinction only applies to asymmetric settings.) The common degree of altruism or spite is viewed as an exogenous parameter, representing, for example, a shared moral value or social attribute. The parameter quantifies the extent to which each individual i internalizes the welfare of each of the other individuals j . Specifically, it is the ratio between the weights attached to j 's and i 's payoffs in i 's "modified payoff", which is linear in the payoffs. An individual's modified payoff determines his preferences over action profiles, and in particular his best responses to the others' actions.

This simple, linear form of interdependent preferences is clearly not the only conceivable one. Other functional forms have been suggested (e.g., Fehr and Schmidt, 1999; Bolton and Ockenfels, 2000), which might better predict people's behavior in certain experimental settings. In the "psychological games" literature (Geanakoplos et al., 1989; Rabin, 1993), preferences for physical outcomes are affected by a person's beliefs about the other's actions and discrepancies between these beliefs and the actual actions. An advantage of the simpler interdependent preferences considered in this paper is that the assumption that

they are determined by a single parameter facilitates comparative statics analysis. It is possible to investigate the conditions under which an increasing or decreasing degree of altruism results in lower or higher social welfare, and the conditions under which such changes of preferences have the opposite effects. This investigation is the main goal of this paper.

The propensity for acting altruistically or spitefully may have a biological basis. In particular, with family members it may be the result of kin selection. According to Hamilton's rule (Hamilton, 1963, 1964; Frank, 1998), natural selection favors acts that maximize the actor's inclusive fitness, which is his own fitness augmented by r times that of each of the other affected individuals, where r is their coefficient of relatedness. This stems from the fact that helping a relative assists the propagation of the actor's own genes; as the coefficient of relatedness increases, so does the number of shared genes. In small populations, the coefficient of relatedness may also take on negative values, which represent less-than-average relatedness. In this case, the possibility of spiteful behavior arises (Hamilton, 1970). The inclusive fitness has the form described above. The fitness of each of the individuals involved in the interaction enters linearly, and the weight attached to it, which is the relevant coefficient of relatedness r , is exogenous, that is, determined by the family tree. Thus, comparative statics analysis might reveal, for example, how the expected consequences of a particular interaction involving two or more individuals depends on their relatedness, e.g., whether they are full or half-siblings.

As it turns out, the question of whether altruism has a positive effect on social welfare has a simple, affirmative answer only in the case of nonstrategic interactions, in which each individual's optimal action does not depend on the others' actions. In strategic interactions, or games, altruism and spite do not necessarily have the effects one would expect. For example, even in a symmetric two-player game with a unique, symmetric equilibrium, the players' equilibrium payoff may be higher if they are both selfish rather than mildly caring, and even higher if they resent each other. Thus, altruism in a strategic interaction may paradoxically result in real, material losses for all parties. A central finding in this paper is that a crucial factor affecting the nature of comparative statics is the stability or instability of the equilibria involved. In particular, continuously increasing the weight that players place on each other's payoff can only increase social welfare or leave it unchanged if the strategies involved are stable, but has the opposite effect if the strategies are definitely unstable. (This term is defined below.) This finding is akin to Samuelson's (1983) "correspondence principle", which maintains that conditions for stability often coincide with those under which comparative statics analysis leads to what are usually regarded as "normal" conclusions, such as the conclusion that an increase in demand for a commodity results in a rise in its equilibrium price (Lindbeck, 1992). Since comparative statics analysis compares equilibria in different games, whereas stability is a property of the equilibrium in a particular game, the finding that the latter conveys information about the former is not at all obvious.¹

¹ Note that this refers to the stability of the equilibrium, and not that of altruism itself, e.g., in the sense of Bester and Güth (1998).

In the works of Samuelson and others, ‘stability’ refers to dynamic, or asymptotic, stability. It therefore depends on the dynamical system used to model the evolution of the economic agents’ off-equilibrium behavior. By contrast, in this paper ‘stability’ means static stability. This arguably more fundamental concept only considers the players’ off-equilibrium *incentives*, and does not involve any assumptions about the translation of these incentives into concrete changes of actions. One example of a static notion of stability, applicable to symmetric $n \times n$ games, is evolutionarily stable strategy, or ESS. Another one, applicable to symmetric two-player games with a unidimensional set of strategies, is continuously stable strategy, or CSS, which is essentially equivalent to the requirement that, at the equilibrium point, the graph of the best-response function, or reaction curve, intersects the forty-five degree line from above. If the intersection is from below, the symmetric equilibrium strategy is definitely unstable. These two examples of static stability are in fact essentially special cases of a much more general notion of local static stability, proposed in Milchtaich (2008), which is applicable to all symmetric and asymmetric two- and multiplayer games with non-discrete strategy spaces. As it turns out, the effects of altruism and spite on social welfare are related to this general notion of static stability, rather than to any special dynamic one.

The layout of the paper is as follows. The next section defines the modified game, which is the tool used in this paper to model altruistic or spiteful preferences. A new parameter, the selfishness coefficient, is introduced, which is an alternative measure of the degree of altruism or spite. Section 3 presents the distinction between local comparative statics, which concern small, continuous changes to the selfishness coefficient and the corresponding equilibria, and global comparative statics, which allow for large, discrete changes. An example of the former is given, which shows that altruism can either increase or decrease the firms’ profits in a symmetric Cournot duopoly game, depending on the stability or instability of the corresponding equilibrium strategies. A general definition of static stability of a symmetric equilibrium strategy in a symmetric two-player game is presented in the first part of Section 4. The subsequent subsections specialize the definition to the modified game, present a general theorem that links the stability of the equilibrium strategy in the modified game with comparative statics, and give another theorem that concerns global comparative statics. Section 5 lays out a comparable analysis for asymmetric two-player games. Section 6 generalizes the definitions and results in the preceding two sections to games with more than two players. As it shows, the move to an arbitrary number of players does not affect the main results, provided that the right notion of stability is used. Some of the assumptions underlying the present model, possible extensions, and the relation between the static stability notion used here and dynamic stability are discussed in Section 7, which concludes the main text of the paper. Appendix A lists some algebraic identities involving payoffs, modified payoffs and average payoffs, which are used elsewhere in the paper. The question of whether the relation between the selfishness coefficient and the equilibrium payoff in a symmetric $n \times n$ game is monotonic (either nonincreasing or nondecreasing) is studied in Appendix B. For the same kind of games, Appendix C presents a useful connection between static stability and stability with respect to perturbations of the game parameters, which has an implication for comparative statics.

2 Altruism and Spite

The model described below is based on a rather standard model of linearly interdependent preferences. However, it is less general than, e.g., Levine's (1998) model, in which different players can be more or less altruistic and their attitudes are reflected in the way other players treat them. Here the players' preferences are all interdependent in the same manner. Specifically, the dependencies are expressible by a single parameter r , called the *coefficient of altruism* (Levine, 1998) or coefficient of effective sympathy (Edgeworth, 1881). The parameter specifies the weight each player i attaches to the payoff h^j of each of the other players j , relative to the weight attached to the player's own payoff h^i . Thus, the quantity player i seeks to maximize is (possibly, a positive multiple of)

$$h^i + r \sum_{j \neq i} h^j. \quad (1)$$

The coefficient of altruism r may be positive, expressing altruism or concern for social welfare, or negative, expressing envy or spite (Morgan et al., 2003). In this paper, it is viewed as exogenously given, e.g., a shared moral standard. Varying the coefficient thus corresponds to a cross-group or cross-society comparison; different values of r represent different groups rather than different kinds of individuals within a group.² An example of such a group is a (human or non-human) family. In the biological theory of kin selection, an expression similar to (1) gives the *inclusive fitness* of an individual interacting with relatives. In this case, the coefficient of altruism r is the *coefficient of relatedness* between i and j , which, for example, is 0.5 for full siblings and 0.25 for half-siblings (Crow and Kimura, 1970). Thus, (1) expresses the inclusive fitness when the interaction involves only equality related individuals, e.g., offspring of a single pair of parents.

The analysis in this paper excludes cases of extreme selflessness or spite. An individual does not care about another individual more than he cares about himself, and he does not hate the others so much that he would enjoy or be indifferent to an equal loss by all players. Mathematically, this means that

$$-\frac{1}{n-1} < r \leq 1, \quad (2)$$

where n is the number of players. With this assumption, and the parameter s defined by³

$$s = \frac{1-r}{1+(n-1)r},$$

expression (1) can be written as $(1+(n-1)r)(sh^i + (1-s)\bar{h})$, where

² The evolution and origin of sympathy and spite are outside the scope of this paper. The model is not an evolutionary one, and it is not suitable for studying the effects that individuals' attitudes towards others have on their own success. The model and corresponding comparative statics analysis may, however, have relevance for *group selection*. See the discussion in Section 7.

³ The definition implies that s is determined by r as a strictly decreasing function in the interval given by (2). The inverse function has an identical form, i.e., $r = (1-s)/(1+(n-1)s)$.

$$\bar{h} = \frac{1}{n} \sum_{j=1}^n h^j$$

is the average payoff. Thus, (1) is a positive multiple of player i 's *modified payoff* h_s^i , defined by

$$h_s^i = sh^i + (1 - s)\bar{h}. \quad (3)$$

The games (with payoff functions) $h = (h^1, h^2, \dots, h^n)$ and $h_s = (h_s^1, h_s^2, \dots, h_s^n)$ (which have identical strategy spaces) will be referred to as the *unmodified* and *modified games*, respectively. The parameter s , which by (2) satisfies $s \geq 0$, will be called the *selfishness coefficient*.⁴ If $s = 0$, the players are completely unselfish, and their only concern is social welfare, i.e., maximization of \bar{h} . If $s = 1$, the players are completely selfish, and only care about their own payoffs. The case $s > 1$ can be interpreted as representing spite, since it corresponds to a negative coefficient of altruism r . Alternatively, a high selfishness coefficient may be interpreted as representing *envy*. An envious person attaches a great weight to the difference

$$e^i = h^i - \bar{h}$$

between his own and the average payoffs. The latter interpretation is based on the decomposition

$$h_s^i = \bar{h} + se^i, \quad (4)$$

which shows that the weight of e^i in the modified payoff h_s^i is precisely the selfishness coefficient s .

Although the expressions (1) and (3) represent the same interdependent preferences, there are some advantages to using the latter rather than the former. The modified payoff representation decomposes the perceived payoff into a “private” component and a “social” one. The main concern of this paper is with the effect of the relative weights of these components on social welfare. If the weight the players attach to the social component increases or decreases at the expense of the private component (i.e., the selfishness coefficient decreases or increases, respectively), what is the effect on the actual social welfare? Note that a modification of the game does not by itself change the average payoff:

$$\bar{h}_s = \bar{h} \quad (5)$$

for all s . Thus, the average payoff may shift only if there is a real change in behavior. Such a change may be expected to occur, since when preferences change, the equilibrium strategies normally change too. The issue then is comparative statics: the effect of s on the average equilibrium payoff.

⁴ Note that $h_1 = h$, and that iterating the modification procedure would not give a new kind of game but only a modified game with a different selfishness coefficient, since $(h_s)_t = h_{s,t}$ for all s and t . This means that the selfishness coefficient acts on games as a monoid, i.e., a semigroup with an identity element.

3 Comparative Statics

The effect of selfishness on social welfare (i.e., on the players' average payoff) can be studied either globally or locally. *Global comparative statics* compare social welfare at the equilibria in the unmodified and modified games, h and h_s , or more generally, the equilibria in h_t and h_s , with $t \neq s$. The comparison is global in that it is not restricted to equilibria in h_s that are close to particular equilibria in h_t or to small changes in the selfishness coefficient (i.e., s close to t). The unrestricted nature of global comparative statics means that meaningful results can be obtained only in special cases. This paper considers such comparative statics mainly in the context of symmetric two-player games (Section 4). The paper's focus is on *local comparative statics*, which concern the way social welfare at a given equilibrium in a given modified game h_t (or, in the special case $t = 1$, the unmodified game h) changes when the selfishness coefficient continuously increases or decreases from t . For this to be meaningful, a continuous mapping has to exist that assigns to every s close to t an equilibrium in h_s , which coincides with the given equilibrium for $s = t$. This effectively rules out games with discrete strategy spaces, since if strategies are isolated, such a continuous mapping is necessarily constant. Assuming that the strategy spaces are non-discrete and that a mapping as above exists,⁵ the question is whether the players' average payoff increases or decreases with increasing (or decreasing) s . As the following example shows, even in a single game both possibilities may occur.

Example 1. *Symmetric Cournot competition.* Firms 1 and 2 produce an identical good at zero cost. They simultaneously decide on their respective output levels q^1 and q^2 and face a downward sloping, convex demand curve given by the price (or inverse demand) function

$$P(q) = [(q + 0.4) \ln(q + 1.4)]^{-\frac{3}{2}},$$

where $q = q^1 + q^2$ is the total output. The profit of each firm i is the revenue $P(q)q^i$. Hence, the firm's modified payoff is given by

$$h_s^i(q^1, q^2) = P(q) \left(s q^i + \frac{1-s}{2} q \right).$$

If the selfishness coefficient s decreases from 1 all the way to 0, the duopoly effectively becomes a monopoly and the firms' profits increase. However, this is not necessarily so for a moderate decrease in s . For every output level q^1 of firm 1, there is a unique, nonzero output level q^2 for firm 2 at which this firm's profit is maximal. The same is also true for the modified payoff, for every s greater than about 1/3 (see Figure 1a). For s close to 1, there are precisely two equilibria, which are both symmetric, i.e., the firms' output levels, and hence also their profits, are equal. In one equilibrium the output level is below 2 and in the other it is above 2. The first equilibrium output level continuously increases with increasing selfishness coefficient, and consequently the profits decrease. However, at the second equilibrium, output increases and profits decrease with *decreasing* s (Figure 1b).

⁵ For a sufficient condition for existence in the special case of symmetric $n \times n$ games, see Section 4.2.

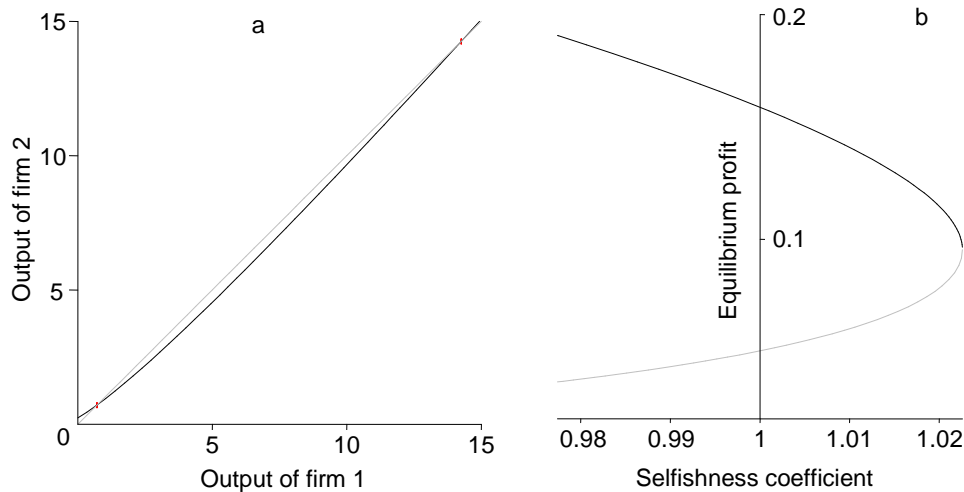


Figure 1. The Cournot duopoly game in Example 1. a. The reaction curve. For every output level for firm 1, the unique output level for firm 2 that maximizes this firm’s modified payoff is shown for the selfishness coefficient $s = 0.9$ (black curve). The two points (marked by short vertical lines) at which the curve intersects the forty-five degree line (grey) are the (symmetric) equilibria. The lower (left) and upper (right) points, respectively, represent stable and unstable equilibrium output levels. When the selfishness coefficient s increases (or decreases), the two points move in opposite directions. b. The equilibrium profits. The firms’ equilibrium profit depends on the selfishness coefficient s . It also depends on whether the equilibrium output level is stable or unstable. The profit for the former (black, upper curve) is higher than for the latter (grey, lower curve), and it decreases rather than increases with increasing s .

The modified payoffs of two competing firms may conceivably represent real profits for owners, e.g., if they own stock in both companies. However, in Example 1 and elsewhere in this paper, the modified payoffs are not *assumed* to be real entities. The question asked is how the firms’ profits would change if each of them were also concerned with the total profit, or equivalently, with the profit of the other firm. The comparative statics seen in this particular example can be understood by examining the geometry of firm 2’s reaction curve, or the graph of its best-response function. Increasing the selfishness coefficient s raises the curve, since it attenuates the negative effect that an increase in firm 2’s output level (which decreases firm 1’s profit) has on 2’s modified payoff. At the lower-output equilibrium, where the upward-sloping reaction curve is less steep than the forty-five degree line (see Figure 1a), raising the curve moves the equilibrium point upwards and to the right. Thus, the equilibrium output increases. At the higher-output equilibrium, where the reaction curve is steeper than the forty-five degree line, a shift upwards has the opposite effect on the equilibrium output level.

Significantly, the same geometrical property of the reaction curve also determines whether the equilibrium output level is stable. As shown below, the equilibrium output is stable if the reaction curve is less steep than the forty-five degree line, and unstable (even definitely unstable, in a sense defined below) if it is steeper than that line. As this paper shows, this connection between comparative statics and stability is in fact a general phenomenon – much more so than the above geometrical demonstration might suggest. It does not depend on any special properties of the above example (e.g., zero production cost), properties of the Cournot duopoly game, or even the unidimensionality of the strategy space. As shown below, it is rather generally true that stable or definitely unstable equilibrium strategies imply “decreasing” or “increasing” local comparative statics, respectively, in the sense of the

direction on which the players' average equilibrium payoff moves as the selfishness coefficient continuously increases.

This general connection between stability and local comparative statics is laid out in the following sections. This is done in several steps, starting with symmetric two-player games and ending with asymmetric multiplayer ones. The notion of stability used here is that introduced in Milchtaich (2008); see that paper of motivation and detailed exposition. This notion of (local) static stability, which is applicable to symmetric as well as asymmetric games with two or more players, essentially generalizes several well-established stability notions that are specific to particular classes of games, such as evolutionarily stability (ESS) and the geometrical criterion for stability used in the above analysis.

4 Symmetric Two-Player Games

A symmetric two-player game is a function⁶ $g: X \times X \rightarrow \mathfrak{R}$, where X , the strategy space, is a finite or infinite topological space and \mathfrak{R} is the real line. If one player uses strategy x and the other uses y , their payoffs are $g(x, y)$ and $g(y, x)$, respectively. The topology on X , which defines a neighborhood system for each strategy x (Kelly, 1955), should in principle be part of the specification of the game. However, in many cases it is unnecessary to specify it explicitly, as this is clear from the context. For example, in (the mixed extension of) a symmetric $n \times n$ game (where n is the number of pure strategies), the strategy space is the unit simplex in \mathfrak{R}^n , with the relative topology. If strategies are numbers, X is by default considered a subspace of \mathfrak{R} , so that a set of strategies is a neighborhood of a strategy x if and only if, for some $\varepsilon > 0$, every $y \in X$ with $|x - y| < \varepsilon$ is in the set.

Definition 1. (Milchtaich, 2008) A strategy x in a symmetric two-player game g is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood where the inequality

$$g(y, x) - g(x, x) + g(y, y) - g(x, y) < 0, \quad (6)$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds for all strategies $y \neq x$.

A stable strategy is not necessarily an equilibrium strategy. An equilibrium strategy x has to satisfy the condition that, for all strategies y ,

$$g(y, x) \leq g(x, x), \quad (7)$$

i.e., (x, x) is a symmetric equilibrium.⁷ An equilibrium strategy x that is also stable is a *stable equilibrium strategy*. It is not difficult to see that, for an equilibrium strategy x , a necessary and sufficient condition for stability is that, for every strategy $y \neq x$ in some neighborhood of x ,

⁶ This formulation identifies a symmetric game with the corresponding payoff function.

⁷ In this paper, an 'equilibrium strategy' in the context of symmetric games always refers to a symmetric equilibrium. For a discussion of asymmetric equilibria, see section 5.

$$(1 - p)g(x, x) + pg(x, y) > (1 - p)g(y, x) + pg(y, y) \quad (8)$$

for all $0 < p \leq 1/2$. This inequality says that x affords a higher expected payoff than y against a rival who randomizes between using y and x with probabilities p and $1 - p$, respectively.

In symmetric two-player games in which the strategy space is a subset of \mathfrak{R} , i.e., strategies are real numbers, the stability of an equilibrium strategy has a simple, intuitive interpretation. If g is twice continuously differentiable, and with the possible exception of certain borderline cases, an equilibrium strategy is stable or definitely unstable if, at the (symmetric) equilibrium point, the reaction curve intersects the forty-five degree line from above or from below, respectively (see Figure 1a). Stability is also very close to the notion of *continuously stable strategy*, or CSS (Eshel and Motro, 1981; Eshel, 1983).

Proposition 1. (Milchtaich, 2008) Let g be a symmetric two-player game with a strategy space X that is a subset of the real line, and x an equilibrium strategy lying in the interior of X such that g has continuous second-order partial derivatives⁸ in a neighborhood of the equilibrium point (x, x) . If

$$g_{11}(x, x) + g_{12}(x, x) < 0, \quad (9)$$

then x is stable and a CSS. If the reverse inequality holds, then x is definitely unstable and not a CSS.

If (x, x) is an interior equilibrium as in the proposition, then (the second-order maximization condition) $g_{11}(x, x) \leq 0$ holds, since strategy x is a best response to itself. If the inequality is strict, then (9) can be written as

$$-\frac{g_{12}(x, x)}{g_{11}(x, x)} < 1.$$

This inequality or the reverse one, respectively, says that at the equilibrium point the slope of the reaction curve is less than or greater than 1, which is the slope of the forty-five degree line.

Another class of games for which the general Definition 1 can be given a more familiar form is the class of (the mixed extensions of) symmetric $n \times n$ games. In such games, both players share a common finite set of n actions. A (mixed) strategy $x = (x_1, x_2, \dots, x_n)$ specifies the probability x_i with which a player chooses the i th action, for $i = 1, 2, \dots, n$. The set of all actions i with $x_i > 0$ is the *support* (or carrier) of x , and a strategy is *pure* or *completely mixed*, respectively, if its support contains only a single action i (in which case the strategy itself may also be denoted by i) or all n actions. A symmetric $n \times n$ game g is bilinear, and is hence completely specified by the $n \times n$ *payoff matrix* $A = (g(i, j))_{i, j=1}^n$. Viewing strategies as column (probability) vectors, the game can be presented also in matrix notation:

⁸ The partial derivatives of (the payoff function) g are denoted by subscripts. For example, g_{12} is the mixed partial derivative.

$$g(x, y) = x^T Ay,$$

where x^T is x transpose.

A standard notion of stability for symmetric $n \times n$ games g is evolutionary stability, which can be defined as follows. A strategy y can *invade* another strategy x in g if either (i) $g(y, x) > g(x, x)$ or (ii) $g(y, x) = g(x, x)$ and $g(y, y) > g(x, y)$. It can *weakly invade* x in g if a similar condition holds with the strict inequality in (ii) replaced by a weak one. Strategy x is an *evolutionarily stable strategy* (ESS; Maynard Smith, 1982) if there is no strategy $y \neq x$ that can weakly invade it in g , and it is a *neutrally stable strategy* (NSS) if there is no strategy that can invade it in g . Clearly, every ESS is an NSS and every NSS is an equilibrium strategy. A completely mixed equilibrium strategy x is *definitely evolutionarily unstable* (Weissing, 1991) if every strategy $y \neq x$ can invade it in g . As the following proposition shows, these notions of evolutionary stability and instability are equivalent to the corresponding ones in Definition 1.

Proposition 2. (Milchtaich, 2008). A strategy in a symmetric $n \times n$ game g is an ESS or an NSS if and only if it is stable or weakly stable, respectively. A completely mixed equilibrium strategy x in g is definitely evolutionarily unstable if and only if it is definitely unstable.

A symmetric $n \times n$ game \bar{g} is *doubly symmetric* if $\bar{g}(x, y) = \bar{g}(y, x)$ for all x and y , or equivalently, if the payoff matrix is symmetric. The following result, due to Hofbauer and Sigmund (1988; see also Weibull, 1995, pp. 56–57), follows from Proposition 2 as an immediate corollary. The proposition can therefore be viewed as a generalization of these authors' result from doubly symmetric to general symmetric $n \times n$ games.

Corollary 1. A strategy x in a doubly symmetric $n \times n$ game \bar{g} is an ESS or an NSS if and only if it has a neighborhood where the inequality

$$\bar{g}(y, y) < \bar{g}(x, x) \tag{10}$$

or a similar weak inequality, respectively, holds for all strategies $y \neq x$.

4.1 Altruism and spite in symmetric two-player games

Corollary 1 may be viewed as a crude comparative statics result. Any symmetric $n \times n$ game g becomes a doubly symmetric game \bar{g} when played by completely unselfish players ($s = 0$), who attach equal weight to their own and the other player's payoffs:

$$\bar{g}(x, y) = \frac{1}{2}g(x, y) + \frac{1}{2}g(y, x). \tag{11}$$

By the corollary, for any equilibrium strategy x in \bar{g} that is *not* weakly stable there are strategies y arbitrarily close to x for which the players' payoffs are greater than for x , i.e., the reverse inequality to that in (10) holds. Like Example 1, this suggests a connection between stability and the welfare effects of altruism.

For a general selfishness coefficient $s (\geq 0)$, the modified game obtained from a symmetric $n \times n$ game g is symmetric, but generally not doubly symmetric. Adapting the general notation in Section 2 to symmetric two-player games, the modified game g_s is given by

$$g_s = sg + (1 - s)\bar{g}, \quad (12)$$

where \bar{g} is the average payoff (11). Note that if both players use the same strategy x , then they received the same modified payoff, which coincides with the unmodified one since

$$g_s(x, x) = g(x, x). \quad (13)$$

By Proposition 2, a strategy x_s in the modified game g_s is stable if and only if it is an ESS in that game. The following proposition presents the stability condition directly in terms of the unmodified payoffs g . Note that this proposition only considers the case of altruistic players.

Proposition 3. (Milchtaich, 2006a) For a symmetric $n \times n$ game g and a selfishness coefficient $0 \leq s < 1$, a strategy x_s is an ESS or an NSS in the modified game g_s if and only if the inequality

$$g(y, (1 - r)x_s + ry) < g(x_s, x_s) \quad (14)$$

or a similar weak inequality, respectively, holds for all strategies $y \neq x_s$ in some neighborhood of x_s , where $r = (1 - s)/(1 + s)$ is the coefficient of altruism.

Proposition 3 shows that a stable strategy x_s in the modified game g_s is characterized by the property that it affords a higher expected (unmodified) payoff than any other strategy close to it for a player whose opponent either mimics him, and uses whatever (mixed) strategy y he uses, or uses strategy x_s , the former with probability $r (= (1 - s)/(1 + s))$ and the latter with probability $1 - r$. This characterization is somewhat similar to Myerson et al.'s (1991) notion of δ -viscous equilibrium. The main difference is that the latter only takes into consideration alternative pure strategies. In particular, if x_s itself is pure, and (14) holds for every pure strategy $y \neq x_s$, then x_s is a δ -viscous equilibrium for $\delta = r$. However, such a strategy x_s is not necessarily even an equilibrium strategy in g_s .

In the special case of complete unselfishness, $s = 0$ (hence, $r = 1$), Proposition 3 gives Corollary 1. That corollary implies that each of the one or more strategies x satisfying

$$g(x, x) = \max_y g(y, y) \quad (15)$$

is an NSS, and hence an equilibrium strategy, in the modified game $g_0 (= \bar{g})$. (Note that 'strategies' here means mixed strategies. A similar assertion does not hold for pure strategies.) This proves the following. (For a somewhat similar result, see Bernheim and Stark, 1988.)

Corollary 2. For every symmetric $n \times n$ game g there is some strategy x that is an equilibrium strategy in g_0 and yields an equilibrium payoff that is as high or higher than that of any equilibrium strategy in any modified game g_s , $s \geq 0$.

Importantly, Corollary 2 does not say anything about the effect of increasing selfishness on the other equilibrium strategies in g_0 , if such strategies exist, or on the equilibria of the other modified games g_s , with $s > 0$. The result is also specific to $n \times n$ games. A much more general and comprehensive analysis is presented in the next subsection.

4.2 Local comparative statics in symmetric two-player games

The connection between local comparative statics and the stability or instability of the strategies involved is not restricted to any particular class of symmetric two-player games. The basic general result is the following theorem, which is proved (for an arbitrary number of players) in Section 6.

Theorem 1. For a symmetric two-player game g , and s_0 and s_1 with $0 \leq s_0 < s_1$, suppose that g is Borel measurable⁹ and there is a continuous and locally one-to-one¹⁰ function that assigns to each $s_0 \leq s \leq s_1$ a stable strategy x_s in the modified game g_s , such that the function $f: [s_0, s_1] \rightarrow \mathfrak{R}$ defined by

$$f(s) = g(x_s, x_s) \tag{16}$$

is absolutely continuous.¹¹ Then f is strictly decreasing. A similar result holds with ‘stable’ and ‘strictly increasing’ replaced by ‘weakly stable’ and ‘nondecreasing’, respectively, or by ‘definitely unstable’ and ‘strictly increasing’, respectively.

If each of the strategies x_s in Theorem 1 is an equilibrium strategy in the corresponding modified game g_s (which the theorem does not require), then $f(s)$, defined in (16), is the (both unmodified and modified; see (13)) equilibrium payoff of both players. An increase in $f(s)$ hence spells a *Pareto improvement*. Thus, Theorem 1 gives that, under the stated assumptions, stability of the equilibria guarantees that both players will benefit from gradually becoming less selfish, if the change in preferences is simultaneous and to the same degree for both players. Conversely, if the equilibria are definitely unstable, a similar change of preferences will have the opposite effect on the players’ equilibrium payoff.

Note that changing the selfishness coefficient may also leave the equilibrium payoffs unchanged. For example, in any doubly symmetric $n \times n$ game (see above), the modified game coincides with the unmodified one, regardless of s , so that changing the latter has no effect whatsoever. In Theorem 1, such examples are excluded by the assumption that the assignment of an equilibrium strategy x_s to each s is locally one-to-one. The assignment is also assumed to be continuous, which means that (unlike for global comparative statics; see below) two equilibrium strategies can be compared only if they are connected in the strategy space by a curve whose points are equilibrium strategies for intermediate values of

⁹ Borel measurability means that the inverse image of every real interval is a Borel set (Rana, 2002, Ex. 7.3.13). A sufficient condition for this is that g is continuous.

¹⁰ A function is locally one-to-one if each point in its domain has a neighborhood where the function does not return any value more than once. Intuitively, this means that the function is nowhere “flat”. Obviously, a one-to-one function is also locally so.

¹¹ A sufficient condition for absolute continuity is that f is continuously differentiable.

the selfishness coefficient. In the case of multiple equilibria, this guarantees that an equilibrium in one modified game is compared with the “right” equilibrium in the other game. Even with all of these assumptions, stability or definite instability, respectively, are not *necessary* conditions for the equilibrium payoff to decrease or increase with increasing selfishness coefficient. In fact, as Example 2 below shows, if the equilibrium strategies are neither stable nor definitely unstable, both kinds of comparative statics are possible.

When considering particular classes of symmetric two-player games, more specific versions of Theorem 1 can be obtained by replacing the general stability or definite instability condition with a condition that is equivalent to, or at least implies, that property in the class under consideration. For example, consider games with a unidimensional strategy space. Proposition 1 suggests that the corresponding version of Theorem 1 would assert that the effects of altruism and spite on payoffs are determined by the sign of an expression similar to that on the left-hand side of (9), with g replaced by the modified game g_s and the partial derivatives evaluated at a symmetric equilibrium in that game. In fact, a stronger result holds. The following proposition presents a *quantitative* relation between the sign and *magnitude* of the above expression and the effect of the selfishness coefficient s on the equilibrium payoff.

Proposition 4. For a symmetric two-player game g with a strategy space X that is a subset of the real line, and s_0 and s_1 with $s_0 < s_1$, suppose that there is a continuously differentiable function that assigns to each $s_0 < s < s_1$ an equilibrium strategy x_s in the modified game g_s such that x_s lies in the interior of X and g has continuous second-order partial derivatives in a neighborhood of the equilibrium point (x_s, x_s) . Then, at each point $s_0 < s < s_1$,

$$\frac{df}{ds} = 2s((g_s)_{11} + (g_s)_{12}) \left(\frac{dx_s}{ds} \right)^2, \quad (17)$$

where f is defined by (16) and the partial derivatives are evaluated at (x_s, x_s) .

Proof. Since x_s is an interior equilibrium strategy in g_s for every $s_0 < s < s_1$, it satisfies the first-order condition

$$(g_s)_1(x_s, x_s) = 0. \quad (18)$$

By the identity

$$(g_s)_1 = sg_1 + (1-s)\bar{g}_1, \quad (19)$$

which follows from (12), differentiation of both sides of (18) with respect to s gives

$$(g_1(x_s, x_s) - \bar{g}_1(x_s, x_s)) + ((g_s)_{11}(x_s, x_s) + (g_s)_{12}(x_s, x_s)) \frac{dx_s}{ds} = 0. \quad (20)$$

By (18) and (19),

$$-s(g_1(x_s, x_s) - \bar{g}_1(x_s, x_s)) = \bar{g}_1(x_s, x_s).$$

The right-hand side is equal to $\bar{g}_2(x_s, x_s)$, since the symmetry of the game implies that the

average payoff \bar{g} is invariant to permutations of its two arguments. Therefore, by (16) and (20),

$$\frac{df}{ds} = (\bar{g}_1(x_s, x_s) + \bar{g}_2(x_s, x_s)) \frac{dx_s}{ds} = 2s((g_s)_{11}(x_s, x_s) + (g_s)_{12}(x_s, x_s)) \left(\frac{dx_s}{ds}\right)^2.$$

■

Consider next symmetric $n \times n$ games. For this class of games, the following somewhat stronger version of Theorem 1 holds.

Theorem 2. (Milchtaich, 2006a) For a symmetric $n \times n$ game g , and s_0 and s_1 with $0 \leq s_0 < s_1$, suppose that there is a continuous function that assigns to each $s_0 \leq s \leq s_1$ a strategy x_s that is an ESS in the modified game g_s , such that $x_{s_0} \neq x_{s_1}$. Then,

$$g(x_{s_1}, x_{s_1}) < g(x_{s_0}, x_{s_0}).$$

As indicated in Section 3, a local comparative statics result like Theorem 2 is significant only if a continuous function $s \mapsto x_s$ as in the theorem exists. For $n \times n$ games, the existence of such a function is guaranteed by a simple and rather weak condition. An ESS x in such a game is said to be a *regular* ESS if every action that is a best response to x is in its support, i.e., (x, x) is a quasi-strict equilibrium (van Damme, 1991). Proposition A4 in Appendix C immediately gives the following.

Corollary 3. For a symmetric $n \times n$ game g and selfishness coefficient $t \geq 0$, and any regular ESS x_t in g_t , there exists a continuous function $s \mapsto x_s$ that assigns to each $s \geq 0$ in a neighborhood of t a regular ESS x_s in the game g_s , which is moreover the only equilibrium strategy in g_s that has the same support as x_t .

It follows from Theorem 2 that, for s close to t such that $x_s \neq x_t$, x_s affords a higher payoff for the players than x_t if $s < t$, and a lower payoff if $s > t$.

As indicated above, stable or definite unstable equilibria are only sufficient conditions for decreasing or increasing local comparative statics, respectively. It is shown below that these conditions are close to being also necessary in the special case of symmetric 2×2 games. However, as the following example demonstrates, this is not so in general, and not even in the 3×3 case.

Example 2. *Generalized rock–scissors–paper games.* A symmetric 3×3 game g has the following payoff matrix:

$$\begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ -2 & 3 & 0 \end{pmatrix}. \quad (21)$$

This game (which is essentially the same one used by Chamberland and Cressman, 2000) and the modified game g_s are generalized rock–scissors–paper games for all $s > 1/3$. Such games have a single equilibrium, which is symmetric and completely mixed (Hofbauer and Sigmund, 1998). The equilibrium strategy, $x_s = ((x_s)_1, (x_s)_2, (x_s)_3)$, can be found by

straightforward computation. It depends continuously on the selfishness coefficient s , and gives the equilibrium payoff $((3s)^2 - 1)/((11s)^2 - 5)$, which increases monotonically as s increases from $1/3$ (Figure 2). By Theorems 1 or 2, this implies lack of stability. In fact, for all $s > 1/3$, the modified game g_s does not have an ESS or an NSS. However, it also does not have a definitely evolutionarily unstable equilibrium strategy. A necessary condition for the unique equilibrium strategy in a generalized rock–scissors–paper game with a payoff matrix $A = (a_{ij})_{i,j=1}^3$ to be evolutionarily stable, neutrally stable or definitely evolutionarily unstable is that, for all $1 \leq i < j \leq 3$, the sum $a_{ij} + a_{ji}$ is positive (Hofbauer and Sigmund, 1998, Theorem 7.7.1; Weissing, 1991, Theorem 4.6), nonnegative or negative, respectively. None of these conditions holds here, since in (21), $a_{13} + a_{31} < 0$ but $a_{23} + a_{32} > 0$. This shows, in particular, that definite instability is not a necessary condition for increasing comparative statics. Similarly, stability is not a necessary condition for decreasing comparative statics. This is demonstrated by the generalized rock–scissors–paper game g with the following payoff matrix, which differs from (21) in a single entry only:

$$\begin{pmatrix} 0 & -3 & 1 \\ 2 & 0 & -2 \\ -2 & 3 & 0 \end{pmatrix}. \quad (22)$$

For $s > 1/3$, the players' payoff at the unique (symmetric and completely mixed) equilibrium in the modified games g_s is $((5s)^2 - 1)/((13s)^2 - 5)$. Similarly to (21), and for the same reason, the corresponding equilibrium strategy is not an ESS, NSS or definitely evolutionarily unstable. However, unlike (21), the equilibrium payoff *decreases* monotonically¹² as s increases from $1/3$ (Figure 2).

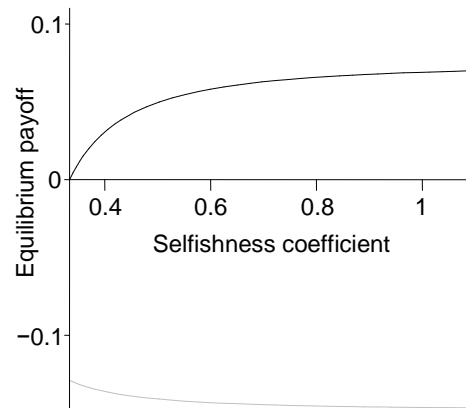


Figure 2. Generalized rock–scissors–paper games in Example 2. For the game with the payoff matrix (21), the payoff at the unique symmetric equilibrium in the modified game increases as the selfishness coefficient s increases from $1/3$ (black, upper curve). For the game with the payoff matrix (22), the payoff decreases (gray, lower curve). In both cases, the equilibrium strategies in the modified games are neither stable nor definitely unstable.

¹² Example 2 raises the question of whether in symmetric $n \times n$ games the equilibrium payoff is always a monotonic (either nonincreasing or nondecreasing) function of the selfishness coefficient. This question is addressed in Appendix B.

Since none of the equilibria in Example 2 is stable or definitely unstable, this example raises the question of whether there are *weaker* properties that imply increasing or decreasing local comparative statics in symmetric $n \times n$ games. The following proposition identifies such properties. This result is more special than Theorem 2 in that it only concerns strategies with identical supports, e.g., completely mixed ones. On the other hand, it is a stronger result in that it gives a *necessary* and sufficient condition for an equilibrium in one modified game to afford a higher payoff than an equilibrium in another such game.

Proposition 5. For a symmetric $n \times n$ game g , and (nonnegative) s and t with $s \neq t$, let x_s and x_t be strategies with identical supports that are equilibrium strategies in g_s and g_t , respectively. The strict inequality

$$(t - s)(g(x_t, x_t) - g(x_s, x_s)) < 0$$

or a similar weak inequality, respectively, holds if and only if x_t cannot weakly invade or cannot invade x_s in the modified game g_s . In particular, the strict or weak inequality, respectively, holds if x_s is stable or weakly stable in g_s and $x_t \neq x_s$.

Proof. The identity of the supports implies that x_t is a best response to x_s in g_s and x_s is a best response to x_t in g_t . Thus, for $x = x_s$ and $y = x_t$, the first two terms on the right-hand side of the identity (62) in Appendix A are zero, and therefore $(t - s)(g(x_t, x_t) - g(x_s, x_s))$ is equal to $(s + t)(g_s(x_t, x_t) - g_s(x_s, x_t))$. The factor $s + t$ is positive, since s and t are nonnegative and distinct. The factor $g_s(x_t, x_t) - g_s(x_s, x_t)$ is negative or nonpositive, respectively, if and only if x_t cannot weakly invade or cannot invade x_s in g_s . By Proposition 2, the former or latter condition, respectively, holds (in particular) if x_s is stable or weakly stable in g_s and $x_s \neq x_t$. ■

Unlike Theorem 2, Proposition 5 does not assume that the strategies compared are connected in the strategy space by a curve consisting of equilibrium strategies corresponding to intermediate values of the selfishness coefficient. Thus, this result is somewhat in the spirit of global comparative statics.

4.3 Global comparative statics in symmetric two-player games

Global comparative statics differ from local comparative statics in that the comparison is not limited to continuous changes in the selfishness coefficient and the corresponding strategies. In particular, the topology on the strategy space is irrelevant. This makes the analysis applicable also to games with discrete strategy spaces, which is not the case for local comparative statics (see Section 3). The essence of global comparative statics is captured by the following.

Definition 2. Selfishness *decreases social welfare* in a symmetric two-player game g if, for every s and t with $0 \leq s \leq t$ and equilibrium strategies x_s and x_t in g_s and g_t , respectively,

$$x_s \neq x_t \text{ implies } g(x_t, x_t) < g(x_s, x_s). \quad (23)$$

Selfishness *weakly decreases social welfare* in g if it satisfies the weaker condition in which (23) is replaced by

$$s \neq t \text{ implies } g(x_t, x_t) \leq g(x_s, x_s).$$

The first condition in Definition 2 is rather demanding. In particular, it implies (in the special case $s = t$) that for every s the modified game g_s has at most one equilibrium strategy. Nevertheless, as the follows theorem shows, that condition is implied by a simple condition on the unmodified payoffs, which (predictably) also implies stability.

Theorem 3. Selfishness decreases social welfare in every symmetric two-player game g in which, for every pair of distinct strategies x and y ,¹³

$$g(y, y) - g(x, y) - g(y, x) + g(x, x) < 0. \quad (24)$$

Selfishness weakly decreases social welfare in every game satisfying the weaker condition in which the strict inequality (24) is replaced by a weak one. The former, stronger, condition or the latter, weaker, one implies that, for every $s \geq 0$, an equilibrium strategy in the modified game g_s is necessarily stable or weakly stable, respectively.

Proof. Suppose that x and y are equilibrium strategies in the modified games g_s and g_t , respectively, and $0 \leq s \leq t$. Then, $g_s(y, x) - g_s(x, x) \leq 0 \leq g_t(y, y) - g_t(x, y)$. If (24) holds, then $s + t > 0$, since (by the previous inequalities) $s = t = 0$ would imply that $(g_0(y, y) - g_0(x, y)) - (g_0(y, x) - g_0(x, x)) \geq 0$, which by (60) contradicts (24). Therefore, if (24) holds, the right-hand side of (63) is negative, which implies that $g(y, y) < g(x, x)$. Similarly, if the weak-inequality version of (24) holds and $s < t$, then the right-hand side of (63) is nonpositive, and therefore $g(y, y) \leq g(x, x)$.

Suppose now only that x is an equilibrium strategy in the modified game g_s . A sufficient condition for x to be stable or weakly stable is that the left-hand side of the identity (61) is negative or nonpositive, respectively, for all $y \neq x$. By that identity, a sufficient condition for this is that (24) or a similar weak inequality, respectively, holds for all $y \neq x$. ■

Inequality (24), which can also be written as

$$g(y, y) - g(x, y) < g(y, x) - g(x, x), \quad (25)$$

may be interpreted as expressing “symmetric substitutability” (Bergstrom, 1995): the profitability of switching from any strategy x to another strategy y is lower if the opponent uses y than if he uses x . By the theorem, this condition or its weak-inequality version implies that selfishness decreases or weakly decreases social welfare, respectively.

An alternative interpretation of (24) is that *coordination* decreases the players’ payoffs. This interpretation is based on the fact that the left-hand side of (24) is equal to four times the difference between (i) each player’s expected payoff if the two players *jointly* randomize 50–50 between x and y , and so always choose the same strategy, and (ii) their expected payoff if they *independently* randomize 50–50 between x and y . More generally, suppose that for some finite list of distinct strategies, x_1, x_2, \dots, x_n , both players use each strategy x_i with the

¹³ Note the differences in signs between the left-hand sides of (24) and (6).

same (marginal) probability $p_i > 0$ (with $\sum_i p_i = 1$). Then, the difference between each player's expected payoff if the strategy choices of the two players are perfectly correlated and the expected payoff if their choices are independent is given by $\sum_{i < j} p_i p_j (g(x_i, x_i) - g(x_j, x_i) - g(x_i, x_j) + g(x_j, x_j))$. A sufficient condition for this to be negative or nonpositive is that the same is true for the expression on the left-hand side of (24) whenever $x \neq y$. Thus, Theorem 3 may be interpreted as saying that selfishness decreases or weakly decreases social welfare in every symmetric two-player game g in which coordination decreases or weakly decreases payoffs, respectively.

The symmetric substitutability condition (25) clearly holds, and selfishness therefore decreases social welfare, in every symmetric two-player game g with a unidimensional strategy space that is *strictly submodular*, i.e., satisfies

$$g(y, y') - g(x, y') < g(y, x') - g(x, x') \quad (26)$$

whenever $y > x$ and $y' > x'$. Selfishness weakly decreases social welfare if g is *submodular*, i.e., a similar condition holds with the strict inequality (26) replaced by a weak one. If the strategy space is an open interval (either finite or infinite) and g has continuous second-order partial derivatives, submodularity is equivalent to $g_{12} \leq 0$ (everywhere), and a sufficient condition for strict submodularity is $g_{12} < 0$ (everywhere). The latter condition implies that (9) holds for every equilibrium strategy x , which is therefore necessarily stable. In fact, as Theorem 3 shows, if g is strictly submodular or submodular, respectively, then stability or weak stability of the equilibrium strategies holds not only in g but also in the modified game g_s , for all $s \geq 0$.

In the special case of a symmetric Cournot duopoly game, a simpler and somewhat weaker condition than strict submodularity is sufficient to guarantee that selfishness decreases social welfare. In such a game, the profit $g(x, y)$ of a producer with output level x competing against an identical producer with output level y is $P(x + y)x - C(x)$, where P is the price (or inverse demand) function and C is the cost function. Therefore, (24) can be written as

$$\frac{1}{2} [P(2x) \cdot 2x + P(2y) \cdot 2y] < P(x + y)(x + y). \quad (27)$$

(Note that this inequality does not involve C . This is because the price is a function of the firm's own production level only.) A sufficient (and, if P is continuous, also necessary) condition for (27) to hold for every pair of distinct x and y in the (finite or infinite) interval of possible output levels is that the total revenue is a strictly concave function of the total output. Similarly, if the total revenue is concave, a weak inequality similar to (27) always holds. By Theorem 3, this weak inequality implies that increasing comparative statics (exemplified by the lower curve in Figure 1b) cannot occur, so that moving from duopoly towards (not necessarily all the way to) effective monopoly cannot hurt the firms' profits. This proves the following.

Corollary 4. In a symmetric Cournot duopoly game, each of the following two conditions, the latter being weaker than the former, implies that selfishness decreases the firms' equilibrium profit:

- (i) A firm's profit is given by a strictly submodular function of the two firms' output levels.
- (ii) The total revenue is a strictly concave function of the total output.

Dropping "strictly", each of the two conditions implies that selfishness weakly decreases welfare.

It is instructive to compare Corollary 4 with the results of Koçkesen et al. (2000). These authors show that, in a symmetric Cournot duopoly game that is strictly submodular, a firm with negatively interdependent preferences obtains a strictly higher profit than does a competitor with independent preferences in any equilibrium. The difference between the two firms' preferences is that the latter is only concerned with its own profit while the former also seeks a high ratio ρ between its own and the average profit. Complete selfishness, $s = 1$, corresponds to independent preferences, whereas weak spite, i.e., s greater than 1 but close to it, gives negatively interdependent preferences if the ratio between the profits ρ is not too small. Thus, with a strictly submodular profit, if only one firm is spiteful, it is likely to do better than its competitor. However, as Corollary 4 shows, if both firms have such preferences, they do not have higher profits than two firms with independent preferences.

Another class of symmetric two-player games for which Theorem 3 can be given a more concrete form is symmetric $n \times n$ games.

Proposition 6. For a symmetric $n \times n$ game g , consider the quadratic form $G: \mathfrak{R}^{n-1} \rightarrow \mathfrak{R}$ defined by

$$G(\zeta_1, \zeta_2, \dots, \zeta_{n-1}) = \sum_{i,j=1}^{n-1} (g(i,j) - g(n,j) - g(i,n) + g(n,n)) \zeta_i \zeta_j. \quad (28)$$

If G is negative definite, then selfishness decreases welfare in g , and for every $s \geq 0$, the modified game g_s has a unique equilibrium strategy, which is stable. If G is negative semidefinite, then selfishness weakly decreases welfare in g , and for every $s \geq 0$, every equilibrium strategy in g_s is weakly stable. If G is positive definite, then for every $s \geq 0$, every completely mixed equilibrium strategy in g_s is definitely unstable.

Proof. Consider two distinct (mixed) strategies $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. The expression on the left-hand side of (24) is equal to $\sum_{i,j=1}^n g(i,j) \zeta_i \zeta_j$, where $\zeta_i = y_i - x_i$. Since $\sum_{i=1}^n \zeta_i = 0$, this sum is equal to that in (28). Therefore, the assertions concerning negative definiteness and semidefiniteness follow from Theorem 3 and the fact that in a symmetric $n \times n$ game an equilibrium strategy always exists. If G is positive definite, then the reverse of inequality (24) holds, and therefore, by (60),

$$g_s(y, y) - g_s(x, y) - g_s(y, x) + g_s(x, x) > 0 \quad (29)$$

for every $s \geq 0$. If in addition the strategy x is a completely mixed equilibrium strategy in g_s , then $g_s(y, x) = g_s(x, x)$, and the expression on the left-hand side of (29) is hence equal to $g_s(y, y) - g_s(x, y) + g_s(y, x) - g_s(x, x)$, which proves that x is definitely unstable. ■

The quadratic form G defined in (28) is particularly simple if $n = 2$. In this case, G is negative definite, negative semidefinite or positive definite if and only if the expression $g(1,1) - g(2,1) - g(1,2) + g(2,2)$ is negative, nonpositive or positive, respectively. This expression is the difference between the sum of the two diagonal entries of the payoff matrix and the sum of the two off-diagonal entries. A negative or positive difference expresses *strategic substitutability* or *complementarity*, respectively (Bulow et al., 1985). This means that the profitability of switching from any of the two pure strategies i to the other strategy j decreases or increases, respectively, as the probability that the other player uses j increases (cf. (25)). Two kinds of symmetric 2×2 games that always exhibit strategic substitutability are the battle-of-the-sexes game, in its symmetric form

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad (30)$$

and Chicken (or hawk–dove game). It follows, by Proposition 6, that in these games the equilibrium payoff can only decrease or remain unchanged with increasing s . By contrast, in games with strategic complementarity, if the equilibria are completely mixed, the equilibrium payoff can only *increase* or remain unchanged with increasing s . In the prisoner’s dilemma, both strategic substitutability and complementarity are possible. This is shown by the next example.

Example 3. Public good game. Two people are presented with the offer to contribute \$5 for the production of some public good. The production function is such that, if the total sum raised is \$0, \$5 or \$10, respectively, 0, 4 or 7 units of the public good are produced. The worth of a unit of public good is \$1 for each person. For a completely unselfish person ($s = 0$), it is clearly a dominant strategy to Contribute the money. For a selfish ($s = 1$) or spiteful ($s > 1$) person, the dominant strategy is to Decline the offer. For intermediate values of s , which are not too close to 0 or 1, the unique symmetric equilibrium is completely mixed, and the probability of making the contribution, as well as the equilibrium payoff (Figure 3b, solid black curve), decrease with increasing s . However, if \$5 only allows 3 units of the public good to be produced, selfishness in the same range has the opposite effect on the probability of contributing and on the equilibrium payoff (Figure 3b, dashed curve). The second case differs from the first in that the production function is convex rather than concave. Correspondingly, the game has strategic complementarity rather than substitutability, and the completely mixed equilibrium strategy is definitely unstable rather than stable. In the borderline case of a linear production function, \$5 allows $3\frac{1}{2}$ units of the public good to be produced, implying that the quadratic form G is identically zero. In this case, selfishness weakly increases social welfare (Figure 3b, gray curve).

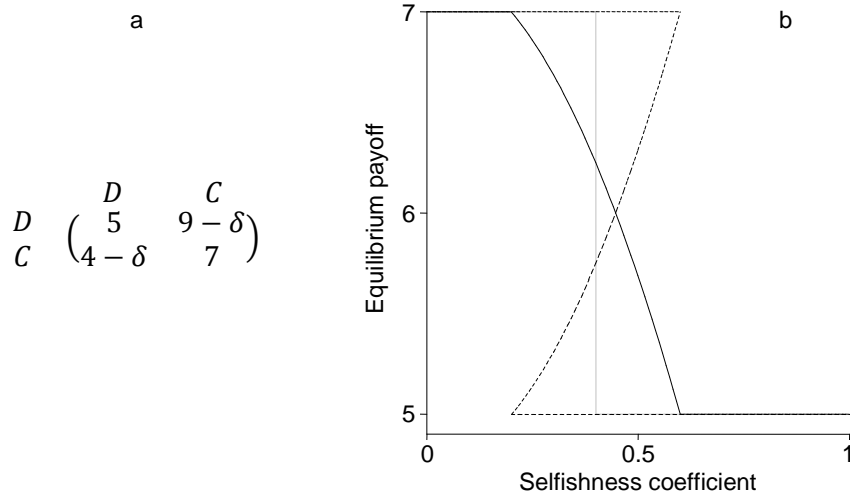


Figure 3. The three kinds of prisoner’s dilemma in Example 3. a. The payoff matrix. The parameter δ is 0, 1/2, or 1. b. The equilibrium payoff(s). The dependence on the selfishness coefficient s is different for $\delta = 0$ (solid black curve), $\delta = 1/2$ (gray curve) and $\delta = 1$ (dashed curve). The non-horizontal portions of the three curves correspond to completely mixed equilibrium strategies, which are stable, weakly stable or definitely unstable, respectively.

5 Asymmetric Two-player Games

Asymmetric equilibria of symmetric two-player games can be equally important or more so than the symmetric equilibria. For example, this is so for symmetric 2×2 mis-coordination games such as (30), in which the asymmetric equilibrium payoffs Pareto dominate the symmetric ones. An asymmetric equilibrium requires some asymmetry between the players: a cue that will tell each of them which strategy to choose. This is *inessential* (or uncorrelated) asymmetry in that it does not directly affect the payoffs. Nevertheless, as argued in Milchtaich (2008), it is highly consequential for stability analysis, since it enables the players to deviate from the equilibrium in a coordinated manner. Inessentially asymmetric games are more similar to “truly” asymmetric games than to symmetric ones. In particular, the comparative statics of asymmetric equilibria in symmetric games is best viewed as a special case of comparative statics of asymmetric games. These comparative statics, like those of symmetric equilibria in symmetric games, turn out to be related to (static) stability. However, in the asymmetric context, stability is a property of strategy profiles rather than strategies.

Definition 3. (Milchtaich, 2008) A strategy profile (x^1, x^2) in an asymmetric two-player game $h = (h^1, h^2): X^1 \times X^2 \rightarrow \mathfrak{R}^2$, where the players’ strategy spaces X^1 and X^2 are topological spaces, is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood in the product space $X^1 \times X^2$ where the inequality

$$\begin{aligned}
 & \frac{1}{2}(h^1(y^1, x^2) - h^1(x^1, x^2) + h^1(y^1, y^2) - h^1(x^1, y^2)) \\
 & + \frac{1}{2}(h^2(x^1, y^2) - h^2(x^1, x^2) + h^2(y^1, y^2) - h^2(y^1, x^2)) < 0
 \end{aligned}$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds for all strategy profiles $(y^1, y^2) \neq (x^1, x^2)$.

Conceptually, this definition of stability, weak stability and definite instability is based on Definition 1. In fact, it is precisely the condition that (x^1, x^2) has the same property as a strategy in the symmetric game g obtained by *symmetrizing* h . Symmetrization means that each player can be assigned either to the role of player 1 in h or to that of player 2, with equal probabilities, and in both cases the opponent is assigned to the other role.¹⁴ A strategy in g is thus a strategy profile $x = (x^1, x^2)$ in h . It is easy to show (see Milchtaich, 2008, Proposition 4) that x is an equilibrium in h if and only if it is an equilibrium strategy in g , and in this case, the equilibrium payoff in g is equal to the players' average equilibrium payoff in h , which is given by

$$\bar{h}(x^1, x^2) = \frac{1}{2}h^1(x^1, x^2) + \frac{1}{2}h^2(x^1, x^2).$$

Thus, stability in an asymmetric game h can be characterized in terms of stability in an auxiliary symmetric game. The implication of this is that a local comparative statics result for asymmetric games can be derived by substituting that auxiliary game for g in Theorem 1. Such a substitution gives the following theorem, which is proved (for an arbitrary number of players) in Section 6.

Theorem 4. For an asymmetric two-player game $h = (h^1, h^2): X^1 \times X^2 \rightarrow \mathfrak{R}^2$, and s_0 and s_1 with $0 \leq s_0 < s_1$, suppose that h is Borel measurable and there is a continuous and locally one-to-one function that assigns to each $s_0 \leq s \leq s_1$ a stable strategy profile $x_s = (x_s^1, x_s^2)$ in the modified game $h_s = (h_s^1, h_s^2)$,¹⁵ such that the function $f: [s_0, s_1] \rightarrow \mathfrak{R}$ defined by

$$f(s) = \bar{h}(x_s^1, x_s^2) \tag{31}$$

is absolutely continuous. Then f is strictly decreasing. A similar result holds with 'stable' and 'strictly increasing' replaced by 'weakly stable' and 'nondecreasing', respectively, or by 'definitely unstable' and 'strictly increasing', respectively.

Theorem 4 is less widely applicable than Theorem 1. This is because stability is in a sense a more stringent requirement for asymmetric games than for symmetric ones. For example, in a bimatrix game, a strategy profile is stable if and only if it is a strict equilibrium (Selten, 1980; Milchtaich, 2008, Proposition 5). Since strict equilibria are pure, a function as in Theorem 4 does not exist.

One class of two-player games to which Theorem 4 is applicable is games in the plane, in which the two players' strategy spaces are intervals or some other subsets of the real line. The differential conditions for stability or definite instability in such games are given by the following proposition.

Proposition 7. (Milchtaich, 2008) Let $h = (h^1, h^2): X^1 \times X^2 \rightarrow \mathfrak{R}^2$ be an asymmetric two-player game in which the players' strategy spaces are subsets of the real line, and (x^1, x^2) an equilibrium lying at the interior of $X^1 \times X^2$ with a neighborhood where h^1 and h^2 have

¹⁴ For a formal definition of symmetrization, see Section 6 or Milchtaich (2008).

¹⁵ The function $h_s^i: X^1 \times X^2 \rightarrow \mathfrak{R}$, which gives player i 's modified payoff ($i = 1, 2$), is defined in (3).

continuous second-order derivatives. A sufficient condition for stability or definite instability of the equilibrium is that the matrix

$$H = \begin{pmatrix} h_{11}^1 & h_{12}^1 \\ h_{21}^2 & h_{22}^2 \end{pmatrix}, \quad (32)$$

with the derivatives evaluated at (x^1, x^2) , is negative definite or positive definite, respectively. A necessary condition for weak stability is that the matrix is negative semidefinite.

Proposition 7 and Theorem 4 together point to a connection between local comparative statics and properties of the matrix

$$H_s = \begin{pmatrix} (h_s^1)_{11} & (h_s^1)_{12} \\ (h_s^2)_{21} & (h_s^2)_{22} \end{pmatrix}, \quad (33)$$

which is obtained by replacing h in (32) with the modified game h_s . A direct, quantitative connection between this matrix and comparative statics is given by the following proposition.

Proposition 8. For an asymmetric two-player game $h = (h^1, h^2): X^1 \times X^2 \rightarrow \mathfrak{R}^2$, with strategy spaces X^1 and X^2 that are subsets of the real line, and s_0 and s_1 with $s_0 < s_1$, suppose that there is a continuously differentiable function that assigns to each $s_0 < s < s_1$ a strategy profile $x_s = (x_s^1, x_s^2)$ that is an equilibrium in the modified game h_s , lies in the interior of $X^1 \times X^2$, and has a neighborhood where h^1 and h^2 have continuous second-order partial derivatives. Then, at each point $s_0 < s < s_1$,

$$\frac{df}{ds} = s \left(\frac{dx_s}{ds} \right)^T H_s \left(\frac{dx_s}{ds} \right), \quad (34)$$

where f is defined in (31) and the matrix H_s is evaluated at x_s .

Proof. Since x_s is an equilibrium strategy in h_s for every $s_0 < s < s_1$, it satisfies the first-order conditions

$$(h_s^i)_i(x_s^1, x_s^2) = 0, \quad i = 1, 2. \quad (35)$$

By the identity

$$(h_s^i)_i = s h_i^i + (1-s) \bar{h}_i, \quad i = 1, 2, \quad (36)$$

which follows from (3), differentiation of both sides of (35) with respect to s gives

$$\left(h_i^i(x_s^1, x_s^2) - \bar{h}_i(x_s^1, x_s^2) \right) + \sum_{j=1}^2 (h_s^i)_{ij}(x_s^1, x_s^2) \frac{dx_s^j}{ds} = 0, \quad i = 1, 2. \quad (37)$$

By (35) and (36),

$$-s \left(h_i^i(x_s^1, x_s^2) - \bar{h}_i(x_s^1, x_s^2) \right) = \bar{h}_i(x_s^1, x_s^2), \quad i = 1, 2. \quad (38)$$

Therefore, by (31) and (37),

$$\frac{df}{ds} = \sum_{i=1}^2 \bar{h}_i(x_s^1, x_s^2) \frac{dx_s^i}{ds} = s \sum_{i,j=1}^2 (h_s^i)_{ij}(x_s^1, x_s^2) \frac{dx_s^i}{ds} \frac{dx_s^j}{ds}.$$

■

Global comparative statics are less relevant to asymmetric two-player games than for symmetric ones (Section 4.3). Definition 2 readily extends to asymmetric games. However, conditions analogous to those in Theorem 3 do not hold for most asymmetric games. The only exception is games in which an expression analogous to that on the left-hand side of (24) is identically zero, i.e.,

$$h^i(y^1, y^2) - h^i(x^1, y^2) - h^i(y^1, x^2) + h^i(x^1, x^2) = 0, \text{ for all } x^1, x^2, y^1, y^2 \text{ and } i = 1, 2. \quad (39)$$

Games satisfying this condition are *non-strategic* in that the change in the payoff of a player switching strategies (from x^1 to y^1 in the case of player 1 or from x^2 to y^2 in the case of player 2) is independent of the opponent's strategy. It is not difficult to see that condition (39) holds if and only if, for all $s \geq 0$, a similar condition holds with h replaced by the modified game h_s . In this case, in any modified game, any equilibrium is weakly stable. The following proposition extends the analogy with Theorem 3 by showing that, in games as above, selfishness weakly decreases social welfare.

Proposition 9. Let $h = (h^1, h^2): X^1 \times X^2 \rightarrow \mathfrak{R}^2$ be an asymmetric two-player game satisfying (39). For every s and t with $0 \leq s < t$, and equilibria $x_s = (x_s^1, x_s^2)$ and $x_t = (x_t^1, x_t^2)$ in h_s and h_t , respectively,

$$\bar{h}(x_t^1, x_t^2) \leq \bar{h}(x_s^1, x_s^2). \quad (40)$$

Proof. It follows immediately from equality (39) that a similar equality holds with h^i replaced by \bar{h} . Therefore, for equilibria x_s and x_t as above, the right-hand side of (65) is nonpositive for $(x^1, x^2) = x_s$ and $(y^1, y^2) = x_t$. This proves (40). ■

A simple example illustrating the last result is *non-strategic altruism*, which is of considerable importance to the theory of kin selection (for references, see Milchtaich, 2006a). An altruistic act confers a benefit b on the recipient at a cost c to the actor, with $b > c > 0$. Therefore, it changes the actor's modified payoff by $-sc + (1-s)(b-c)/2$. If the selfishness coefficient s is low enough to make this change positive, acting altruistically maximizes the actor's payoff in the modified game. This implies that lowering the selfishness coefficient can only increase social welfare or leave it unchanged.

If the interaction described above is symmetric in that both individuals have an equal chance to be in the position of a potential actor or receiver, an increase in social welfare actually benefits them both. Such a symmetric interaction is still non-strategic, since each player's payoff is *additively separable*. Specifically, the payoff is the sum of a nonpositive term (cost) that is 0 or $-c$, depending on the individual's own decision of whether to act altruistically, and a nonnegative term (benefit) that is 0 or b , depending on the other individual's decision.

There is no interaction term. The third case in Example 3 (i.e., linear production function) is an example of such symmetric non-strategic altruism.

6 Multiplayer Games

The connection between comparative statics and stability holds not only for two-player games but for any number of players n . However, establishing this connection requires using the proper notion of stability. As for two-player games, the first step is to define stability of strategies in symmetric games $g: X \times X \times \dots \times X \rightarrow \mathfrak{R}$, where the strategy space X (common to all n players) is a topological space. The difficulty here is that there are several non-equivalent reasonable extensions of Definition 1 to multiplayer games, which differ in the assumptions they make about the correlations (or lack thereof) between the strategies used by a player's opponents (Milchtaich, 2008). In the two-player case, there is only one opponent, so that the question of correlations does not arise and the only assumption is that the opponent uses each of the two strategies x and y under consideration with probability $1/2$. Consistency requires that the definition in the multiplayer case assume the same for each single opponent. This assumption implies that, regardless of correlation, the expected number of opponents using x is equal to that using y . Equivalently,

$$\sum_{j=1}^n (j-1)p_j = \frac{n-1}{2}, \quad (41)$$

where, for $1 \leq j \leq n$, p_j is the probability that $n-j$ of the opponents use x and $j-1$ use y (and $\sum_{j=1}^n p_j = 1$). These considerations lead to the following.

Definition 4. (Milchtaich, 2008) For a probability vector $p = (p_1, p_2, \dots, p_n)$, a strategy x in a symmetric n -player game $g: X \times X \times \dots \times X \rightarrow \mathfrak{R}$ is *p-stable*, *weakly p-stable* or *definitely p-unstable*, respectively, if it has a neighborhood where

$$\sum_{j=1}^n p_j \left(g(\underbrace{y, x, \dots, x}_{n-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}}) - g(\underbrace{x, x, \dots, x}_{n-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}}) \right) < 0, \quad (42)$$

a similar weak inequality or the reverse (strict) inequality holds for all strategies $y \neq x$. Strategy x is *stable*, *weakly stable* or *definitely unstable* if the corresponding condition holds for every probability vector p satisfying (41).

Special cases of p -stability are *dependent-stability*, defined by

$$p_j = \begin{cases} \frac{1}{2}, & j = 1, n \\ 0, & j = 2, \dots, n-1 \end{cases}, \quad (43)$$

independent-stability, defined by

$$p_j = \frac{1}{2^{n-1}} \binom{n-1}{j-1}, \quad j = 1, 2, \dots, n, \quad (44)$$

and *uniform-stability*, defined by

$$p_j = \frac{1}{n}, \quad j = 1, 2, \dots, n. \quad (45)$$

In each case, the corresponding notions of weak stability and definite instability are similarly defined. The probability vector (43) describes the distribution of the number of opponents using strategy x (and the number using the alternative strategy y) if either all of them use x or they all use y , and both possibilities have probability $1/2$. The probability vector (44) corresponds to independent randomizations by the opponents between x and y with half-half probabilities. That in (45) assigns to the number of opponents using strategy x the uniform distribution, i.e., all numbers are equally likely. Note that, in all three cases, (42) can be simplified to

$$\sum_{j=1}^n p_j \left(g(\underbrace{y, \dots, y}_j, x, \dots, x) - g(\underbrace{x, \dots, x}_j, y, \dots, y) \right) < 0. \quad (46)$$

The next step is to define stability of strategy *profiles* in *asymmetric* multiplayer games $h = (h^1, h^2, \dots, h^n): X^1 \times X^2 \times \dots \times X^n \rightarrow \mathfrak{R}^n$, where the strategy space X^i of each player i is a topological space. As in the two-player case (Section 5), a natural way to do this is to use the familiar procedure of symmetrization. An asymmetric game h is symmetrized by allowing the players to take turns playing the different roles in h , which entails that each player i has to choose a strategy profile $x_i = (x_i^1, x_i^2, \dots, x_i^n)$ in h . An assignment of the n players to the n roles in h is described by a permutation π of $(1, 2, \dots, n)$: player i is assigned to role $\pi(i)$, or equivalently, the player assigned to the role j is $\pi^{-1}(j)$. Symmetrization involves averaging a player's payoff as π varies over the set Π of all $n!$ permutations.

Definition 5. (Milchtaich, 2008) The game obtained by *symmetrizing* an asymmetric n -player game $h = (h^1, h^2, \dots, h^n): X^1 \times X^2 \times \dots \times X^n \rightarrow \mathfrak{R}^n$ is the symmetric n -player game $g: X \times X \times \dots \times X \rightarrow \mathfrak{R}$ where the strategy space X is the product space $X^1 \times X^2 \times \dots \times X^n$ and

$$g(x_1, x_2, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in \Pi} h^{\pi(1)}(x_{\pi^{-1}(1)}^1, x_{\pi^{-1}(2)}^2, \dots, x_{\pi^{-1}(n)}^n) \quad (47)$$

for all $x_1 = (x_1^1, x_1^2, \dots, x_1^n), x_2 = (x_2^1, x_2^2, \dots, x_2^n), \dots, x_n = (x_n^1, x_n^2, \dots, x_n^n) \in X$. A strategy profile $x = (x^1, x^2, \dots, x^n)$ in h is *stable*, *weakly stable* or *definitely unstable* if it has the same property as a strategy in g . Similar definitions apply to p -stability, dependent-stability, independent-stability, uniform-stability and the other related notions of stability and instability.

A strategy profile $x = (x^1, x^2, \dots, x^n)$ in an asymmetric n -player game h is an equilibrium if and only if it is a (symmetric) equilibrium strategy in the symmetric game g obtained by symmetrizing h (Milchtaich, 2008, Proposition 9). In this case, the equilibrium payoff in g is equal to the players' average equilibrium payoff in h . In fact, it follows as a special case from (47) that for *any* strategy profile x ,

$$g(x, x, \dots, x) = \bar{h}(x). \quad (48)$$

This equality and the following proposition together entail that symmetrization in a sense preserves comparative statics. The proposition says that the operations of symmetrization and modification commute.

Proposition 10. Let $h = (h^1, h^2, \dots, h^n): X^1 \times X^2 \times \dots \times X^n \rightarrow \mathfrak{R}^n$ be an asymmetric n -player game, and g the game obtained by symmetrizing h . For given s , let h_s be the corresponding modified game. Then the game obtained by symmetrizing h_s coincides with the modified game g_s , which is defined in (12) (with \bar{g} denoting the average payoff).

Proof. For strategies $x_1 = (x_1^1, x_1^2, \dots, x_1^n)$, $x_2 = (x_2^1, x_2^2, \dots, x_2^n)$, ..., $x_n = (x_n^1, x_n^2, \dots, x_n^n) \in X^1 \times X^2 \times \dots \times X^n$ in g , the players' average payoff can be computed by averaging the payoff of a single player over all strategy profiles with these strategies. Thus,

$$\begin{aligned} \bar{g}(x_1, x_2, \dots, x_n) &= \frac{1}{n!} \sum_{\sigma \in \Pi} g(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \\ &= \frac{1}{n!} \sum_{\sigma \in \Pi} \frac{1}{n!} \sum_{\pi \in \Pi} h^{\pi(1)}(x_{\sigma(\pi^{-1}(1))}^1, x_{\sigma(\pi^{-1}(2))}^2, \dots, x_{\sigma(\pi^{-1}(n))}^n) \\ &= \frac{1}{n!} \sum_{\sigma \in \Pi} \frac{1}{n!} \sum_{\pi \in \Pi} h^{\pi(\sigma(1))}(x_{\pi^{-1}(1)}^1, x_{\pi^{-1}(2)}^2, \dots, x_{\pi^{-1}(n)}^n) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi} \frac{1}{n!} \sum_{\sigma \in \Pi} h^{\pi(\sigma(1))}(x_{\pi^{-1}(1)}^1, x_{\pi^{-1}(2)}^2, \dots, x_{\pi^{-1}(n)}^n) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi} \bar{h}(x_{\pi^{-1}(1)}^1, x_{\pi^{-1}(2)}^2, \dots, x_{\pi^{-1}(n)}^n). \end{aligned}$$

The second equality uses the definition of symmetrization, and the third one is obtained by replacing π in the summand by the composed permutation $\pi \circ \sigma$. (This replacement leaves the inner sum unchanged, since as π varies over all permutations, so does $\pi \circ \sigma$.)

Substituting the above expression for \bar{g} in (12) gives

$$\begin{aligned} g_s(x_1, x_2, \dots, x_n) &= s g(x_1, x_2, \dots, x_n) + (1 - s) \bar{g}(x_1, x_2, \dots, x_n) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi} s h^{\pi(1)}(x_{\pi^{-1}(1)}^1, x_{\pi^{-1}(2)}^2, \dots, x_{\pi^{-1}(n)}^n) \\ &\quad + (1 - s) \bar{h}(x_{\pi^{-1}(1)}^1, x_{\pi^{-1}(2)}^2, \dots, x_{\pi^{-1}(n)}^n) = \frac{1}{n!} \sum_{\pi \in \Pi} h_s^{\pi(1)}(x_{\pi^{-1}(1)}^1, x_{\pi^{-1}(2)}^2, \dots, x_{\pi^{-1}(n)}^n), \end{aligned}$$

which proves the proposition's assertion. \blacksquare

Proposition 10 is used in the proof of the following theorem, which is the basic local comparative statics result for multiplayer games. In the special case $n = 2$, it gives Theorems 1 and 4.

Theorem 5. For a symmetric n -player game $g: X \times X \times \dots \times X \rightarrow \mathfrak{R}$, and s_0 and s_1 with $0 \leq s_0 < s_1$, suppose that g is Borel measurable and there is a continuous and locally one-to-one function that assigns to each $s_0 \leq s \leq s_1$ a stable strategy x_s in the modified game g_s , such that the function $f: [s_0, s_1] \rightarrow \mathfrak{R}$ defined by

$$f(s) = g(x_s, x_s, \dots, x_s) \quad (49)$$

is absolutely continuous. Then f is strictly decreasing.

For an asymmetric n -player game $h = (h^1, h^2, \dots, h^n): X^1 \times X^2 \times \dots \times X^n \rightarrow \mathfrak{R}^n$, and s_0 and s_1 with $0 \leq s_0 < s_1$, suppose that h is Borel measurable and there is a continuous and locally one-to-one function that assigns to each $s_0 \leq s \leq s_1$ a stable strategy profile $x_s = (x_s^1, x_s^2, \dots, x_s^n)$ in the modified game h_s , such that the function $f: [s_0, s_1] \rightarrow \mathfrak{R}$ defined by

$$f(s) = \bar{h}(x_s) \quad (50)$$

is absolutely continuous. Then f is strictly decreasing.

In both the symmetric and asymmetric cases, a similar result holds with ‘stable’ and ‘strictly increasing’ replaced by ‘weakly stable’ and ‘nondecreasing’, respectively, or by ‘definitely unstable’ and ‘strictly increasing’, respectively. Moreover, all of the above is true with ‘stable’, ‘weakly stable’ or ‘definitely unstable’ replaced by ‘uniformly-stable’, ‘weakly uniformly-stable’ or ‘definitely uniformly-unstable’, respectively.

Proof. Let g , s_0 , s_1 and the function $s \mapsto x_s$ be as in the first part of the theorem. For $-(s_1 - s_0)/2 < \epsilon < (s_1 - s_0)/2$, consider the (Borel) set $U_\epsilon \subseteq [s_0, s_1]$ defined by

$$(51)$$

$$U_\epsilon = \left\{ s_0 + |\epsilon| \leq s \leq s_1 - |\epsilon| \left| \sum_{j=1}^n \left(g_s(\underbrace{x_{s+\epsilon}, \dots, x_{s+\epsilon}}_{j \text{ times}}, x_s, \dots, x_s) - g_s(\underbrace{x_s, \dots, x_s}_{j \text{ times}}, x_{s+\epsilon}, \dots, x_{s+\epsilon}) \right) < 0 \right. \right\}.$$

For each $s_0 < s < s_1$, x_s is by assumption a stable, and in particular uniformly-stable, strategy in g_s , and $x_t \neq x_s$ for all $t \neq s$ in some neighborhood of s . Therefore, for $\epsilon \neq 0$ sufficiently close to 0 (including negative ϵ), $s \in U_\epsilon$ (cf. (46)). It follows that the Lebesgue measure of U_ϵ tends to $s_1 - s_0$ as ϵ tends to 0. The same is clearly true for the set $U_{-\epsilon}$, hence also for the set $U_{-\epsilon} - \epsilon = \{s - \epsilon \mid s \in U_{-\epsilon}\} \subseteq [s_0, s_1]$ (which has the same measure as $U_{-\epsilon}$, since it is obtained from it by translation), and hence also for the set

$$V_\epsilon = U_\epsilon \cap (U_{-\epsilon} - \epsilon).$$

Therefore, for (Lebesgue-)almost every $s_0 < s < s_1$, $s \in V_{1/k}$ for infinitely many positive integers k . For each such k , the strict inequality in (51) holds for $\epsilon = 1/k$ (since $s \in U_{1/k}$) and a similar inequality holds for $\epsilon = -1/k$ with s replaced by $s + 1/k$ (since $s + 1/k \in U_{-1/k}$). These inequalities imply that

$$\begin{aligned}
& \left(s + \frac{1}{k} \right) \sum_{j=1}^n \left(g_s \left(\underbrace{x_{s+\frac{1}{k}}, \dots, x_{s+\frac{1}{k}}}_{j \text{ times}}, x_s, \dots, x_s \right) - g_s \left(\underbrace{x_s, \dots, x_s}_{j \text{ times}}, x_{s+\frac{1}{k}}, \dots, x_{s+\frac{1}{k}} \right) \right) \\
& + s \sum_{j=1}^n \left(g_{s+\frac{1}{k}} \left(\underbrace{x_s, \dots, x_s}_{j \text{ times}}, x_{s+\frac{1}{k}}, \dots, x_{s+\frac{1}{k}} \right) - g_{s+\frac{1}{k}} \left(\underbrace{x_{s+\frac{1}{k}}, \dots, x_{s+\frac{1}{k}}}_{j \text{ times}}, x_s, \dots, x_s \right) \right) < 0.
\end{aligned} \tag{52}$$

It follows from (64), with $t = s + 1/k$, $x = x_s$ and $y = x_t$, that the expression on the left-hand side of inequality (52) is equal to $(1/k)(g(x_{s+1/k}, \dots, x_{s+1/k}) - g(x_s, \dots, x_s))$.

Therefore, that inequality is equivalent to

$$f\left(s + \frac{1}{k}\right) - f(s) < 0. \tag{53}$$

If (53) holds for infinitely many k 's and f is differentiable at s , then $f'(s) \leq 0$. Since f , being an absolutely continuous function, is differentiable almost everywhere in $[s_0, s_1]$ and satisfies

$$f(t) = f(s_0) + \int_{s_0}^t f'(s) ds, \quad s_0 \leq t \leq s_1$$

(Yeh, 2006, Theorem 13.17), this proves that f is nonincreasing. To prove that f is in fact strictly decreasing it suffices to show that there is no subinterval of (s_0, s_1) in which f is constant. Any point s lying in such a subinterval satisfies $f(s + 1/k) - f(s) = 0$ for all large enough k , which implies that $s \in V_{1/k}$ for at most finitely many positive integers k . Since it is shown above that the latter property does *not* hold for almost all $s \in (s_0, s_1)$, this proves that such a subinterval does not exist.

If, for each $s_0 < s < s_1$, x_s is a weakly stable strategy in g_s , then the conclusion that f is nonincreasing still holds, and the only change required in the above proof is changing the strict inequalities in (51), (52) and (53) to weak ones. (Strict inequalities are only required for the part showing strict monotonicity.) If each x_s is definitely unstable, then a proof very similar to that above shows that f is strictly increasing. The only change required here is reversing the strict inequalities in (51), (52) and (53).

Let h , s_0 , s_1 and the function $s \mapsto x_s$ now be as in the second part of the theorem. By Definition 5, for each $s_0 < s < s_1$ the strategy profile x_s is a stable strategy in the symmetric game obtained by symmetrizing the modified game h_s . By Proposition 10, that symmetric game is the modified game g_s , where g is the game obtained by symmetrizing h . By (48), $g(x_s, x_s, \dots, x_s) = \bar{h}(x_s)$. It therefore follows immediately from the first part of the theorem that the last expression is a strictly decreasing function of s .

The proofs for the cases of weakly stable or definitely unstable strategies are similar.

Finally, it remains to note that the assumption that each x_s is stable, weakly stable or definitely unstable is nowhere fully used in the above proof, which only uses the implication

that x_s is uniformly-stable, weakly uniformly-stable or definitely uniformly-unstable, respectively. Hence, to obtain the corresponding result, it suffices to assume that. ■

For $n = 2$, stability and uniform-stability are equivalent, since the only probability vector p satisfying (41) is (45). This is not so for general n , where stability (and even dependent- or independent-stability) can be a more stringent requirement than uniform-stability. For example, this is so for certain symmetric four-player $3 \times 3 \times 3 \times 3$ games (Milchtaich, 2008, Example 2).

There are, however, classes of games for which stability and uniform-stability are essentially equivalent for any number of players n . One such class is games with unidimensional strategy spaces. In the two-player case, the differential condition for stability or definite instability of an equilibrium strategy in a symmetric such game is given by Proposition 1. A very similar result holds for all $n \geq 2$; the only difference is that the left-hand side of (9) is replaced by

$$g_{11}(x, x) + (n - 1)g_{12}(x, x).$$

Importantly, the same result holds with ‘stable’ and ‘definitely unstable’ replaced by ‘ p -stable’ and ‘definitely p -unstable’, respectively, for *any* probability vector p satisfying (41) (Milchtaich, 2008, Theorem 2). This shows, in particular, that stability, dependent-stability, independent-stability and uniform-stability are essentially equivalent for this class of symmetric games. The same is true for asymmetric games, for which the obvious n -player generalization of Proposition 7 holds for all $n \geq 2$, and ‘stability’, ‘weak stability’ and ‘definite instability’ can again be replaced by ‘ p -stability’, ‘weak p -stability’ and ‘definitely p -instability’, respectively, for any probability vector p satisfying (41) (Milchtaich, 2008, Theorem 4).

The similarity between the two-player case and the multiplayer case of games with unidimensional strategy spaces extends to comparative statics. This is true for both symmetric and asymmetric games. That is, the obvious n -player generalizations of Propositions 4 and 8 hold, with Eq. (17) in the former replaced by

$$\frac{df}{ds} = ns((g_s)_{11} + (n - 1)(g_s)_{12}) \left(\frac{dx_s}{ds} \right)^2. \quad (54)$$

The proofs of these generalizations are virtually identical to those given in the $n = 2$ case. The following example illustrates the result concerning symmetric games.

Example 4. *Symmetric multiplayer public good game.* Each of the $n (\geq 2)$ players is endowed with a unit amount of private good, and can contribute any part $0 \leq x \leq 1$ of it for the production of a public good. The amount of public good produced depends on the players’ total contribution of private good, or equivalently on the average contribution \bar{x} . Specifically, it is given by $\phi(\bar{x})$, where the production function $\phi: [0,1] \rightarrow \mathfrak{R}$ is increasing and has a negative and continuous second derivative. The payoff of a player contributing x is given by

$$g = 1 - x + \phi(\bar{x}).$$

Therefore, the average payoff is $1 - \bar{x} + \phi(\bar{x})$, and for every s the corresponding modified payoff is given by

$$g_s = 1 - (sx + (1 - s)\bar{x}) + \phi(\bar{x}).$$

This satisfies $(g_s)_{11} + (n - 1)(g_s)_{12} = (1/n)\phi'' < 0$, which by Eq. (54) suggests that the equilibrium payoff in the modified game decreases with increasing s . A direct proof for this can be given as follows. Since ϕ is strictly concave, a necessary and sufficient condition for the existence of a symmetric equilibrium in the modified game in which all the players contribute the same amount $0 < x_s < 1$ is that x_s satisfies the first-order condition

$$\phi'(x_s) = 1 + (n - 1)s. \quad (55)$$

If this equation has a solution x_s in $(0,1)$ (which may or may not be the case, depending on the extreme values of ϕ'), then it is unique. Moreover, in this case, a unique solution exists for every value of the selfishness coefficient sufficiently close to s , the solution depends on the selfishness coefficient as a differentiable function, and its derivative satisfies

$$\frac{dx_s}{ds} = \frac{n - 1}{\phi''(x_s)} < 0.$$

This shows that selfishness reduces the players' contributions. This reduction decreases social welfare, since by (55), $\phi'(x_s) > 1$ for $s > 0$, which means that the social gain from converting private good into public good exceeds the loss for the contributing player. More specifically, by (54), for $s > 0$

$$\frac{df}{ds} = (n - 1)^2 \frac{s}{\phi''(x_s)} < 0, \quad (56)$$

where $f(s)$ is the players' equilibrium payoff for the selfishness coefficient s .

The effect of selfishness on social welfare in the case of a *convex* production function cannot be examined in a setting as in Example 4, since convexity entails that interior equilibria do not exist. However, interior equilibria may exist in a related all-or-nothing model, in which a player randomizes between contributing his entire endowment of private good or nothing at all. This model, which is described in the following example, is an n -player generalization of the one in Example 3.

Example 5. *Symmetric multilinear public good game.* The example is similar to Example 4 except that the players' actual contributions of private good can only be 0 or 1 and the second derivative of the production function ϕ is either negative everywhere (so that ϕ is strictly concave) or positive everywhere (so that ϕ is strictly convex). The players' (mixed) strategies x, x', x'', \dots , which can be viewed as independent Bernoulli random variables, specify the contribution probabilities. The player using strategy x contributes or does not contribute if $x = 1$ or $x = 0$, respectively. Thus, the probability of contribution is equal to the expectation $E[x]$. The strategy is pure if $x = 1$ almost surely or $x = 0$ almost surely; otherwise, it is completely mixed. The average contribution $\bar{x} = (1/n)(x + x' + x'' + \dots)$ and the amount of public good produced $\phi(\bar{x})$ are also random variables. For a selfishness coefficient s , the (expected) payoff and modified payoff of the player using strategy x are

given by

$$g = 1 - E[x] + E[\phi(\bar{x})]$$

and

$$g_s = 1 - E[sx + (1 - s)\bar{x}] + E[\phi(\bar{x})]. \quad (57)$$

Suppose now that x is a completely mixed equilibrium strategy in the modified game. Whether local comparative statics is increasing or decreasing depends on the stability of x . To check whether x is stable, take a strategy $y \neq x$ and replace g in (42) by g_s . For $j = 1$, the expression in parenthesis in (42) is 0, since in a completely mixed equilibrium any strategy is a best response. This implies that, for $j > 1$, the expression is *not* 0. This can be seen by writing it as

$$(E[y] - E[x]) \left(- \left(s + \frac{1-s}{n} \right) + E \left[\phi \left(\bar{x} - \frac{x}{n} + \frac{1}{n} \right) - \phi \left(\bar{x} - \frac{x}{n} \right) \right] \right), \quad (58)$$

where the random variable $\bar{x} - x/n$ is the average contribution of $n - j$ players who use strategy x and $j - 1$ players who use y . If $E[y] > E[x]$, then for $j > 1$ the distribution of $\bar{x} - x/n$ strictly first-order stochastically dominates the distribution for $j = 1$, and therefore (58), which is 0 for $j = 1$, is negative for $j > 1$ if ϕ is strictly concave (i.e., decreasing marginal product) and positive if ϕ is strictly convex (increasing marginal product). If $E[y] < E[x]$, the signs are reversed. It follows that, regardless of whether $E[y]$ is more or less than $E[x]$, and for any probability vector p with $p_1 < 1$, inequality (42) or the reverse inequality holds if the production function is strictly concave or strictly convex, respectively. This proves, in particular, that if a completely mixed equilibrium strategy in g_s exists, then it is stable in the former case and definitely unstable in the latter.

Parenthetically, these considerations show that for a completely mixed equilibrium strategy x in the modified game, stability, dependent-stability, independent-stability and uniform-stability are equivalent: they all hold if ϕ is concave but not if it is convex. It follows from a stochastic dominance argument similar to that in the previous paragraph that all these kinds of stability have the same meaning. Namely, if one or more of the players deviate from x to a strategy or strategies prescribing a lower (higher) probability of contribution, then the unique best response for each of the remaining players is to contribute (respectively, not to contribute) with probability 1. It can be shown that this is also the meaning of x being an evolutionarily stable strategy (ESS) in the modified game (in the sense of Broom et al., 1997; see also Milchtaich, 2008). However, these equivalences are specific to the game considered in this example; they do not hold in general. For an equilibrium strategy x in a general symmetric multilinear game with more than two players, the conditions for stability, dependent-stability, uniform-stability and being an ESS are progressively less stringent (Milchtaich, 2008, Theorem 1).

Stability or definite instability of the completely mixed equilibrium strategies suggests that local comparative statics in Example 5 are decreasing or increasing, respectively. This suggestion can be directly verified. Specifically, it can be shown that if a completely mixed equilibrium strategy x in g_s exists for some $s \geq 0$, then a unique such strategy exists for

every value of the selfishness coefficient close to s , and the derivative of the equilibrium payoff with respect to this coefficient is given by

$$\frac{df}{ds} = \frac{n-1}{n} \frac{s}{\sum_{k=0}^{n-2} \binom{n-2}{k} E[x]^k (1-E[x])^{n-2-k} \left[\phi\left(\frac{k+2}{n}\right) - 2\phi\left(\frac{k+1}{n}\right) + \phi\left(\frac{k}{n}\right) \right]}$$

Since ϕ'' is by assumption a continuous function, standard calculus gives that the expression on the right-hand side is equal to

$$n(n-1) \frac{s}{\phi''(\xi)}, \quad (59)$$

for some $0 < \xi < 1$. This proves that, as suggested by stability considerations, local comparative statics are decreasing or increasing if the production function is strictly concave or strictly convex, respectively. Figure 3 illustrates this result in the two-player case.

Interestingly, the coefficient in (59) is greater by a factor of $n/(n-1)$ than the corresponding one in (56). This implies, for example, that if the production function of the public good is quadratic and concave, then the effect of changing the selfishness coefficient on social welfare is greater (if $n = 2$, twice as large) here than in the model in Example 4. This, however, refers only to the rates of change. For a *fixed* degree of altruism, the equilibrium payoff in one environment may or may not be greater than in the other. For a review of the experimental economics literature concerning both kinds of public good environments, see Ledyard (1995).

7 Discussion

It has long been known that altruism may theoretically lead to socially inefficient outcomes in asymmetric two-player strategic interactions, even if both individuals are equally altruistic towards each other (Lindbeck and Weibull, 1988; Corts, 2006). As the examples in this paper show, altruism may result in Pareto inferior outcomes also in symmetric two-player interactions. That is, the players' material gains may be higher if they are both selfish (i.e., have $s = 1$) rather than mildly caring ($0 < s < 1$), and even higher if they resent one another ($s > 1$). The main message of the paper is that, in both symmetric and asymmetric games, such increasing, "paradoxical" comparative statics are unlikely if the equilibria or equilibrium strategies involved are statically stable, i.e., stable in the sense of Definitions 1, 3, 4 or 5 (whichever is relevant for the kind of games considered).

A corollary of this finding is that increasing comparative statics are unlikely in groups or societies in which the dynamics of strategy choices tend to exclude statically unstable equilibria or equilibrium strategies. In other words, if *dynamic* stability implies static stability, then increasing selfishness is only likely to make the group members better off. This result may be particularly significant for groups that compete with each other, so that the effect of altruism on social welfare may affect the group's ability to survive in the long run. In this case, the above finding suggests that altruism may be favored by group selection. Thus, dynamic stability, which refers to *intragroup* dynamics, may be consequential for *intergroup* dynamics.

Whether or not dynamic stability implies static stability generally depends on the particular dynamics (Milchtaich, 2008). This is illustrated by the case of symmetric $n \times n$ games, for which the notion of static stability considered in this paper coincides with evolutionary stability (Proposition 2). In an animal population in which such a game g is played between pairs of related individuals with the same coefficient of relatedness r (e.g., full siblings, with $r = 0.5$, which corresponds to the selfishness coefficient $s = (1 - 0.5)/(1 + 0.5) = 1/3$; see Section 2), the dynamics are governed by mutation and natural selection. A strategy may be considered dynamically stable if it is *uninvadable* in the sense that, if all members of the population adopt it, no mutant strategy can invade. An uninvadable strategy is necessarily an ESS in the corresponding modified game g_s (but not conversely; see Hines and Maynard Smith, 1979; Milchtaich, 2006a). As explained above, this means that in games between relatives in nature, increasing comparative statics are unlikely. That is, if in a different population the same game g is played between somewhat less closely related individuals, the outcome is likely to be either the same as or worse than in the first population.

An alternative notion of dynamic stability in symmetric $n \times n$ games, which is weaker (rather than stronger) than the static notion of evolutionary stability, is asymptotic stability under the continuous-time replicator dynamics (Hofbauer and Sigmund, 1998). This refers, more precisely, not to the stability of the strategies used by individual players, which (in the simple version considered here) are pure strategies, but to the stability of their frequencies in the population, x_1, x_2, \dots, x_n . For a game with a payoff matrix $A = (a_{ij})$, the rate of change \dot{x}_i of (pure) strategy i ($= 1, 2, \dots, n$) is determined by the difference between the expected payoff from using strategy i and the mean payoff, according to the (replicator) equation

$$\dot{x}_i = x_i \left(\sum_{j=1}^n a_{ij} x_j - \sum_{j,k=1}^n a_{jk} x_j x_k \right).$$

Asymptotic stability with respect to the replicator dynamics does not preclude increasing comparative statics, and instability does not preclude decreasing comparative statics. For example, in a generalized rock–scissors–paper game, the equilibrium strategy is globally asymptotically stable under the continuous-time replicator dynamics if and only if the equilibrium payoff is positive (Hofbauer and Sigmund, 1998, Theorem 7.7.2; Weissing, 1991, Theorem 5.6). In this case, the population converges to the equilibrium strategy from any initial interior point (i.e., a completely mixed strategy). If the equilibrium payoff is negative, the equilibrium is unstable, and the system converges to the boundary of the strategy space from any initial point other than the equilibrium strategy itself. For the payoff matrix (21), the equilibrium payoff in the modified game g_s is positive for all $s > 1/3$, and for (22) it is negative. Hence, for (21) the corresponding equilibrium strategy is stable under the replicator dynamics (as well as under other natural dynamics; see Chamberland and Cressman, 2000), and for (22) it is unstable. However, as seen in Figure 2, the equilibrium payoff increases with increasing selfishness coefficient in the former case and decreases in the latter. This demonstrates the point made above: depending on the dynamics, the notion of static stability used in Theorem 1 and the other results in this paper may or may not be

implied by dynamic stability. If the implication does not hold, then statically unstable equilibria, and hence also “paradoxical” comparative statics, are not necessarily unlikely.

This paper does not explicitly consider the case of extreme selflessness, corresponding to a negative selfishness coefficient, $s < 0$. Players with such preferences would prefer their own share of the pie, i.e., of the total payoff, to be as small as possible, which is arguably not very realistic. Some of the paper’s results do not use the assumption that $s \geq 0$, and are therefore applicable to all s . In particular, Propositions 4 and 8 suggest that the effect of selfishness on social welfare for negative s is a mirror image of the effect for positive s – which implies that the two effects are qualitatively different. Although this finding holds only under certain conditions, e.g., those expressed in these two propositions, it does indicate that the assumption $s \geq 0$ cannot be dropped in the results concerning the quantitative properties of local comparative statics, e.g., Theorems 1, 4 and 5.

A noteworthy aspect of these quantitative properties is that they are evident for s arbitrarily close to 1, i.e., for low levels of altruism or spite. This may indicate that the assumption of linearity of the modified payoff in the players’ (unmodified) payoffs is not overly restrictive. Rather, as suggested by Levine (1998), the functional form (1) (or the equivalent one (3)) may be viewed as a linearization, valid in the limit of weak altruism or spite, of a potentially more complex function. According to this view, the coefficient of altruism r expresses the ratio between the *marginal* contributions to a player’s modified payoff of (i) the material payoff of any of the opponents and (ii) the player’s own material payoff. It may be conjectured that if this ratio is close to zero and is the same for all players, its effect on the average material equilibrium payoff is as described in this paper, which means that the effect is largely determined by the stability of the equilibria involved.

The model presented in Section 2 can easily be extended to games with a continuum of players. In this extension, \bar{h} in (3) is the integral average of the payoffs with respect to a specified probability measure on the set of players. This so-called *population measure*, which should be part of the specification of the game, gives the total weight, or “size”, of each set of players. A number of strategic interactions involving a large number of individuals, e.g., network congestion games (Milchtaich, 2006c), are naturally modeled as games with a continuum of players. Altruism may significantly affect the equilibrium outcomes in such games (Chen and Kempe, 2007). A systematic extension of the results in this paper to games with a continuum of players would require a corresponding extension of the notion of static stability, which is not currently available.

Appendix A: Identities

The following is a list of useful identities pertaining to modified games. The identities hold for all symmetric games g (with the number of players indicated by the number of arguments), asymmetric two-player games h , strategies x and y , strategy profiles (x^1, x^2) and (y^1, y^2) , and numbers r , s and t . All the identities follow from (3), (12) and (13) after some algebra.

$$g_s(y, y) - g_s(x, y) - g_s(y, x) + g_s(x, x) = g(y, y) - g(x, y) - g(y, x) + g(x, x) \quad (60)$$

$$g_s(y, x) - g_s(x, x) + g_s(y, y) - g_s(x, y) \quad (61)$$

$$= 2(g_s(y, x) - g_s(x, x)) + (g(y, y) - g(x, y) - g(y, x) + g(x, x))$$

$$(t - s)(g(y, y) - g(x, x)) \quad (62)$$

$$= (t - s)(g_s(y, x) - g_s(x, x)) + 2s(g_t(x, y) - g_t(y, y))$$

$$+ (s + t)(g_s(y, y) - g_s(x, y))$$

$$(t - s)(g(y, y) - g(x, x)) \quad (63)$$

$$= 2t(g_s(y, x) - g_s(x, x)) + 2s(g_t(x, y) - g_t(y, y))$$

$$+ (s + t)(g(y, y) - g(x, y) - g(y, x) + g(x, x))$$

$$(t - s)(g(y, y, \dots, y) - g(x, x, \dots, x)) \quad (64)$$

$$= t \sum_{j=1}^n \left(g_s(\underbrace{y, \dots, y}_j, x, \dots, x) - g_s(\underbrace{x, \dots, x}_j, y, \dots, y) \right)$$

$$+ s \sum_{j=1}^n \left(g_t(\underbrace{x, \dots, x}_j, y, \dots, y) - g_t(\underbrace{y, \dots, y}_j, x, \dots, x) \right)$$

$$(t - s)(\bar{h}(y^1, y^2) - \bar{h}(x^1, x^2)) \quad (65)$$

$$= t(h_s^1(y^1, x^2) - h_s^1(x^1, x^2)) + t(h_s^2(x^1, y^2) - h_s^2(x^1, x^2))$$

$$+ s(h_t^1(x^1, y^2) - h_t^1(y^1, y^2)) + s(h_t^2(y^1, x^2) - h_t^2(y^1, y^2))$$

$$+ (s + t)(\bar{h}(y^1, y^2) - \bar{h}(x^1, y^2) - \bar{h}(y^1, x^2) + \bar{h}(x^1, x^2))$$

$$g_s(x, y) - g_t(y, x) = \frac{s + t}{2}(g(x, y) - g(y, x)) \quad (66)$$

$$(t - s)g_r(x, y) + (s - r)g_t(x, y) + (r - t)g_s(x, y) = 0 \quad (67)$$

Appendix B: Monotonicity

As Example 2 shows, in different symmetric $n \times n$ games social welfare may decrease or increase with increasing selfishness. This raises the question of whether the change is always monotonic or whether, for example, social welfare can first decrease and then increase. If the modified games involved have multiple symmetric equilibria, the answer may depend on which equilibria are looked at. Two possibilities are: (i) the efficient symmetric equilibria, i.e., those giving the highest equilibrium payoff in each modified game, or (ii) the completely mixed symmetric equilibria. The answers to these two versions of the above question turn out to depend on whether the number of pure strategies n is two, three or more. The next proposition concerns the efficient symmetric equilibria and the one following it concerns the completely mixed ones.

Proposition A1. Let g be a symmetric $n \times n$ game. For $s \geq 0$, define

$$F(s) = \max_{x_s} g(x_s, x_s),$$

where the maximum is taken over all equilibrium strategies x_s in the modified game g_s . If $n = 2$, the function $F(s)$ is monotonic, indeed nonincreasing. However, if $n \geq 3$, the function is not necessarily monotonic.

Proof. The payoff matrix of a symmetric 2×2 game g can always be given the following normalized form, by subtracting a constant from all entries and relabeling pure strategies if necessary:

$$\begin{pmatrix} 0 & b \\ -c & b - c + d \end{pmatrix}, \quad (68)$$

for some b, c and d with $b - c + d \geq 0$. The normalization does not affect the difference between the sums of the two diagonal entries and the two off-diagonal entries of the payoff matrix, which is equal to d in both the normalized and unnormalized forms. For $s \geq 0$, the payoff matrix of g_s is

$$\begin{pmatrix} 0 & b_s \\ -c_s & b_s - c_s + d \end{pmatrix}, \quad (69)$$

where $b_s = sb + (1 - s)(b - c)/2$ and $c_s = sc + (1 - s)(c - b)/2$. By Proposition 6 and the remarks following it, if $d < 0$ (i.e., strategic substitutability), then selfishness decreases social welfare, and hence $F(s)$ is determined by s as a nonincreasing function. Suppose that $d \geq 0$. If both players use the same mixed strategy $x = (x_1, x_2)$ (with $x_1 + x_2 = 1$), their payoff can be written as

$$(b - c)x_2 + dx_2^2. \quad (70)$$

Since this is a convex, quadratic function of x_2 , it attains its maximum in the unit interval at one of the endpoints. Since by assumption $b - c + d$ is nonnegative, *this* is the maximum, and it is attained at $x_2 = 1$. If s is such that $c_s \leq d$, then there is a pure equilibrium strategy in g_s for which the equilibrium payoff is $b - c + d$. If $c_s > d$ (≥ 0), then the same pure strategy is strictly dominated, and the payoff at the unique equilibrium in g_s is 0. Since c_s is given by an affine function of s , and $c_0 = (c - b)/2 \leq d/2 \leq d$, either $c_s \leq d$ for all $s \geq 0$ or there is some $M \geq 0$ such that $c_s \leq d$ for all $0 \leq s \leq M$ and $c_t > d$ for all $t > M$. As shown above, for every such s and t , $F(s) = b - c + d \geq 0 = F(t)$. This proves that, for $d \geq 0$ as well as $d < 0$, the function $F(s)$ is monotonically nonincreasing (as illustrated by Figure 3).

The rock–scissors–paper game with the payoff matrix (21) is an example of a symmetric 3×3 game for which $F(s)$ is not monotonic. For $s > 1/3$ (in which range the equilibria are unique), $F(s)$ is increasing (Figure 2). However, by Corollary 2, $F(0) \geq F(s)$ for all s (which implies that for $s > 1/3$ the inequality is strict). ■

Proposition A2. For a symmetric $n \times n$ game g , and s_0 and s_1 with $0 \leq s_0 < s_1$, suppose that there is a continuous function that assigns to each $s_0 \leq s \leq s_1$ a completely mixed equilibrium strategy x_s in the modified game g_s . If $n = 2$, the equilibrium payoff $f(s)$ (defined in (16)) is a monotonic function of s in the interval $[s_0, s_1]$. The same is true if $n = 3$ and the payoff matrix is non-singular. However, if $n \geq 4$, then $f(s)$ is not necessarily monotonic even if the payoff matrix is non-singular.

Proof. Consider, first, the case $n = 2$. Suppose that the payoff matrix is normalized as in (68) and that not all entries are zero. (If $b = c = d = 0$, then $f(s)$ is identically zero, and hence monotonic.) Using (69), it is not difficult to show that for all but at most one value of $s_0 \leq s \leq s_1$ (namely, that for which $c_s = 0$), the completely mixed equilibrium strategy in g_s is given by $x_s = (1 - c_s/d, c_s/d)$. Therefore, (70) gives

$$f(s) = (b - c) \frac{c_s}{d} + d \left(\frac{c_s}{d} \right)^2 = \frac{(b + c)^2}{4d} s^2 - \frac{(b - c)^2}{4d}.$$

This is a monotonic, quadratic function of s .

Next, suppose that $n = 3$ and that g has a non-singular payoff matrix A . For $s_0 \leq s \leq s_1$, the payoff matrix A_s of the modified game g_s is given by $A_s = (1/2)(1 + s)A + (1/2)(1 - s)A^T$, where the matrix A^T is A transpose. The determinant of A_s is determined by s as a polynomial $P(s)$ with a degree of three at most. In fact, the degree is even: Since the determinant of a matrix is equal to that of the transpose matrix, $P(s) = P(-s)$ for all s , which implies that $P(s) = \alpha s^2 + \beta$, for some coefficients α and β . These coefficients are not both zero, since the assumption that A is non-singular gives $P(1) = \det A \neq 0$. Therefore, $P(s) \neq 0$ for all but at most two values of s . By similar reasoning, the polynomial $Q(s) = \det(A_s - E) = \det((1/2)(1 + s)(A - E) + (1/2)(1 - s)(A - E)^T)$, where E is the 3×3 matrix with all the entries equal to 1, has the form $Q(s) = \gamma s^2 + \delta$, for some coefficients γ and δ . If s is such that $\det A_s = P(s) \neq 0$, then it follows as a special case from the Lemma in Milchtaich (2006b) that $f(s)$, the completely mixed equilibrium payoff in the game with the payoff matrix A_s , satisfies the equation

$$\frac{f(s) - 1}{f(s)} = \frac{\det(A_s - E)}{\det A_s} = \frac{Q(s)}{P(s)}.$$

Solving for $f(s)$ and using the above expressions for $P(s)$ and $Q(s)$ gives:

$$f(s) = \frac{\alpha s^2 + \beta}{(\alpha - \gamma)s^2 + (\beta - \delta)}.$$

This expression is a monotonic function of s . In fact, it is decreasing, increasing, or constant if the product $\alpha\delta$ is greater than, less than, or equal to $\beta\gamma$, respectively.

The last assertion of the proposition is demonstrated by the 4×4 game g with the following (non-singular) payoff matrix:

$$\begin{pmatrix} 2 & 5 & 1 & 0 \\ -7 & -2 & 9 & 8 \\ -3 & 7 & 9 & -9 \\ 9 & 2 & -4 & -5 \end{pmatrix}$$

It can be shown that the modified game g_s has a unique completely mixed equilibrium strategy for every $0 \leq s \leq 1$. Straightforward computation shows that the equilibrium payoff decreases for $0 < s < 0.263$ but increases for $0.263 < s < 1$. Thus, it is not monotonic. ■

Proposition A2 shows that, for general symmetric $n \times n$ games, the payoff at the completely mixed symmetric equilibria in the modified games is not necessarily monotonically related to the selfishness coefficient. Nevertheless, the finding that monotonicity does apply for 2×2 games can be generalized. The key is the fact that all symmetric 2×2 games are *potential games*. A game with a finite number of players is called a potential game if each player's payoff can be expressed as the sum of two functions, such that the first function (called the potential) is the same for all players and the second function only depends on the strategies of the other players. A necessary and sufficient condition for this is that, for every simple closed path of length four in the space of strategy profiles (in which a particular player switches from some strategy x to another strategy y , then another player switches from some strategy x' to another strategy y' , then the first player returns to x , and finally the second player returns to x' , completing the cycle), the changes in the deviating players' payoffs sum up to 0 (Monderer and Shapley, 1996, Corollary 2.9). For a symmetric $n \times n$ game g , this condition reads:

$$\begin{aligned} & (g(y, x') - g(x, x')) + (g(y', y) - g(x', y)) \\ & + (g(x, y') - g(y, y')) + (g(x', x) - g(y', x)) = 0. \end{aligned} \quad (71)$$

In fact, for g to be a potential game, it is necessary and sufficient that (71) holds for all *pure* strategies x, y, x' and y' (Monderer and Shapley, 1996, Lemma 2.10). This condition implies that the $(n - 1) \times (n - 1)$ matrix defining the quadratic form G in (28) is symmetric. In fact, it is not difficult to show that the above condition is *equivalent* to the symmetry of that matrix, and therefore holds trivially if $n = 2$ (but not if $n = 3$). The following proposition shows that if g is a potential game, then completely mixed equilibrium strategies corresponding to different values of the selfishness coefficient lie side-by-side along a straight line in the strategy space. In other words, the line segment connecting any pair of such strategies consists of equilibrium strategies corresponding to intermediate values of the selfishness coefficients. The players' payoff along that line changes monotonically.

Proposition A3. For a symmetric $n \times n$ potential game g , and s_0 and s_1 with $0 \leq s_0 < s_1$, let x_{s_0} and x_{s_1} be completely mixed equilibrium payoffs in the modified games g_{s_0} and g_{s_1} , respectively. For every $s_0 \leq s \leq s_1$, the completely mixed strategy

$$x_s = \frac{s_1 - s}{s_1 - s_0} x_{s_0} + \frac{s - s_0}{s_1 - s_0} x_{s_1} \quad (72)$$

is an equilibrium strategy in g_s . The equilibrium payoff is equal to

$$\frac{s^2 - s_0^2}{s_1^2 - s_0^2} (g(x_{s_1}, x_{s_1}) - g(x_{s_0}, x_{s_0})) + g(x_{s_0}, x_{s_0}), \quad (73)$$

and is thus determined by s as a monotonic, quadratic function in the interval $[s_0, s_1]$.

Proof. For every s , strategy x and pure strategy j , it follows from the identity (66) (used twice, with $t = s_1$) that

$$\begin{aligned}
& 2 \left(g_s(x_{s_1}, j) - g_{s_1}(j, x_{s_1}) \right) + 2 \left(g_s(j, x) - g_{s_1}(x, j) \right) \\
& = (s + s_1) \left(g(x_{s_1}, j) - g(j, x_{s_1}) + g(j, x) - g(x, j) \right).
\end{aligned} \tag{74}$$

It follows from (71), applied to $y = x_{s_1}$, $x' = j$ and $y' = 1$, that the right-hand side of (74) is equal to

$$(s + s_1) \left(g(x_{s_1}, 1) - g(1, x_{s_1}) + g(1, x) - g(x, 1) \right).$$

This expression does not involve j . Since x_{s_1} is a completely mixed equilibrium strategy, $g_{s_1}(j, x_{s_1})$ on the left-hand of (74) is equal to $g_{s_1}(x_{s_1}, x_{s_1})$, which also does not involve j . Therefore, it follows from (74) that $g_s(j, x)$ does not depend on j if and only if the same is true for the difference

$$g_s(x_{s_1}, j) - g_{s_1}(x, j).$$

Clearly, a sufficient condition for this is that the difference is zero, i.e.,

$$g_s(x_{s_1}, j) = g_{s_1}(x, j), \quad j = 1, 2, \dots, n. \tag{75}$$

This condition is also necessary, since if the above difference does not depend on j , then

$$\begin{aligned}
\sum_{j=1}^n \left(g_s(x_{s_1}, j) - g_{s_1}(x, j) \right) (x_{s_1})_j &= g_s(x_{s_1}, x_{s_1}) - g_{s_1}(x, x_{s_1}) = g_{s_1}(x_{s_1}, x_{s_1}) - g_{s_1}(x, x_{s_1}) \\
&= 0,
\end{aligned}$$

where the second equality follows from (13) and the third one follows from the assumption that the equilibrium strategy x_{s_1} is completely mixed. This shows that the condition that $g_s(j, x)$ does not depend on j is equivalent to (75), which proves the following.

Claim. For $s_0 \leq s \leq s_1$, a strategy x has the property that *every* strategy is a best response to it in g_s if and only if it satisfies (75). A completely mixed strategy x has that property if and only if it is an equilibrium strategy in g_s . In particular, strategies x_{s_0} and x_{s_1} satisfy

$$g_{s_0}(x_{s_1}, j) = g_{s_1}(x_{s_0}, j), \quad j = 1, 2, \dots, n. \tag{76}$$

To complete the proof of the proposition, fix some $s_0 \leq s \leq s_1$. For every pure strategy j ,

$$\begin{aligned}
g_s(x_{s_1}, j) &= \frac{s_1 - s}{s_1 - s_0} g_{s_0}(x_{s_1}, j) + \frac{s - s_0}{s_1 - s_0} g_{s_1}(x_{s_1}, j) \\
&= \frac{s_1 - s}{s_1 - s_0} g_{s_1}(x_{s_0}, j) + \frac{s - s_0}{s_1 - s_0} g_{s_1}(x_{s_1}, j) \\
&= g_{s_1} \left(\frac{s_1 - s}{s_1 - s_0} x_{s_0} + \frac{s - s_0}{s_1 - s_0} x_{s_1}, j \right),
\end{aligned} \tag{77}$$

where the first equality follows from the identity (67), with $r = s_0$ and $t = s_1$, the second equality follows from (76), and the third from the bilinearity of g_{s_1} . By the Claim, (77) proves that the completely mixed strategy x_s defined in (72) is an equilibrium strategy in g_s . The equilibrium payoff can be computed as follows. Since $x_s = (1 - \theta)x_{s_0} + \theta x_{s_1}$, with

$$\theta = (s - s_0)/(s_1 - s_0),$$

$$\begin{aligned} g(x_s, x_s) &= (\theta^2 - \theta) \left(g(x_{s_1}, x_{s_1}) - g(x_{s_0}, x_{s_1}) - g(x_{s_1}, x_{s_0}) + g(x_{s_0}, x_{s_0}) \right) \\ &\quad + \theta \left(g(x_{s_1}, x_{s_1}) - g(x_{s_0}, x_{s_0}) \right) + g(x_{s_0}, x_{s_0}) \\ &= (\theta^2 - \theta) \frac{s_1 - s_0}{s_0 + s_1} \left(g(x_{s_1}, x_{s_1}) - g(x_{s_0}, x_{s_0}) \right) \\ &\quad + \theta \left(g(x_{s_1}, x_{s_1}) - g(x_{s_0}, x_{s_0}) \right) + g(x_{s_0}, x_{s_0}) \\ &= \frac{s^2 - s_0^2}{s_1^2 - s_0^2} \left(g(x_{s_1}, x_{s_1}) - g(x_{s_0}, x_{s_0}) \right) + g(x_{s_0}, x_{s_0}), \end{aligned}$$

where the second equality follows from the assumption that the equilibrium strategies x_{s_1} and x_{s_0} are completely mixed and an identity similar to (63), in which s is replaced by s_0 and t by s_1 . ■

Appendix C: Strong Stability

A stable equilibrium strategy in a symmetric $n \times n$ game is normally also “strongly” stable in the sense that a continuous deformation of the payoff matrix changes the equilibrium strategy in a continuous manner. Specifically, the following lemma, which is essentially due to Selten (1983), shows that every *regular* ESS has this property. An ESS is said to be regular if every action that is a best response to it is in its support (equivalently, if the corresponding symmetric equilibrium is quasi-strict; see van Damme, 1991). The proposition has a corollary for local comparative statics; see Section 4.2.

Proposition A4. Let x be a regular ESS in a symmetric $n \times n$ game with a payoff matrix A . There is a neighborhood V of x in the strategy space and a neighborhood U of A in \mathfrak{R}^{n^2} such that:

- (i) every symmetric $n \times n$ game g with a payoff matrix in U has a unique equilibrium strategy in V ,
- (ii) that strategy is a regular ESS, its support is equal to that of x , and it is the only equilibrium strategy in g with that support, and
- (iii) the mapping from U to V thus defined is continuous.

Proof. According to the regularity assumption, every action i that is not in the support of x (i.e., $x_i = 0$) is not a best response to x . Therefore, there are neighborhoods V' and U' of x and A , respectively, such that every such action i is also not a best response to any strategy $y \in V'$ in any symmetric $n \times n$ game with a payoff matrix $B \in U'$. In particular, if y is an equilibrium strategy for B , its support is necessarily contained in that of x . Let $V \subseteq V'$ be a closed neighborhood of x that includes only strategies whose support *contains* that of x (i.e., $y_i > 0$ for every $y \in V$ and every action i with $x_i > 0$).

The regularity of x implies that it is an *essential* ESS (van Damme, 1991, Theorem 9.3.6). That is, every symmetric $n \times n$ game with a payoff matrix close to A has an ESS close to x . In particular, there is a neighborhood $U \subseteq U'$ of A such that every symmetric $n \times n$ game with a payoff matrix $B \in U$ has some ESS $y \in V$. As shown above, the support of y – indeed, of

any equilibrium strategy for B lying in V – coincides with that of x . Since the support of an ESS cannot coincide with that of any other equilibrium strategy in the same game (van Damme, 1991, Lemma 9.2.4), conditions (i) and (ii) in the proposition hold. The mapping assigning to each element of U the set of its equilibrium strategies in V is clearly upper semicontinuous. Since by (i) this mapping is singleton-valued, condition (iii) also holds. ■

References

- Bergstrom, T. C. (1995). On the evolution of altruistic ethical rules for siblings. *American Economic Review* **85**, 58–81.
- Bernheim, B. D. and Stark, O. (1988). Altruism within the family reconsidered: Do nice guys finish last? *American Economic Review* **78**, 1034–1045.
- Bester, H. and Güth, W. (1998). Is altruism evolutionarily stable? *Journal of Economic Behavior and Organization* **34**, 193–209.
- Bolton, G. E. and Ockenfels, A. (2000). ERC: A theory of equity, reciprocity and competition. *American Economic Review* **90**, 166–193.
- Broom, M., Cannings, C., and Vickers, G. T. (1997). Multi-player matrix games. *Bulletin of Mathematical Biology* **59**, 931–952.
- Bulow, J. I., Geanakoplos, J. D., and Klemperer, P. D. (1985). Multimarket oligopoly: Strategic substitutes and complements. *Journal of Political Economy* **93**, 488–511.
- Chamberland, M., and Cressman, R. (2000). An example of dynamic (in)consistency in symmetric extensive form evolutionary games. *Games and Economic Behavior* **30**, 319–326.
- Chen, P.-A. and Kempe, D. (2007). Altruism and selfishness in traffic routing. The 45th Annual Allerton Conference on Communication, Control, and Computing, September 26–28, 2007. University of Illinois at Urbana-Champaign, IL, USA.
- Corts, K. S. (2006). When altruism lowers total welfare. *Economics and Philosophy* **22**, 1–18.
- Crow, J. F. and Kimura, M. (1970). *An Introduction to Population Genetics Theory*. Harper & Row, New York.
- Edgeworth, F. Y. (1881). *Mathematical Psychics*. Kegan Paul, London.
- Eshel, I. (1983). Evolutionary and continuous stability. *Journal of Theoretical Biology* **103**, 99–111.
- Eshel, I. and Motro, U. (1981). Kin selection and strong evolutionary stability of mutual help. *Theoretical Population Biology* **19**, 420–433.
- Fehr, E. and Schmidt, K. M. (1999). A theory of fairness, competition, and cooperation. *The Quarterly Journal of Economics* **114**, 817–868.

- Frank, R. H. (1988). *Passions within Reason*. Norton, New York.
- Frank, S. A. (1998). *Foundations of Social Evolution*. Princeton University Press, Princeton.
- Geanakoplos, J., Pearce, D., and Stacchetti, E. (1989). Psychological games and sequential rationality. *Games and Economic Behavior* **1**, 60–79.
- Hamilton, W. D. (1963). The evolution of altruistic behavior. *American Naturalist* **97**, 354–356.
- Hamilton, W. D. (1964). The genetical evolution of social behaviour. I and II. *Journal of Theoretical Biology* **7**, 1–52.
- Hamilton, W. D. (1970). Selfish and spiteful behaviour in an evolutionary model. *Nature* **228**, 1218–1220.
- Hines, W. G. S. and Maynard Smith, J. (1979). Games between relatives. *Journal of Theoretical Biology* **79**, 19–30.
- Hofbauer, J. and Sigmund, K. (1988). *The Theory of Evolution and Dynamical Systems*. London Mathematical Society Students Texts 7. Cambridge University Press, Cambridge, UK.
- Hofbauer, J. and Sigmund, K. (1998). *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge, UK.
- Kelly, J. L. (1955). *General Topology*. Springer, New York.
- Koçkesen, L., Ok, E. A., and Sethi, R. (2000). The strategic advantage of negatively interdependent preferences. *Journal of Economic Theory* **92**, 274–299.
- Ledyard, J. O. (1995). Public goods: A survey of experimental research. In J. H. Kagel and A. E. Roth (Eds.), *The Handbook of Experimental Economics*, pp. 111–194. Princeton University Press, Princeton, NJ.
- Levine, D. K. (1998). Modeling altruism and spitefulness in experiments. *Review of Economic Dynamics* **1**, 593–622.
- Lindbeck, A. (1992). Presentation Speech, 1970. In A. Lindbeck (Ed.), *Nobel Lectures in Economic Sciences (1969–1980) – The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel*, pp. 53–55. World Scientific, Singapore.
- Lindbeck, A. and Weibull, J. W. (1988). Altruism and time consistency: The economics of fait accompli. *Journal of Political Economy* **96**, 1165–1182.
- Maynard Smith, J. (1982). *Evolution and the Theory of Games*. Cambridge University Press, Cambridge, UK.
- Milchtaich, I. (2006a). [Comparative statics of games between relatives](#). *Theoretical Population Biology* **69**, 203–210.

- Milchtaich, I. (2006b). [Computation of completely mixed equilibrium payoffs in bimatrix games](#). *International Game Theory Review* **8**, 483–487.
- Milchtaich, I. (2006c). [Network topology and the efficiency of equilibrium](#). *Games and Economic Behavior* **57**, 321–346.
- Milchtaich, I. (2008). [Static stability in games](#). Bar-Ilan University, Department of Economics Working Paper 2008-04.
- Monderer, D. and Shapley, L. S. (1996). Potential games. *Games and Economic Behavior* **14**, 124–143.
- Morgan, J., Steiglitz, K., and Reis, G. (2003). The spite motive and equilibrium behavior in auctions. *Contributions to Economic Analysis & Policy* Volume 2, Issue 1, Article 5.
- Myerson, R. B., Pollock, G. B., and Swinkels, J. M. (1991). Viscous population equilibria. *Games and Economic Behavior* **3**, 101–109.
- Rabin, M. (1993). Incorporating fairness into game theory and economics. *American Economic Review* **83**, 1281–1302.
- Rana, I. K. (2002). *An Introduction to Measure and Integration*, Second Edition. Graduate Studies in Mathematics Vol. 45. American Mathematical Society, Providence, RI.
- Ridley, M. (1997). *The Origins of Virtue*. Viking, New York.
- Samuelson, P. A. (1983). *Foundations of Economic Analysis*, Enlarged Edition. Harvard University Press, Cambridge, MA.
- Selten, R. (1980). A note on evolutionarily stable strategies in asymmetric animal conflicts. *Journal of Theoretical Biology* **84**, 93–101.
- Selten, R. (1983). Evolutionary stability in extensive two-person games. *Mathematical Social Sciences* **5**, 269–363.
- van Damme, E. (1991). *Stability and Perfection of Nash Equilibria*, Second Edition. Springer-Verlag, Berlin.
- Weibull, J. W. (1995). *Evolutionary Game Theory*. MIT Press, Cambridge, MA.
- Weissing, F. J. (1991). Evolutionary stability and dynamic stability in a class of evolutionary normal form games. In R. Selten (ed.), *Game Equilibrium Models I: Evolution and Game Dynamics*, pp. 29–97. Springer-Verlag, Berlin.
- Yeh, J. (2006). *Real Analysis: Theory of Measure and Integration*, Second Edition. World Scientific, NJ.

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