

Berti, Patrizia; Pratelli, Luca; Rigo, Pietro

**Working Paper**

## Exchangeable Sequences Driven by an Absolutely Continuous Random Measure

Quaderni di Dipartimento, No. 142

**Provided in Cooperation with:**

University of Pavia, Department of Economics and Quantitative Methods (EPMQ)

*Suggested Citation:* Berti, Patrizia; Pratelli, Luca; Rigo, Pietro (2011) : Exchangeable Sequences Driven by an Absolutely Continuous Random Measure, Quaderni di Dipartimento, No. 142, Università degli Studi di Pavia, Dipartimento di Economia Politica e Metodi Quantitativi (EPMQ), Pavia

This Version is available at:

<https://hdl.handle.net/10419/95286>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

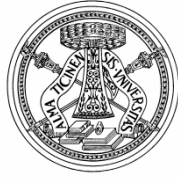
Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*



**Quaderni di Dipartimento**

**Exchangeable Sequences Driven by an Absolutely  
Continuous Random Measure**

Patrizia Berti  
(Università di Modena e Reggio Emilia)

Luca Pratelli  
(Accademia Navale di Livorno)

Pietro Rigo  
(Università di Pavia)

# 142 (03-11)

Dipartimento di economia politica  
e metodi quantitativi  
Università degli studi di Pavia  
Via San Felice, 5  
I-27100 Pavia

Marzo 2011

# EXCHANGEABLE SEQUENCES DRIVEN BY AN ABSOLUTELY CONTINUOUS RANDOM MEASURE

PATRIZIA BERTI, LUCA PRATELLI, AND PIETRO RIGO

ABSTRACT. Let  $S$  be a Polish space and  $(X_n : n \geq 1)$  an exchangeable sequence of  $S$ -valued random variables. Let  $\alpha_n(\cdot) = P(X_{n+1} \in \cdot \mid X_1, \dots, X_n)$  be the predictive measure and  $\alpha$  a random probability measure on  $S$  such that  $\alpha_n \xrightarrow{\text{weak}} \alpha$  a.s.. Two (related) problems are addressed. One is to give conditions for  $\alpha \ll \lambda$  a.s., where  $\lambda$  is a (non random)  $\sigma$ -finite Borel measure on  $S$ . Such conditions should concern the finite dimensional distributions  $\mathcal{L}(X_1, \dots, X_n)$ ,  $n \geq 1$ , only. The other problem is to investigate whether  $\|\alpha_n - \alpha\| \xrightarrow{\text{a.s.}} 0$ , where  $\|\cdot\|$  is total variation norm. Various results are obtained. Some of them do not require exchangeability, but hold under the weaker assumption that  $(X_n)$  is conditionally identically distributed, in the sense of [2].

## 1. TWO RELATED PROBLEMS

Throughout,  $S$  is a Polish space and

$$X = (X_1, X_2, \dots)$$

a sequence of  $S$ -valued random variables on the probability space  $(\Omega, \mathcal{A}, P)$ . We let  $\mathcal{B}$  denote the Borel  $\sigma$ -field on  $S$  and  $\mathbb{S}$  the set of probability measures on  $\mathcal{B}$ . A random probability measure on  $S$  is a map  $\alpha : \Omega \rightarrow \mathbb{S}$  such that  $\sigma(\alpha) \subset \mathcal{A}$ , where  $\sigma(\alpha)$  is the  $\sigma$ -field on  $\Omega$  generated by  $\omega \mapsto \alpha(\omega)(B)$  for all  $B \in \mathcal{B}$ .

For each  $n \geq 1$ , let  $\alpha_n$  be the  $n$ -th *predictive measure*. Thus,  $\alpha_n$  is a random probability measure on  $S$  and  $\alpha_n(\cdot)(B)$  is a version of  $P(X_{n+1} \in B \mid X_1, \dots, X_n)$  for all  $B \in \mathcal{B}$ . Define also  $\alpha_0(\cdot) = P(X_1 \in \cdot)$ .

If  $X$  is *exchangeable*, as assumed in this section, there is a random probability measure  $\alpha$  on  $S$  such that

$$\alpha_n(\omega) \xrightarrow{\text{weak}} \alpha(\omega) \quad \text{for almost all } \omega \in \Omega.$$

Such an  $\alpha$  also grants the usual representation

$$P(X \in B) = \int \alpha(\omega)^\infty(B) P(d\omega) \quad \text{for every Borel set } B \subset S^\infty$$

where  $\alpha(\omega)^\infty = \alpha(\omega) \times \alpha(\omega) \times \dots$

Let  $\lambda$  be a  $\sigma$ -finite measure on  $\mathcal{B}$ . Our *first problem* is to give conditions for

$$(1) \quad \alpha(\omega) \ll \lambda \quad \text{for almost all } \omega \in \Omega.$$

The conditions should concern the finite dimensional distributions  $\mathcal{L}(X_1, \dots, X_n)$ ,  $n \geq 1$ , only.

---

2000 *Mathematics Subject Classification.* 60G09, 60G42, 60G57, 62F15.

*Key words and phrases.* Conditional identity in distribution, Exchangeability, Predictive measure, Random probability measure.

While investigating (1), one meets another problem, of possible independent interest. Let  $\|\cdot\|$  denote total variation norm on  $(S, \mathcal{B})$ . Our *second problem* is to give conditions for

$$\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0.$$

## 2. MOTIVATIONS

Again, let  $X = (X_1, X_2, \dots)$  be exchangeable.

Reasonable conditions for (1) look of theoretical interest. They are of practical interest as well, as regards Bayesian nonparametrics. In this framework, the starting point is a prior  $\pi$  on  $\mathbb{S}$ . Since  $\pi = P \circ \alpha^{-1}$ , condition (1) means that the prior is supported by those  $\nu \in \mathbb{S}$  such that  $\nu \ll \lambda$ . This is a basic information for the subsequent statistical analysis. Roughly speaking, it means that the "underlying statistical model" consists of absolutely continuous laws.

From a foundational point of view, according to de Finetti, only assumptions on observable facts make sense. This is why the conditions for (1) have been requested to concern  $\mathcal{L}(X_1, \dots, X_n)$ ,  $n \geq 1$ , only. See [3], [5], [6], [7], [8].

A condition of this type is

$$(2) \quad \mathcal{L}(X_1, \dots, X_n) \ll \lambda^n \quad \text{for all } n \geq 1,$$

where  $\lambda^n = \lambda \times \dots \times \lambda$ . Clearly, (2) is necessary for (1). A (natural) question, thus, is whether (2) suffices for (1) as well.

The answer is yes provided  $\alpha$  can be approximated by the predictive measures  $\alpha_n$  in some stronger sense. In fact, condition (2) can be written as

$$\alpha_n(\omega) \ll \lambda \quad \text{for all } n \geq 0 \text{ and almost all } \omega \in \Omega.$$

Hence, if (2) holds and  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ , the set

$$A = \{\|\alpha_n - \alpha\| \rightarrow 0\} \cap \{\alpha_n \ll \lambda \text{ for all } n \geq 0\}$$

has probability 1. And, for each  $\omega \in A$ , one obtains

$$\alpha(\omega)(B) = \lim_n \alpha_n(\omega)(B) = 0 \quad \text{whenever } B \in \mathcal{B} \text{ and } \lambda(B) = 0.$$

Therefore, (1) follows from (2) and  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ . In addition, a martingale argument implies the converse implication, that is

$$\alpha \ll \lambda \text{ a.s.} \iff \|\alpha_n - \alpha\| \xrightarrow{a.s.} 0 \text{ and } \mathcal{L}(X_1, \dots, X_n) \ll \lambda^n \text{ for all } n;$$

see Theorem 1. Thus, our first problem turns into the second one.

The question of whether  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$  is of independent interest. Among other things, it is connected to Bayesian consistency. Surprisingly, however, this question seems not answered so far. To the best of our knowledge,  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$  in every example known so far. And in fact, for some time, we conjectured that  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$  under condition (2). But this is not true. As shown in Example 5, when  $S = \mathbb{R}$  and  $\lambda =$  Lebesgue measure, it may be that  $\mathcal{L}(X_1, \dots, X_n)$  is absolutely continuous for all  $n$  and yet  $\alpha$  is singular continuous a.s.. Indeed, the (topological) support of  $\alpha(\omega)$  has Hausdorff dimension 0 for almost all  $\omega \in \Omega$ .

Thus, (2) does not suffice for (1). To get (1), in addition to (2), one needs some growth conditions on the conditional densities. We refer to forthcoming Theorem

4 for such conditions. Here, we mention a result on the second problem. Actually, for  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$ , it suffices that

$$P\{\omega : \alpha_c(\omega) \ll \lambda\} = 1$$

where  $\alpha_c(\omega)$  denotes the continuous part of  $\alpha(\omega)$ ; see Theorem 2.

Finally, some results mentioned above do not need exchangeability of  $X$ , but the weaker assumption

$$(X_1, \dots, X_n, X_{n+2}) \sim (X_1, \dots, X_n, X_{n+1}) \quad \text{for all } n \geq 0.$$

Those sequences  $X$  satisfying the above condition, investigated in [2], are called *conditionally identically distributed* (c.i.d.).

### 3. MIXTURES OF I.I.D. ABSOLUTELY CONTINUOUS SEQUENCES

In this section,  $\mathcal{G}_0 = \{\emptyset, \Omega\}$ ,  $\mathcal{G}_n = \sigma(X_1, \dots, X_n)$  for  $n \geq 1$  and  $\mathcal{G}_\infty = \sigma(\cup_n \mathcal{G}_n)$ . If  $\mu$  is a random probability measure on  $S$ , we write  $\mu(B)$  to denote the real random variable  $\mu(\cdot)(B)$ ,  $B \in \mathcal{B}$ . Similarly, if  $h : S \rightarrow \mathbb{R}$  is a Borel function, integrable with respect to  $\mu(\omega)$  for almost all  $\omega \in \Omega$ , we write  $\mu(h)$  to denote  $\int h(x) \mu(\cdot)(dx)$ .

**3.1. Preliminaries.** Let  $X = (X_1, X_2, \dots)$  be c.i.d., as defined in Section 2. Since  $X$  needs not be exchangeable, the representation  $P(X \in \cdot) = \int \alpha(\omega)^\infty(\cdot) P(d\omega)$  can fail for any  $\alpha$ . However, there is a random probability measure  $\alpha$  on  $S$  such that

$$(3) \quad \sigma(\alpha) \subset \mathcal{G}_\infty \quad \text{and} \quad \alpha_n(B) = E\{\alpha(B) \mid \mathcal{G}_n\} \quad \text{a.s.}$$

for all  $B \in \mathcal{B}$ . In particular,  $\alpha_n \xrightarrow{weak} \alpha$  a.s.. Also, letting

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

be the empirical measure, one obtains  $\mu_n \xrightarrow{weak} \alpha$  a.s.. Such an  $\alpha$  is of interest for one more reason. There is an exchangeable sequence  $Y = (Y_1, Y_2, \dots)$  of  $S$ -valued random variables on  $(\Omega, \mathcal{A}, P)$  such that

$$(X_n, X_{n+1}, \dots) \xrightarrow{d} Y \quad \text{and} \quad P(Y \in \cdot) = \int \alpha(\omega)^\infty(\cdot) P(d\omega).$$

See [2] for details.

We next recall some known facts about vector-valued martingales; see [9]. Let  $(\mathcal{Z}, \|\cdot\|_*)$  be a separable Banach space. Also, let  $\mathcal{F} = (\mathcal{F}_n)$  be a filtration and  $(Z_n)$  a sequence of  $\mathcal{Z}$ -valued random variables on  $(\Omega, \mathcal{A}, P)$  such that  $E\|Z_n\|_* < \infty$  for all  $n$ . Then,  $(Z_n)$  is an  $\mathcal{F}$ -martingale in case  $(\phi(Z_n))$  is an  $\mathcal{F}$ -martingale for each linear continuous functional  $\phi : \mathcal{Z} \rightarrow \mathbb{R}$ . If  $(Z_n)$  is an  $\mathcal{F}$ -martingale,  $(\|Z_n\|_*)$  is a real-valued  $\mathcal{F}$ -submartingale. So, Doob's maximal inequality yields

$$E\{\sup_n \|Z_n\|_*^p\} \leq \left(\frac{p}{p-1}\right)^p \sup_n E\{\|Z_n\|_*^p\} \quad \text{for all } p > 1.$$

The following martingale convergence theorem is available as well. Let  $Z : \Omega \rightarrow \mathcal{Z}$  be  $\mathcal{F}_\infty$ -measurable and such that  $E\|Z\|_* < \infty$ , where  $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$ . Then,  $Z_n \xrightarrow{a.s.} Z$  provided  $\phi(Z_n) = E\{\phi(Z) \mid \mathcal{F}_n\}$  a.s. for all  $n$  and all linear continuous functionals  $\phi : \mathcal{Z} \rightarrow \mathbb{R}$ .

**3.2. Results.** In the sequel,  $\lambda$  is a  $\sigma$ -finite measure on  $\mathcal{B}$  and  $\alpha$  a random probability measure on  $S$  such that  $\alpha_n \xrightarrow{weak} \alpha$  a.s.. Equivalently, if  $X$  is c.i.d. (in particular, exchangeable),  $\alpha$  is a random probability measure on  $S$  such that  $\mu_n \xrightarrow{weak} \alpha$  a.s.. It can (and will) be assumed  $\sigma(\alpha) \subset \mathcal{G}_\infty$ .

**Theorem 1.** *Suppose  $X = (X_1, X_2, \dots)$  is c.i.d.. Then,  $\alpha \ll \lambda$  a.s. if and only if  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$  and  $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$  for all  $n$ .*

*Proof.* The "if" part can be proved exactly as in Section 2. Conversely, suppose  $\alpha \ll \lambda$  a.s.. It can be assumed  $\alpha(\omega) \ll \lambda$  for all  $\omega \in \Omega$ . We let  $L_p = L_p(S, \mathcal{B}, \lambda)$  for each  $1 \leq p \leq \infty$ .

Let  $f : \Omega \times S \rightarrow [0, \infty)$  be such that  $\alpha(\omega)(dx) = f(\omega, x) \lambda(dx)$  for all  $\omega \in \Omega$ . Since  $\mathcal{B}$  is countably generated,  $f$  can be taken  $\mathcal{A} \otimes \mathcal{B}$ -measurable (see [4], V.5.58, page 52) so that

$$1 = \int 1 dP = \int \int f(\omega, x) \lambda(dx) P(d\omega) = \int E\{f(\cdot, x)\} \lambda(dx).$$

Thus, given  $n \geq 0$ ,  $E\{f(\cdot, x) \mid \mathcal{G}_n\}$  is well defined for  $\lambda$ -almost all  $x \in S$ . Since  $X$  is c.i.d., condition (3) also implies

$$\begin{aligned} \int_B E\{f(\cdot, x) \mid \mathcal{G}_n\} \lambda(dx) &= E\left\{ \int_B f(\cdot, x) \lambda(dx) \mid \mathcal{G}_n \right\} \\ &= E\{\alpha(B) \mid \mathcal{G}_n\} = \alpha_n(B) \quad \text{a.s. for fixed } B \in \mathcal{B}. \end{aligned}$$

Since  $\mathcal{B}$  is countably generated, the previous equality yields

$$\alpha_n(\omega)(dx) = E\{f(\cdot, x) \mid \mathcal{G}_n\}(\omega) \lambda(dx) \quad \text{for almost all } \omega \in \Omega.$$

This proves that  $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$  for all  $n$ . In particular, up to modifying  $\alpha_n$  on a  $P$ -null set, it can be assumed  $\alpha_n(\omega)(dx) = f_n(\omega, x) \lambda(dx)$  for all  $n \geq 0$ , all  $\omega \in \Omega$ , and suitable functions  $f_n : \Omega \times S \rightarrow [0, \infty)$ .

Regard  $f, f_n : \Omega \rightarrow L_1$  as  $L_1$ -valued random variables. Then,  $f : \Omega \rightarrow L_1$  is  $\mathcal{G}_\infty$ -measurable for  $\int h(x) f(\cdot, x) \lambda(dx) = \alpha(h)$  is  $\mathcal{G}_\infty$ -measurable for all  $h \in L_\infty$ . Clearly,  $\|f(\omega, \cdot)\|_{L_1} = \|f_n(\omega, \cdot)\|_{L_1} = 1$  for all  $n$  and  $\omega$ . Finally,  $X$  c.i.d. implies

$$\begin{aligned} E\left\{ \int h(x) f(\cdot, x) \lambda(dx) \mid \mathcal{G}_n \right\} &= E\{\alpha(h) \mid \mathcal{G}_n\} = \alpha_n(h) \\ &= \int h(x) f_n(\cdot, x) \lambda(dx) \quad \text{a.s. for all } h \in L_\infty. \end{aligned}$$

By the martingale convergence theorem (see Subsection 3.1)  $f_n \xrightarrow{a.s.} f$  in the space  $L_1$ , that is

$$\|\alpha_n(\omega) - \alpha(\omega)\| = \frac{1}{2} \int |f_n(\omega, x) - f(\omega, x)| \lambda(dx) \longrightarrow 0 \quad \text{for almost all } \omega \in \Omega.$$

□

In the exchangeable case, the argument of the previous proof yields a little bit more. Indeed, if  $X$  is exchangeable and  $\alpha \ll \lambda$  a.s., then

$$\sup_{B \in \mathcal{B}^k} \left| P\{(X_{n+1}, \dots, X_{n+k}) \in B \mid \mathcal{G}_n\} - \alpha^k(B) \right| \xrightarrow{a.s.} 0,$$

where  $k \geq 1$  is any integer and  $\alpha^k = \alpha \times \dots \times \alpha$ .

Next result deals with the second problem of Section 1. For each  $\nu \in \mathbb{S}$ , let  $\nu_c$  and  $\nu_d$  denote the continuous and discrete parts of  $\nu$ , that is,  $\nu_d(B) = \sum_{x \in B} \nu\{x\}$  for all  $B \in \mathcal{B}$  and  $\nu_c = \nu - \nu_d$ .

**Theorem 2.** *Suppose  $X = (X_1, X_2, \dots)$  is c.i.d. and  $P\{\omega : \alpha_c(\omega) \ll \lambda\} = 1$ . Then,  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$  if and only if*

$$(4) \quad \begin{aligned} & \text{there is a set } A_0 \in \mathcal{A} \text{ such that } P(A_0) = 1 \text{ and} \\ & \alpha_n(\omega)\{x\} \longrightarrow \alpha(\omega)\{x\} \text{ for all } x \in S \text{ and } \omega \in A_0. \end{aligned}$$

*In particular,  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$  if  $X$  is exchangeable and  $\alpha_c \ll \lambda$  a.s. (in fact, condition (4) is automatically true if  $X$  is exchangeable).*

*Proof.* The "only if" part is trivial. Suppose condition (4) holds. For each  $n \geq 0$ , take functions  $\beta_n$  and  $\gamma_n$  on  $\Omega$  such that  $\beta_n(\omega)$  and  $\gamma_n(\omega)$  are measures on  $\mathcal{B}$  for all  $\omega \in \Omega$  and

$$\beta_n(B) = E\{\alpha_c(B) \mid \mathcal{G}_n\}, \quad \gamma_n(B) = E\{\alpha_d(B) \mid \mathcal{G}_n\}, \quad \text{a.s.},$$

for all  $B \in \mathcal{B}$ . Since  $X$  is c.i.d., condition (3) yields  $\alpha_n = \beta_n + \gamma_n$  a.s..

We first prove  $\|\beta_n - \alpha_c\| \xrightarrow{a.s.} 0$ . It can be assumed  $\alpha_c(\omega) \ll \lambda$  for all  $\omega \in \Omega$ , so that  $\alpha_c(\omega)(dx) = f(\omega, x) \lambda(dx)$  for all  $\omega \in \Omega$  and some function  $f : \Omega \times S \rightarrow [0, \infty)$ . For fixed  $B \in \mathcal{B}$ , arguing as in the proof of Theorem 1, one has

$$\beta_n(B) = E\left\{\int_B f(\cdot, x) \lambda(dx) \mid \mathcal{G}_n\right\} = \int_B E(f(\cdot, x) \mid \mathcal{G}_n) \lambda(dx) \quad \text{a.s..}$$

By standard arguments, it follows that  $\beta_n \ll \lambda$  a.s.. Again, it can be assumed  $\beta_n(\omega)(dx) = f_n(\omega, x) \lambda(dx)$  for all  $\omega \in \Omega$  and some function  $f_n : \Omega \times S \rightarrow [0, \infty)$ . Define  $L_1 = L_1(S, \mathcal{B}, \lambda)$  and regard  $f_n, f : \Omega \rightarrow L_1$  as  $L_1$ -valued random variables. By the same martingale argument used for Theorem 1, one obtains  $f_n \xrightarrow{a.s.} f$  in the space  $L_1$ . That is,  $\|\beta_n - \alpha_c\| \xrightarrow{a.s.} 0$ .

We next prove  $\|\gamma_n - \alpha_d\| \xrightarrow{a.s.} 0$ . Take  $A_0$  as in condition (4) and define

$$A_1 = \left\{ \lim_n \|f_n - f\|_{L_1} = 0 \text{ and } \alpha_n = \beta_n + \gamma_n \text{ for all } n \geq 0 \right\}.$$

Then,  $P(A_0 \cap A_1) = 1$  and

$$\begin{aligned} \alpha_d(\omega)\{x\} &= \alpha(\omega)\{x\} - \alpha_c(\omega)\{x\} = \alpha(\omega)\{x\} - f(\omega, x) \lambda\{x\} \\ &= \lim_n (\alpha_n(\omega)\{x\} - f_n(\omega, x) \lambda\{x\}) = \lim_n (\alpha_n(\omega)\{x\} - \beta_n(\omega)\{x\}) = \lim_n \gamma_n(\omega)\{x\} \end{aligned}$$

for all  $\omega \in A_0 \cap A_1$  and  $x \in S$ . Define also

$$A = A_0 \cap A_1 \cap \{\gamma_n(S) \longrightarrow \alpha_d(S)\}.$$

Since  $\gamma_n(S) = 1 - \beta_n(S) \xrightarrow{a.s.} 1 - \alpha_c(S) = \alpha_d(S)$ , then  $P(A) = 1$ . Fix  $\omega \in A$  and let  $D_\omega = \{x \in S : \alpha(\omega)\{x\} > 0\}$ . Then,

$$\alpha_d(\omega)(D_\omega) \leq \liminf_n \gamma_n(\omega)(D_\omega)$$

since  $D_\omega$  is countable and  $\alpha_d(\omega)\{x\} = \lim_n \gamma_n(\omega)\{x\}$  for all  $x \in D_\omega$ . Further,

$$\limsup_n \gamma_n(\omega)(D_\omega) \leq \limsup_n \gamma_n(\omega)(S) = \alpha_d(\omega)(S) = \alpha_d(\omega)(D_\omega).$$

Therefore,  $\lim_n \|\gamma_n(\omega) - \alpha_d(\omega)\| = 0$  is an immediate consequence of

$$\begin{aligned} \gamma_n(\omega)\{x\} &\longrightarrow \alpha_d(\omega)\{x\} \quad \text{for each } x \in D_\omega, \\ \alpha_d(\omega)(D_\omega) &= \lim_n \gamma_n(\omega)(D_\omega), \quad \alpha_d(\omega)(D_\omega^c) = \lim_n \gamma_n(\omega)(D_\omega^c) = 0. \end{aligned}$$

Finally, suppose  $X$  exchangeable. We have to prove condition (4). If  $S$  is countable, condition (4) is trivial for  $\alpha_n(B) \xrightarrow{a.s.} \alpha(B)$  for fixed  $B \in \mathcal{B}$ . If  $S = \mathbb{R}$ , Glivenko-Cantelli theorem yields  $\sup_x |\mu_n(I_x) - \alpha(I_x)| \xrightarrow{a.s.} 0$ , where  $I_x = (-\infty, x]$  and  $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$  is the empirical measure. Hence, (4) follows from

$$\sup_x |\alpha_n(I_x) - \mu_n(I_x)| \xrightarrow{a.s.} 0;$$

see Corollary 3.2 of [1]. If  $S$  is any uncountable Polish space, take a Borel isomorphism  $\psi : S \rightarrow \mathbb{R}$ . (Thus,  $\psi$  is bijective with  $\psi$  and  $\psi^{-1}$  Borel measurable). Then,  $(\psi(X_n))$  is an exchangeable sequence of real random variables and condition (4) is a straightforward consequence of

$$P\{\psi(X_{n+1}) \in B \mid \psi(X_1), \dots, \psi(X_n)\} = P\{\psi(X_{n+1}) \in B \mid \mathcal{G}_n\} = \alpha_n(\psi^{-1}B) \quad \text{a.s.}$$

for each Borel set  $B \subset \mathbb{R}$ . This concludes the proof.  $\square$

When  $X$  is c.i.d. (but not exchangeable)  $\|\alpha_n - \alpha\| \xrightarrow{a.s.} 0$  needs not be true even if  $\alpha_c \ll \lambda$  a.s..

**Example 3.** Let  $(Z_n)$  and  $(U_n)$  be independent sequences of independent real random variables such that  $Z_n \sim \mathcal{N}(0, b_n - b_{n-1})$  and  $U_n \sim \mathcal{N}(0, 1 - b_n)$ , where  $0 = b_0 < b_1 < b_2 < \dots < 1$  and  $\sum_n (1 - b_n) < \infty$ . As shown in Example 1.2 of [2],

$$X_n = \sum_{i=1}^n Z_i + U_n$$

is c.i.d. and  $X_n \xrightarrow{a.s.} V$  for some real random variable  $V$ . Since  $\mu_n \xrightarrow{weak} \delta_V$  a.s., then  $\alpha = \delta_V$  and  $\alpha_c \ll \lambda$  a.s. (in fact,  $\alpha_c = 0$  a.s.). However, condition (4) fails. In fact,  $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$  for all  $n$ , where  $\lambda$  is Lebesgue measure. Hence,  $\alpha_n(\omega)\{V(\omega)\} = 0$  while  $\alpha(\omega)\{V(\omega)\} = 1$  for all  $n$  and almost all  $\omega \in \Omega$ .

We now turn to the first problem of Section 1. Recall that condition (2) amounts to  $\alpha_n \ll \lambda$  a.s. for all  $n \geq 0$ . Therefore, up to modifying  $\alpha_n$  on a  $P$ -null set, under condition (2) one can write

$$\alpha_n(\omega)(dx) = f_n(\omega, x) \lambda(dx)$$

for each  $\omega \in \Omega$ , each  $n \geq 0$ , and some function  $f_n : \Omega \times S \rightarrow [0, \infty)$ .

**Theorem 4.** *Suppose  $X = (X_1, X_2, \dots)$  is c.i.d. and  $\mathcal{L}(X_1, \dots, X_n) \ll \lambda^n$  for all  $n$ . Fix a constant  $p > 1$  and define*

$$I_n^B(\omega) = \int_B f_n(\omega, x)^p \lambda(dx), \quad B \in \mathcal{B}.$$

*Then,  $\alpha \ll \lambda$  a.s. provided, for every compact  $K \subset S$ ,*

$$(5) \quad \sup_n I_n^K(\omega) < \infty \quad \text{for almost all } \omega \in \Omega.$$

*In particular,  $\alpha \ll \lambda$  a.s. whenever  $\sup_n E\{I_n^K\} < \infty$  for each compact  $K \subset S$ .*



*Proof.* Fix a nondecreasing sequence  $B_1 \subset B_2 \subset \dots$  such that  $B_n \in \mathcal{B}$ ,  $\lambda(B_n) < \infty$ , and  $\cup_n B_n = S$ . Since  $\lambda(B_1) < \infty$  and  $S$  is Polish, there is a compact set  $K_1 \subset B_1$  satisfying  $\lambda(B_1 \cap K_1^c) < 1$ . By induction, for each  $n \geq 2$ , there is a compact set  $K_n$  such that  $K_{n-1} \subset K_n \subset B_n$  and  $\lambda(B_n \cap K_n^c) < 1/n$ . Since  $X$  is c.i.d., condition (3) implies

$$\alpha(K_m) = \lim_n E\{\alpha(K_m) \mid \mathcal{G}_n\} = \lim_n \alpha_n(K_m) \quad \text{a.s. for all } m \geq 1.$$

Define  $H = \cup_m K_m$  and  $A_H = \{\alpha(H) = 1\}$ . If  $\omega \in A_H$ , then

$$\alpha(\omega)(B) = \alpha(\omega)(B \cap H) = \sup_m \alpha(\omega)(B \cap K_m) \quad \text{for all } B \in \mathcal{B}.$$

Moreover,  $P(A_H) = 1$ . In fact,  $\lambda(H^c) = 0$  and  $\alpha_n \ll \lambda$  a.s. for all  $n$ , so that

$$\alpha(H) = \lim_n E\{\alpha(H) \mid \mathcal{G}_n\} = \lim_n \alpha_n(H) = 1 \quad \text{a.s.}$$

Thus, it suffices to prove  $\alpha(\cdot \cap K_m) \ll \lambda$  a.s. for all  $m$ .

Suppose (5) holds. Fix  $m \geq 1$  and define  $K = K_m$  and  $\lambda_K(\cdot) = \lambda(\cdot \cap K)$ . By (5) and  $p > 1$ , the sequence  $(f_n(\omega, \cdot) : n \geq 1)$  is uniformly integrable in  $(S, \mathcal{B}, \lambda_K)$  for almost all  $\omega \in \Omega$ . Take a set  $A \in \mathcal{A}$  such that  $P(A) = 1$  and, for each  $\omega \in A$ ,

$$\begin{aligned} \alpha(\omega)(K) &= \lim_n \alpha_n(\omega)(K), \quad \alpha_n(\omega) \xrightarrow{weak} \alpha(\omega), \\ (f_n(\omega, \cdot) : n \geq 1) &\text{ is uniformly integrable in } (S, \mathcal{B}, \lambda_K). \end{aligned}$$

Fix  $\omega \in A$ . Since  $\lambda_K(S) = \lambda(K) \leq \lambda(B_m) < \infty$  and  $(f_n(\omega, \cdot) : n \geq 1)$  is uniformly integrable, there is a subsequence  $(n_j)$  and a function  $\psi_\omega \in L_1(S, \mathcal{B}, \lambda_K)$  such that  $f_{n_j}(\omega, \cdot) \rightarrow \psi_\omega$  in the weak-topology of  $L_1(S, \mathcal{B}, \lambda_K)$ . This means that

$$\int_{B \cap K} \psi_\omega(x) \lambda(dx) = \lim_j \int_{B \cap K} f_{n_j}(\omega, x) \lambda(dx) = \lim_j \alpha_{n_j}(\omega)(B \cap K) \quad \text{for all } B \in \mathcal{B}.$$

Therefore,

$$\begin{aligned} \int_K \psi_\omega(x) \lambda(dx) &= \lim_j \alpha_{n_j}(\omega)(K) = \alpha(\omega)(K) \quad \text{and} \\ \int_{F \cap K} \psi_\omega(x) \lambda(dx) &= \lim_j \alpha_{n_j}(\omega)(F \cap K) \leq \alpha(\omega)(F \cap K) \quad \text{for each closed } F \subset S. \end{aligned}$$

By standard arguments, the previous two relations yield  $\alpha(\omega)(B \cap K) = \int_{B \cap K} \psi_\omega(x) \lambda(dx)$  for all  $B \in \mathcal{B}$ . Thus,  $\alpha(\omega)(\cdot \cap K) \ll \lambda$ . This concludes the proof of the first part.

It remains to see that condition (5) follows from  $\sup_n E\{I_n^K\} < \infty$  for each compact  $K$ . Fix  $B \in \mathcal{B}$  and suppose  $\sup_n E\{I_n^B\} < \infty$ . Let  $\lambda_B(\cdot) = \lambda(\cdot \cap B)$  and  $L_r = L_r(S, \mathcal{B}, \lambda_B)$  for all  $r$ . It can be assumed  $I_n^B(\omega) < \infty$  for all  $\omega \in \Omega$  and  $n \geq 0$ . Thus, each  $f_n : \Omega \rightarrow L_p$  can be seen as an  $L_p$ -valued random variable such that

$$E\|f_n\|_{L_p} = E\{(I_n^B)^{1/p}\} \leq (E\{I_n^B\})^{1/p} < \infty.$$

Further,  $\int f_n(\cdot, x) h(x) \lambda_B(dx) = \alpha_n(I_B h)$  is  $\mathcal{G}_n$ -measurable for all  $h \in L_q$ , where  $q = p/(p-1)$ . Since  $X$  is c.i.d., condition (3) also implies

$$\begin{aligned} E\left\{\int f_{n+1}(\cdot, x) h(x) \lambda_B(dx) \mid \mathcal{G}_n\right\} &= E\{\alpha_{n+1}(I_B h) \mid \mathcal{G}_n\} \\ &= E\left\{E(\alpha(I_B h) \mid \mathcal{G}_{n+1}) \mid \mathcal{G}_n\right\} \\ &= E\{\alpha(I_B h) \mid \mathcal{G}_n\} = \alpha_n(I_B h) \\ &= \int f_n(\cdot, x) h(x) \lambda_B(dx) \quad \text{a.s. for all } h \in L_q. \end{aligned}$$

Hence,  $(f_n)$  is a  $(\mathcal{G}_n)$ -martingale. By Doob's maximal inequality,

$$E\left\{\sup_n I_n^B\right\} = E\left\{\sup_n \|f_n\|_{L^p}^p\right\} \leq q^p \sup_n E\{\|f_n\|_{L^p}^p\} = q^p \sup_n E\{I_n^B\} < \infty.$$

In particular,  $\sup_n I_n^B < \infty$  a.s., and this concludes the proof.  $\square$

Some remarks on Theorem 4 are in order. First,

$$f_n(\omega, \cdot) = \frac{g_{n+1}(X_1(\omega), \dots, X_n(\omega), \cdot)}{g_n(X_1(\omega), \dots, X_n(\omega))} \quad \text{for almost all } \omega \in \Omega,$$

where each  $g_n : S^n \rightarrow [0, \infty)$  is a density of  $\mathcal{L}(X_1, \dots, X_n)$  with respect to  $\lambda^n$ . Thus, more concretely,  $I_n^B$  can be written as

$$I_n^B = \frac{\int_B g_{n+1}(X_1, \dots, X_n, x)^p \lambda(dx)}{g_n(X_1, \dots, X_n)^p} \quad \text{a.s..}$$

Second, as apparent from the proof, condition (5) can be slightly weakened as follows. For each compact  $K$ , the sequence  $(f_n(\omega, \cdot) : n \geq 1)$  is uniformly integrable, in the space  $(S, \mathcal{B}, \lambda_K)$ , for almost all  $\omega \in \Omega$ .

Third, suppose  $X$  exchangeable and fix *any* random probability measure  $\gamma$  on  $S$  such that  $P(X \in \cdot) = \int \gamma(\omega)^\infty(\cdot) P(d\omega)$ . Then,  $\gamma \ll \lambda$  a.s. under the assumptions of Theorem 4. In fact,  $\alpha$  and  $\gamma$  have the same probability distribution, when regarded as  $\mathbb{S}$ -valued random variables.

A last (and important) remark deals with condition (2). Indeed, even if  $X$  is exchangeable, condition (2) is not enough for  $\alpha \ll \lambda$  a.s.. When  $S = \mathbb{R}$  and  $\lambda =$  Lebesgue measure, it may be that  $X$  is exchangeable,  $\mathcal{L}(X_1, \dots, X_n)$  is absolutely continuous for all  $n$ , and yet the support of  $\alpha(\omega)$  has Hausdorff dimension 0 for almost all  $\omega \in \Omega$ . We close the paper showing this fact.

**Example 5.** Let  $U_m$  and  $Y_{m,n}$  be independent real random variables, on the probability space  $(\Omega, \mathcal{A}, P)$ , such that:

- $U_m$  is uniformly distributed on  $(\frac{1}{m+1}, \frac{1}{m})$  for each  $m \geq 1$ ;
- $P(Y_{m,n} = 0) = P(Y_{m,n} = 1) = \frac{1}{2}$  for all  $m, n \geq 1$ .

Define  $V_m = U_m^m$  and

$$X_n = \sum_{m=1}^{\infty} U_m^m Y_{m,n} = \sum_{m=1}^{\infty} V_m Y_{m,n}.$$

Then,  $X = (X_1, X_2, \dots)$  is conditionally i.i.d. given  $\mathcal{V} = \sigma(V_1, V_2, \dots)$ . Precisely, for  $\omega \in \Omega$  and  $B \in \mathcal{B}$ , define

$$\alpha(\omega)(B) = P\left\{u \in \Omega : \sum_m V_m(\omega) Y_{m,1}(u) \in B\right\}.$$

Then,  $\alpha(B)$  is a version of  $P(X_1 \in B \mid \mathcal{V})$  and  $P(X \in \cdot) = \int \alpha(\omega)^\infty(\cdot) P(d\omega)$ . In particular,  $X$  is exchangeable. Moreover,  $\mu_n \xrightarrow{weak} \alpha$  a.s. for

$$P(\mu_n \xrightarrow{weak} \alpha \mid \mathcal{V}) = 1 \quad \text{a.s.}$$

Next, the (topological) support of  $\alpha(\omega)$  has Hausdorff dimension 0 for almost all  $\omega \in \Omega$ . Suppose in fact  $b_1 > b_2 > \dots > 0$  are real numbers such that  $\sum_m b_m < \infty$  and  $Z_1, Z_2, \dots$  i.i.d. random variables with  $P(Z_1 = 0) = P(Z_1 = 1) = 1/2$ . Then, by Theorem 8 of [10], the support of  $\mathcal{L}(\sum_m b_m Z_m)$  has Hausdorff dimension 0 whenever  $\lim_m (\sum_{j>m} b_j)^{-1} b_m = \infty$ . Thus, letting  $b_m = V_m(\omega)$  and  $Z_m = Y_{m,1}$ , it suffices to verify that

$$(6) \quad \lim_m \frac{V_m(\omega)}{\sum_{j>m} V_j(\omega)} = \infty \quad \text{for almost all } \omega \in \Omega.$$

And condition (6) follows immediately from

$$(j+1)^{-j} < V_j < j^{-j} \quad \text{and} \quad \sum_{j>m} V_j \leq \sum_{j>m} j^{-j} \leq \sum_{j>m} (m+1)^{-j} = \frac{(m+1)^{-m}}{m} \quad \text{a.s.}$$

We finally prove that  $\mathcal{L}(X_1, \dots, X_n)$  is absolutely continuous, with respect to Lebesgue measure on  $\mathbb{R}^n$ , for all  $n$ . Given the array  $y = (y_{m,n} : m, n \geq 1)$ , with  $y_{m,n} \in \{0, 1\}$  for all  $m, n$ , define

$$X_{n,y} = \sum_m V_m y_{m,n}.$$

Fix  $n \geq 1$  and denote  $I_n$  the  $n \times n$  identity matrix. If  $y$  satisfies

$$(7) \quad \begin{pmatrix} y_{m+1,1} & \cdots & y_{m+1,n} \\ \cdots & \cdots & \cdots \\ y_{m+n,1} & \cdots & y_{m+n,n} \end{pmatrix} = I_n \quad \text{for some } m \geq 0,$$

then

$$(X_{1,y}, \dots, X_{n,y}) = (V_{m+1}, \dots, V_{m+n}) + (R_1, \dots, R_n) \\ \text{with } (R_1, \dots, R_n) \text{ independent of } (V_{m+1}, \dots, V_{m+n}).$$

In this case, since  $(V_{m+1}, \dots, V_{m+n})$  has an absolutely continuous distribution,  $(X_{1,y}, \dots, X_{n,y})$  has an absolutely continuous distribution as well. Hence, letting  $Y = (Y_{m,n} : m, n \geq 1)$ , the conditional distribution of  $(X_1, \dots, X_n)$  given  $Y = y$  is absolutely continuous as far as  $y$  satisfies (7). To conclude the proof, it suffices noting that

$$P(Y = y \text{ for some } y \text{ satisfying (7)}) = 1.$$

## REFERENCES

- [1] Berti P., Mattei A., Rigo P. (2002) Uniform convergence of empirical and predictive measures, *Atti Sem. Mat. Fis. Univ. Modena*, L, 465-477.
- [2] Berti P., Pratelli L., Rigo P. (2004) Limit theorems for a class of identically distributed random variables, *Ann. Probab.*, 32, 2029-2052.
- [3] Cifarelli D.M., Regazzini E. (1996) de Finetti's contribution to probability and statistics, *Statist. Sci.*, 11, 253-282.
- [4] Dellacherie C., Meyer P.A. (1982) *Probabilities and Potential B*, North-Holland.
- [5] Diaconis P., Freedman D. (1984) Partial exchangeability and sufficiency, *Proc. of the Indian Statistical Institute Golden Jubilee International Conference on Statistics: Applications and New Directions* (J. K. Ghosh and J. Roy, eds.) 205-236. Indian Statistical Institute, Calcutta.
- [6] Diaconis P., Freedman D. (1988) Conditional limit theorems for exponential families and finite versions of de Finetti's theorem, *J. Theoret. Probab.*, 1, 381-410.
- [7] Diaconis P., Freedman D. (1990) Cauchy's equation and de Finetti's theorem, *Scandinavian J. Statist.*, 17, 235-274.
- [8] Fortini S., Ladelli L., Regazzini E. (2000) Exchangeability, predictive distributions and parametric models, *Sankhya A*, 62, 86-109.
- [9] Neveu J. (1975) *Discrete parameter martingales*, North-Holland.
- [10] Pratsiovytyi M.V., Feshchenko O.YU. (2007) Topological, metric and fractal properties of probability distributions on the set of incomplete sums of positive series, *Theory Stoch. Proc.*, 13, 205-224.

PATRIZIA BERTI, DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA "G. VITALI", UNIVERSITA' DI MODENA E REGGIO-EMILIA, VIA CAMPI 213/B, 41100 MODENA, ITALY  
*E-mail address:* `patrizia.berti@unimore.it`

LUCA PRATELLI, ACCADEMIA NAVALE, VIALE ITALIA 72, 57100 LIVORNO, ITALY  
*E-mail address:* `pratel@mail.dm.unipi.it`

PIETRO RIGO (CORRESPONDING AUTHOR), DIPARTIMENTO DI ECONOMIA POLITICA E METODI QUANTITATIVI, UNIVERSITA' DI PAVIA, VIA S. FELICE 5, 27100 PAVIA, ITALY  
*E-mail address:* `prigo@eco.unipv.it`