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**COMPARATIVE STATICS IN DYNAMIC
PROGRAMMING MODELS OF ECONOMICS**

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Harald Lang

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OF ECONOMICS**

by

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1 INTRODUCTION

A dynamic programming problem of economics typically has the following formal form:

$$v(x) = \max_s T(s, \theta, v)(x) \quad (1)$$

where v is the "value function", s some policy parameter, θ some parameter of the problem and $T(s, \theta, \cdot)$ is a mapping taking functions of x into new functions of x . For a comparative statics analysis, the differentiability properties of (1) are of interest.

Our main result (theorem 4) essentially states that as long as the optimal policy parameter s is unique, if one can formally differentiate (1) with respect to θ , treating the policy s as being fixed and v as being a priori differentiable, then this differentiation is a posteriori justified.

We also give a version of the "envelope theorem" which sometimes can be used to differentiate (1) with respect to the variable x . (The formal distinction in this context between variables and parameters of v is that the right-hand side of (1)

* Most of this work was done while I was visiting researcher at the Institute for Economic Studies, University of Stockholm.

depends on θ and $v(\theta, \cdot)$ for the current value of θ only, whereas it depends on $v(\theta, x')$ also for x' different from the current value of x .) A result in this direction has also been presented by Benveniste and Scheinkman (1979).

2 SOME MATHEMATICAL RESULTS

Lemma 1. Let D be an open subset of R^n , $f(x)$ a continuous function on D , $g(x, z)$ a function on $D \times D$, and assume that

$$f(x) \geq g(x, z) \text{ for } x \text{ close enough to } z, \text{ and}$$

$$f(x) = g(x, x) \text{ for all } x \in D.$$

Assume further that $g(x, z)$ is differentiable w.r.t. x and that $g_1(x, x)$ (the subscript denotes differentiation w.r.t. the first n variables) is continuous on D .

Then $f(x)$ is continuously differentiable, and

$$f_x(x) = g_1(x, x).$$

Proof. Since we differentiate w.r.t. one variable at a time, we may without loss of generality set $n=1$. Now, for any $x_0 \in D$

$$\liminf_{0 < h \rightarrow 0} [f(x_0+h) - f(x_0)]/h \geq$$

$$\liminf_{0 < h \rightarrow 0} [g(x_0+h, x_0) -$$

$$-g(x_0, x_0)]/h = g_1(x_0, x_0). \quad (2)$$

Now take $h > 0$ arbitrarily and define

$$m(h) \equiv h + \max\{g_1(x, x) \mid x_0 \leq x \leq x_0 + h\}.$$

For h fixed, define

$$F(x) \equiv f(x) - m(h)(x-x_0), \quad x \in [x_0, x_0+h].$$

$F(x)$ is continuous, and its maximum in $[x_0, x_0+h]$ must be attained at x_0 . Indeed, for any $x \in [x_0, x_0+h]$

$$\limsup_{0 < k \rightarrow 0} [F(x-k) - F(x)]/(-k) =$$

$$\limsup_{0 < k \rightarrow 0} [f(x-k) - f(x)]/(-k) - m(h) \leq$$

$$\limsup_{0 < k \rightarrow 0} [g(x-k, x) - g(x, x)]/(-k) - m(h) =$$

$$g_1(x, x) - m(h) \leq -h < 0,$$

which is impossible at a maximum point. Hence $F(x_0+h) \leq F(x_0)$, i.e. $f(x_0+h) - f(x_0) \leq m(h)h$, thus

$$\limsup_{0 < h \rightarrow 0} [f(x_0+h) - f(x_0)]/h \leq \limsup_{0 < h \rightarrow 0} m(h) =$$

$$g_1(x_0, x_0). \quad (3)$$

Of course, (2) and (3) show that the right-hand derivative of f at x_0 exists and equals $g_1(x_0, x_0)$. The left-hand derivative is treated similarly. Q.E.D.

Theorem 1 ("envelope theorem"). Let D be an open subset of R^n , $A: D \rightarrow R^m$ a continuous, convex- and compact-valued correspondence, $h(x, y)$ a continuous function on $D \times R^m$ and define

$$f(x) = \max_{y \in A(x)} h(x, y), \quad x \in D.$$

Then $f(x)$ is continuous. If we further assume that the maximizer $y^* = y^*(x)$ is uniquely determined by x , then $y^*(x)$ is continuous. If further $h_1(x,y)$ exists and is continuous and $y^*(x)$ is an interior point of $A(x)$ then $f(x)$ is differentiable and

$$f'_x(x) = h_1(x, y^*(x)).$$

It is important to note that $h(x,y)$ is not assumed to be differentiable w.r.t. y .

Proof. The continuity of $f(x)$ and of $y^*(x)$ is well known, and we do not repeat the arguments here. Now define

$$g(x,z) \equiv h(x, y^*(z))$$

The differentiability conclusion of $f(x)$ now follows from Lemma 1 applied to the pair f and g . Q.E.D.

Theorem 2 (Blackwell, 1965). Let D be some subset of R^n , and $B(D)$ the Banach space of bounded, real-valued continuous functions on D , normed by the supremum norm.

Let $T:B(D) \rightarrow B(D)$ be a mapping with the following two properties (Blackwell's conditions, abbreviated B.C. in the sequel):

$$\text{(monotonicity) } f \geq g \text{ implies } T(f) \geq T(g)$$

$$\text{(discounting) there is a number } \beta < 1 \text{ (the modulus of } T \text{) such for all } f \in B(D) \text{ and all constants } c > 0, T(f+c) \leq T(f) + \beta c.$$

Then T is a contraction mapping with modulus β . In particular, the equation

$$f = T(f) \tag{4}$$

has a unique solution $f \in B(D)$, and if S is a closed subset of $B(D)$ such that T maps S into S , then the solution f lies in S .

Theorem 3 ("Bellman's principle"). With the notation of Theorem 2, let $T = T(s; \cdot)$ depend continuously on some parameter $s \in KCR^m$ and assume that the modulus β can be chosen independently of s . Let $A: D \rightarrow K$ be a compact-valued correspondence, and consider the two equations

$$f(x) = \max_{s \in A(x)} T(s; f)(x) \equiv T^*(f)(x)$$

$$g(x) = T(s_0(x); f)(x)$$

where $s_0(x): D \rightarrow A(x)$ is any continuous function (it is easy to see that both right-hand sides define mappings satisfying B.C., so both equations have unique solutions). Then

$$g(x) \leq f(x) \text{ for all } x \in D.$$

Proof. Let S be the closed subset of $B(D)$ consisting of functions $\geq g$. Then for any $h \in S$ we have

$$T^*(h)(x) \geq T(s_0(x); h)(x) \geq T(s_0(x); g)(x) = g(x)$$

where in the second relation we used the monotonicity of T . Hence, by theorem 2, $f \in S$. Q.E.D.

For the rest of this section we will adopt notions from differential calculus for mappings between Banach spaces. We refer to Dieudonné (1960) as a general reference.

Lemma 2. With the notation of Theorem 2, let $T=T(\theta; \cdot)$ depend continuously on some real parameter $\theta \in (\theta_0, \theta_1)$, and assume that the modulus β of T can be chosen independently of θ . Assume further that $T(\cdot; \cdot): (\theta_0, \theta_1) \times B(D) \rightarrow B(D)$ is differentiable. Then the solution $f=f(\theta; \cdot)$ to

$$f = T(\theta; f) \tag{5}$$

is differentiable w.r.t. θ , and f_θ is the unique solution to

$$f_\theta = T_\theta(\theta; f) + T_f(\theta, f; f_\theta) = T^*(f_\theta). \tag{6}$$

Here the mapping T^* satisfies B.C.

Proof. Equation (4) may be written

$$f - T(\theta; f) = 0$$

so the conclusion that f is differentiable, as well as formula (6), follows from the implicit function theorem. The only non-trivial thing to check is that the derivative of $f-T(\theta; f)$ w.r.t. f is invertible. But this derivative is $I-T_f(\theta, \cdot)$, where I is the identity mapping. We show below that the linear mapping T_f satisfies B.C., so its norm is at most $\beta < 1$, and the invertibility of $I-T_f$ follows.

To prove monotonicity of T^* , take $g \geq 0$ in $B(D)$. For any $h \in B(D)$ and $\epsilon > 0$

$$\begin{aligned} T_f(\theta, h; \epsilon g) &\geq T(\theta; h + \epsilon g) - T(\theta; h) - \\ &\| T(\theta; h + \epsilon g) - T(\theta; h) - T_f(\theta, h; \epsilon g) \| \geq 0 + o(\epsilon) \end{aligned}$$

by the monotonicity of $T(\theta; \cdot)$ and the definition of differentiability. But $T_f(\theta, h; \cdot)$ is linear, so dividing by ϵ and letting $\epsilon \rightarrow 0$ gives $T_f(\theta, h; g) \geq 0$. Since $T_f(\theta, h; \cdot)$ is linear, this provides monotonicity of T_f , and hence of T^* .

Discounting is proved similarly. Q.E.D.

Theorem 4. With the notation of Theorem 2, let $T=T(s, \theta; \cdot)$ depend continuously on $s \in KCR^m$ and $\theta \in (\theta_0, \theta_1)$ with a modulus of T being independent of s and of θ . Let $D(\theta)$ denote $(\theta_0, \theta_1) \times D$ and let $A: D(\theta) \rightarrow K$ be a continuous, compact-valued correspondence and consider the equation

$$f(x) = \max_{s \in A(\theta, x)} T(s, \theta; f)(x) \tag{7}$$

Assume that the maximizer $s=s^*(\theta; x)$ is uniquely determined by θ and x and that $s^*(\theta, x) \in A(\theta_1, x)$ for all $x \in D$ if θ_1 is close to θ^1

Then f is differentiable w.r.t. θ , and f_θ is the unique solution to

$$f_\theta = T_\theta(s^*(x), \theta; f) + T_f(s^*(x), \theta, f; f_\theta) \tag{8}$$

where the right-hand side defines a mapping in f_θ satisfying B.C.

Proof. First we must prove that f is jointly continuous in θ and x . To this end, we may temporarily think of the right-hand side of (7) being a mapping $B(D(\theta)) \rightarrow B(D(\theta))$. Indeed, for any $g \in B(D(\theta))$

$$\max_{s \in A(x, \theta)} T(s, \theta; g)(x)$$

¹ E.g., if $A(\theta, x) = A(x)$ is independent of θ .

is continuous on $D(\theta)$ by Theorem 1 (although the maximizer need not be unique for arbitrary g). Now, by Theorem 2, $f \in B(D(\theta))$, which proves the joint continuity of f in θ and x .

Now define $g(\theta, \theta'; x)$ on $(\theta_0, \theta_1) \times (\theta_0, \theta_1) \times D$ by the equation

$$g(\theta, \theta'; x) = T(s^*(\theta'; x), \theta; g)(x)$$

where $s^*(\cdot; \cdot)$ is continuous by Theorem 1. By Theorem 3,

$$f(\theta; x) \geq g(\theta, \theta'; x), \quad \theta \text{ close to } \theta'$$

and by definition

$$f(\theta; x) = g(\theta, \theta; x).$$

By Lemma 2, $g(\theta, \theta'; x)$ is differentiable w.r.t. θ , so by Lemma 1, f is differentiable w.r.t. θ , $f_\theta(\theta; x) = g_1(\theta, \theta; x)$ and substituting f_θ for g_1 in the equation for g_1 given by (6), gives (8). Q.E.D.

3 AN EXAMPLE

To illustrate the results of Section 2, let us consider the following simple dynamic problem;

$$v(x) = \max_{0 \leq y \leq f(x)} \{u(f(x)-y) + \beta v(y)\}. \quad (9)$$

Here u is utility flow of consumption, x capital stock and $f(x)$ a production function. The discount factor is $\beta < 1$. This period's production $f(x)$ is split into consumption c and next period's capital y ; $c+y=f(x)$.

Assume that u and f are continuously differentiable, increasing and concave, and denote the right-hand side of (9) by $T(v)$. Obviously T satisfies B.C., and the value function v is increasing and concave by Theorem 2, since T maps the set of (non-strictly) increasing, concave functions into itself.

The correspondence $x \rightarrow [0, f(x)]$ is continuous and by concavity, optimal $y=y^*$ is unique, so if y^* is not a corner solution, $v(x)$ is differentiable by Theorem 1 and

$$v'(x) = u'(f(x)-y^*(x))f'(x).$$

(Observe that in order to apply Theorem 1 we need only a priori know that $v(y)$ is continuous.)

Now we may write the first order condition for y (assuming away corner solutions, for simplicity)

$$-u'(f(x)-y) + \beta v'(y) = 0.$$

Using this equality it is possible to show that both $y^*(x)$ and $f(x)-y^*(x)$ are increasing in x (we omit the details), which we will exploit below.

Now assume that $f=f(\theta;x)$ is parametrized by θ such that $f_{\theta} > 0$ and $f_{\theta x} \leq 0$. We may now use Theorem 4 to differentiate (9) w.r.t. θ to get

$$v_{\theta}(x) = u'(f(x)-y^*(x))f_{\theta}(x) + \beta v_{\theta}(y^*(x)). \quad (10)$$

As anticipated by Theorem 4, the right-hand side of (10) is a mapping in v_{θ} satisfying B.C.; we denote it by $T^*(v_{\theta})$. Hence we can use Theorem 2 to derive properties of v_{θ} ; for instance v_{θ} is decreasing in x . Indeed, using u' decreasing, $f_{x\theta} \leq 0$, the increas-

ingness of $f(x)-y^*(x)$ and of $y^*(x)$, we see that T^* maps decreasing functions into decreasing functions. Using this fact one can show (we omit the details) that optimal consumption c^* increases with increasing θ ; i.e., an improvement in production according to the specification $f_{\theta} > 0$, $f_{\theta x} \leq 0$ will (not surprisingly) increase consumption during the first consumption period for any initial capital stock x .

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