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**COMPARATIVE STATICS IN DYNAMIC  
PROGRAMMING MODELS OF ECONOMICS**

by

Harald Lang

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OF ECONOMICS**

by

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**1 INTRODUCTION**

A dynamic programming problem of economics typically has the following formal form:

$$v(x) = \max_s T(s, \theta, v)(x) \quad (1)$$

where  $v$  is the "value function",  $s$  some policy parameter,  $\theta$  some parameter of the problem and  $T(s, \theta, \cdot)$  is a mapping taking functions of  $x$  into new functions of  $x$ . For a comparative statics analysis, the differentiability properties of (1) are of interest.

Our main result (theorem 4) essentially states that as long as the optimal policy parameter  $s$  is unique, if one can formally differentiate (1) with respect to  $\theta$ , treating the policy  $s$  as being fixed and  $v$  as being a priori differentiable, then this differentiation is a posteriori justified.

We also give a version of the "envelope theorem" which sometimes can be used to differentiate (1) with respect to the variable  $x$ . (The formal distinction in this context between variables and parameters of  $v$  is that the right-hand side of (1)

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\* Most of this work was done while I was visiting researcher at the Institute for Economic Studies, University of Stockholm.

depends on  $\theta$  and  $v(\theta, \cdot)$  for the current value of  $\theta$  only, whereas it depends on  $v(\theta, x')$  also for  $x'$  different from the current value of  $x$ .) A result in this direction has also been presented by Benveniste and Scheinkman (1979).

## 2 SOME MATHEMATICAL RESULTS

Lemma 1. Let  $D$  be an open subset of  $R^n$ ,  $f(x)$  a continuous function on  $D$ ,  $g(x, z)$  a function on  $D \times D$ , and assume that

$$f(x) \geq g(x, z) \text{ for } x \text{ close enough to } z, \text{ and}$$

$$f(x) = g(x, x) \text{ for all } x \in D.$$

Assume further that  $g(x, z)$  is differentiable w.r.t.  $x$  and that  $g_1(x, x)$  (the subscript denotes differentiation w.r.t. the first  $n$  variables) is continuous on  $D$ .

Then  $f(x)$  is continuously differentiable, and

$$f_x(x) = g_1(x, x).$$

Proof. Since we differentiate w.r.t. one variable at a time, we may without loss of generality set  $n=1$ . Now, for any  $x_0 \in D$

$$\liminf_{0 < h \rightarrow 0} [f(x_0+h) - f(x_0)]/h \geq$$

$$\liminf_{0 < h \rightarrow 0} [g(x_0+h, x_0) -$$

$$-g(x_0, x_0)]/h = g_1(x_0, x_0). \quad (2)$$

Now take  $h > 0$  arbitrarily and define

$$m(h) \equiv h + \max\{g_1(x, x) \mid x_0 \leq x \leq x_0 + h\}.$$

For  $h$  fixed, define

$$F(x) \equiv f(x) - m(h)(x-x_0), \quad x \in [x_0, x_0+h].$$

$F(x)$  is continuous, and its maximum in  $[x_0, x_0+h]$  must be attained at  $x_0$ . Indeed, for any  $x \in [x_0, x_0+h]$

$$\limsup_{0 < k \rightarrow 0} [F(x-k) - F(x)]/(-k) =$$

$$\limsup_{0 < k \rightarrow 0} [f(x-k) - f(x)]/(-k) - m(h) \leq$$

$$\limsup_{0 < k \rightarrow 0} [g(x-k, x) - g(x, x)]/(-k) - m(h) =$$

$$g_1(x, x) - m(h) \leq -h < 0,$$

which is impossible at a maximum point. Hence  $F(x_0+h) \leq F(x_0)$ , i.e.  $f(x_0+h) - f(x_0) \leq m(h)h$ , thus

$$\limsup_{0 < h \rightarrow 0} [f(x_0+h) - f(x_0)]/h \leq \limsup_{0 < h \rightarrow 0} m(h) =$$

$$g_1(x_0, x_0). \quad (3)$$

Of course, (2) and (3) show that the right-hand derivative of  $f$  at  $x_0$  exists and equals  $g_1(x_0, x_0)$ . The left-hand derivative is treated similarly. Q.E.D.

Theorem 1 ("envelope theorem"). Let  $D$  be an open subset of  $R^n$ ,  $A: D \rightarrow R^m$  a continuous, convex- and compact-valued correspondence,  $h(x, y)$  a continuous function on  $D \times R^m$  and define

$$f(x) = \max_{y \in A(x)} h(x, y), \quad x \in D.$$

Then  $f(x)$  is continuous. If we further assume that the maximizer  $y^* = y^*(x)$  is uniquely determined by  $x$ , then  $y^*(x)$  is continuous. If further  $h_1(x,y)$  exists and is continuous and  $y^*(x)$  is an interior point of  $A(x)$  then  $f(x)$  is differentiable and

$$f'_x(x) = h_1(x, y^*(x)).$$

It is important to note that  $h(x,y)$  is not assumed to be differentiable w.r.t.  $y$ .

Proof. The continuity of  $f(x)$  and of  $y^*(x)$  is well known, and we do not repeat the arguments here. Now define

$$g(x,z) \equiv h(x, y^*(z))$$

The differentiability conclusion of  $f(x)$  now follows from Lemma 1 applied to the pair  $f$  and  $g$ . Q.E.D.

Theorem 2 (Blackwell, 1965). Let  $D$  be some subset of  $R^n$ , and  $B(D)$  the Banach space of bounded, real-valued continuous functions on  $D$ , normed by the supremum norm.

Let  $T:B(D) \rightarrow B(D)$  be a mapping with the following two properties (Blackwell's conditions, abbreviated B.C. in the sequel):

$$\text{(monotonicity) } f \geq g \text{ implies } T(f) \geq T(g)$$

$$\text{(discounting) there is a number } \beta < 1 \text{ (the modulus of } T) \text{ such for all } f \in B(D) \text{ and all constants } c > 0, T(f+c) \leq T(f) + \beta c.$$

Then  $T$  is a contraction mapping with modulus  $\beta$ . In particular, the equation

$$f = T(f) \tag{4}$$

has a unique solution  $f \in B(D)$ , and if  $S$  is a closed subset of  $B(D)$  such that  $T$  maps  $S$  into  $S$ , then the solution  $f$  lies in  $S$ .

Theorem 3 ("Bellman's principle"). With the notation of Theorem 2, let  $T = T(s; \cdot)$  depend continuously on some parameter  $s \in KCR^m$  and assume that the modulus  $\beta$  can be chosen independently of  $s$ . Let  $A: D \rightarrow K$  be a compact-valued correspondence, and consider the two equations

$$f(x) = \max_{s \in A(x)} T(s; f)(x) \equiv T^*(f)(x)$$

$$g(x) = T(s_0(x); f)(x)$$

where  $s_0(x): D \rightarrow A(x)$  is any continuous function (it is easy to see that both right-hand sides define mappings satisfying B.C., so both equations have unique solutions). Then

$$g(x) \leq f(x) \text{ for all } x \in D.$$

Proof. Let  $S$  be the closed subset of  $B(D)$  consisting of functions  $\geq g$ . Then for any  $h \in S$  we have

$$T^*(h)(x) \geq T(s_0(x); h)(x) \geq T(s_0(x); g)(x) = g(x)$$

where in the second relation we used the monotonicity of  $T$ . Hence, by theorem 2,  $f \in S$ . Q.E.D.

For the rest of this section we will adopt notions from differential calculus for mappings between Banach spaces. We refer to Dieudonné (1960) as a general reference.

Lemma 2. With the notation of Theorem 2, let  $T=T(\theta; \cdot)$  depend continuously on some real parameter  $\theta \in (\theta_0, \theta_1)$ , and assume that the modulus  $\beta$  of  $T$  can be chosen independently of  $\theta$ . Assume further that  $T(\cdot; \cdot): (\theta_0, \theta_1) \times B(D) \rightarrow B(D)$  is differentiable. Then the solution  $f=f(\theta; \cdot)$  to

$$f = T(\theta; f) \tag{5}$$

is differentiable w.r.t.  $\theta$ , and  $f_\theta$  is the unique solution to

$$f_\theta = T_\theta(\theta; f) + T_f(\theta, f; f_\theta) = T^*(f_\theta). \tag{6}$$

Here the mapping  $T^*$  satisfies B.C.

Proof. Equation (4) may be written

$$f - T(\theta; f) = 0$$

so the conclusion that  $f$  is differentiable, as well as formula (6), follows from the implicit function theorem. The only non-trivial thing to check is that the derivative of  $f-T(\theta; f)$  w.r.t.  $f$  is invertible. But this derivative is  $I-T_f(\theta, \cdot)$ , where  $I$  is the identity mapping. We show below that the linear mapping  $T_f$  satisfies B.C., so its norm is at most  $\beta < 1$ , and the invertibility of  $I-T_f$  follows.

To prove monotonicity of  $T^*$ , take  $g \geq 0$  in  $B(D)$ . For any  $h \in B(D)$  and  $\epsilon > 0$

$$\begin{aligned} T_f(\theta, h; \epsilon g) &\geq T(\theta; h + \epsilon g) - T(\theta; h) - \\ &\| T(\theta; h + \epsilon g) - T(\theta; h) - T_f(\theta, h; \epsilon g) \| \geq 0 + o(\epsilon) \end{aligned}$$



by the monotonicity of  $T(\theta; \cdot)$  and the definition of differentiability. But  $T_f(\theta, h; \cdot)$  is linear, so dividing by  $\epsilon$  and letting  $\epsilon \rightarrow 0$  gives  $T_f(\theta, h; g) \geq 0$ . Since  $T_f(\theta, h; \cdot)$  is linear, this provides monotonicity of  $T_f$ , and hence of  $T^*$ .

Discounting is proved similarly. Q.E.D.

Theorem 4. With the notation of Theorem 2, let  $T=T(s, \theta; \cdot)$  depend continuously on  $s \in KCR^m$  and  $\theta \in (\theta_0, \theta_1)$  with a modulus of  $T$  being independent of  $s$  and of  $\theta$ . Let  $D(\theta)$  denote  $(\theta_0, \theta_1) \times D$  and let  $A: D(\theta) \rightarrow K$  be a continuous, compact-valued correspondence and consider the equation

$$f(x) = \max_{s \in A(\theta, x)} T(s, \theta; f)(x) \tag{7}$$

Assume that the maximizer  $s=s^*(\theta; x)$  is uniquely determined by  $\theta$  and  $x$  and that  $s^*(\theta, x) \in A(\theta_1, x)$  for all  $x \in D$  if  $\theta_1$  is close to  $\theta^1$

Then  $f$  is differentiable w.r.t.  $\theta$ , and  $f_\theta$  is the unique solution to

$$f_\theta = T_\theta(s^*(x), \theta; f) + T_f(s^*(x), \theta, f; f_\theta) \tag{8}$$

where the right-hand side defines a mapping in  $f_\theta$  satisfying B.C.

Proof. First we must prove that  $f$  is jointly continuous in  $\theta$  and  $x$ . To this end, we may temporarily think of the right-hand side of (7) being a mapping  $B(D(\theta)) \rightarrow B(D(\theta))$ . Indeed, for any  $g \in B(D(\theta))$

$$\max_{s \in A(x, \theta)} T(s, \theta; g)(x)$$

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<sup>1</sup> E.g., if  $A(\theta, x) = A(x)$  is independent of  $\theta$ .

is continuous on  $D(\theta)$  by Theorem 1 (although the maximizer need not be unique for arbitrary  $g$ ). Now, by Theorem 2,  $f \in B(D(\theta))$ , which proves the joint continuity of  $f$  in  $\theta$  and  $x$ .

Now define  $g(\theta, \theta'; x)$  on  $(\theta_0, \theta_1) \times (\theta_0, \theta_1) \times D$  by the equation

$$g(\theta, \theta'; x) = T(s^*(\theta'; x), \theta; g)(x)$$

where  $s^*(\cdot; \cdot)$  is continuous by Theorem 1. By Theorem 3,

$$f(\theta; x) \geq g(\theta, \theta'; x), \quad \theta \text{ close to } \theta'$$

and by definition

$$f(\theta; x) = g(\theta, \theta; x).$$

By Lemma 2,  $g(\theta, \theta'; x)$  is differentiable w.r.t.  $\theta$ , so by Lemma 1,  $f$  is differentiable w.r.t.  $\theta$ ,  $f_\theta(\theta; x) = g_1(\theta, \theta; x)$  and substituting  $f_\theta$  for  $g_1$  in the equation for  $g_1$  given by (6), gives (8). Q.E.D.

### 3 AN EXAMPLE

To illustrate the results of Section 2, let us consider the following simple dynamic problem;

$$v(x) = \max_{0 \leq y \leq f(x)} \{u(f(x)-y) + \beta v(y)\}. \quad (9)$$

Here  $u$  is utility flow of consumption,  $x$  capital stock and  $f(x)$  a production function. The discount factor is  $\beta < 1$ . This period's production  $f(x)$  is split into consumption  $c$  and next period's capital  $y$ ;  $c+y=f(x)$ .

Assume that  $u$  and  $f$  are continuously differentiable, increasing and concave, and denote the right-hand side of (9) by  $T(v)$ . Obviously  $T$  satisfies B.C., and the value function  $v$  is increasing and concave by Theorem 2, since  $T$  maps the set of (non-strictly) increasing, concave functions into itself.

The correspondence  $x \rightarrow [0, f(x)]$  is continuous and by concavity, optimal  $y=y^*$  is unique, so if  $y^*$  is not a corner solution,  $v(x)$  is differentiable by Theorem 1 and

$$v'(x) = u'(f(x)-y^*(x))f'(x).$$

(Observe that in order to apply Theorem 1 we need only a priori know that  $v(y)$  is continuous.)

Now we may write the first order condition for  $y$  (assuming away corner solutions, for simplicity)

$$-u'(f(x)-y) + \beta v'(y) = 0.$$

Using this equality it is possible to show that both  $y^*(x)$  and  $f(x)-y^*(x)$  are increasing in  $x$  (we omit the details), which we will exploit below.

Now assume that  $f=f(\theta;x)$  is parametrized by  $\theta$  such that  $f_{\theta} > 0$  and  $f_{\theta x} \leq 0$ . We may now use Theorem 4 to differentiate (9) w.r.t.  $\theta$  to get

$$v_{\theta}(x) = u'(f(x)-y^*(x))f_{\theta}(x) + \beta v_{\theta}(y^*(x)). \quad (10)$$

As anticipated by Theorem 4, the right-hand side of (10) is a mapping in  $v_{\theta}$  satisfying B.C.; we denote it by  $T^*(v_{\theta})$ . Hence we can use Theorem 2 to derive properties of  $v_{\theta}$ ; for instance  $v_{\theta}$  is decreasing in  $x$ . Indeed, using  $u'$  decreasing,  $f_{x\theta} \leq 0$ , the increas-

ingness of  $f(x)-y^*(x)$  and of  $y^*(x)$ , we see that  $T^*$  maps decreasing functions into decreasing functions. Using this fact one can show (we omit the details) that optimal consumption  $c^*$  increases with increasing  $\theta$ ; i.e., an improvement in production according to the specification  $f_{\theta} > 0$ ,  $f_{\theta x} \leq 0$  will (not surprisingly) increase consumption during the first consumption period for any initial capital stock  $x$ .

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