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**LESSONS FROM LEARNING TO HAVE
RATIONAL EXPECTATIONS**

by

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LESSONS FROM LEARNING TO HAVE RATIONAL EXPECTATIONS^{*}

by

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Abstract

This paper reviews a growing literature investigating how economic agents may learn rational expectations. Fully rational learning requires implausible initial information assumptions, therefore some form of bounded rationality has come into focus. Such learning models often converge to rational expectations equilibria within certain bounds. Convergence analysis have been much simplified by methods from adaptive control theory. Learning stability as a correspondence principle show some promise in common macro models. A new selection problem arises since differences in initial information and learning methods give rise to many different equilibria, making economic modelling sensitive to assumptions on information and information processing.

Section 1.

Introduction

When will economic agents learn enough about their economic environment to end up in a rational expectations equilibrium(REE)? That question has been the focus of much theoretical work since the end of the 70s. It is still a rapidly evolving research field that is hard to summarize and unify. Lacking the competence to do so, it may still be worthwhile to attempt a presentation of some important papers and results and how they seem to fit in with one another. The selection presented here is not exhaustive and the significance of the results is still a matter of controversy. Therefore the aim is not so much to evaluate but to point out common trends and divergences in the research. Before doing so, some general comments on rational expectations and the learning issue may serve to give a broader perspective in which to fit the models of RE learning.

The concept of rational expectations has become familiar to all economists over the last two decades. No doubt there is considerable intuitive appeal in the idea that systematic deviations in expectations from outcomes ought to be corrected. Any rational economic agent would be expected to at least try learning from observations in order to correct mistakes in forecasts. And it could well be argued that those who fail to do so will be disadvantaged and perish in the economic competition. There are, however, several difficulties with these arguments.

One of these is the main theme of this paper, viz. when and how aggregate information can be used to learn the environment of economic action good enough to support expectations on the future that do not deviate systematically from outcomes. To delimit the scope of this theme it may be useful to start by noting three difficulties that the literature about RE learning mostly do not consider.

One is that it is hard to imagine actual economic agents really forming the rational

expectations belonging to a model they have never heard of. Interestingly John Muth(1961) in his original paper actually motivates RE by the empirical observation that agents often seem to anticipate changes in key variables better or as good as predictions from economic models. As Arrow(1978) remarks, that might very well be because agents have access to more relevant information than the economist, which is a rather less than convincing argument to assume that their expectations are consistent with economic models based on considerably less or at least different information. Muth's argument then is that the modeller should anyway assume predictions to be the best possible within the model framework¹. But then, of course, the Lucas critique apply, since even rational expectations in that case is a parametrically reduced model of the underlying structural expectations and information structure. This is hardly mentioned in the RE learning literature.

The survival argument for rational expectations suffers from another difficulty, as pointed out by Richard Day(1990), viz. that the adaptive success of an economic agent is not equivalent to economic success, since the optimal satisfaction of an agents desires given his means is not the same thing as optimizing the survival in a given environment². Although exit and entry of learning agents clearly are interesting features of learning models, this issue has not yet been treated in the context of RE learning.

A third difficulty is the costs associated with expectations formation. As Radner(1982) remarks, such information costs would be likely to introduce non-convexities in choice sets, due to fixed set-up costs and dependence of the production set on the informational structure. Such costs are mostly neglected in RE learning models³ but there are some recent papers that indicate that this difficulty may receive more attention in future research.

The above problems as well as others not mentioned here may well give rise to scepticism about the realism in the RE hypothesis. We discontinue the list here, although it would certainly be possible to go on. But, whatever the arguments are to question the empirical

relevance of the rational expectations hypothesis, it might still be a useful theoretical device when we want to compress exceedingly complex real individual behaviour into theoretical representatives. It is then a long tradition in economics that a more or less reasonable adjustment process should be assumed and use the stability of that process as a correspondence principle. The conditions that are necessary for a learning process to converge to an REE serve to weed out models where the equilibrium is unstable in the sense that a perturbation of the equilibrium will lead agents to revise their expectations in such a way that the REE cannot be reestablished. Such stability also provides a criterion by which the number of REEs can be cut down when there are several. Models of the learning of REs are, I think, generally intended as a Samuelsonian correspondence principle rather than as attempts to describe how economic agents really learn. For the latter purpose other models of learning based on psychological research and purely adaptive algorithms are no doubt a better choice⁴.

In the last few years a unified approach of analyzing the learning process in RE models based on methods from the theory of adaptive control (Ljung(1977), Ljung and Söderström(1983)) have been developed. These methods are considerably more adapted to the problem than the martingale theorems used initially to establish convergence results. Formidable difficulties in the technical tractability of the learning problem have been overcome and substantial progress made in the understanding of how and when such processes converge to REEs. That is in itself an important feat regardless of the still remaining difficulties in interpreting the diverse results of such learning models.

The literature on RE learning is often classified according to whether the agents are fully rational or only boundedly rational, following an article in *Journal of Economic Theory*, 1982, by Blume, Bray and Easley. Fully rational learning takes place when agents know the model specification well enough to learn by estimating the parameter values consistently. Essentially the whole model specification is known, excepting only a few parameter values. Since this begs the question how the model specification came to be

known — not only its fundamental equilibrium form but also including the updating procedures used by other agents — Bray and Kreps(1987) has evaluated this approach as a "sterile benchmark". The concept of bounded rationality has therefore become widely accepted as a more fruitful approach for this kind of models. Most of the papers, although not all, referred to in section 2 below concern learning based on some form of bounded rationality. The agents are assumed to have some reasonable initial belief about the model but lack information needed to guarantee consistent estimation. It seems that most researchers try to narrow down their discussion to assumptions restricting agents to use commonly accepted econometric estimation methods in their learning. But there are many variations and an important recent paper prefer the computability concept from computer theory as a criterion for learnability instead of the convergence of estimation procedures. More general search models incorporating experimental learning strategies have also been used lately to investigate the RE learning issue.

From the research on RE learning it is clear that assumptions on initial information sets and the procedures of learning that agents use is very important. That is only to be expected, but not only are convergence properties dependent on these assumptions, the REE outcome itself is contingent on the informational assumptions⁵. It is remarkable how very simple, not to say primitive, learning rules often converge under fairly reasonable conditions to REEs conditional on limited amounts of information which sometimes may even be irrelevant. Although it has to be admitted that there are plenty of non-convergence results as well. On the bright side it should be noted that the use of learning stability as a selection criterion itself to weed out sunspots and bubble equilibria from commonly used macro models has been at least partly successful. One problematic aspect is, that it remains an open and hardly researched question how to find any generally applicable selection criteria among the multitude of information assumptions associated with distinct REEs.

Published results so far seem to support the boundedly rational learning as a reasonable

correspondence principle for maintaining the RE hypothesis in a limited set of model types, in general stationary models where information assumptions are highly stylized. Since that set includes many commonly used macroeconomic models and hence, if the methodology based on the correspondence principle is accepted, means that the attempts to give credibility to the RE hypothesis by learning arguments have met with at least some success. On the other hand the identifying restrictions provided by rational expectations is derived from the informational structure imposed on the models rather than from any rationality *per se*. This is a feature of RE models that in general does not directly hit the eye, but which becomes very obvious as it is emphasized by learning models. Thereby the *ad hoc* character of these assumptions comes into focus.

How should rational agents choose their learning procedures when they lack the information and capabilities necessary to make a fully rational choice? Learning necessarily means committing errors and correcting them. Hence the optimal procedure will depend on how costly errors are and how easily they can be corrected. That, however, is information only available in a precise form after learning has taken place. Agents then have to form conjectures based on insufficient information. This opens the possibility that they may get stuck on non-rational equilibria in the learning process.⁶

In applying the RE hypothesis to real developing economies where non-stationarity and insufficient or even false information is common and totally unexpected economic events take place, learning is considerably more complex. In "experimentally organized economies" (Eliasson(1989a)), the learning process itself becomes more important than any (temporary) convergence point. In such a setting, where information is scarce and localized and behaviour is experimental and testing rather than optimizing, the path followed by the economy will depend more on the dynamics of the learning process than on any characteristics of a long run REE of the economy.

In a modern economy where information and information handling is of primary

importance⁷ this indicates that the conclusions derived from the rational expectations hypothesis may be very sensitive to implicit or explicit information assumptions. While adaptive learning in general are centered directly on goal achievement and treats the environment essentially as a black box, the rational expectations learning in contrast aims at specifying the framework within which to optimize. However, treating the parameters of the relevant framework as the facts to be learned, this is an adaptive learning process and subject to all the problems of such processes when the underlying structure changes while the learning is still going on. Being cautious about the applicability and interpretation of RE models in real economic contexts should however not prevent us from learning the lessons of theoretical research on RE learning. There can be little doubt that this research has provided a richer and deeper insight in the workings of expectations in economic modelling.

Of course, space limits as well as subtle shifts of meaning buried in different usages of terminology prevent any really deep probing of the problems of RE learning in the context of a short overview like this. The next section, which also is the main part of this paper, will describe and organize some important parts of the literature on learning about rational expectations as well as try to substantiate some of the assertions made above. As will be seen the adaptive — or even adoptive, in the sense of Alchian(1950) — character of these learning processes is a prominent feature. In the third and concluding section some tentative connections are made between the literature surveyed and more general adaptive models as well as game theoretical concepts. I have chosen to avoid formalization on the whole, with one trivial exception, in order to avoid squeezing slightly disparate equilibrium and learning definitions into any common framework, which still awaits general agreement.

Section 2.

Learning rational expectations.

This main section will be subdivided into seven subsections. The first will discuss the distinction between fully rational and boundedly rational learning. The second will review some results concerning econometric learning algorithms. The third subsection reviews papers concentrated on the issue of econometric learning stability as selection criterion among multiple REEs. The fourth subsection treats work on general learning stability in a temporary equilibrium framework. The fifth cursorily reviews a more diverse collection of papers. These diverge from the papers in the earlier subsections in assumptions about state spaces and learning algorithms in ways that is hard to generalize under any common heading. The sixth subsection treats some computability results that differ considerably in spirit from the main trend within the RE learning literature. The seventh and final subsection contains a summary and some tentative and partial conclusions.

2.1 Fully rational versus boundedly rational learning

The distinction between fully and boundedly rational learning commonly used in the literature is based on Blume, Bray and Easley(1982). It will be used here in a not quite equivalent form.

Fully rational learning means that agents know the model specification well enough to learn by estimating the parameter values consistently. More exactly, when they estimate they use likelihood functions that are correct specifications, conditional on available information, for data generated by stochastic processes where agents do use these likelihood functions. Townsend(1978) introduces an early model of fully rational learning by a Nash equilibrium concept. The points of convergence of the learning process may be considered as Nash equilibria in learning strategies, where each agent's market model specification, including parameter values, is the correct one for the market information generated *ex post*,

if he and everybody else use this specification. The main point in fully rational learning is that agents learn this correct specification by using correctly specified estimation models.

Under rather mild regularity assumptions these models will converge to an REE where the expectations of the agents will be verified by the outcomes, apart from some residual stochastic noise, from which the learning methods of the agents can extract no further information. Bayesian learning will in general converge given some coordinating common knowledge assumption. Mark Feldman(1987a) shows this for the case with homogeneous beliefs where the distribution of equilibrium outcomes is assumed to be a continuous function of forecasts and homogeneity of beliefs is common knowledge and Feldman(1987b) proves it for heterogeneous beliefs in a partial equilibrium model based on Townsend(1978), who conjectured this. Margaret Bray and David Kreps(1987) show similar results for somewhat more general information assumptions, but they emphasize that problems of convergence to the REE can still persist even if beliefs converge, if there are multiple market equilibria of the basic model. The problem of coordinating beliefs to one specific equilibrium is by no means trivial, cf. Crawford and Haller(1990). Blume and Easley(1984) investigates Bayesian learning in a model where some agents update beliefs over a common finite set of probability measures and some agents are fully informed. The updating is recursive, i.e. current observations do not enter in current beliefs. The structural model can be learned if the estimated parameters are sufficient to identify the structure. It bears stressing that all of these models rely on a non-trivial common choice of prior beliefs. Bray and Kreps characterizes this as learning *within* a grand rational expectations equilibrium.

Townsend(1983b), as many others, argues that rational learning must incorporate some assumption of common knowledge on some level. It may be with regard to forecast functions used, or updating procedures or on some deeper level. But it is needed to truncate the infinite regress involved in the structure of models where agents try to forecast the forecasts of others. I.e. on some level consensus has to be imposed on behaviour to make it determinate. Blume and Easley(1984) compare this to Harsanyi's(1967) Bayesian games

where it is presumed that the space of player types is common knowledge. However, assuming agents to agree on essential features of the model seems no good starting point for an answer to the question how they came to learn this model. The strong dependence on common knowledge assumptions is not unique to this brand of modelling. It becomes very explicit though since it is common knowledge of other agents learning behaviour that is assumed. The question naturally arises: How was that common knowledge established? It could hardly be inferred from market signals *before* the parameters were learned. Michael Bacharach(1989) argues emphatically that such learning should not even be called rational, because with that much information available, optimizing agents do not act consistently if they converge to the REE fixed point of the model. They ought instead to take advantage of their knowledge to act strategically when they know that parameters depend on their actions. Of course, this is less of a problem when the set of agents is large, but then again: assumptions of common knowledge also become rather less attractive as the number of independent agents with incomplete information increases.

But fully rational learning can be useful in other ways than as approximations to real markets. Xavier Vives(1990) uses a signalling model of fully rational Bayesian learning to characterize the speed of learning when information is asymmetrically distributed. Convergence speeds turn out to depend crucially on the precision of private information. These results can then be extended to somewhat less rational learning. Using fully rational learning as a modelling tool in order to get a handle on questions in more general models may be a more fruitful way of exploiting this particular line of research than as a correspondence principle for RE models.

The unattractive prior information assumptions of fully rational models have made the concept of bounded rationality more widely accepted as a starting point for models of RE learning. Most of the papers referred to below concerns learning based on some form of bounded rationality. Bounded rationality here can mean just about anything that is not fully rational. The common denominator is only that there is something in the model that

prevents agents from being fully rational in a well defined sense. It may be lack of information on the model that necessitates the use of more or less misspecified learning techniques, e.g. Bray(1982). It may be that the set of models to choose from is too restricted, e.g. Blume and Easley(1982), or it could be that only local information is available for some variables, e.g. Frydman(1982). Heterogeneity in information sets and instrumental variables regression is another example where learning must be considered boundedly rational, like in e.g. Fourgeaud, Gouriéroux and Pradel (1986). Restrictions on the calculating abilities of agents is another possibility, e.g. Spear(1989). In fact it is one of the main difficulties of a theory of bounded rationality that it can take so many different forms. The classification used in this paper is somewhat broader than the one used in e.g. Blume, Bray and Easley(1982) where Frydman(1982) i.a. have been classified as a fully rational model. I have not considered knowledge of the "true" model as a sufficient condition for fully rational learning if the ability of agents to make full use of that knowledge is restricted by e.g. in Frydman's case limited information on the realizations of the model.

The boundaries between full and bounded rationality models in the literature are necessarily somewhat fuzzy, due to subtle differences in definitions and approaches. For example, J.S. Jordan(1985) defines the REE somewhat unconventionally as the outcome of a kind of informational tâtonnement process. Thereby the learning process can be dealt with recursively, avoiding the problematic simultaneity in expectations and price determination of the conventional formulation. The expectations of the agents therefore need not be conditioned on the expectations of other agents. Within this REE definition learning could then be considered fully rational. However, the definition as such prevent agents from trying to predict how other agents change their expectations, thereby placing a bound on their rationality. The main feature of boundedly rational learning models in general is that agents are supposed to use in some sense misspecified estimation models due to lack of information.

2.2 Econometrically based learning

The mainstream of the RE learning literature concerns learning of parameters in linear models by means of least squares regression or, occasionally, Bayesian estimation. The early contributions in this area — e.g. DeCanio(1979), Bray(1982), Bray and Savin(1986) and Frydman(1982) — used particular models, like cobweb and asset trading models, and proved convergence of learning within some limited range of structural parameter values. Techniques of proof were often complicated and specially tailored to the specific market at hand.

The stability of learning processes with misspecified models often depends critically on parameter values, and one of the recent developments in the area is the application of a general method from control theory to determine such stable parameter values in relatively more simple ways. This method was developed by L. Ljung(1977) and L. Ljung and T. Söderström(1983) for use in recursive estimation for adaptive control. The first published application, to my knowledge, of this theory to the RE learning problem is a short note by D. Margaritis(1987) analyzing the Bray(1982) model. But it was two papers by Albert Marcet and Thomas J. Sargent(1989a,1989b) that were widely circulated before publishing that adapted the technique to the RE learning problem in general terms and introduced it into the mainstream of research. Marcet and Sargent(1989a) applies the Ljung theorems to the learning problem of self-referential REE models (in the sense that the actual law of motion depends on the perceived law of motion), exemplifying the approach on the models of Bray(1983) and Bray and Savin(1986) as well as a model used by Fourgeaud, Gourieroux and Pradel(1986).

In the Marcet and Sargent(1989a) paper the essentials are outlined for learning processes where information is symmetrically distributed and encompasses all state variables. The behaviour of the system of the stochastic difference equations arising from a linear learning rule may under certain general, but rather messy, regularity conditions be inferred from the

behaviour of associated ordinary differential equations. The perceived behaviour (summarized by a parameter vector, β) induces the real behaviour of the system by the mapping $T(\beta)$ back into the parameter space. Local stability of the stationary point of the associated differential equation system

$$d\beta/dt = T(\beta) - \beta \quad (2.1)$$

then implies local convergence with probability one of the corresponding least squares learning process based on β . Global convergence properties can be inferred from analysis of a larger differential equations system that incorporates changes in the updating procedure for β . More general updating procedures than ordinary least squares can thus be analyzed and fit into this framework. In order to guarantee almost sure convergence some rather complicated boundedness conditions must be fulfilled. A drawback of the technique is that these may sometimes be difficult to verify. To prevent the updating procedure from going outside the verifiable attraction areas a projection facility is often needed that prevents outliers from throwing the estimation out of bounds. Essentially it is an assumption that learning agents throw away certain outlier observations but the projection facility may be a little more sophisticated in order to extract at least some information from the discarded observation.

In Marcet and Sargent(1989b) the approach is extended to models with hidden state variables and asymmetric information like those in e.g. Bray(1982) and Frydman(1982). In models with these characteristics it may be very complicated to compute the actual REE state. Either numerical solution of the associated differential equation governing convergence or simulation of the least squares learning model then offers alternative ways of computing REE values. Marcet and Sargent(1988) summarizes the arguments for this.

When there are hidden state variables and/or private information the approach described above must be modified. The associated differential equation governing local convergence is changed to

$$d\beta/dt = S(\beta) - \beta \quad (2.2)$$

where $S(\beta)$ is a composition of the mapping $T(\beta)$ in (2.1) with certain partitions of the covariance matrix of the system, the partitions depending on which variables are hidden or private. Given that $T(\cdot)$ has its eigenvalues in the unit disc, and the system otherwise is well defined, the mapping $S(\cdot)$ too will be well defined. Under regularity assumptions similar to the above it can then be used much in the same way as $T(\cdot)$ when there are no hidden variables. Apart from the above papers Sargent(1991) applies the apparatus to a more general class of models proposed by Townsend(1983a).

This approach gives a unifying and more general framework for analyzing and comparing convergence properties of a great variety, though not all, of the different learning processes in the literature. The differential equation approach parallels and confirms stability results by Evans and Honkapohja in a series of papers described in the next subsection. The rest of this subsection will be devoted to a more detailed, but still sketchy, verbal description of some important and often cited papers within the mainstream of econometric RE learning literature. Papers focussed on stability issues are deferred to the next subsection.

Margaret Bray (1982) uses an asset market model (based on an infinitely repeated version of the model used in the Grossman and Stiglitz(1980) paper on efficient markets) with two classes of traders, one informed and one uninformed. The informed traders act on rational expectations⁸ all along while the uninformed forecast asset return from current price on the basis of OLS—regressions on past observations of the relation. The underlying stochastic process consists of the variables: information private to the informed traders, the asset returns and supply; that form a sequence of independent identically distributed multivariate normal random variables.

Two different learning rules are investigated. In the first uninformed traders regress asset returns on price, while they are using an initial conjecture as forecast. Trade then takes place at market-clearing prices. When the estimates have converged to a probability limit all uninformed traders simultaneously shift their forecasts to this limit estimate.

Continuing in that way by periodic revisions they will eventually arrive at rational expectations, provided the ratio of informed to uninformed demand is high enough to dominate price effects on asset returns. The degree of coordination in the switch required by agents in such two-stage learning processes is rather formidable.

The second and considerably more complex but also more realistic case is when agents are assumed to update their forecasts every time a new data item is reached. By the assumption that uninformed traders know the means of prices and returns, it can be shown that expectations also in this case will converge to an REE under a similar but less stringent condition as the one in the first case. This recourse to an assumption effectively meaning that average forecasts are known at the time of forecasting seems rather artificial and circular. But Marcet and Sargent(1987), using the differential equations approach, confirm the result without this assumption.

Margaret Bray and N.E. Savin (1986) use a cobweb-model with a continuum of agents learning by Bayesian methods, of which OLS is a special case. The basic stochastic process is a sequence of independent, identically distributed random variables, one of which is unobservable at decision time while the others are assumed observable. This learning process converges to a stable REE with probability one provided some economically reasonable conditions, essentially to ensure that the supply curve cuts the demand curve from below. Maybe the more significant result is that it can be shown that the probability is zero for convergence to any non-rational equilibria. This is a property the authors feel should hold for every reasonable learning process⁹, since estimation methods give consistent estimates if the data are generated by a stationary model. The cobweb model is stationary when expectations have converged and hence non-rational equilibrium expectations ought to be ruled out. However, the authors claim that convergence can be proved for models with non-stationary stochastic processes, too, (p. 1137) although they then need stronger conditions on permissible parameter values, and A. Marcet and T. J. Sargent(1989a) also show the assumption of independently identically distributed variables

to be unnecessarily restrictive in this case.

Just as in the preceding paper agents estimate a standard linear model that obviously is misspecified during the learning phase. Therefore the main theme of this paper is simulation experiments trying to determine how fast convergence of the learning process must be to prevent Bayesian agents from spotting the misspecification of the learning model in relation to the data generating process by standard statistical tests. The closer parameter values are to unstable regions, and the more initial confidence agents have in an incorrect prior conjecture, the more probable such spotting of misspecification becomes, because learning will take place at a slower pace. I.e. if the slopes of demand and supply schedules are sufficiently separated and the agents open-minded about their initial guesses, especially if they use ordinary least squares estimation, learning will take place at a rate that makes it very difficult to obtain a significant misspecification test. What will happen if the misspecification nevertheless is spotted by a test is not clear because there is no precise econometric rules available when it comes to correcting the model specification. I.e. there is no formal rule available to determine rational action in this case.

Roman Frydman (1982) explores another aspect of bounded rationality in information handling. The model is a product market where the agents, on the supply side, are equipped with correct model specifications of the demand and supply structure but lack information on parameters. Cost information is only locally available. A trader will know the market price and the stochastic realizations of his own cost function but not the realizations of other traders' cost functions. Then, to be able to form optimal forecasts in the minimum mean square error sense, they will need information on the "average opinion", i.e. they need to know the average of other agents' forecasts up to a white noise disturbance. If some institution external to the market provides such information, rational learning estimation of parameters may converge to REE. Frydman distinguishes two cases. One where forecasts are modified by the information before supplies are finally determined. The information thus relates to a preliminary average opinion. In this case the probability

is strictly positive that convergence of forecasts *does not* take place even if a very large number of updating rounds are allowed to take place and divergence of opinions take place with probability one if demand is sufficiently inelastic. In the other case supplies are determined before information on average opinion is received. Then model parameters can be consistently estimated with *ex post* information on average opinion. If all firms forecast price in the next period by using the rule that the current parameter estimates are substituted into the relation that holds between parameters and price in the REE, then prices will converge to the REE price. Frydman stresses that such a consensus-rule would in general be suboptimal for the individual to use.

Marcet and Sargent(1987) analyze this model using a learning procedure that disregard the forecasts of other agents and obtain a strong global convergence result. This is corroborating the conclusion of Evans and Honkapohja(1990b) and Grandmont and Laroque(1990) in another context that "oversophisticated" agents that try to be more rational than data allow them to be, may disturb the stability of RE learning.

Fourgeaud, Gourieroux and Pradel(1986) emphasize that no prior knowledge of the model ought to be assumed. Therefore they model the forecast procedure as a regression on a predetermined set of instrumental variables that may or may not be included among the exogenous variables of the structural model. They show that such an automatic forecast procedure converges to REE in a cobweb model under assumptions of stationary regression coefficients and some mild regularity in the asymptotic behaviour of the instrumental variables chosen for prediction. The same holds for a Cagan model of hyperinflation and hence for a model including expectations on future values of the endogenous variable. The surprising feature is that convergence holds independently of how strong the correlation is between instruments and exogenous variables. Hence an essentially *ad hoc* regression will yield rational expectations in the long run. However, the rate of convergence will depend on the choice of instruments as well as how close parameters are to the stability boundaries of the model. Furthermore the REE will depend on this choice of information set. Different

choices of instrumental variables will in general result in different REEs, as the conditional expectations will be dependent on the information sets.

In a conference volume from 1983 (edited by Frydman and Phelps) several contributions are centered around the average opinion problem and how to handle expectations on expectations. Edmund Phelps and J. C. Di Tata discuss short run effects arising from assuming that the individual agent does not believe that the average opinion is the same as his own expectation. George Evans shows how different sets of initial conjectures about the average opinion by learning from experience may lead to different individual rational expectations and outcomes. The REEs of this model may be interpreted as Nash equilibria in strategies dependent on these initial conjectures and the learning rules used. Expectations of others' expectations necessarily entails strategic considerations. Then knowledge of the fundamental model parameters is not enough to guarantee the stability of an REE. It is also necessary that collective expectations of expectations converge when they are updated out of equilibrium. Evans shows how such processes may be unstable in a Goodwin business cycle model as well as in a simple macro model where static expectations imply stationarity. Frydman derives conclusions resembling those of his 1982 paper in an island model of Lucas type.

Only a selection of results, that I find representative of the literature, have been mentioned here. In summary, econometric learning on basis of misspecified models converges probabilistically to a unique REE within certain ranges of structural parameter values in most of the studied models. When agents recognize that outcomes depend on the expectations of other agents problems arise, unless they are short-circuited by some common knowledge assumption. Those problems are very similar to the corresponding interaction in oligopolistic market models. Once agents recognize their strategic interdependence we are in a game situation where a much more sophisticated and detailed modelling is required giving considerably less general results. Whenever more than one fixed point of the mapping $T(\beta)$ exists the problem of choosing among the possible REEs

arise. That is the theme of the next subsection.

2.3 Learning stability as selection criterion

R. Lucas(1986) in a wellknown article proposed that stability of learning processes should be used as a criterion to decide which of multiple REEs to choose as the fundamental one. This subsection reviews some central papers dealing with this issue.

Michael Woodford(1990) investigates if learning can be used as a selection criterion among multiple REEs, i.e. which, if any, equilibrium will a learning process converge to. His framework is an overlapping generations model with fiat money as the only asset where stationary sunspot equilibria can be shown to exist. Unlike most other authors his learning scheme is not based on least squares estimation. He uses a non-parametric adaptive learning rule, stochastic approximation, that is analyzed with the same technique from control theory that Marcet and Sargent use.

Though the results are somewhat complicated to describe, they clearly indicate that the REE that would obtain in the absence of sunspot beliefs may not be the one that learning processes converge to. If agents are willing to consider sunspot variables, uncorrelated with the predicted variable, as nevertheless influencing the outcome, then adaptive learning may lead to sunspot equilibria and the REE that most economists would regard as the fundamental one may even be unstable with respect to the dynamics induced by the learning rule. If sunspot equilibria exist there will generally be multiple locally stable equilibria, but it is not in general possible to determine by initial conditions which of these a particular realization of a stochastic learning process will tend to because the stochastic element means that domains of attraction need not be disjoint.

Woodford also points out that if agents have different choices of sunspot variables or, more realistically, weakly correlated exogenous variables, the situation becomes

increasingly complex. Not only does the set of REEs multiply but stability results may be reversed for a former stable equilibrium point by the introduction of another sunspot variable believed to be possibly relevant by some significant fraction of the agent population. It seems clear then that learning processes *per se* cannot be relied on to single out a reasonable REE even if they do converge to some REE. However, as will be seen below, less ambitious targets may be accomplished by the use of learning processes.

George W. Evans(1985) uses a learning process similar to Bray(1982) as a "natural revision rule" for analyzing expectational stability (E-stability) of REEs in a general model where the current state of the model depends on last periods prediction of both the current and next periods state variables. In these models so called bubble equilibria may appear, i.e. REEs that are deemed as less fundamental than another in some sense. Evans mostly use the definition of bubbles advanced by Bennett McCallum(1983), viz. REEs that do not satisfy the minimum state variable criterion for selection of the fundamental REE. See Evans(1986) for a detailed exposition on the relation between the minimum state variable criterion and E-stability. Evans interest in the adjustment process is, like Woodfords, not primarily the learning aspect but the use of stability of learning as a selection criterion among multiple REEs. He finds the bubble equilibria to be robust with respect to small perturbations in the parameters of the expectation function used by agents for forecasting. Evans refers to this as weak E-stability. However, the bubbles can be shown to be unstable in a strong sense if the learning process admits the use of irrelevant lags of the state variables, i.e. lags which are not included in the bubble RE solution in question.

In Evans(1989) this is followed up to include analysis of stability against inclusion of sunspot variables, and it is shown that Woodfords(1990) stability results on sunspot variables, for one class of the utility functions involved, are not E-stable in this strong sense. Evans initially conjectured that rational bubbles in general should not be strongly E-stable i.e. locally stable to overparametrization. This conjecture , however, seems to be

refuted by himself and Seppo Honkapohja(1990c) in a paper analyzing solutions of a general linear model including expectations on future values, where they find that for some parameter values isolated bubble equilibria indeed are strongly E-stable with respect to inclusion of irrelevant lags in the expectation function. These parameter values are shown to be within reasonable economic bounds in a macro model with real balance effects used in the literature.

In a recent paper, Evans and Honkapohja(1990a), a general class of linear models is analyzed. The class is characterized by one endogenous lag and expectations extending to three future periods. Within this class it can be shown that continua of REEs are at best weakly E-stable and if current period information is used to form expectations, not even weakly E-stable. But the instability with respect to overparametrization is one-sided so there might be convergence for some initial conditions. A close connection is established between E-stability and convergence of adaptive learning algorithms in the differential equation approach used by Marcet and Sargent.

Strong E-stability hence shows some promise as a correspondence principle for RE models in the specific sense that there are at least in some cases adaptive learning algorithms that will converge to a unique strongly E-stable REE. Evans and Honkapohja are confident that the results can be extended to more general classes of models.

However, Woodford(1990), points out that his stability concept is related to but not equivalent to E-stability and in some cases yield different conclusions. He claims that even if no sunspot equilibria are strongly E-stable there are reasonable learning processes which will not converge to the solution commonly regarded as fundamental. This is of importance since it means that scope is left for a multiplicity of "fundamental" equilibria depending on assumptions about the learning procedure.

Evans and Honkapohja(1990b) extends E-stability results to some simple classes of

non-linear models with and without stochastic disturbances. These models exhibit periodic solutions and have been studied in a more general deterministic context by i.a. Grandmont and Laroque, see below. The precise way that stochastic disturbances enter the model is shown to affect stability conditions. For isolated equilibria of the model it is proved that these equilibria alternate between E-stable and E-unstable solutions. Hence when there are several such equilibria E-stability would partition them into one stable and one unstable set of about equal size, give or take the odd one. The convergence results of DeCanio(1979) and Bray and Savin(1986) on the cobweb model are extended to the case where demand and supply may be non-linear. It turns out that the equilibrium can be rendered unstable for some parameter ranges if agents are considering periodic solutions to be possible. Hence, the earlier remark above about "oversophisticated" agents contributing to instability.

To conclude this section a short note by N. Gottfries(1985) could be mentioned. He uses a deterministic overlapping generation model with asymmetric information and a tâtonnement process of revisions of demand and supply before trade takes place. By the revision process private information can be disclosed and learned by others. It is found that only the unique stationary perfect foresight equilibrium may be stable, though it need not be.

Typically learning processes may be used to rule out equilibria by instability, but it seems more doubtful whether they really lend support to any specific equilibrium. Alternative reasonable learning processes may very well reverse stability results and thus call into question how "fundamental" a chosen equilibrium really is. Without a criterion that singles out some learning process as more reasonable than others multiplicity will remain a problem.

2.4 Temporary equilibrium stability

Most models described so far have started out by assuming some specific kind of learning process, Bayesian, least squares or (Woodford(1990)) stochastic approximation, and then trying to determine conditions when a more or less general model will converge to an REE. Even if the Marcet and Sargent framework has a more general potential it seems so far only to have been applied to variant forms of least squares and Bayesian learning. J.–M. Grandmont(1985), Grandmont and Laroque(1986,1988,1990) takes another route by trying to characterize the set of learning processes that are compatible with a stable temporary equilibrium in the neighbourhood of a perfect foresight equilibrium. This work is closely related to earlier contributions by Fuchs(1976,1977,1979a and 1979b).

In Grandmont(1985) the general dynamics of a non–stochastic overlapping generations model is discussed with only a small part discussing learning as a fixed function of a finite sample of past prices. He finds that stability of backward perfect foresight equilibria implies forward learning stability, but not the converse in general. This, at first sight, surprising connection between forward and backward dynamics has a natural explanation since the learning process uses backdated variables to predict the future. Grandmont and Laroque(1986) extend these results in a one–dimensional non–linear model and also give a class of expectation functions for which the converse also holds. Equilibria in these models may be periodic cycles and hence the expectation function itself must be able to "detect" the cycles, in fact forecasts must be able to detect cycles with period $2k$ to generate stability of a k cycle equilibrium. Grandmont and Laroque(1988) further extends this to a multi–dimensional framework where the temporary equilibrium may depend on lagged variables.

Grandmont and Laroque(1990) criticize the use of projection facilities in the multiple equilibria context. They, rightly it seems, deems it contrary to the spirit of enquiry, since it presumes that agents have some consensus on which domain of attraction to project

estimates into. But how could such consensus arise before any learning have taken place? Using the same temporary equilibrium framework as in earlier papers they find the temporary equilibrium locally unstable for almost all initial conditions when the forecasting function is continuous and agents attach positive prior probability to the possibility of divergence. When the forecasting function is allowed to be discontinuous there are open sets of initial conditions that may result in convergence.

It remains somewhat obscure how this approach ties in with the mainstream of the literature. The assumption of a fixed memory bound makes comparisons with econometric learning models difficult, since it may considerably change equilibrium properties, as shown by Fourgeaud, Gouriéroux and Pradel(1985). Evans and Honkapohja(1990b) make some connections to their own work, where the condition on the expectation function to detect higher order cycles is viewed as a condition for strong E-stability against overparametrization in the sense of allowing for longer period cycles.

2.5 Some other approaches to learning

Lawrence Blume and David Easley(1982) develop a learning model where each agent considers a finite set of possible models of the economy, and learns by updating a prior distribution over these models. The true joint signal becomes known *ex post* while the agents *ex ante* had knowledge only of their own contribution to the signal. They then learn according to the simple rule: increase the weight of the model if its prediction is better than average and vice versa. Blume and Easley conclude that an REE will be locally stable with this kind of learning. However, there are also non-rational expectations equilibria and even cycles that are locally stable. None of the admissible models include predictions of other agents' predictions and the set of economic models the agents choose from thus is too small to describe the full behaviour of the economy. Therefore some sets of data will induce model choices that are non-rational and still locally stable. Hence learning in this way may, but does not necessarily, lead to REE. Curiously, an extremely simplistic learning

procedure actually does guarantee convergence to REE. If the agents use one point distributions and update by choosing randomly a new point distribution whenever a prediction fails, they almost surely converge to an REE. The authors reject this result, partly because it depends heavily on the finiteness of the model space, and partly because such behaviour has "no trace of rationality attached to it"(ibid. p. 350).

J. E. Foster and M. Frierman(1990) use the Blume and Easley(1982) model to investigate conditions of global stability of the RE learning process. They employ a graphical representation that makes the model considerably more transparent to intuition and shows that gross substitutability is a sufficient condition for the revealing REE to be globally stable under Bayesian learning. Gross substitutability in this context is conditioned on the state of the world and includes the effect on total demand that a price change induces by updating of beliefs. It is interesting that sufficient conditions for a unique REE stable under learning are analogue to the common Walrasian conditions for a static equilibrium. Essentially the gross substitutability condition requires that the adjustment of beliefs affect decisions relatively slowly in the sense that these effects do not dominate the ordinary income and substitution effects. Foster and Frierman points out that the stability conditions in the Bray and Savin(1986) cobweb model (described above) also are analogue to commonly stated stability conditions for the static model.

Another approach to the learning issue takes its departure in the wellknown "two-armed bandit" problem, where a gambler, choosing between two slot machines, one with known and the other with unknown pay-off probabilities, may with positive probability end up playing the machine with the lower pay-off probability for ever. Michael Rothschild(1974) has applied this to the price setting problem when demand is unknown and found the choice of final price to be undetermined. Nicholas M. Kiefer(1989) has worked out a variation on this to the case of a monopolist trying to establish which of two possible demand curves is the true one. In this case, too, the monopolist may get stuck on the wrong conclusion.

The key mechanism behind these results are that agents are supposed to optimize their learning behaviour, actively generating information. If an initial sequence of experiments leads to beliefs about expected payoff from continued experiments that are too low the agent will discontinue active learning. The idea is that learning entails a cost or at least a possible cost in terms of sacrificing short term profits in order to learn about the environment. Other papers in the same vein of thought are Easley and Kiefer(1988), using a more general model, Kiefer and Nyarko(1989), analyze a similar setup restricted to linear models. Kiefer(1989) provides a summary and introduction to this area of learning models. Bala and Kiefer(1990) introduce investment in calculation abilities, e.g. computers, in the same type of models.

Conceptually similar but technically rather different is a paper by Evans and Ramey(1988) where explicit calculation costs and myopic agents induce non-rational equilibria for some parameter values and REEs for others. While the Kiefer *et alia* papers posit an agent actively seeking to generate information, Evans and Ramey keep to the mainstream paradigm of agents passively receiving market generated information but achieve much the same effect by making learning costly so that information may not be used even if it is available.

The papers treated above and in the preceding subsections are based on some adaptive learning mechanism and are only a sample from a thriving branch of the economic literature. There are several other papers in a similar vein. To mention a few other results and views in short: S. J. DeCanio(1979) concludes, in a simple cobweb model, that the existence of a rational forecasting function is no guarantee that agents will ever discover it. In a deterministic overlapping generations model G. Tillman(1985) concludes that self-fulfilling expectations equilibria exist and are stable only under homothetic preferences and small elasticities of substitution, when the model is rigorously derived from utility-maximizing behaviour. In a similar overlapping generation model J.-P. Benassy and M. C. Blad(1989) show that rational expectations will almost never be learned. Their

learning process is, however, extremely simple. It uses only the second last observation at every updating. The argument for such simplistic behaviour is the great complexity arising because of non-linearities in the system governing dynamical behaviour of optimizing individuals.

It seems an open question whether the instability results by Benassy and Blad(1989) and Grandmont and Laroque(1990) may have something to do with the fixed memory length they use. Evans and Honkapohja(1990b) make a remark in that direction. A somewhat peripheral paper that may be relevant on this specific question is Gates, Rickard and Wilson(1977) which analyzes the adjustment process on oligopoly markets and finds that updating processes placing high weights on the most recent observations increases the risk for instability. Fixed memory learning as compared to accumulating memory learning, e.g. ordinary least squares, in the long run weights recent observations relatively less. However, Fuchs(1976) in a deterministic context finds that too high weighting of observations in the past decreases stability when the expectations function is fixed. Results on memory length and weighting schemes thus are rather context dependent and more general results on this seems to be lacking.

2.6 Computability and decidability

There are a few papers concerned not with convergence of adaptive learning but with the question: Is market information sufficient for agents to make the necessary calculations for a consistent updating? In this subsection two different approaches will be described.

A recent contribution by Stephen Spear(1989) imposes the constraint that any forecast function used must be computable by a finite algorithm. Using results from computer science he shows that with perfect information about the state space (which is finite) the rational expectations forecast function can be recursively identified in a two-stage learning process. Two-stage means that agents first collect observations of the outcome using a

fixed forecast function and then switch to the identified function conditional on the old. Then they repeat the process until they eventually arrive at the fixed point like in Bray(1982). At least that is what happens if the functional mapping from forecast functions to price functions as well as the price functions themselves are primitive recursive, a not very restrictive requirement in practice. Lacking perfect information, however, the rational expectations function cannot be learned by inferring the functional mapping from forecast functions to price functions in the two-stage process because that would require knowledge of correspondences, i.e. multi-valued functions, that cannot be recursively calculated. The same obstacle arises if agents only try to determine whether they are using a fixed point function or not. This problem is in general undecidable because of an analogue in recursion theory to the Gödel theorem. When agents update the forecast function in every period Spear finds that even if it is assumed that they arrive at an equilibrium, where the forecast function is consistent with the forecast function selected by the updating procedure, it still cannot be determined whether this is an REE in the sense that it is the same as the true price function. More concretely, agents may receive information signals such that their updating of the forecast function cease to change it, but this forecast function may still differ from the price function of the economy. The point is that the agents are unable to tell the difference because they cannot calculate which updating procedures that converge to non-rational equilibria and which converge to REEs.

At first glance these results seem to contradict e.g. the Fourgeaud, Gourieroux and Pradel(1986) results where no knowledge of the fundamental state variables is presumed. But it should be noticed that these strong computability results really relate to the possibility for agents to completely specify how the economy transforms forecast functions into price functions within a fairly wide class of computable functions. Ordinarily it is only required that information signals from the model does not controvert the models used by agents for forecasting. That does not in general imply that the models used are identical with the theoretical model of the economy where agents use such forecasting models. Furthermore most of the models used in the learning literature have a linear structure as

well as the learning rules used. This considerably limits the possibilities among which to learn. It remains to be seen what significance these computability results really have for the question whether learning agents end up in REE. The learning impossibility results in Spear's sense is actually a statement to the effect that there is no way for the agent to *decide* whether a model equilibrium is REE or not. From the standpoint of economic theory that seems less relevant than asking if the model used by the agent is consistent with the equilibrium information the economy will provide him with.

Jonathan Thomas(1989) provides a very simple and concrete, though rather non-economic example, of an economy with infinitely many REEs none of which are computable.

Mordecai Kurz(1989) provides a quite different angle on whether agents really are capable of computing REE processes. He assumes agents with no restrictions whatsoever on calculating abilities. He also assumes away the feature that to most researchers have seemed the main difficulty in learning, namely that actions of the agent depend on beliefs about the beliefs of other agents, and suggests a non-participant learner as e.g. an economist. Giving a rigorous definition of complexity of stochastic processes he shows that these cannot be learned generically by Bayesian methods. Kurz argues that real economic processes typically are of a kind satisfying his definition of complexity, for example dependent on a large number of parameters, and the set of these parameters continually changing with time. Rather than Spear's dependence on intrinsic logical limits to inference Kurz points to non-reducible complexity as a reason why agents should not be expected to learn completely the parameters of the processes they need to forecast.

2.7 Summary

In all the above models some market clearing mechanism is assumed. Hence the information received from the market prices is not confounded by quantity constraints,

though that would not seem to be any really critical feature. It seems reasonable to assume that the inclusion of such constraints should not in any essential way change the results on learning. More crucial is that the definition of rational expectations is contingent on whatever information sets agents are endowed with by the modeller. Especially learning in asymmetric information models seems then to be rather *ad hoc*. In e.g. Bray(1982) it remains obscure how the informed agents came to learn the correct specification. In Fourgeaud, Gourieroux and Pradel(1986) it remains unclear why a certain choice of instrumental variable is made, by all agents *nota bene*. Some preliminary results of R. Frydman(1987) points out the possibility that diversity of opinion as to the correct model specification may in some cases actually enhance convergence to REE¹⁰. On the other hand there are results like Brusco(1988) in a similar model concluding that there will be no convergence with heterogeneous information when no group of agents is perfectly informed.

Anyway the REE will in general be dependent on the specifications of information sets used. This may lead, in the case of heterogeneity in initial beliefs, to rapid multiplication of possible REEs contingent on information assumptions. Such dependence is of course very troublesome for the predictive value of the rational expectations hypothesis since the information sets actually used by agents are only rarely observable.

The picture emerging is somewhat complex. On the one hand stable REE often emerge from simple learning rules in linear models, at least within some parameter ranges. On the other hand, those rules could generally be improved upon by an optimizing individual agent. But attempts to such improvements would often render the REE unstable. In the case of only locally available information we may have to assume the existence of some external information dissemination and even individually sub-optimal consensus rules may be necessary for its almost certain convergence to REE. Moreover the actual REE achieved will be sensitive not only to assumptions about initial information sets, but also the length of memory and how learning rules weight past observations, as well as the confidence agents have in their beliefs about the appropriate models and learning rules. Extensions to

more general model spaces and elaborate updating rules seem to undermine convergence results for simpler, fundamentally linear models. When the cost of information processing is taken into consideration it may further modify conclusions. If learning is too slow to prevent agents from discovering that their models are misspecified, it is far from clear what will happen, but many results point in the direction that if they try to be *too* clever the REEs will lose stability.

To me it seems to be at least two related sets of questions that need be answered before the relevance of RE learning to economic theory becomes reasonably clear.

1. Which one of several competing model specifications should a rational agent use when even economists disagree? In what sense might learning based on consensus rules be a rational economic choice? Is there an optimal choice of model specification given incomplete information on the form of the model? When and how should the basic model choices and learning rules be revised?

2. If agents learn by misspecified models such that they end up in REE, then there ought to be a pay-off to detecting such misspecification or even in some cases a pay-off to maintaining uncertainty by misleading signals or by experimenting to find out more about the system. How speedy need convergence be to prevent detection of misspecification? What happens to an economy where agents deliberately take sub-optimal decisions either to learn or to deceive?¹¹ To what extent should other rational agents take such possibilities into consideration?

No doubt there are more questions and perhaps more relevant, these are just two areas that strike me as important. In part this is due to my own work regarding the intertemporal consistency of conjectural variations in oligopoly models, i.e. the correctness of the conjectured dependence between the decision variables of oligopolists (cf. Lindh(1991) and section 3.). The problems encountered in this branch of modelling leads me to suspect that especially the second set of questions might be very tough to answer in

any general way. Experiments could be conclusive only if you knew to what extent other agents engaged in experimenting. Hence all information is contingent on other agents' behaviour, which in turn depends on the information these agents possess and their informed guesses as well as the extent of their knowledge of each others' knowledge, etc. That means questions of strategic interaction in learning introduces a potential circularity in the definition of information sets that may prevent learning from being even boundedly rational in any reasonable sense. A. Kirman(1983) provides a simple duopoly illustration of how such circularity makes the outcomes of learning procedures dependent on initial conditions and hence generally indeterminate without an argument for the specification of initial conditions. Townsend(1983a and 1983b) is clearly more optimistic in this regard, a view that seems based on belief in the reasonableness of common knowledge assumptions. In the game theory literature similar problems have been extensively investigated in connection with common knowledge assumptions, cf. e.g. Ken Binmore and Adam Brandenburger (1988) who go so far as to assert that Bayesian learning can be no more than a tiny part of genuine learning behaviour because it leaves the choice of priors unexplained. In a much earlier paper J. Marschak(1963) cautions us to observe that the optimal updating procedure cannot be chosen independently of the actions to be taken.

The hints above about "oversophisticated" agents disturbing stability also adds to this picture of complex interaction between beliefs, information and optimality at all levels. The information generated by the economy will depend on optimization dependent on beliefs. Beliefs that in turn are modified by the new information by updating procedures that themselves may be dependent on information and beliefs, etc. The potential circularity is obvious, how to handle it is considerably less obvious. Good answers to the second set of questions therefore require good answers to the first set. The self-referential character of guessing about other agents' guesses demands some restrictions to be answerable. Such restrictions can be given by answers pointing out how rational model choices should be made in the absence of certain knowledge and to what extent other rational people can be expected to abide by the rules of the model and refrain from experiments and deceptions

that create circularity in the learning process. But surely we still know very little about this in economics. The next section attempts to widen the perspective from RE learning to related problems in other areas in order to provide a wider perspective on these issues.

Section 3.

An attempt at perspective

In this section the problems of RE learning will be at least superficially related to more general adaptive learning in economics and to strategic interaction and game theory. Some interesting connections and references are pointed out without any ambition to discuss the deep issues involved. First some remarks are made on adaptive learning, then some issues of oligopolistic competition naturally leading to game theoretic aspects is considered. The section concludes by some brief comments on common knowledge assumptions.

In the models treated in section 2. "learning" means "learning the model" in order to optimize. A different approach to learning is the behavioural models in the spirit of R. Cyert and J. March(1963). The distinctive mark of this literature is that agents require much less information than is commonly assumed in rational, even boundedly rational learning models. That does in no way imply that outcomes differ, though of course they may. But "learning" in these models means "learning to be successful" in terms of whatever goal one wants to achieve.

Such learning models often have very simple rule mechanisms by which agents learn. The simplicity of the rules, however, does not necessarily impede their effectiveness. In e.g. Richard H. Day(1967) and R. H. Day and E. H. Tinney(1968) all it takes to converge to optimal solutions is some regularity, essentially convexity properties, in the postulated environment and some restrictions on how agents interaction takes place. The adaptive mechanism is extremely simple, just repeat successful behaviour and avoid unsuccessful and moderate responses according to the short history of the last two decisions made. If response moderation is avoiding extremes, convergence ordinarily results.¹²

Optimization over very large information sets can sometimes be replaced by very simple rules of thumb in stable enough environments. Rational economic agents should in many

cases prefer simple rules of thumb to optimization even if perfect information sets were available at reasonable costs (cf. Baumol and Quandt (1964) or Winston(1989)). Uncertainties are associated with all real world information, e.g. measurement errors and transmission losses. The models used to process the information is subject to considerable uncertainty regarding their relevance. That is especially true of many economic models.

However, some coordination in behaviour as well as sufficiently informative feed-back is necessary, R. B. Archibald and C. S. Elliott(1989) show in a learning model formally equivalent with expected utility models that individuals may easily "learn" false hypotheses, i.e. commit Type II errors to use statistical terminology, if their sampling of the environment is biased or incomplete. This is closely related to the two-armed bandit problem in section 2.4 above. There is always the possibility of getting stuck at inefficient or non-rational equilibria or confounding signals generated by the structure of the model with signals generated by erratic or strategic behaviour of other agents. Sidney G. Winter(1970, 1975) emphasizes that, although the equilibrium of the optimizing model may be obtained as a special long run equilibrium of an evolutionary adaptive model, this requires rather special assumptions on the adaptive mechanism. Day, Morley and Smith(1974) show how very small changes in the environment can radically change outcomes. Day(1975) cautions that the potential complexity of adaptive models essentially is limitless and that in real life there will always remain scope for error and misjudgement.

In Marcet and Sargent(1988) the processes of boundedly rational learning are formulated in a way that clarifies their adaptive character. The difference is essentially in the state space. While adaptive processes in general imply movements in a space of available actions, the RE learning processes move in a space of conditional expectations or more generally model specifications. We may see the similarity by considering how to find, given a model specification including expectations, the forecasting rule making a certain action the optimal choice for an agent using this forecasting rule in this model. Delimiting the allowable set of forecasting rules should provide restrictions on the set of possibly optimal

actions, but in general one would conjecture that variation in information assumptions and learning procedures would allow a fairly wide choice of model equilibria to be optimal. The results described in the preceding section also indicate this.

The concept of a learning process intrinsically includes some element of misperception on some level, because if agents had no misperceptions whatsoever they would have nothing to learn. Every learning process is in some sense an adaptive process where the outcome by definition cannot be precisely known a priori. Agents faced with the problem of finding out just what mistakes, beliefs and learning strategies others use can easily render economic processes unstable by trying to be overly rational¹³ or by attaching too great confidence in faulty prior beliefs, and even if equilibrium is attained it may be a sunspot or a bubble resting on irrelevant common beliefs. This of course adds a very high degree of complexity to economic models unless we are prepared to resort to some restrictions on what may and may not be allowable learning procedures and common knowledge. The question then is what should be presumed in order to keep things tractable.

In a rudimentary form that problem was considered already by Augustin Cournot (1838). Although more or less completely neglected at the time¹⁴, his solution to a simple duopoly model, being a special case of the general Nash equilibrium concept of non-cooperative games, has become famous. It has been the basis for the well known reaction function approaches to oligopoly problems of which the traditional conjectural variation models are an early example. It is interesting that Cournot takes the impossibility to exclude mistakes and deception as an argument in favour of optimizing as if the rival's action was independent of your own. The resulting equilibrium of his own simple adaptive model of duopoly was for a long time regarded as inferior to the cartel optimizing solution, see i.a. Fellner(1949).

Based on Fellner's "right for the wrong reasons" argument — that agents ought to perceive that their conjectures about the reactions of other agents are wrong — these

models quite recently gave rise to a strand of literature exploring a "consistent" conjectural variation concept (Bresnahan(1981) and Perry(1982) among many others). Consistency in this context referred to the property that conjectural variations should be consistent with the optimal reactions of the oligopolists. The learning character of strategic interaction here becomes quite explicit through the motivation that agents ought to learn by experience how their rivals will react to changes¹⁵.

The Fellner critique obviously is very close in spirit to the common motivation for rational expectations, that agents learn by experience to avoid all systematic mistakes in their forecasts. But as the literature on RE learning shows, systematic misperceptions in the learning process itself does not necessarily prevent convergence to REEs. Likewise conjectural variation models normally exhibit stability if agents do not take discrepancies in actual and expected values as a reason to revise their a priori beliefs before equilibrium is reached.

The problems of these failed attempts to rationalize conjectural variations by consistency are closely related to the problems in defining stable equilibrium concepts in non-cooperative game theory. Ken Binmore and Partha Dasgupta(1986) regard the above models as well as the somewhat related conjectural equilibrium (F. Hahn (1977,1978)) as premature. They argue that game theory has not yet developed concepts precise enough to describe rationality or consistency in this setting without ambiguity¹⁶. Maybe, but it remains to be seen whether such precision can be obtained. As the Gödel theorem warns us, the techniques of formal proofs do not necessarily generate all true statements.

One may view strategic game solutions as mimicking the outcome of learning processes by the selection of strategies that will prove stable in a certain environment of game rules and actions of other players. Striking similarities can be seen between economic equilibrium concepts and concepts arrived at by biologists modelling evolutionary games. In evolutionary games the player does not choose among different strategies, receiving a

pay-off which he tries to maximize. Instead the set of strategies are seen as a population of single-minded players that reproduce themselves into the next stage of the game according to success. Though the dynamic processes are very different, the concepts of equilibria are often quite similar in games where rational players choose optimal strategies and in evolutionary games where the strategies survive. The results are in general very sensitive to assumptions on how new entrants choose their strategies, and also to the exact characteristics of the pay-off structure.¹⁷

The problem of strategic games is more ordinarily perceived as finding an acceptable solution to two or several conflicting optimization problems (O. Morgenstern and J. von Neumann (1947)) or making rational choices in situations where the outcome depends on the actions of other agents or players. That point of view leads to another aspect of the general problem of economic learning. K. Arrow(1986) stresses that rationality, although often presented as a property of the individual alone, in fact is mainly dependent on the social context of the individual. One can easily agree with Arrow that the comprehensive common knowledge and sophisticated rational calculations of fully rational learning goes contrary to the spirit of viewing market processes as efficient informational institutions¹⁸.

However, bounded rationality can take many different forms and yield many different results. If such a route should be followed economic theory cannot establish those bounds on rationality on an *ad hoc* basis. It seems inescapable that boundedly rational learning requires explicit modelling of the institutional and informational environment of economic agents. F. Hahn(1989) argues that the definition of economic equilibrium should explicitly recognize that learning implies that the historical path of the economy and the information specific to this path is decisive. The hypothesis of individual rationality then cannot be sufficient to resolve the issue of what agents will learn in a market system. In order to draw general conclusions on how to improve real economies we not only must consider their specific histories but also to prescribe how institutions and information arrangements should change. Thus there seems to be much need for a theory of *collective* rationality

governing this choice of institutions and information arrangements to provide stabilizing common knowledge to learning agents.¹⁹

The lessons of RE learning research may be potentially revolutionary in economics by bringing path dependence and institutional information arrangements into focus. The demonstration that the RE hypothesis relies so heavily on implicit assumptions in this respect must surely have consequences for how economists think about economic equilibria in the future.

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¹ A historical remark here is that the optimal properties of some rational predictors used by Muth actually was investigated already by Herman Wold in his doctoral thesis of 1938, cf. H. Lang(1989).

² Mark E. Schaffer(1989) shows that explicit modelling of firm competition by evolutionary games may lead to the fittest survivors being not, as economists would expect, the profit maximizers. That is except in the case when firms lack all market power, i.e. do not influence the profit of each other. Not even optimizing relative profits needs be a viable survival strategy.

³ Cf. Kirman(1983), Evans and Ramey(1988) and Bala and Kiefer(1990) for some exceptions.

⁴ Cf. e.g. Day(1975,1990), Cross(1983), Baumol and Quandt(1964), Cyert and March(1963) among many others for discussion and references.

⁵ If the REE is fully revealing in Radner's(1982) sense that all private information can be inferred from market outcomes, it would seem the REE is independent of informational assumptions, and in some sense that is so, since all sources of information postulated in the model have been exhausted. However, it still is the case that the REE is dependent on assumptions about what the set of total private information is and the scope for active information production.

⁶ Alan Kirman(1983) provides an example of how this may lead to indeterminacy even in very simple duopoly models. Frank Hahn(1977, 1978) has explored the issue of more general conjectural equilibrium models. Two-armed bandit models provide further examples, cf. section 2.5.

⁷ According to G. Eliasson(1989b) some 60 percent or more of total labour input in Swedish firms are devoted to some kind of information handling.

⁸ In the sense of correct conditional expectations given their private information.

⁹ Cf. section 2.4 and the Blume and Easley(1982) model for one example where this does not hold.

¹⁰ Diversity of opinion is then defined as negative correlations among the sets of variables used for forecasting by different groups of agents.

¹¹ S. J. Grossman, R. E. Kihlstrom, and L. J. Mirman(1977) deals with a similar problem in the learning by doing context. G. Eliasson(1989a) also tries to give some answers to this. The optimal control aspect without strategic interaction has been dealt with by Kiefer et alia, see above.

¹² Crain, Shughart and Tollison(1984) have attempted an empirical test of Day's satisficing model and found that it seemed to fit data nicely with the exception that expansive responses did not seem to be moderated by past failures.

¹³ An interesting paper in this context is Crawford(1985) using a coordinated updating procedure to show that mixed-strategy Nash equilibria are unstable for a wide class of learning mechanism. Randomizing agents could be considered "overly" rational.

¹⁴ It was only forty five years after publication and six years after his own death that Cournot's "Recherches sur les principes Mathématiques de la Théorie des Richesses" was reviewed by the statistician Joseph Bertrand(1883).

¹⁵ The existence of such conjectures, however, hinges critically on assumptions of linear conjectures and often results in a negative pay-off to learning. That is to say, the agent would do better by not trying to learn about the reactions of other agents. There are also

logical problems with the interpretation of "consistency" in this context that leads to semantic paradoxes (cf. Lindh(1991)).

¹⁶ Cf. Bernheim(1986) for a discussion of how different rationality concepts affect the choice of relevant equilibrium concept. An enlightening discussion on behaviour out of equilibrium in a game theoretic context can be found in Kreps(1989).

¹⁷ For some short notes on this with further references, see Dasgupta(1989) and Hammerstein(1989).

¹⁸ Common knowledge or consensus rules are central to any definition of rational, or boundedly rational, behaviour of the individual. For a thorough but easily accessible discussion on common knowledge, game theory and Bayesian learning, see K. Binmore and A. Brandenburger(1988).

¹⁹ In fact there is a recent paper by M. Cripps(1991) investigating the optimal monetary policy in a game with agents learning about inflation concluding that the optimal policy in fact delays learning.