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by
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and
Albert G Hart

April, 1982
Revised

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Abstract:

This paper uses a simple model to explore the effects of "increasing demand risk" on business fixed investment. We show that within a putty-clay framework an increase in demand uncertainty can be expected to have two countervailing effects. On the one hand increasing risk tends to induce a firm to increase its capacity, but on the other hand the optimal capital-intensity of that capacity decreases.

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* This paper is related to an unpublished piece by Hart (1973). The conclusions of that earlier paper are basically unchanged, but the methods of demonstrating those conclusions are quite different. A suggestion by Guillermo Calvo that aided in the proof of the final proposition is gratefully acknowledged.

1. Introduction

This paper presents a simple model of the effect of "increasing demand risk" on investment. A primary motivation for developing this model comes from the financial pages of the daily press in which the depressive effect of (increasing) uncertainty on investment is continually asserted.

Such an effect is not at all obvious; nonetheless, one can find ample support for this position in the academic journals. The standard model in the literature examines the question of how output price uncertainty affects the competitive firm's demand for labor and capital within a stylized, single-period framework and derives the result that a risk-averse competitive firm will decrease output in response to increased price uncertainty with a corresponding decrease in factor demands, excepting the case of inferior factors. (See, e g, Batra and Ullah (1974).) The behavior of a risk-neutral firm is not affected by increased price uncertainty.

The intuition that increased uncertainty operating through the risk preferences of entrepreneurs has a general contractionary effect may have some validity, but it is hard to believe that this is more than a second-order effect. The existence of uncertainty has more direct and obvious effects on the production decisions of firms; in particular, there is a need for flexibility in the presence of uncertainty.

Our model is built around this idea of flexibility as a response to uncertainty. We consider a firm which has an ex ante constant returns to scale production function relating capacity to capital and labor services. Ex post the firm faces a strict clay relationship -- output is proportional to employment up to the capacity limit. Input and output prices are fixed and known with certainty, but the quantity the firm will be able to sell is assumed to be constrained by the level of effective demand, a random

variable. The firm thus faces two types of risk. If demand expands by more than expected, the firm risks losing profitable sales due to insufficient capacity; while if demand expands by less than expected, the firm risks paying for capital services which are not fully utilized. The interpretation that comes out of this putty-clay framework is that the firm can simultaneously hedge against both of these risks. The firm can hedge against underexpansion relative to demand by increasing capacity beyond the level which would otherwise be optimal while at the same time hedging against overexpansion by biasing its production techniques towards flexibility in the form of reduced capital intensity.

Our model in effect intergrates two existing approaches to capturing the direct effect of uncertainty on investment. In one type of model (e g, Rothschild and Stiglitz (1971) or Hartman (1976)) uncertainty has a direct effect on expected profits through the short-run fixity of capital services. Ex ante the firm faces a production function in capital and labor services; ex post the firm faces the same production function, but with the level of capital services fixed. In a second class of model (e g, Ch 5 of Nickell (1978)) an ex ante choice of investment is identical to a choice of capacity, and the question is one of the degree of excess capacity needed to cope with uncertainty.

Our model is in the same spirit as these papers in the sense that uncertainty produces real effects without recourse to the rather artificial device of risk aversion, yet the putty-clay approach represents a quite different way of introducing these effects. The putty-clay approach emphasizes the flexibility inherent in excess capacity, while at the same time allowing scope for input substitution via variations in the capital-intensity of capacity.

In the next two sections we develop these ideas more precisely within

the context of a simple model. However, before proceeding a strong caveat should be registered. This whole genre of model might well be dismissed on the grounds that what is being investigated is the effect of an increase in uncertainty about an endogenous variable. The response of risk-averse competitive firms to an increase in price uncertainty (from whatever exogenous source) will obviously have an effect on price itself, and an analogous feedback effect will operate in our model. However, one must recognize the pervasiveness of "partial-partial" thinking, and for didactic reasons we express our disagreement with the conventional wisdom within that framework. One hopes of course that some of the intuition suggested by these models carries over to a more fully specified equilibrium context.

2. The Model

Consider a firm taking decisions for an about-to-commence installation period which bear upon production during an ensuing operating period. The firm faces an ex ante constant returns to scale production function relating capacity to labor services and investment. Ex post, in the operating period, the firm faces a clay production function that equates the output/capacity ratio to the ratio of employment to labor requirements at full capacity. Of course, this clay relationship only holds for output/capacity ratios between zero and one. These production assumptions should be interpreted as applying to the installation of new capacity, ie, the model abstracts from any interactions between this new capacity and any pre-existing capacity. Either "independence of vintages" or investment in completely new operations (ie, the absence of any pre-existing capacity) is assumed.

The firm is assumed to face fixed prices for its output and inputs which are denominated in operating period terms and which are known with certainty. However, the firm faces a demand constraint in the sense that the quantity it will be able

to sell will be equal to the lesser of capacity and an amount imposed by effective demand, and the level of this effective demand constraint is assumed to be a random variable.

The following notation will be used throughout:

c = capacity

n = labor requirement at c

k = investment (= capital requirement at c)

y = actual (ex post) output

e = actual (ex post) employment

p = output price

w = wage rate

r = implicit price of investment

The putty production function, $c = c(n,k)$ is assumed to be twice differentiable and strictly concave with positive marginal products. Therefore $c(n,k)$ can be inverted to work with labor requirements as a function of c and k, ie,

(1) $n = n(c,k)$.

This inverse function exhibits constant returns to scale and its partial derivatives satisfy

(2) $n_c > 0$, $n_k < 0$, $n_{cc} > 0$, $n_{kk} > 0$.

In interpreting these derivatives it is useful to note that n_c is the reciprocal of the marginal product of labor and that $-n_k$ is the marginal rate of substitution. The ex post production relationship is

(3) $y/c = e/n$, $0 \leq y \leq c$, implying

(4) $e = n(c,k)y/c$, $0 \leq e \leq n$.

The demand constraint is introduced via a continuous random variable, X, which is subjectively distributed over $[0,1]$ (or any closed interval) with distribution function $F(x)$. The realization, x, of this random variable is interpreted as the level of effective demand. We assume that this demand constraint is always binding in the sense that

$$(5)\quad y = \min[x,c].$$

Thus, for a given x, *ex post* profits may be expressed as a function of the *ex ante* decision variables, c and k, as

$$(6)\quad \Pi(c,k) = px - wn(c,k)x/c - rk,\ 0 \leq x \leq c$$
$$= pc - wn(c,k) - rk \quad c \leq x \leq 1,$$

implying

$$(7)\quad E[\Pi(c,k)] = \int_0^c [px - wn(c,k)\frac{x}{c} - rk]f(x)dx + (1-F(c))[pc - wn(c,k) - rk], \text{ or}$$

$$(8)\quad E[\Pi(c,k)] = (1 - \frac{1}{c}\int_0^c F(x)dx)(pc - wn(c,k)) - rk.$$

Taking expected *ex post* profits as the maximand, the problem is to examine how increasing demand risk affects the optimizing values of c and k, where increasing risk is defined in the now-standard Rothschild/Stiglitz (1970) sense of mean-preserving spread. Rothschild and Stiglitz prove that if the random variables X and Y have the same mean, then "Y is riskier than X" can be expressed in three equivalent ways: (i) any risk averse decision-maker will prefer X to Y, (ii) Y can be derived from X via a sequence of "mean preserving spreads," and (iii) the distribution function of Y is "fatter in the tails" than the distribution function of X. A more precise statement of this last expression is that if the points of increase of F and G, the distribution functions of X and Y, are confined to a closed

interval , $[a,b]$, then

$$T(y) \equiv \int_a^y [G(z) - F(z)]dz \geq 0 \text{ for all } y \text{ and } T(b) = 0.$$

Thus, increasing risk decreases $1 - \frac{1}{c}\int_0^c F(x)dx$ for any fixed c. One can therefore interpret expected profits (8) as a "risk co-efficient" times the operating surplus at full capacity less fixed investment costs.

To best exploit this interpretation, it is useful to adopt a last bit of notation. Write the distribution function as $F(x,\theta)$, where θ is defined by the condition (cf, Diamond and Stiglitz (1974))

$$(9) \quad \frac{\partial}{\partial\theta}\int_0^y F(x,\theta)dx \geq 0 \text{ for } 0 \leq y \leq 1,$$

and define

$$(10) \quad g(c,\theta) \equiv 1 - \frac{1}{c}\int_0^c F(x,\theta)dx.$$

Then

$$(11a) \quad 0 \leq g(c,\theta) \leq 1$$

$$(11b) \quad g_c(c,\theta) \equiv \frac{\partial g(c,\theta)}{\partial c} = \frac{1}{c}\left[\frac{1}{c}\int_0^c F(x,\theta)dx - F(c,\theta)\right] \leq 0$$

$$(11c) \quad g_\theta(c,\theta) \equiv \frac{\partial g(c,\theta)}{\partial \theta} = -\frac{1}{c}\frac{\partial}{\partial\theta}\int_0^c F(x,\theta)dx \leq 0.$$

The maximand (8) can thus be re-written as

$$(12) \quad E(\Pi(c,k;\theta)) = g(c,\theta)[pc - wn(c,k)] - rk$$

3. Results

We want to investigate the effect of increasing demand risk on the firm's operations. First, note that increasing demand risk is indeed "bad for business"; i e, increasing demand risk will always have a deleterious effect on expected profits. Under no circumstances can the firm adjust c and k to more than offset the increase in risk.

Proposition 1: Expected profits are non-increasing in θ.

Proof: (i) Let $\Pi^*(\theta) = \max_{c,k} E[\Pi(c,k;\theta)]$. By the "envelope theorem"

$$\frac{d\Pi^*(\theta)}{d\theta} = g_\theta(c,\theta)[pc - wn(c,k)].$$

But because $\Pi^* > 0$ we have $pc - wn(c,k) > 0$, and $g_\theta(c,\theta) \leq 0$ by (11c).

(ii) Alternatively, equation (6) reveals that profits are concave in x. Therefore, the Rothschild/Stiglitz (1971) results based on Jensen's Inequality imply that expected profits are non-increasing in θ.

To investigate the effect of increasing demand risk on investment we decompose the effect of increasing risk into two effects. At the optimum, k may be expressed as a function of θ and of the optimal value of c (itself a function of θ); ie, $k = k(\theta, c(\theta))$. This leads to the natural decomposition

$$(13)\quad \frac{dk}{d\theta} = \left.\frac{\partial k}{\partial \theta}\right|_c + \left.\frac{\partial k}{\partial c}\right|_\theta \frac{dc}{d\theta};$$

that is, the effect of increasing demand risk on investment can be broken into a capital-intensity effect $\left(\left.\frac{\partial k}{\partial \theta}\right|_c\right)$ and a capacity-expansion effect $\left(\left.\frac{\partial k}{\partial c}\right|_\theta \frac{dc}{d\theta}\right)$. We prove below that the capital-intensity effect is unambiguously non-positive but, given reasonable conditions, that the capacity-expansion effect is non-negative.

Proposition 2: The capital-intensity effect is non-positive, ie, $\left.\frac{\partial k}{\partial \theta}\right|_c \leq 0$.

Proof: (i) Differentiating (6) with respect to k reveals that $\partial\Pi(c,k)/\partial k$ is concave in x. Therefore, for any given value of c, an increase in risk implies a decrease in k.

(ii) An alternative proof is useful since it reveals the terms comprising (13). Maximizing $E[\Pi(c,k;\theta)]$ with respect to c and k yields the first-order conditions

(14a) $\partial(\cdot)/\partial c = g_c(c,\theta)[pc - wn(c,k)] + g(c,\theta)[p - wn_c(c,k)] = 0$

(14b) $\partial(\cdot)/\partial k = -g(c,\theta)wn_k(c,k) - r = 0.$

Differentiating either (14a) or (14b) with respect to θ allows us to identify the terms of (13). In particular, differentiating (14b) with respect to θ yields

$$(15)\quad \frac{dk}{d\theta} = \frac{-g_\theta(c,\theta)n_k(c,k)}{g(c,\theta)n_{kk}(c,k)} + \frac{-[g_c(c,\theta)n_k(c,k) + g(c,\theta)n_{ck}(c,k)]}{g(c,\theta)n_{kk}(c,k)}\frac{dc}{d\theta}.$$

Therefore, $\left.\frac{\partial k}{\partial \theta}\right|_c = -g_\theta(c,\theta)n_k(c,k)/g(c,\theta)n_{kk}(c,k)$; and so by (2) and (11), the capital-intensity effect is non-positive.

To investigate the capacity-expansion effect we will impose an additional assumption; namely, that the increase in demand risk does not increase the probability that the benchmark capacity will be adequate to meet demand, That is, we assume that $\frac{\partial F(c,\theta)}{\partial \theta} \leq 0$.

Proposition 3: Assuming that an increase in demand risk does not increase the probability that the benchmark capacity will be adequate to meet demand, an increase in demand risk cannot decrease the optimal capacity. That is, $\partial F(c,\theta)/\partial\theta \leq 0$ implies $dc/d\theta \geq 0$.

Proof: The proof consists of differentiating the first-order conditions, (14a,b), and solving for $dc/d\theta$. The details are given in the appendix.

The above result establishes that under plausible conditions increasing demand risk implies an increase in the optimal level of capacity. This need not, however, lead to the conclusion that increasing demand risk implies a positive capacity-expansion effect on investment since for such to be the case it needs also to be true that an increase in capacity induces an increase in investment, all else equal. Our final proposition gives a simple sufficient condition that ensures this.

Proposition 4: If the elasticity of substitution (σ) between capital and labor is no greater than 1, then $\left.\frac{\partial k}{\partial c}\right|_{\theta} \geq 0$.

Proof: The proof consists of examining the term in (15) corresponding to $\left.\frac{\partial k}{\partial c}\right|_{\theta}$ and showing that $\sigma \leq 1$ implies that this term is non-negative. Again, the details are given in the appendix.

The reason that an additional assumption is required to ensure $\left.\frac{\partial k}{\partial c}\right|_{\theta} \geq 0$ can be explained as follows. If the level of effective demand were known with certainty, then $\partial k/\partial c \geq 0$ would be equivalent to $n_{ck}(c,k) \leq 0$. That is, it would only be required that an increase in investment not induce a decrease in the marginal product of labor at the optimal c. Introducing uncertainty about the level of effective demand makes matters more problematic. For fixed θ, as c increases, the level of "demand risk" remains unchanged, but the probability of incurring a cost as a consequence of that risk is increasing. This argument simply recognizes that the fact of $\partial F(c,\theta)/\partial c \geq 0$ introduces a "secondary consideration" into the question of whether an expansion of capacity necessarily increases investment. The condition of $\sigma \leq 1$ ensures that this consideration cannot dominate, regardless of the form of $F(x,\theta)$.

4. Conclusion

The major part of the existing literature predicts that increasing risk will decrease investment or at best leave levels of investment unchanged, pathological

cases aside.[1] The notion that "uncertainty is bad for investment" is also prevalent in the business press. The practical importance of the putty-clay approach to this problem is to challenge this conventional wisdom by revealing the potential expansionary effects of increasing risk.

These results ought not to seem counter-intuitive. As demonstrated in Stigler (1939), the question of how a firm's behavior varies with respect to uncertainty about the level of demand at any given point in time is formally quite similar to the question of how a firm's behavior varies with respect to known fluctuations in the level of demand over time. The proposition that an anticipated increase in the amplitude of cyclical fluctuations in demand will induce firms to carry more capacity albeit at a lessened capital-intensity, would not seem foreign to most economists.

Finally, we should like to conclude with a moral. Since investment is an inherently forward-looking process, investment decisions must be taken on the basis of expectations about the future. Introducing capacity constraints is a method of introducing a cost to faulty expectations, and thereby introducing a motive for entrepreneurs to hedge against those errors. Our results indicate that this hedging can produce outcomes quite different from what one might expect were all mistakes correctable ex post. Those who ignore capacity considerations in modelling investment decisions -- whether at the theoretical level or at the empirical level -- may therefore do so at considerable risk.

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[1] One might similarly suspect that an increase in investment in response to increasing demand risk would only represent a pathological outcome in our model. To see that such is not the case, consider the example of a fixed coefficients ex ante technology. In this case the capital-intensity effect is completely absent, and investment necessarily increases so long as $\partial F(c,\theta)/\partial\theta < 0$. Small deviations from fixed coefficients, ie, limited substitution possibilities, obviously will produce the same result. Note that with a fixed coefficients technology our model essentially reduces to Example 5a of Nickell (1978).

Appendix:

Proof of Proposition 3: The first-order conditions (14a,b) implicitly give c and k as functions of θ. Differentiation with respect to θ shows that $dc/d\theta$ is of the same sign as

$$[g_{c\theta}(pc - wn) + g_{\theta}(p - wn_c)]gwn_{kk} + [g_c wn_k + gwn_{ck}]g_{\theta}wn_k .$$

(Note that the arguments of $g(\cdot)$ and $n(\cdot)$ have been suppressed to save space.)

Since $w > 0$, to show $dc/d\theta \geq 0$ requires showing

$$[g_{c\theta}(pc - wn) + g_{\theta}(p - wn_c)]gn_{kk} + [g_c wn_k + gwn_{ck}]g_{\theta}n_k \geq 0.$$

Using the fact that $g_{c\theta} = -\frac{1}{c}[g_{\theta} + \frac{\partial F(c,\theta)}{\partial \theta}]$ gives

$$g_{c\theta}(pc - wn) + g_{\theta}(p - wn_c) = g_{\theta}w(\frac{n}{c} - n_c) - \frac{1}{c}\frac{\partial F(c,\theta)}{\partial \theta}(pc - wn).$$

Then, given $\partial F(c,\theta)/\partial\theta \leq 0$, $dc/d\theta \geq 0$ if

$$g_{\theta}[\{(\frac{n}{c} - n_c)n_{kk} + n_k n_{ck}\}g + g_c n_k^2] \geq 0.$$

Finally, since $g_{\theta}g_c n_k^2 \geq 0$, $dc/d\theta \geq 0$ if

$$\xi \equiv (\frac{n}{c} - n_c)n_{kk} + n_k n_{ck} \leq 0.$$

But, $\xi = 0$ follows from the linear homogeneity of $n(c,k)$, as will now be demonstrate

Linear homogeneity implies $n = n_c c + n_k k$. Differentiating with respect to k yields

$$n_k = n_{ck}c + n_{kk}k + n_k \text{ , ie, } n_{ck} = -\frac{k}{c}n_{kk}.$$

Substituting back then yields

$$\xi = (\frac{n}{c} - n_c)n_{kk} - \frac{k}{c}n_k n_{kk} = (n - n_c c - n_k k)n_{kk}/c = 0,$$

Proof of Proposition 4: From (15) we want to show that $\sigma \leq 1$ implies

$$g_c(c,\theta)n_k(c,k) + g(c,\theta)n_{ck}(c,k) \leq 0.$$

But, $g_c(c,\theta) = \frac{1}{c}[1 - g(c,\theta) - F(c,\theta)]$, so

$$g_c(c,\theta)n_k(c,k) + g(c,\theta)n_{ck}(c,k) = [1 - F(c,\theta)]\frac{n_k(c,k)}{c} - g(c,\theta)[\frac{n_k(c,k)}{c} - n_{ck}(c,k)]$$

Therefore, $n_k(c,k)/c \geq n_{ck}(c,k)$ is sufficient for $\left.\frac{\partial k}{\partial c}\right|_{\theta} \geq 0$.

Since $n_{ck}(c,k) = -\frac{k}{c}n_{kk}(c,k)$ (from the proof of Proposition 3)

$$n_k(c,k)/c \geq n_{ck}(c,k) \text{ iff } n_k(c,k) \geq -kn_{kk}(c,k).$$

Next, $\sigma \leq 1$ implies $\partial(\frac{-n_k(c,k)k}{n(c,k)})/\partial k \leq 0$. But,

$$\partial(\frac{-n_k(c,k)k}{n(c,k)})/\partial k = \frac{-[n_{kk}(c,k)k + n_k(c,k)]n(c,k) + kn_k^2(c,k)}{n^2(c,k)},$$

ie, $\sigma \leq 1$ implies $n_k(c,k) \geq -kn_{kk}(c,k)$.

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