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## What is a Reasonable Rate of Return for an Undiversified Investor?

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## What is a Reasonable Rate of Return for an Undiversified Investor?

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### **Abstract**

The valuation of illiquid or non-marketable assets is complicated by the fact that the discount rate cannot be computed by using the risk attributes of the asset along with market parameters. Rather, individual attitudes toward risk affect the discount rate. Some recent research has avoided this difficulty by adopting an “opportunity cost approach”, arguing that an undiversified holder of a risky asset will require *at least* the return that they could have earned by leveraging the market portfolio to achieve the same level of risk. We evaluate this claim in a model with an explicit utility function. It turns out not to be true that the opportunity cost necessarily understates the required rate of return, unless we also restrict the holder of an illiquid asset to invest all her liquid wealth in the market portfolio. When the holder can also invest in a riskless asset, the opportunity cost method actually overstates the required rate of return for investors with sufficiently low risk-aversion. In general, however, the opportunity cost approach provides a reasonable approximation to the exact required rate of return over a wide range of risk-aversion levels provided the investor can also borrow and lend.

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The valuation of illiquid assets presents special challenges to practitioners and academics. There are two complementary ways to go beyond ad hoc “haircuts” based on history or intuition. The first is to use comparables; to somehow locate two securities that differ only in their marketability and then track the empirical discount (see, e.g., Bajaj et al, 2000 for a recent discussion of the costs and benefits of this approach). The alternative is to explicitly model and evaluate the disadvantages posed by illiquidity. The disadvantage that is best grounded in finance principles is the loss of diversification.<sup>1</sup>

While it is clear that investors should require a premium for forgoing diversification, standard valuation frameworks rely on the assumption that investors can freely diversify their holdings.<sup>2</sup> When investors are restricted from trading, the discount rate is determined by individual attitudes toward risk in addition to objective, market-level data (for a recent example, see Hall and Murphy, 2000). Recent work by Smith and Smith (2000), Meulbroek (2001) and Kerins, Smith and Smith (2001) avoid this problem and apply CAPM principles by adopting an opportunity cost approach. Specifically, these papers reason that the undiversified holder of a risky asset will require *at least* the return that she could have earned by leveraging the market portfolio to achieve the same level of risk. Put another way, the opportunity cost approach requires the undiversified position to provide the same Sharpe ratio as the market portfolio.

This approach has the appealing property of delivering preference-free results, but as the authors recognize, it cannot exactly capture the cost of lost diversification. The reason is that few investors would voluntarily choose to take on exactly the same amount of risk if their investment were in fact tradable. In order to assess the choices that investors would actually make, we must specify a utility or objective function. That is the approach taken here. We explicitly model a utility-maximizing investor in the setting of Meulbroek (2001) and Kerins, Smith and Smith (2001).

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<sup>1</sup> Other discounts for minority status or for control apply even to freely tradable securities. Models that attempt to capture the valuation effect of transaction costs (e.g., Amihud and Mendelson, 1986) rely on relatively ad hoc assumptions about the motivation for trading. Constantinides (1986) and Vayanos (1998) find that transaction costs are relatively unimportant in a fully specified general equilibrium setting (but see also Swan, 2000). Longstaff (1995) characterizes the cost of illiquidity as foregone insider trading profits, but his model relies on the assumption that liquidity providers for traded securities are willing to bear arbitrarily large losses to an informed trader without any compensation.

<sup>2</sup> This is obviously the case with CAPM but is true a fortiori for arbitrage methods such as Black-Scholes which require complete continuous hedging.

The drawback to our approach is that the illiquidity discount now depends on our assumptions about the specific form of the utility function and the degree of risk-aversion. An obvious countervailing benefit is that we can check how closely the preference-free opportunity cost approach approximates the exact premium, which requires us to know the investor's utility function. There are two other advantages that emerge during the analysis

1. We identify and overcome an ambiguity with how the opportunity cost method treats the other investments taken on by the holder of an illiquid asset. To implement the opportunity cost approach we must identify the total risk the investor bears and therefore make some assumption about her other investments. Smith and Smith (2000), Meulbroek (2000), and Kerins, Smith and Smith (2001) assume that any wealth not tied up in the liquid asset is invested in the market portfolio, so that the total risk equals that of a portfolio that is  $X\%$  in the illiquid asset and  $100-X\%$  in the market. By specifying a utility function, we are able to allow for the more realistic possibility that the investor may also wish to place some of her wealth into a risk-free asset.
2. Once we allow the investor some choice in the allocation of her liquid wealth, it is also the case that her choice reveals her risk-aversion. In applications, we can work backwards from her holding in the risk-free asset to impute her risk-aversion and thereby the premium that she requires from the illiquid asset. Put another way, to implement the opportunity cost approach we must know the fraction of wealth represented by the illiquid asset. If we also know the extent of her holdings in either the risk-free asset or in liquid risky asset, we can impute her risk-aversion and further improve our estimate of the required rate of return.<sup>3</sup>

The analysis proceeds under three progressively more realistic alternative assumptions about the restrictions placed on the entrepreneur's portfolio. For brevity, we call the investor an "entrepreneur" when she holds the illiquid asset and an "investor" if she does not. In all cases, we assume that the investor cannot trade the illiquid asset at all. We require the return on the illiquid asset to make the investor as well off as if she instead could freely allocate her wealth in classical CAPM fashion, between the market and the

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<sup>3</sup> Strictly speaking, in the context of the model we can figure out the exact required rate of return. But the model makes a host of assumptions that are only approximations of reality; exponential utility, normally distributed returns, and noiseless estimates of beta, variances, and the market risk premium.

risk-free asset.<sup>4</sup> We begin with the case where the entrepreneur's entire wealth must be devoted to the non-tradable asset. We then model the case of partial commitment where the entrepreneur has wealth in the market as well as the non-tradable asset. Finally, we allow the entrepreneur to optimally allocate her tradable wealth between the market portfolio and the risk-free asset.

The results are as follows:

1. In the first two cases where the entrepreneur is forced to allocate all liquid wealth to the market portfolio, we confirm Meulbroek's (2001) contention that the opportunity cost approach understates the required rate of return on the illiquid asset. The bad news is that the understatement can exceed 100% for reasonable parameter values.
2. When we allow the entrepreneur to choose her own most preferred holding in the risk-free asset and the market portfolio, we can no longer say in general whether the opportunity cost approach under or overstates the cost of illiquidity. When the entrepreneur's risk-aversion is sufficiently low, the opportunity cost actually overstates the required rate of return. The good news is that the absolute difference between the opportunity cost and the explicit utility-based premium is significantly reduced when we allow the investor a more realistic set of choices.

The economic forces driving these results are as follows:

1. If the entrepreneur is highly risk-averse, she requires a large premium to compensate for the risk implied by a holding of  $X\%$  in a risky venture and  $100-X\%$  in the market portfolio. The required premium is significantly reduced by allowing her to allocate some of her non-venture wealth to the risk-free asset.
2. If the entrepreneur has very low risk-aversion, she will desire more risk than that provided by the combination of the venture of the risky venture and the market portfolio. This is the case where the opportunity cost approach overstates rather than understates the required rate of return. The reason is that at the assumed holding of  $X\%$  in the venture and  $100-X\%$  in the market, the entrepreneur can boost her utility by reducing her holding in the risk-free asset to invest more in the

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<sup>4</sup> One could ensure that the opportunity cost approach is always correct by stipulating that the diversified investor chooses a holding in the risk-free asset and the market that exactly mimics her risk as an entrepreneur. However, in so doing the entire reason for two-fund separation is also lost.

market. This increase in utility reduces the required rate of return on the other portion of her portfolio, i.e., the illiquid asset.

The paper is organized as follows. Section I deals with the simplest case where the entrepreneur's wealth is fully committed to the venture. Here there is no ambiguity as to her portfolio; she has a choice only when she does not do the venture and becomes an investor. Section II briefly considers the case where the entrepreneur has liquid wealth but is forced to hold it all in the market portfolio. We then turn to our own main case where the entrepreneur has a choice of where to allocate her non-venture wealth.

## I. Full-Commitment Case

### A. Opportunity cost approach

The risk-free rate is denoted  $r_f$  and the expected market return is  $r_m > r_f$ . Market risk is denoted  $\sigma_m$ . We refer to the illiquid asset as a venture and denote its return by  $r_v$  and risk by  $\sigma_v$ . We normalize wealth to one. If the entrepreneur does not do the venture but chooses to take on as much risk as is implied by the venture, she would expect the return  $r_v^O = r_f + (r_m - r_f)\sigma_v/\sigma_m$ . This is the required rate of return in the full-commitment opportunity cost case.

### B. Utility-based approach

In order to stay with the mean-variance framework, we will assume the investor has exponential utility of the form  $-e^{-kc}/k$  where  $c$  is terminal consumption and  $k$  is the coefficient of absolute risk-aversion. The gain of using a utility-based approach is that instead of just postulating that the entrepreneur would choose to bear the same risk as the venture, we can solve for her optimal exposure to the market,  $w$ , with the remainder being invested in the risk-free asset. The weakness is that our answers depend on our assumption of her degree of risk-aversion,  $k$ .

If the entrepreneur does the full-commitment venture, she has no further choices and receives an expected utility of

$$U_v = r_v - \frac{k\sigma_v^2}{2}$$

If the entrepreneur does not do the venture, she is a diversified investor and can choose the weight on the market to maximize:

$$U_d = r_f + w(r_m - r_f) - \frac{k w^2 \sigma_m^2}{2}$$

Solving for her optimal holding we obtain  $w^*=(r_m-r_f)/k\sigma_m^2$ . Substituting this optimal market exposure back into the utility function, we can express the entrepreneur's utility if she does not do the venture as

$$U_d^* = r_f + w^*(r_m - r_f) - \frac{k w^{*2} \sigma_m^2}{2} = r_f + \frac{(r_m - r_f)^2}{2k\sigma_m^2}$$

Thus, the venture should promise a return  $r_v^*$  that satisfies

$$U_v = r_v^* - \frac{k\sigma_v^2}{2} = U_d^* \quad r_v^* = r_f + \frac{(r_m - r_f)^2}{2k\sigma_m^2} + \frac{k\sigma_v^2}{2}.$$

Proposition 1 summarizes the comparison between the opportunity cost and the utility-based approach.

*Proposition 1:*

*The opportunity cost and the utility-based discount rates are equal if the coefficient of absolute risk-aversion  $k=k^*=(r_m-r_f)/\sigma_v\sigma_m$ . For all  $k \neq k^*$ , the utility-based approach yields a higher discount rate.*

The critical level of risk-aversion  $k^*$  has a natural interpretation. It is that level of risk-aversion for which the investor would voluntarily choose to leverage the market portfolio to achieve the same level of risk as the venture. For all other levels of risk-aversion, Proposition 1 confirms the contention of Meulbroek (2001) and Kerins, Smith and Smith (2001) that the opportunity cost approach provides only a lower bound on the required rate of return for an undiversified investor.

Figures 1.1-1.3 show clearly how the results come about. An investor with high risk-aversion ( $k>k^*$ ) is forced to bear risk beyond the point where she is willing to take just the market risk premium. An investor with low risk-aversion ( $k<k^*$ ) is denied her desire to take on additional risk at the market risk premium. The venture's expected return must compensate for these losses as well as the lost diversification.

## **II. Partial commitment, entrepreneur forced to hold liquid wealth in the market portfolio**

### ***A. Opportunity cost approach***

Now assume that the venture only comprises a fraction  $v$  of the entrepreneur's wealth, and that the remaining  $1-v$  is invested in the market portfolio. This is somewhat artificial as the non-venture wealth could also be invested in other vehicles. Of course, to characterize this choice we would have to assume a utility function. We deal with this

case in detail in section III. What we will not deal with is the fact that the entrepreneur might not want to invest in exactly the market portfolio of risky assets. She might want to downweight assets that are highly correlated with her venture in favor of those that are less correlated with it, assuming that such assets do not themselves have significant amounts of idiosyncratic risk. We do capture the flavor of the results, however, by progressively freeing up the entrepreneur's investment opportunities.

If the entrepreneur has  $v$  of her wealth tied up in the venture and  $1-v$  in the market, her risk is now:

$$\sigma_{vp} = \left( v^2 \sigma_v^2 + (1-v)^2 \sigma_m^2 + 2v(1-v)Cov(r_v, r_m) \right)^{1/2} = \left( v^2 \sigma_v^2 + (1-v)\sigma_m^2[(1-v) + 2v\beta] \right)^{1/2}$$

The opportunity cost approach asks what expected return the entrepreneur could receive if she were diversified and chose to bear the same amount of risk,  $\sigma_{vp}$ . This return is just  $r_f + (r_m - r_f)\sigma_{vp}/\sigma_m$ , while if she does do the venture she gets the expected return  $vr_{vp} + (1-v)r_m$ . Solving for the critical value of  $r_{vp}$  we obtain

$$r_{vp}^O = \frac{r_m \left( \frac{\sigma_{vp}}{\sigma_m} - (1-v) \right) - r_f \left( \frac{\sigma_{vp}}{\sigma_m} - 1 \right)}{v} = r_m + \left( \frac{\frac{\sigma_{vp}}{\sigma_m} - 1}{v} \right) (r_m - r_f)$$

### ***B. Utility-based approach***

In this section, the entrepreneur is forced to hold her non-venture wealth in the market portfolio. This means her utility is::

$$U_{vp} = vr_{vp} + (1-v)r_m - \frac{k\sigma_{vp}^2}{2}$$

If she does not do the venture and is instead an investor, her utility is the same as in the above full-commitment analysis as she has zero commitment to the venture in this case.

Proposition 2 shows how the results carry over to the partial commitment case.

*Proposition 2:*

*As with the full-commitment case, the opportunity cost and the utility-based discount rates are equal if the coefficient of absolute risk-aversion  $k=k^*=(r_m-r_f)/\sigma_{vp}\sigma_m$ . For all  $k \neq k^*$ , the utility-based approach yields a higher discount rate.*

The intuition is exactly the same as for the full-commitment case. The weakness with the result is that part of the premium does not reflect lost diversification but rather the fact

that we have denied her the ability to allocate her wealth between the market and the risk-free asset. That is what we address in the next section.

### III. Full range of choices: entrepreneur can allocate non-venture wealth between market and risk-free asset whether or not she does the venture

#### A. General Results

As indicated above, we have made a rather inconsistent assumption. The entrepreneur can only invest her liquid wealth in the market if she does the venture. If she does not do the venture, by contrast, she is also allowed to freely allocate her wealth between the market and the risk-free asset. We now allow the entrepreneur optimally allocate her liquid wealth between the market and the risk-free asset. In addition to internal consistency, a practical benefit of allowing the entrepreneur more choice if she does the venture is that we can use the entrepreneur's investment choices to infer her risk-aversion and thereby her required rate of return.

If the entrepreneur does the venture she has  $v$  of her wealth in the venture,  $w_e$  of her wealth in the market, and  $(1-v-w_e)$  of her wealth in the risk-free asset. We can write her utility as:

$$U_{vp} = r_f + v(r_v - r_f) + w_e(r_m - r_f) - \frac{k}{2}(v^2\sigma_v^2 + w_e\sigma_m^2(w_e + 2v\beta))$$

Her optimal market holding satisfies the first-order condition  $r_m - r_f - k\sigma_m^2[w_e + \beta v] = 0$ , so the optimal market holding for the entrepreneur can be written: as  $w_e^* = (r_m - r_f)/k\sigma_m^2 - \beta v$ . Recall that if she does not do the venture, she chooses a market holding of  $w^* = (r_m - r_f)/k\sigma_m^2$  so we also know that  $w_e^* = w^* - \beta v$ . We can impute the risk-aversion parameter from the entrepreneur's holding of the market and the venture along with the correlation between the two;  $k = (r_m - r_f)/\sigma_m^2(w_e^* + \beta v)$ .

We now have to solve for the return on the venture that equates the entrepreneur's maximized utility with and without the venture. The most convenient way to do this is to re-express her expected utility as an investor using the fact that  $w^* = w_e^* + \beta v$ . Specifically, we can write her expected utility as

$$U_d^* = r_f + (r_m - r_f)(w_e^* + \beta v) - \frac{k\sigma_m^2(w_e^{*2} + 2w_e^*\beta v + \beta^2 v^2)}{2}$$

Now, if the entrepreneur does do the venture, we can write her utility as:

$$U_{vp}^* = r_f + w_e^*(r_m - r_f) + v(r_v - r_f) - \frac{k(v^2\sigma_v^2 + w_e^*\sigma_m^2[w_e^* + 2\beta v])}{2}$$

Now solve for the value of  $r_v$  that equates  $U_d^*$  and  $U_{vp}^*$ . Separating out the risk and expected return terms we have  $r_v v - r_m \beta v = 0.5kv^2[\sigma_v^2 - \beta^2\sigma_m^2]$ . Finally, we can express the required return on the venture as :

$$r_v = r_f + \beta(r_m - r_f) + 0.5kv[\sigma_v^2 - \beta^2\sigma_m^2].$$

This expression is completely intuitive in that the investor requires the diversified investor's return plus an additional premium for firm-specific risk which has a variance exactly equal to  $[\sigma_v^2 - \beta^2\sigma_m^2]$ . The result is not particularly insightful, however, because it depends on the level of risk-aversion.

Proposition 3 provides a more useful perspective. It shows that the opportunity cost approach may now overstate rather than understate the premium if the entrepreneur is not very risk-averse.

*Proposition 3*

*The opportunity cost approach strictly **overstates** the required rate of return at  $k=k^*$  if the returns on the illiquid asset are not perfectly correlated with the market.*

The reason for the result is clearly illustrated in Figure 2.1. By allowing the entrepreneur to allocate her liquid wealth between the market and the risk-free asset, she now gains access to the entire dashed frontier rather than just to the point  $(\sigma_{vp}, vr_v^* + (1-v)r_m)$ . At the level of risk-aversion  $k^*$ , her personal price of risk is exactly equal to the market price of risk,  $(r_m - r_f)/\sigma_m$ . But so long as the venture is not perfectly correlated with the market, the dashed frontier is everywhere steeper than the market price of risk, meaning that her own total risk increases by less than  $\sigma_m$  when she increases her holding in the market and decreases her holding in the risk-free asset. Thus she takes on more risk, thereby increasing her utility and reducing the required rate of return on the venture.

The last observation driving Proposition 3 also implies that the opportunity cost and the utility-based approach tend to be closer together when we allow the entrepreneur to choose her holding of the risk-free asset. For most reasonable values of risk-aversion, the investor would choose significantly less risk than the entrepreneur. While the entrepreneur would love to diversify away her risk from the venture, she also values the ability to scale back her investment in the market portfolio in favor of the risk-free asset. When she can do so, her required rate of return on the venture drops, as illustrated in Figure 2.2.

## ***B. Assessment of alternative discount rates***

We are now in a position to evaluate our results quantitatively for alternative degrees of risk-aversion. The first and most obvious question relates to credible degrees of risk-aversion. Hall and Murphy (2000) use values of risk-aversion in the neighborhood of three to simulate the value of an option to an undiversified executive, based on estimates in Friend and Blume (1978). We can provide some more insight into the choice of risk-aversion by indicating the portfolio choices that the investor would make. Table 1 illustrates these choices for various degrees of risk-aversion for the case where the illiquid asset is 40% of the investor's wealth, the volatility of the illiquid asset is 50% with a beta of 1, the volatility of the market portfolio is 20%, and the risk-free rate and the market risk premium are both 6%. Since the investment in the market portfolio can be expressed as  $w_e^* = (r_m - r_f) / k\sigma_m^2 - \beta v$ , the dependence of the results on different parameter values is quite transparent. The raw volatility of the illiquid asset is irrelevant, while investment in the market falls linearly in either beta or the amount of wealth tied up in the venture. It is also worth noting that for these parameter values, the critical level of risk-aversion  $k^*$  is approximately 1.11, which represents a low degree of risk-aversion. The reason is quite straightforward. The combination of the market and the illiquid asset has more risk than the market portfolio, so if the diversified investor is to choose to bear this level of risk, she must short the risk-free asset and leverage the market portfolio. This is not an appealing choice for a highly risk-averse investor. However, it is also worth noting that allowing her to borrow and lend at the risk-free rate if she holds an illiquid asset does not seem any less realistic than the opportunity cost method's required assumption that the investor can leverage the market portfolio to achieve the same level of risk as the venture. Specifically, Table 1 reminds us that the entrepreneur and the investor have exactly the same desired holding of the risk-free asset when beta is one. For betas less than one, the desired holding of the risk-free asset is actually greater when she holds the illiquid asset.

Figure 3 summarizes the results of our three alternative techniques for ascertaining the required rate of return on the illiquid venture. By design, the opportunity cost technique does not depend on risk-aversion, and consistent with the arguments of Smith and Smith (2000) and Meulbroek (2001), it *generally* understates the required premium. This is not always the case, however, once we take the internally consistent approach of allowing the investor to choose between the market and the risk-free asset whether or not she holds an illiquid asset. For the selected parameter values, at risk-aversion levels below approximately 1.2, the opportunity cost overstates the required rate of return. More

important perhaps, the differences are relatively small when risk-aversion is low so long as we do not restrict the entrepreneur from borrowing or lending at the risk-free rate as required by the opportunity cost approach. Even more important, when the investor has higher and more reasonable degrees of risk-aversion, the opportunity cost method appears to *dramatically* understate the entrepreneur's required rate of return if we do not allow her to borrow and lend. The understatement is significantly smaller, sometimes cut in half, if we do allow her to choose her own most preferred holding in the market and the risk-free asset regardless of whether or not she holds an illiquid asset.

### **III. Conclusions**

To use the standard mean-variance approach with optimizing investors, it was necessary to assume normally distributed returns and constant absolute risk-aversion. As is well-known, the alternative is either to abandon the mean-variance setting or to make the even less palatable assumption of quadratic utility. An additional limitation is that we restrict attention to a single period. This helps in making the assumption of constant absolute risk-aversion more palatable, but leaves out any consideration of wealth dynamics. Equally important, we restrict our notion of liquidity versus illiquidity to the once-off opportunity to diversify. A richer setting might allow for dynamic portfolio rebalancing and perhaps also the opportunity to trade on private information. Finally, we restrict investment choices to the market and the risk-free asset. Neither we nor Meulbroek (2001) and Kerins, Smith and Smith (2001) allow the undiversified investor to invest in a subset of risky assets that might have a lower correlation with the non-marketable position. We are, however, encouraged by the fact that the opportunity cost approach seems more accurate when we allow the undiversified investor to adjust the liquid portion of her portfolio. For investors with moderate or high levels of risk-aversion, allowing a richer set of investment choices could increase utility and therefore reduce the required rate of return still closer to that generated by the intuitive and preference-free opportunity cost approach.

### Proof of Proposition 1

The venture must promise an expected return that satisfies:

$$U_v = r_v^* - \frac{k\sigma_v^2}{2} = U_d^* \quad r_v^* = r_f + \frac{(r_m - r_f)^2}{2k\sigma_m^2} + \frac{k\sigma_v^2}{2}$$

The difference between the required return under the opportunity-cost and the utility-based approach can then be written:

$$r_v^* - r_v^O = \frac{(r_m - r_f)^2}{2k\sigma_m^2} + \frac{k\sigma_v^2}{2} - \frac{(r_m - r_f)\sigma_v}{\sigma_m}$$

This difference is exactly zero at  $k=k^*$ . To show that it is strictly positive otherwise, it suffices to show that the difference is minimized at  $k=k^*$ . This in turn follows from the facts that (i) the derivative:

$$\frac{\partial(r_v^* - r_v^O)}{\partial k} = -\frac{(r_m - r_f)^2}{2k^2\sigma_m^2} + \frac{\sigma_v^2}{2} = 0 \text{ if } k = k^*,$$

and (ii) from the fact that  $k^*$  is in fact a global minimum by the second-order condition:

$$\frac{\partial^2(r_v^* - r_v^O)}{\partial k^2} = \frac{(r_m - r_f)^2}{k^3\sigma_m^2} > 0$$

### Proof of Proposition 2

Under the utility-based approach, the venture must provide a rate of return that satisfies:

$$U_{vp} = vr_{vp}^* + (1-v)r_m - \frac{k\sigma_{vp}^2}{2} = U_d^* = \frac{r_m^2}{2k\sigma_m^2}$$

$$\Rightarrow r_{vp}^* = \frac{\left( \frac{(r_m - r_f)^2}{2k\sigma_m^2} + \frac{k\sigma_{vp}^2}{2} - (1-v)r_m \right)}{v} = r_m + \frac{1}{v} \left( r_f - r_m + \frac{(r_m - r_f)^2}{2k\sigma_m^2} + \frac{k\sigma_{vp}^2}{2} \right)$$

We can therefore express the difference between the two discount rates as:

$$r_{vp}^* - r_{vp}^O = \frac{1}{v} \left( r_f - r_m + \frac{(r_m - r_f)^2}{2k\sigma_m^2} + \frac{k\sigma_{vp}^2}{2} - \left[ \frac{\sigma_{vp}}{\sigma_m} - 1 \right] (r_m - r_f) \right)$$

$$= \frac{1}{v} \left( \frac{(r_m - r_f)^2}{2k\sigma_m^2} + \frac{k\sigma_{vp}^2}{2} - \frac{\sigma_{vp}}{\sigma_m} (r_m - r_f) \right)$$

The analysis of Proposition 1 now applies exactly if we multiply through by  $v > 0$  and replace  $\sigma_v$  with  $\sigma_{vp}$ .

### Proof of Proposition 3

First, note that since  $r^*_{vp}$  equates  $U^*_{vp}$  and  $U^*_d$  and  $U^*_{vp}$  strictly increases in  $r_{vp}$ , an increase in  $U_{vp}$  will strictly decrease  $r^*_{vp}$ . Second, note that if the entrepreneur *chooses* to deviate from holding  $1-v$  in the market and bearing risk  $\sigma_{vp}$  when her risk-aversion is  $k^*$ , she will strictly increase her utility  $U_{vp}$  by so doing.

Now,  $k^*$  is defined as that level of risk-aversion that would lead the entrepreneur to choose the market exposure  $1-v$  and the risk level  $\sigma_{vp}$  if she faced the market price of risk,  $-\partial(\text{expected return})/\partial w / \partial(\text{risk})/\partial w = r_m/\sigma_m$ . But when the venture is imperfectly correlated with the market, an increase in her market holding provides her with enjoys compensation for risk that is strictly greater than  $r_m/\sigma_m$ . Therefore at  $k=k^*$ , she will take on more risk and increase her utility above  $U^*_d$ . The reason she faces a trade-off more favorable than the market price of risk is that her expected return is  $r_f + w r_m + v r_v$  so  $\partial(\text{expected return})/\partial w = r_m$ . But her risk is now:

$$\sigma_{vp} = \left( v^2 \sigma_v^2 + \sigma_m^2 [w^2 + 2wv\beta] \right)^{1/2}$$

If the venture is perfectly correlated with the market,  $\beta = \sigma_v/\sigma_m$  and  $\sigma_{vp} = v\sigma_v + w\sigma_m$  and  $\partial(\text{risk})/\partial w = r_m$  so the entrepreneur faces exactly the market price of risk and will maintain her holding in the market at  $1-v$ . For all  $\beta < \sigma_v/\sigma_m$ , the entrepreneur faces less than the market price of risk and will increase her holding in the market and more importantly her utility,  $U^*_{vp}$ .

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Figure 1.1: Required rates of return with high risk-aversion

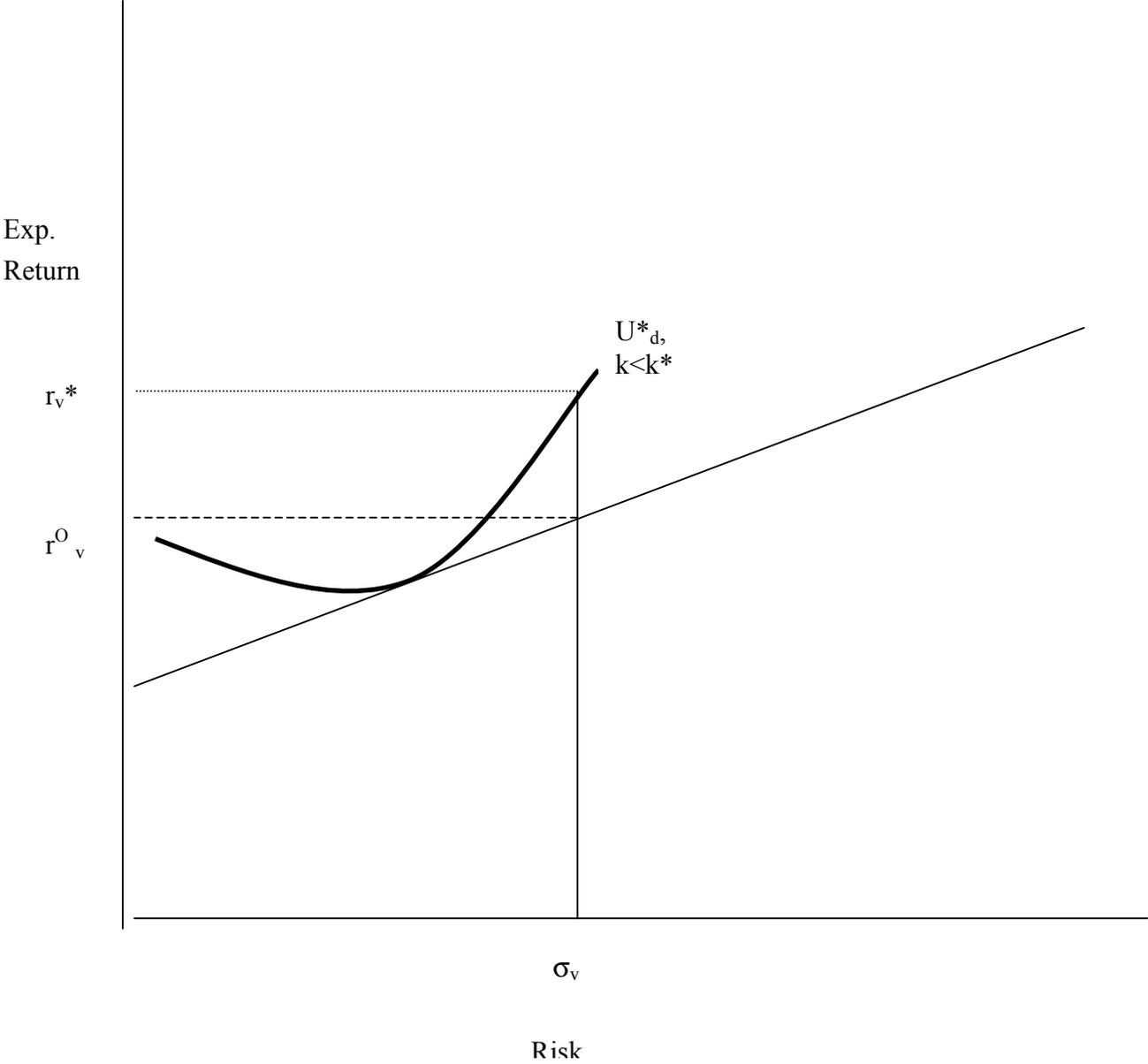


Figure 1.2: Required rates of return when  $k=k^*$

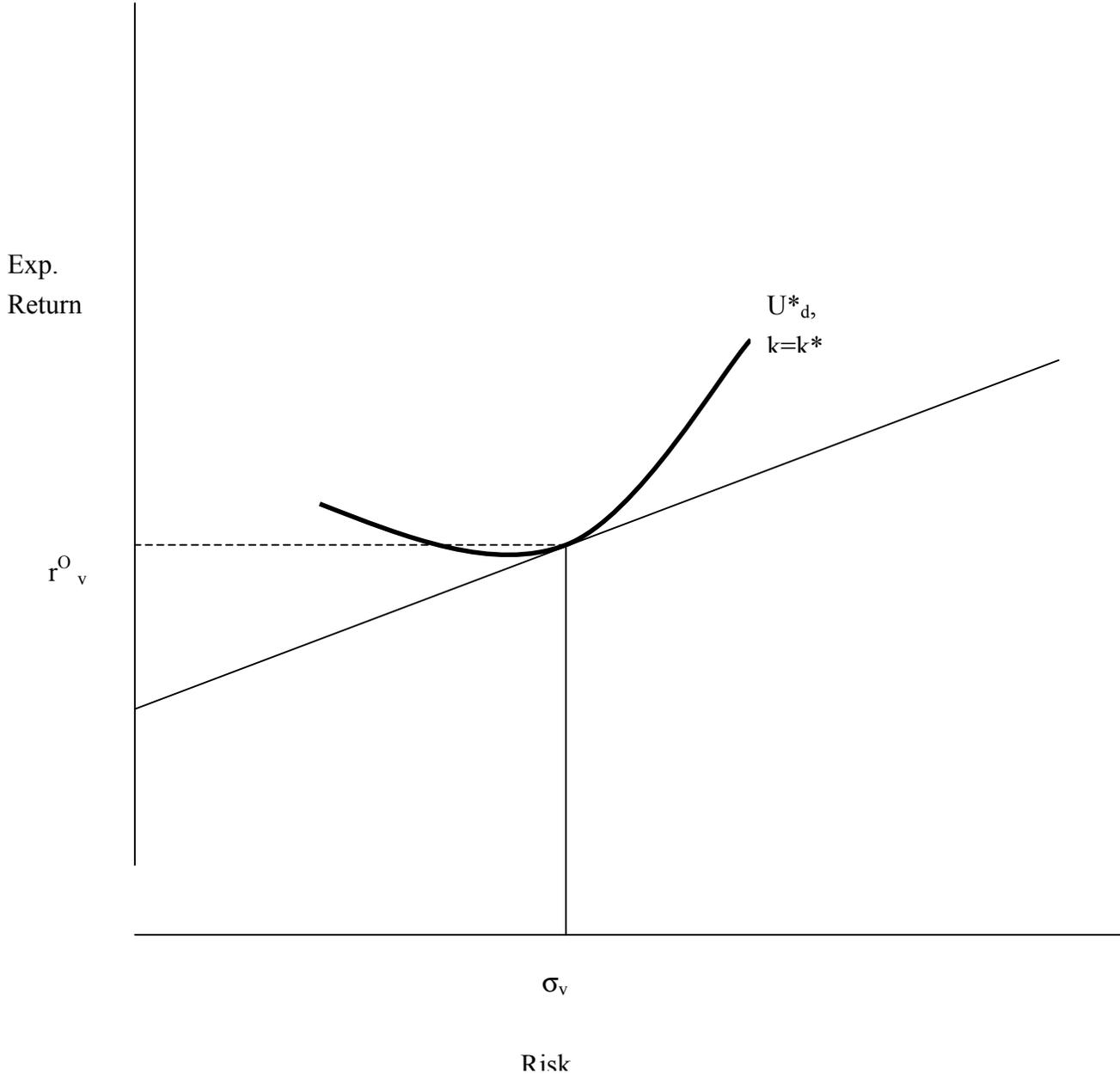


Figure 1.3: Required rates of return with low risk-aversion

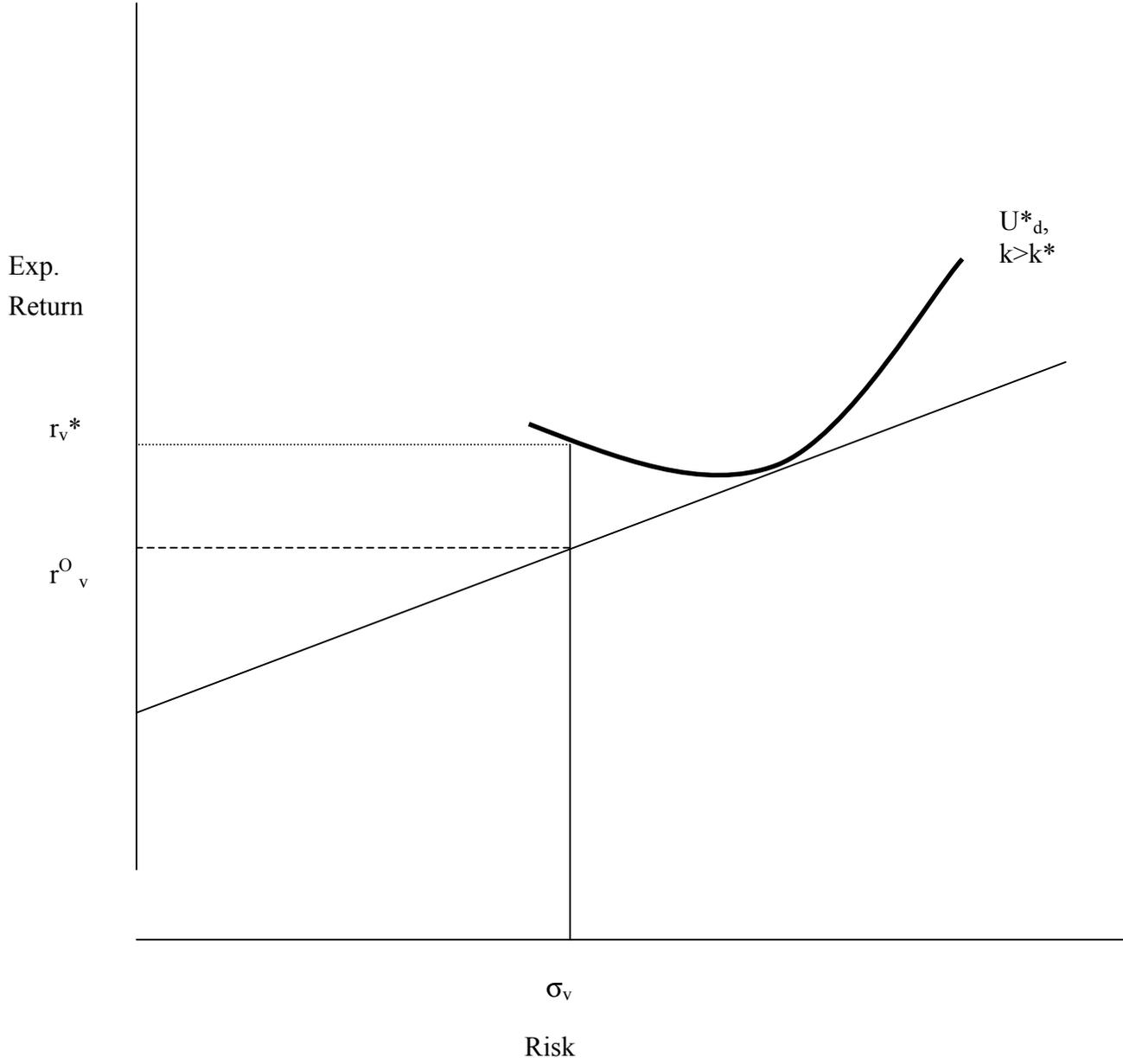


Figure 2.1: How the opportunity cost approach overstates the required rate of return when  $k=k^*$

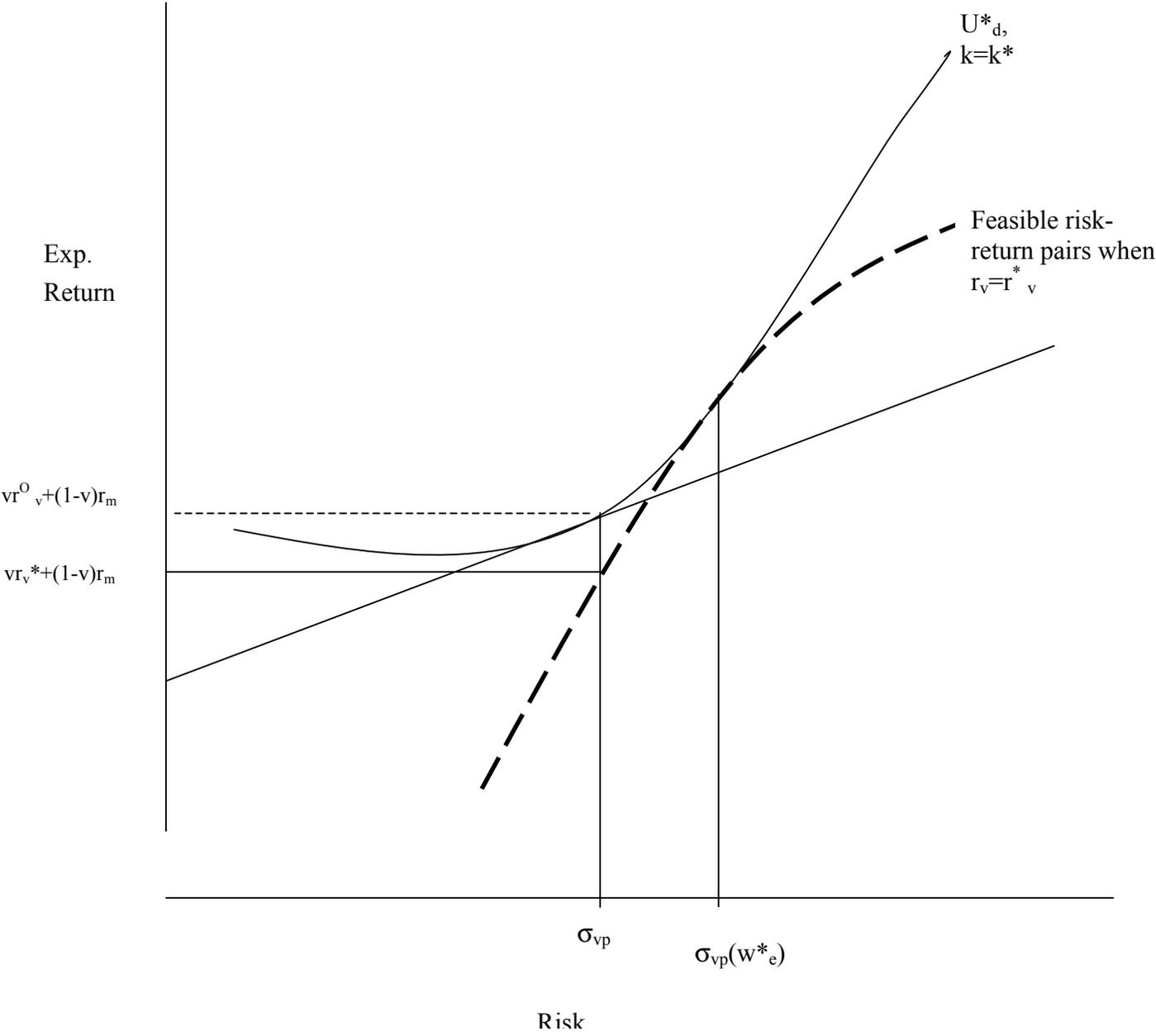


Figure 2.2: Why allowing for choice of risk-free investment reduces the apparent error in the opportunity cost discount rate for high risk-aversion

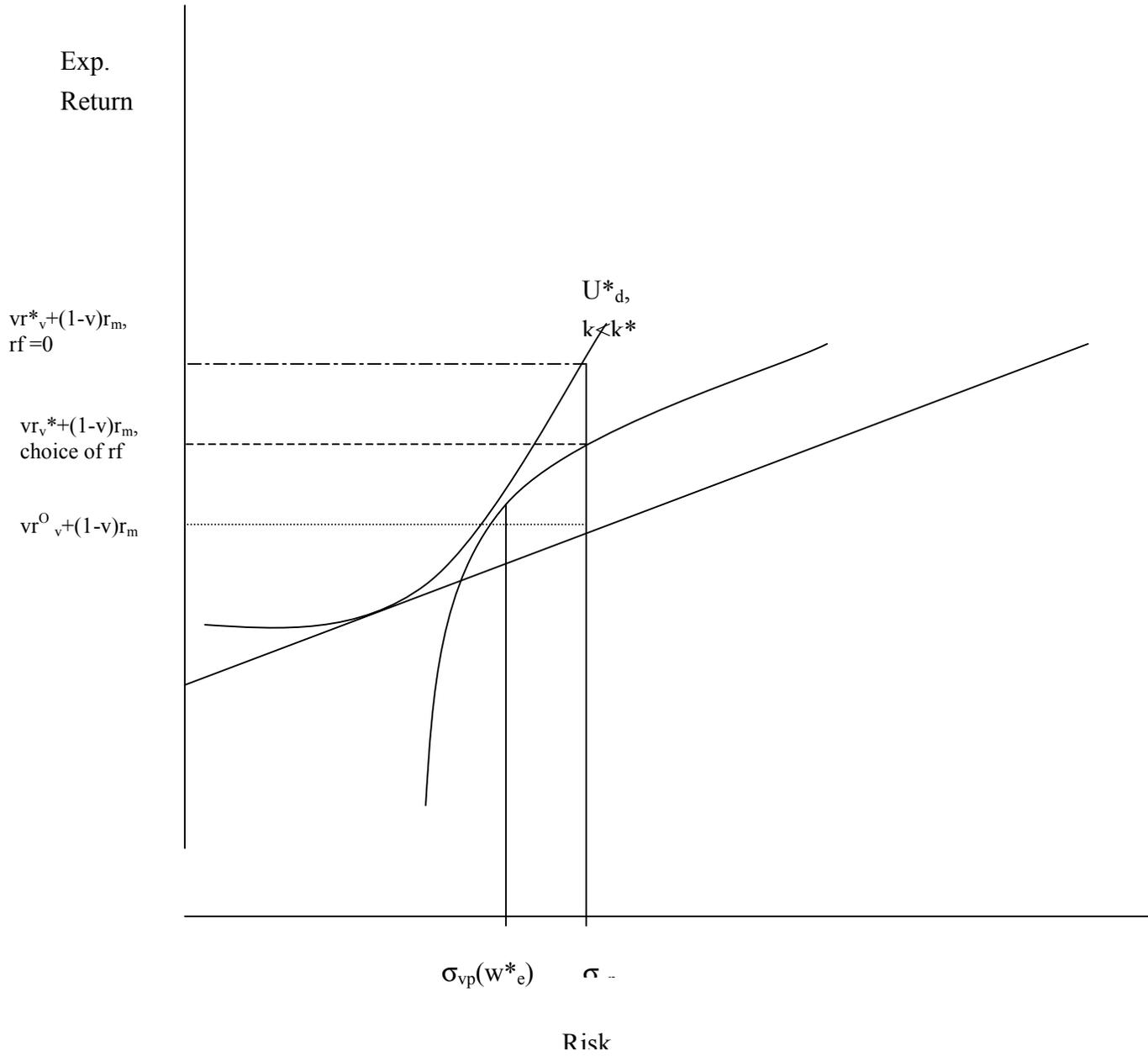


Table 1: Portfolio choices for  $\sigma(v)=50\%$ ,  $v=40\%$ ,  $\sigma(m)=20\%$ ,  $\beta=1$ ,  $r_m - r_f = r_f = 6\%$

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Coefficient of absolute risk- aversion	Entrepreneur's rm weight	Entrepreneur's rf weight	Investor's rm weight	Investor's rf weight
0.5	2.60	-2	3	-2.00
1	1.10	-0.5	1.5	-0.50
1.5	0.60	0	1	0.00
2	0.35	0.25	0.75	0.25
2.5	0.20	0.4	0.6	0.40
3	0.10	0.5	0.5	0.50
3.5	0.03	0.57	0.43	0.57
4	-0.03	0.63	0.38	0.63
4.5	-0.07	0.67	0.33	0.67
5	-0.10	0.7	0.3	0.70

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Figure 3: Required rates of return when  $\sigma(v)=50\%$ ,  $\sigma(m)=20\%$ ,  $v=40\%$ ,  $\beta=1$ ,  $r_f=r_m-r_f=6\%$

