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# Paths and Consistency in Additive Cost Sharing

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## Abstract

Using a new representation theorem for additive cost sharing methods as sums of path methods, we show that many of the standard additive cost sharing methods (Aumann-Shapley, Shapley Shubik, and Serial Cost) are consistent. These results follow directly from a simple sufficient condition for consistency: being generated by associative paths, which can be used to show consistency for many other methods.

We introduce a new axiom, dummy consistency, which is quite mild. Nonetheless there is an important relationship between dummy consistency and consistency. For example, we show that all additive cost sharing methods which are dummy consistent and demand monotonic are consistent.

Using dummy consistency, we also show that the Aumann-Shapley and Serial Cost methods are the *unique* (additive) consistent extension of their restriction on all two agent problems, while the Shapley-Shubik method has multiple consistent extensions but a unique symmetric one. In fact, these results are unchanged when we replace consistency with dummy consistency. Our characterization of the set of dummy-consistent cost sharing methods provides a simple framework for analyzing consistent extensions and is useful for constructing nonsymmetric methods.

## 1 Introduction

The principle of consistency has a long history in many economic allocation problems. (See [22] for a detailed review and bibliography.) For example, under standard assumptions, the

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Walrasian allocation is consistent in the appropriate sense and indeed can be characterized by consistency in combination with other basic axioms. The basic idea behind consistency in any allocation problem is that, if an agent leaves the problem and takes her share with her, then the allocation among the remaining agents is unchanged.

The application of consistency to cost sharing methods<sup>1</sup> (CSMs) with binary demands was introduced by Young [26] who showed that the Shapley value is the unique strongly consistent method. However, Young’s definition of consistency is quite demanding. Recently, Hart and Mas-Colell [12] showed, using a more standard notion of consistency, that the Shapley value is the unique consistent extension which “divides the surplus equally” in 2 person games, using a new tool, the potential of a game.

More recently, Mclean, Pazgal and Sharkey [13] extended this result to the case with continuous demands, showing that the Aumann-Shapley method is the unique consistent extension of the Aumann-Shapley method defined on all 2 agent problems. Their proof of this result is quite involved and also uses a notion of potential.

In this paper we show that, when restricted to additive methods, the existence and uniqueness of consistent extensions can be analyzed much more directly. First we provide a sufficient condition for a CSM to be consistent. Using this we show that the Aumann-Shapley, Shapley-Shubik and Serial Cost methods are all consistent as are the weighted, asymmetric, versions of all of these.

Next we introduce an extremely weak version of consistency, dummy consistency, and pro-

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<sup>1</sup>For a general introduction to cost sharing see e.g., [25] and for details of the approach taken in this paper see [8].

vide a complete characterization of the set of dummy consistent CSMs. Dummy consistency is an extremely natural requirement and thus may be applicable in a wider variety of situations than ordinary consistency. Nonetheless, in many cases the existence and uniqueness of dummy-consistent extensions implies the existence and uniqueness of consistent extensions. This allows us to easily prove the existence and uniqueness of consistent extensions from 2 agent problems to multi-agent problems for the Aumann- Shapley and Serial Cost methods, while the Shapley-Shubik method may have multiple extensions, but a unique symmetric one.<sup>2</sup>

Our results and method of analysis are based on a new representation for additive CSMs based on the idea of a path method, which is a natural generalization of the Aumann-Shapley formula. This representation method is also of independent interest for other applications (see, e.g., [5, 7, 6]).

The paper is organized as follows: Section 2 reviews the structure of the class of additive CSMs and describes their representation as sums of path methods, the key technique used in this paper. Section 3 provides a sufficient condition for a CSM to be consistent and a complete characterization of the dummy-consistent methods. We conclude in Section 4 with the analysis of consistent extensions. Several proofs are in the appendix.

## 2 Additive Cost Sharing

We will consider the problem of dividing the cost of production among a group of agents, where each agent has a demand for an idiosyncratic good. Thus, we will identify agents with

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<sup>2</sup>These results also provide a useful tool for analyzing asymmetric CSMs, a topic explored in [5].

their good, and since we will be interested in the consistency of various methods we will allow for the set of agents to vary.

Let the set of potential agents be identified with the set of nonnegative integers,  $Z_+$ , and let  $\mathcal{N}$  be the set of nonempty finite subsets of  $Z_+$ . For some finite set of agents  $N \in \mathcal{N}$ , their allowed demands will be denoted by  $q \in Q(N, \bar{q}) = [0, \bar{q}]^N$ , where often  $\bar{q} = \infty$ . To save on notation, we will think of  $\bar{q}$  as fixed and therefore it will only be implicit in the notation. For example we will usually write  $Q(N)$  instead of  $Q(N, \bar{q})$ . The cost of serving these demands is  $C(q)$  with  $C \in \mathcal{C}(N)$ , where  $\mathcal{C}(N)$  is the set of nondecreasing, continuously differentiable functions from  $Q(N)$  to  $\mathbb{R}_+$ , satisfying  $C(0) = 0$ .<sup>3</sup> A cost sharing mechanism provides a method for computing the cost shares allocated to each of the agents. The following definition is introduced in [8].

**Definition 1** *Given  $N \in \mathcal{N}$ , an additive cost sharing method  $x$  is a mapping  $x : Q(N) \times \mathcal{C}(N) \rightarrow \mathbb{R}_+^N$  satisfying:*

- 1) *Efficiency:  $\sum_{n \in N} x_n(q; C) = C(q)$ ,*
- 2) *Additivity: for all  $C, D \in \mathcal{C}(N)$  the following holds:  $\forall n \in N, \quad x_n(q; C + D) = x_n(q; C) + x_n(q; D)$ .*
- 3) *Dummy: For any  $C \in \mathcal{C}(N)$  such that  $\partial_n C(p) = 0 \quad \forall p \in Q(N)$ , then  $x_i(q; C) = 0$  for all  $q \in Q(N)$ .*

*Let  $CS(N)$  denote the set of all such CSMs.*

Note that we use the notation  $\partial_n C(p)$  to represent the partial derivative of  $C(q)$  with respect to  $q_n$  evaluated at  $p$ . Let  $CS(q, N)$  be the projection of  $CS(N)$  onto its second

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<sup>3</sup>Note that the generic vector of 0's will be denoted by 0 and the unit vector of 1's by 1, where the dimension of these vectors will be obvious from the context.

component, i.e., if  $x(\cdot; \cdot) \in CS(N)$  then  $x(q; \cdot) \in CS(q, N)$ . Also let  $CS = \bigcup_{N \in \mathcal{N}} CS(N)$ .

The following representation theorem is given in [8].

**Proposition 1 (Friedman and Moulin, 1998)** *For any  $N \in \mathcal{N}$  and  $x \in CS(q, N)$ , there exists a vector of nonnegative Radon measures indexed by  $q$ ,  $\{\rho_n(q)\}_{n \in N}$  each defined on  $[0, q]$  such that for all  $q \in Q(N)$ ,  $C \in C(N)$  and  $n \in N$ :*

$$x_n(q; C) = \int_{[0, q]} \partial_n C(p) d\rho_n(q)(p)$$

*and for each  $q \in Q(N)$  and  $n \in N$  the projection of  $\rho_n(q)$  onto the  $q_n$  axis is the Lebesgue measure, i.e.,  $\rho_n(q)([a, b] \times [0, q_{-n}]) = b - a$  for  $0 \leq a \leq b \leq q_n$ .*

Note that, as written, this is not a complete characterization, since not all vectors of measures lead to valid CSMs. In particular, budget balance need not be satisfied. In [8], a set of restrictions on the measures was given which leads to a proper characterization theorem; however, those conditions are quite complicated. We now present a second representation theorem that is much more straightforward and intuitive. It is based on an important class of CSMs, the path methods.

**Definition 2** *A path function  $\gamma$  is a mapping  $\gamma : [0, \infty] \times Q(N) \rightarrow Q(N)$  satisfying the following for each  $q \in Q(N)$ :*

- 1)  $\gamma(t; q)$  is continuous and nondecreasing in  $t$ .
- 2)  $\gamma(0; q) = 0$  and there exists a  $\hat{t} > 0$  such that for all  $t \geq \hat{t}$ ,  $\gamma(t; q) = q$ .

*Let the set of all such path functions be denoted  $\gamma(N)$ .*

Also, for each  $q \in Q(N)$  let  $\pi_q : (q, N) \rightarrow (q, N)$  be the projection of  $\gamma$  onto its second component, for fixed  $q$ . Given such a path function, it is straightforward to construct a related CSM.

**Definition 3** *Given a path  $\gamma \in \Gamma(N)$  the “path method generated by  $\gamma$ ” is given by*

$$x_n^\gamma(q; C) = \int_{t=0}^{\infty} \partial_n C(\gamma(t; q)) d\gamma_i(t; q).$$

As proven in [8], a path method is a valid CSM. A path method is the natural generalization of the well known formula for the Aumann-Shapley method (discussed in the following section). Note also that any convex combination of two CSMs is a valid CSM, and thus any convex combination of path methods is also a CSM. In fact, as the following theorem demonstrates, any CSM can be constructed as a convex combination of path methods.

**Theorem 1** *For any  $N \in \mathcal{N}$  the following are equivalent:*

- i)  $x \in CS(N)$ .
- ii) *There exists a family of probability measures, indexed by  $q \in Q(N)$ ,  $\mu^q$ , each on  $\Gamma_q(N)$  such that*

$$x = \int_{\gamma(\cdot; q) \in \Gamma_q(N)} x^{\gamma(\cdot; q)} d\mu^q(\gamma(\cdot; q)).$$

The proof is given in the appendix. It is based on Wang’s [23] representation theorem for CSMs with discrete demands; independently, Haimanko [10] proved an analogous theorem for the values of nonatomic games.

Thus, we have a simple representation of the additive CSM’s as a sum of path generated methods.

## 2.1 Examples of path methods

One of the best known CSMs is the Aumann-Shapley method. This method was first used as a CSM in Billera, Heath and Ranaan[1] to allocate the cost of a shared telephone system and was subsequently characterized in terms of economic axioms in Billera and Heath [2] and Mirman and Tauman [15]. For any  $N \in \mathcal{N}$ , it is a path method,  $x^{AS}$ , generated by the path  $\gamma_n(t; q) = \min[tq_n, q_n]$ , which is the well known “diagonal path.” (Figure 1)

A set of nonsymmetric variants of this method has been proposed by Mclean and Sharkey [14]. These are constructed with a weight function  $w : N \rightarrow \mathbb{R}_+ \setminus \{0\}$  and are the path methods given by  $\gamma_n(t; q) = \min[t^{w(n)}q_n, q_n]$ . For example when  $N = \{1, 2\}$  and  $w(n) = n$  the path is shown in Figure 2.

Another well known CSM was proposed by Shubik [20], and is known as the Shapley-Shubik method, since it is the Shapley value of the cooperative game generated from the cost function. Before constructing this method we first consider the class of incremental methods. An incremental method is constructed from an ordering  $\omega \in \Omega(N)$ , where  $\Omega(N)$  is the set of all orderings, i.e., bijective functions  $\omega : N \rightarrow (1, 2, \dots, |N|)$ . The incremental method for order  $\omega$  is computed as follows.

For any  $n \in \mathcal{N}$  and  $q \in Q(N, \infty)$  define  $S^+(n; \omega) = \{m \in N \mid \omega(m) \leq \omega(n)\}$  and  $S^-(n; \omega) = S^+(n; \omega) \setminus n$  then  $x_n^\omega(q; C) = C(q_{S^+(n; \omega)}, 0_{-S^+(n; \omega)}) - C(q_{S^-(n; \omega)}, 0_{-S^-(n; \omega)})$ . This is equivalent to the path method generated by the function  $\gamma_n^\omega(t; q)$  which is 0 for  $t < \omega(n) - 1$ ,  $q_i$  for  $t > \omega(n)$  and  $(t - \omega(n))q_i$  otherwise. For example, let  $N = \{1, 2\}$  and let  $\omega(n) = n$ ; the path for this CSM is as shown in Figure 3.



We can generalize this to a random order method [24] by choosing a set of weights on each ordering  $w : \Omega(N) \rightarrow \mathbb{R}_+ \setminus \{0\}$  which must satisfy  $\sum_{w \in \Omega(N)} w(\omega) = 1$ . Then the random order method with weights  $w$  is given by  $x^w = \sum_{w \in \Omega(N)} w(\omega) x^\omega$ . The Shapley-Shubik method is given by the Random Order method with equal weights:

$$x^{SS} = (|N|!)^{-1} \sum_{w \in \Omega(N)} x^\omega.$$

The generalized serial cost method was introduced in [8] and is based on the serial cost method for homogeneous goods which was analyzed in [17]. The serial cost method for homogeneous goods which was motivated by Fair Queuing [4] in the networking literature and is interesting both axiomatically and strategically.<sup>4</sup> The serial cost method,  $x^{SC}$  is generated by the path  $\gamma_n(t; q) = \min[t, q_n]$ . (See Figure 4.)

Given a weight function  $w : P \rightarrow \mathbb{R} \setminus \{0\}$  we can define the linearly weighted serial cost method as the CSM generated by the path  $\gamma_n(t; q) = \min[w(n)t, q_n]$  and the exponentially weighted serial cost method as the method generated by the path  $\gamma_n(t; q) = \min[t^{w(n)}, q_n]$ , two new classes of methods which are applied in [7].

## 2.2 Demand Monotonicity and Scale Invariance

So far, we have imposed no restrictions on the relationship of cost shares for different demands, i.e., given a CSM  $x$  and a cost function  $C$  we have imposed no restrictions on the relationship between  $x(q; C)$  and  $x(q'; C)$  even if  $q$  and  $q'$  are very close. For example, one

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<sup>4</sup>Shenker [19], and Moulin and Shenker [17], have shown that games induced by serial cost are dominance solvable, while Friedman and Shenker [9] have shown that such games are also solvable in overwhelmed actions, and thus are robustly learnable for a wide class of learning algorithms and information structures. The generalized serial cost method [8] has similar axiomatic and strategic characterizations.

might want to require that  $x(q; C)$  be continuous in  $q$ . The two important axioms which we now introduce are stronger than continuity.

The first axiom, scale invariance, is well known and was used in the original axiomatizations of the Aumann-Shapley method.

Given  $\lambda \in \mathbb{R}_{++}^N$ , define  $\tau_\lambda(q)$  by  $\tau_\lambda(q)_n = \lambda_n q_n$  for  $n \in N$  and define  $\tau_\lambda(C)$  by  $\tau_\lambda(C)(q) = C(\tau_\lambda(q))$ , for  $C \in \mathcal{C}(N, \infty)$ .

**Definition 4 (Scale Invariance)** *For all  $N \in \mathcal{N}$  a CSM,  $x \in CS(N, \infty)$ , is scale invariant if  $x(\tau_\lambda(q); C) = x(q; \tau_\lambda(C))$ , for all  $\lambda \in \mathbb{R}_{++}^N$  and  $C \in \mathcal{C}(N, \infty)$ .*

Scale invariance is the statement that different goods are not comparable, e.g. the units used to measure  $q_n$  cannot be meaningfully compared with those used to measure  $q_m$  when  $n \neq m$ . We now show that the set of scale invariant CSMs are generated by the scale invariant paths. First however, we need a weak version of continuity, called “dummy continuity”, to simplify the presentation.

**Definition 5** *Given any  $N \in \mathcal{N}$ , a CSM  $x \in CS(N)$  is dummy continuous<sup>5</sup> if for all  $C \in \mathcal{C}(N)$ ,  $n, m \in N$  and  $q \in Q(N)$ ,  $\lim_{q'_n \rightarrow 0} x_n(q_{-n}, q'_n; C) = x_n(q_{-n}, 0; C)$ .*

Let  $SI(N)$  be the set of scale invariant CSMs over  $N \in \mathcal{N}$  in  $CS(N, \infty)$ . It is easy to characterize this set of scale invariant CSMs. Define  $\gamma_{SI}(N)$  to be the set of paths such that  $\gamma(t; q) = \tau_q(\gamma(t; 1))$ , thus each component is  $\gamma_i(t; q) = \gamma_i(t, 1)q_i$ , where 1 represents unit vector  $(1, 1, \dots, 1)$ .

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<sup>5</sup>Note that while dummy continuity is quite strong on its own, it is a relatively weak addition to ordinary dummy and additivity. See [8] for a discussion on the need for “dummy continuity” and the changes that arise without it.

**Theorem 2** *For any  $N \in \mathcal{N}$  the following are equivalent:*

*i)  $x \in SI(N)$  and is dummy continuous.*

*ii) There exists a non-negative probability measure  $\mu$  on  $SI(N)$  such that*

$$x = \int_{\gamma \in \Gamma_{SI(N)}} x^\gamma d\mu(\gamma).$$

Proof: This follows upon noting that a scale invariant CSM is completely determined by its behavior on any particular  $q$ , such as  $q = 1$ . Thus  $SI(N)$  is isomorphic to  $CS(1, N, \infty)$ .  $\square$

Another important axiom is demand monotonicity [16, 8].

**Definition 6 (Demand Monotonicity)** *A CSM in  $CS(N)$  is demand monotonic if for all  $n \in N$  and  $q, q' \in Q(N)$  such that  $q_n \leq q'_n$  and  $q_{-n} = q'_{-n}$  and all  $C \in CS(N)$ :*

$$x_n(q; C) \leq x_n(q'; C).$$

Demand monotonicity can be viewed as an extremely weak incentive constraint, since if a CSM is not demand monotonic then there are obvious incentives for agents to overstate their demands. We do not know the precise characterization of demand monotonic paths, but will study a closely related problem in Section 3.1, when we combine demand monotonicity with a weak notion of consistency.

### 3 Consistent Cost Sharing Methods

In this section we provide sufficient conditions for a CSM to be consistent. We begin with a formal definition of consistency.

McLean, Pazgal, and Sharkey [13] define a notion of consistency which is based on that defined by Hart and Mas-Colell [11] for TU games. Their definition is analogous to many versions of consistency found in the literature.<sup>6</sup> However this definition is still quite strong, and we will introduce a weaker condition (dummy consistency) below.

**Definition 7 (Consistency)** *A CSM  $x \in CS$  is consistent if for any  $N \in \mathcal{N}$ ,  $q \in Q(N)$ ,  $C \in \mathcal{C}(N)$ , and  $m, n \in N$ , with  $m \neq n$  :*

$$x_n(q, C) = x_n(q_{-m}, R_m^{q_m}(C))$$

*when  $R_m^{q_m}(C)(p_{-m}) = C(p_{-m}, q_m) - x_m(p_{-m}, q_m; C)$  is an element of  $CS(N \setminus m)$ .*

Thus, any agent and her cost share can be removed from the cost function without affecting the cost shares of the remaining agents. We will denote the subset of  $CS$  which is consistent by  $CON$ .

For example, consider the Aumann-Shapley method with  $|N|$  agents when  $C(p) = |p_n|^2$ , where  $|p| = \sum_{n \in N} p_n$ . Then  $\partial_n C(p) = 2|p_n|$  and thus

$$x_n(q; C) = \int_0^1 2t|q|q_n dt = |q|q_n.$$

Now if we remove agent  $m$ , we get

$$R_m^{q_m}(C)(p_{-m}) = |(q_m, p_{-m})|^2 - |(q_m, p_{-m})|q_m.$$

Recomputing  $x_n$ , by first removing agent  $m$  yields

$$x_n(q; C) = \int_0^1 (2|(q_m, tq_{-m})| - q_m)q_n dt = |q|q_n,$$

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<sup>6</sup>See, e.g. Thomson [22] for a survey of consistency.

as expected if Aumann-Shapley is consistent, which will be proven below and was also shown in [13].

By contrast, we now introduce the property of dummy consistency which only requires that dummies can be removed from a cost sharing problem without affecting the cost shares, a much milder requirement.

**Definition 8 (Dummy Consistency)** *A CSM  $x \in CS$  is dummy-consistent if for all  $N \in \mathcal{N}$ ,  $C \in CS(N)$  such that agent  $n \in N$  is a dummy agent ( $\partial_n C(p) \equiv 0$ ),  $x_m(q; C) = x_m(q_{-n}, R_n^{q_n}(C))$  for all  $m \neq n$ .*

Note that when  $i$  is a dummy agent, then  $R_i^{q_i}(C)(q_{-i}) = C(q_{-i}, 0_i)$ , and thus we do not need to impose any conditions on  $R_i^{q_i}(C)(q_{-i})$ . Let  $DCON$  denote the subset of  $CS$  which is dummy consistent.

Dummy consistency seems (to us) to be an extremely natural requirement for CSMs, while strong consistency, which may be desirable, is not be as fundamental.<sup>7</sup>

For example, consider the Aumann-Shapley method when  $C(q) = |q_{-m}|^2$ . Computing directly for  $n \neq m$  yields  $x_n(q; C) = |q_{-m}|q_n$ , as before, while assuming dummy consistency yields  $x_n(q; C) = \int_0^1 2t|q_{-m}|q_n dt$ , which gives the same result.

### 3.1 Consistent Cost Sharing Methods

In this section we develop partial representation theorems (based on Theorem 1) for consistency and dummy consistency. This will allow us to easily show that many well known

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<sup>7</sup>Several readers of earlier drafts of this paper have suggested that dummy consistency should actually be called “strong dummy” as it is essentially a version of the dummy axiom and much weaker than a consistency action. We retain the (perhaps misleading) title of dummy consistency to stress its relationship with ordinary consistency.

CSMs are consistent, and to allow us to construct other consistent CSMs. We first introduce a new collection of paths.

**Definition 9** *A path  $\gamma \in \mathcal{P}(N)$ , for  $N \in \mathcal{N}$  is separable if for each  $n \in N$ ,  $\gamma_n(t; q)$  is independent of  $q_{-n}$ , which we write as (with a slight abuse of notation)  $\gamma_n(t; q_n)$ .*

Let  $\mathcal{P}_s(N)$  be the subset of  $\mathcal{P}(N)$  containing all the separable paths and  $\mathcal{P}_s = \bigcup_{N \in \mathcal{N}} \mathcal{P}_s(N)$ . Note that there is a natural projection from  $\mathcal{P}_s(N)$  to  $\mathcal{P}_s(N \setminus m)$  for  $m \in N$ ,  $\pi_m^N$  which simply removes the component  $\gamma_m$  from  $\gamma \in \mathcal{P}_s(N)$ . This is well defined, since none of the components in  $\mathcal{P}_s(N \setminus m)$  depend on  $q_m$ .

**Theorem 3** *The following are equivalent:*

- i)  $x \in DCON$ .
- ii) For all  $N \in \mathcal{N}$  there exists a probability measure  $\mu$  on  $\mathcal{P}_s(N)$  such that for any  $m \in N$ ,

$$x^N = \int_{\gamma \in \mathcal{P}_s(N)} x^\gamma d\mu(\gamma)$$

and

$$x^{N \setminus m} = \int_{\gamma \in \mathcal{P}_s(N)} \pi_m^N[x^\gamma] d\mu(\gamma).$$

Proof: In appendix.

Thus, the separable paths generate all of  $DCON$ . As the next result demonstrates, all of the previously mentioned CSMs are generated by separable paths and therefore are dummy consistent.<sup>8</sup>

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<sup>8</sup>The boundedness of  $q$  in the above theorem does not affect these results, since we need only check at each finite value of  $q$ .

**Corollary 1** *The following CSMs are dummy consistent:*

- 1) *Aumann Shapley and weighted Aumann-Shapley.*
- 2) *Serial Cost and both linearly and exponentially weighted Serial cost.*
- 3) *Random order methods and Shapley-Shubik.*

However, it is easy to show that not all CSMs are dummy consistent. Consider the case when  $N = \{1, 2, 3\}$ ,  $\bar{q} = 1$  and the CSM is  $x^\gamma$  where  $\gamma(t; q) = (tq_1, tq_2, (1 - q_1)q_3t + q_1q_3t)$ , which is not separable.<sup>9</sup> Let the cost function be  $C(q_1, q_2, q_3) = q_2q_3$  which does not depend on  $q_1$ . Then  $x_2(q; C) = \int_0^\infty [(1 - q_1)q_3t + q_1q_3t]q_2 = q_1q_3(1/2 - q_1/6)$  which depends on  $q_1$  even though agent 1 is a dummy agent. Thus it is not dummy consistent.

Ordinary consistency imposes a more stringent requirement on the paths. We now demonstrate a dummy consistent CSM which is not strongly consistent. Consider the CSM  $x^\gamma$ , for  $N = \{1, 2, 3\}$  and  $\bar{q} = 1$ , where  $\gamma(t; q) = (tq_1, (t + t^2)q_2/2, tq_3)$ , which is separable and scale invariant. Let  $C(q_1, q_2, q_3) = q_1q_2q_3$ . Then

$$x_\gamma^1(q; C) = \int_0^\infty dt(q_2(t + t^2)/2)(tq_3)q_1 = 17q_1q_2q_3/24,$$

while  $R_3^{q_3}(C)(p_1, p_2) = 17p_1p_2/24$ . If  $x^\gamma$  were strongly consistent, then the cost share to agent 1 could be computed by removing agent 3 yielding  $\int_0^\infty dt 17(q_2(t + t^2)/2)]/24q_3q_1 = 85q_1q_2q_3/288$  which differs from the cost share which was computed directly, e.g. when  $q = (1, 1, 1)$  the direct computation shows that  $x_\gamma^1(q; C) = 17/24$  while the computation applying strong consistency computes a value of  $85/288$  which are unequal. Thus  $x^\gamma$  is not strongly consistent. This is because the path which defines this CSM is not associative.

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<sup>9</sup>Note that in these examples we are assuming that the domain of  $t$  is  $[0, 1]$ . This simplifies the appearance of the path since it allows us to write, e.g.,  $tq_1$  instead of  $\min[1, t]q_1$ .

**Definition 10** A path  $\gamma \in \mathcal{S}(N)$ , for  $N \in \mathcal{N}$  is associative if it is separable and right differentiable in  $t$ , and there exists a continuous and right differentiable function  $\phi(t, s)$  which is nondecreasing in both arguments such that,  $\gamma_n(t; \gamma_n(s; q_n)) = \gamma_n(\phi(t, s); q_n)$  for all  $n \in N$ .

Let  $\mathcal{S}_A(N)$  be the subset of paths in  $\mathcal{S}(N)$  which are associative and  $\mathcal{S}_A = \bigcup_{N \in \mathcal{N}} \mathcal{S}_A(N)$ .

**Theorem 4** Assume that for all  $N \in \mathcal{N}$  there exists a probability measure  $\mu$  on  $\mathcal{S}_A(N)$  such that for any  $m \in N$ ,

$$x^N = \int_{\gamma \in \mathcal{S}_A(N)} x^\gamma d\mu(\gamma)$$

and

$$x^{N \setminus m} = \int_{\gamma \in \mathcal{S}_A(N)} \pi_m^N[x^\gamma] d\mu(\gamma),$$

then  $x \in CON$ .

Proof: In appendix.

Note that above characterization is only partial. We conjecture the complete characterization is true when  $\bar{q} < \infty$ <sup>10</sup>: all  $x \in CON$  can be written as a combination of CSMs generated by associative paths, subject to a few technical provisos.<sup>11</sup>

An important simplifying result, which we now show, is that all demand monotonic methods which are dummy consistent are also strongly consistent, since the paths associated with such methods are associative. Thus it is easy to see that, in this case, ordinary consistency reduces to dummy consistency, which is typically much easier to check.

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<sup>10</sup>When  $\bar{q} = \infty$  is not bounded, the set of associative paths is not closed in the appropriate topology.

<sup>11</sup>For example, right differentiability is probably not necessary, nor perhaps is monotonicity.



There are some technical difficulties involved with this characterization, when the domain of  $q$ 's is unbounded, since the limiting behavior of the class paths, which are both demand monotonic and dummy consistent, is not well behaved. Thus we will consider the class of CSMs which are defined only on a bounded domain.

Let  $DM$  be the set of demand monotonic CSMs in  $CS$ , for  $\bar{q} < \infty$ . Now, the extreme CSMs are constructed from a single infinitely long path. For any  $N \in \mathcal{N}$  let  $\alpha(t)$  be a path defined on  $t \in \mathfrak{R}_+$  which is nondecreasing such that there exists a  $\hat{t} > 0$  such that for all  $t > \hat{t}$ ,  $\alpha(t) \geq \bar{q}$ . Let  $\Gamma_{DM,DCON}(N)$  be the set of all such paths and  $\Gamma_{DM,DCON} = \bigcup_{N \in \mathcal{N}} \Gamma_{DM,DCON}(N)$ . Given  $\alpha \in \Gamma_{DM,DCON}(N)$  define  $\gamma^\alpha$  by  $\gamma^\alpha(t, q)_n = \min[\alpha_n(t), q_i]$ , for all  $n \in N$ . (Figure 5.)

**Theorem 5** *For any  $\bar{q} < \infty$  the following are equivalent:*

- i)  $x \in DM \cap DCON$ .
- ii) *For all  $N \in \mathcal{N}$  there exists a probability measure  $\mu$  on  $\Gamma_{DM,DCON}(N)$  such that for any  $m \in N$ ,*

$$x^N = \int_{\gamma \in \Gamma_{DM,DCON}(N)} x^\gamma d\mu(\gamma)$$

and

$$x^{N \setminus m} = \int_{\gamma \in \Gamma_{DM,DCON}(N)} \pi_m^N[x^\gamma] d\mu(\gamma).$$

Proof: In the Appendix.

Thus, we can now present a simple sufficient condition for consistency:

**Corollary 2** *For any  $\bar{q} < \infty$ , if  $x \in CS$  is dummy consistent and demand monotonic, then it is strongly consistent.*

Proof: From Theorem 5 we know that all CSMs which are demand monotonic and dummy consistent are generated by paths of the form  $\gamma^n(t; q) = \min[\alpha_i(t), q_i]$ , where  $\alpha_i(t)$  is nondecreasing. Thus, these paths are associative with  $\phi(t, s) = \min[t, s]$ . Since CSMs which are demand monotonic and dummy consistent are sums of such paths this implies that they are consistent.  $\square$

We now show that all of the previously discussed CSMs are consistent.

**Theorem 6** *The following CSMs are consistent:*

- 1) *Aumann Shapley and weighted Aumann-Shapley.*
- 2) *Serial Cost and weighted Serial cost.*
- 3) *Random order methods and Shapley-Shubik.*

Proof: Consistency of Serial cost and Random order methods follow immediately from Corollary 2 and the fact that they are demand monotonic. For the weighted Aumann-Shapley Mechanism it follows upon noting that the path which generates the mechanism is associative with  $\phi(t, s) = ts$ , since  $t^{w_i} s^{w_i} q_i = (ts)^{w_i} q_i$ .  $\square$

## 4 Consistent Extensions

As mentioned in the introduction, one important question about consistency is whether two agent CSMs can be extended, using consistency, to general CSMs with an arbitrary number of agents. In [11], Hart and Mas-Collel, showed that in the case of binary demands, if we assume that a consistent CSM when restricted to two agents “divides the surplus equally” then it must be the Shapley Method. For the continuous demand setting considered in this

paper, Mclean, Pazgal and Sharkey [13], showed that if a consistent CSM is the Aumann-Shapley method on all 2 agent problems, then it must be the Aumann-Shapley method. Both proofs require the use of a potential, and do not appear to be easily extendable for studying consistency questions related to extensions of other CSMs.

In this section, we show that when restricted to the case of additive CSMs, problems of this sort are straightforward to analyze by using the representation theorems in the previous section. In particular, we will often be able to use dummy consistency to simplify the analysis. Furthermore, the results for dummy consistency also apply to the case with a fixed population where we might have conditions on the CSM when there are a group of dummy agents. For example, consider a three agent cost sharing problem for which we know that when agent one agent is a dummy, the other 2 agents should use the Aumann-Shapley method, but when a different agent is a dummy, then the remaining two should use serial cost. Using the tools developed in this section for dummy consistency, we can show that there is a unique CSM which satisfies these conditions. (This approach is explored in more detail in [5].)

Thus, our goal in this section is to understand when a CSM defined for two agent problems extends to a consistent CSM for an arbitrary number of agents. Let  $\mathcal{N}_2 = \{N \in \mathcal{N} \mid |N| = 2\}$  and define  $CS_2 = \bigcup_{N \in \mathcal{N}_2} CS(N)$ , which is the set of CSMs defined for all two agent methods.

The basic problem that we now consider is this: given an  $y \in CS_2$ , what is the set of consistent extension of  $y$ , i.e., what methods  $x \in CON$  (or  $DCON$ ) coincide with  $y$  on all  $N \in \mathcal{N}_2$ .

From the previous section we know that if  $y$  cannot be generated by associative (resp. separable) paths there can be no consistent (resp. dummy consistent) extensions of  $y$  to  $CS$ . However, even when  $y$  is generated in this way, there may be no extensions or many extensions. We first show that in a common setting there is at most one consistent extension. (We give examples of methods with multiple extensions in the following section.)

**Theorem 7** *Let  $x \in CS_2$  be generated by a single associative (resp. separable) path. Then there exists at most one  $\hat{x} \in CS$  such that  $x = \hat{x}$  on  $\mathcal{N}_2$  and is consistent (resp. dummy consistent).*

Proof: Let  $\hat{x}$  be one such extension. Then  $x$  is generated by a sum of separable paths  $\gamma^k$ ; however, if the sum cannot be written as a single path, then there must be some  $N \in \mathcal{N}_2$  for which the associated CSM is not constructed from a single path, violating the assumption. Thus, all extensions must be path methods, but clearly if there were two paths that differed, then they must differ on some  $N \in \mathcal{N}_2$ .  $\square$

This theorem is applicable to almost all of the previously discussed methods.

**Corollary 3** *The following methods defined on  $CS_2$  extend uniquely to  $DCON$  (resp.  $CON$ ): Random order methods, Aumann-Shapley, Serial cost, and their weighted versions.*

Note that the above theorem does not include Shapley-Shubik, since it is a sum of multiple path generated methods. As we show in the next section Shapley-Shubik has many (nonsymmetric) extensions; however, it has only one symmetric extension.

**Theorem 8** *The Shapley-Shubik CSM defined on  $CS_2$  has a unique consistent (resp. dummy consistent) symmetric extension.*

Proof: Fix  $N = \{1, 2, \dots, n\} \subset \mathbb{Z}^+$  such that  $n \geq 3$ . For simplicity assume that  $t \in [0, n]$ . For any  $i, j \in N$  with  $i \neq j$ , the Shapley-Shubik CSM on  $\{i, j\}$ ,  $CS(\{i, j\})$  can be written as  $\sum_{\sigma \in \Sigma} x^{\gamma^\sigma} / n!$ , where for  $n \in N$ ,  $\gamma_n^\sigma(t; q) = \phi^{\sigma(i)}(t) q_{\sigma(i)}$ . It is easy to see that when viewed as a CSM over  $N$ , these are the only paths (up to equivalence of reparameterization) which “project” for all  $i, j \in N$ ,  $i \neq j$ , to a two agent random order value. Thus any extension to  $N$  can be written as  $\sum_{\sigma \in \Sigma} \lambda(\sigma) x^{\gamma^\sigma}$  where  $\sum_{\sigma \in \Sigma} \lambda(\sigma) = 1$  and for all  $\sigma \in \Sigma$ ,  $\lambda(\sigma) \geq 0$ . However, unless all the  $\lambda$ ’s are equal the method is not be symmetric.  $\square$

## 4.1 Nonsymmetric extensions

Somewhat surprisingly, a nonsymmetric CSM may be symmetric when “projected” onto  $N_2$ . For example, as we now demonstrate, the Shapley-Shubik mechanism has (many) nonsymmetric extensions.

Consider the case when  $N = \{1, 2, 3\}$ . We construct a CSM which is a convex combination of random order values with the following orderings:  $\omega^1 = (1, 2, 3)$  and  $\omega^2 = (3, 2, 1)$ . Define  $x^i$  to be the random order value for order  $\sigma^i$ . Then consider the CSM defined by  $(x^1 + x^6)/2$ . It is straightforward to check that all two agent methods induced by this method coincide with the Shapley-Shubik CSM. Thus, this provides an example of a nonsymmetric extension of the Shapley-Shubik CSM. (Agents 1 and 3 are interchangeable, but agent 2 is not with either 1 or 3.) Note that there are other nonsymmetric extension which are combinations of

various random order values.<sup>12</sup>

In general there are even more complex examples of CSMs with non-unique consistent extensions. Note that generically, if  $x \in CS_2$  is generated by a finite number of paths then the extension (if it exists) will be unique, since the paths can not be ‘mixed’ together; however, in particular cases they can have multiple extensions as shown here. Consider the following asymmetric CSM in  $CS_2$  which is generated by the average of the following associative and scale invariant paths:

$$\gamma_a(t; q) = (tq_1, tq_2, t^2q_3, tq_4, tq_5, \dots),$$

$$\gamma_b(t; q) = (tq_1, t^{1.5}q_2, t^6q_3, tq_4, tq_5, \dots),$$

$$\gamma_c(t; q) = (tq_1, t^{0.5}q_2, t^3q_3, tq_4, tq_5, \dots),$$

where for simplicity we require that  $t \in [0, 1]$ . By this we mean that for  $N = \{1, 2\}$  the CSM is the average of the methods generated by the three paths,  $(tq_1, tq_2)$ ,  $(tq_1, t^{1.5}q_2)$ , and  $(tq_1, t^{0.5}q_2)$ .

Clearly, this element of  $CS_2$  has the “natural” extension  $[x_{\gamma_a} + x_{\gamma_b} + x_{\gamma_c}]/3$  (see Figure 6a). However, it also has another extension which is given by  $[x_{\gamma_d} + x_{\gamma_e} + x_{\gamma_f}]/3$ , where

$$\gamma_d(t; q) = (tq_1, tq_2, t^6q_3, tq_4, tq_5, \dots),$$

$$\gamma_e(t; q) = (tq_1, t^{1.5}q_2, t^3q_3, tq_4, tq_5, \dots),$$

$$\gamma_f(t; q) = (tq_1, t^{0.5}q_2, t^2q_3, tq_4, tq_5, \dots),$$

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<sup>12</sup>We only considered a world with 3 agents in this example to simplify notation. This can be easily “embedded” in the world with an infinite number of agents which we have been considering using the same technique as demonstrated in the next example.

which is shown in Figure 6b; however they both have the same projections onto 2 agent methods (see Figure 6c for the projection onto the first two coordinates).

Finally, any convex combination of these two methods is dummy consistent. Thus, there are infinitely many “nontrivial” extensions. In general, extensions of arbitrary CSMs can be quite complex.

## A Appendix: Proofs of theorems

### A.1 Theorem 1

**Theorem 1:** For any  $N \in \mathcal{N}$  and  $q \in Q(N)$ , the following are equivalent:

- i)  $x \in CS(q, N)$ .
- ii) There exists a probability measure  $\mu$  on  $\Gamma(q, N)$  such that

$$x = \int_{\gamma(\cdot; q) \in \Gamma(q, N)} x_{\gamma(\cdot; q)} d\mu(\gamma(\cdot; q)).$$

**Proof:** Fix  $N \in \mathcal{N}$  and  $q < \bar{q}$ . First we discuss some mathematical preliminaries. Let  $x, y \in CS(q, N)$  and let  $\rho^x, \rho^y$  be their representing vector measures, as in Proposition 1. Then define  $\hat{d}(x, y) = \sum_{n \in N} d(\rho_n^x, \rho_n^y)$ , where  $d(\cdot, \cdot)$  is the Prohorov metric [3]. Under this metric, convergence for  $CS(q, N)$  is given by  $\lim_{t \rightarrow \infty} x^t = x$  iff for all  $C \in \mathcal{C}(N)$ ,  $\lim_{t \rightarrow \infty} x^t(q; C) = x(q; C)$ . Note that this is the metric of weak convergence and thus  $CS(q, N)$  is a closed and compact metric space, as it is isomorphic to a product of closed convex subsets of the space of probability measures. (See [18] for the standard construction.)

Using this, we define the following metric on  $\Gamma(q, N)$ : if  $\gamma, \gamma' \in \Gamma(q, N)$  then  $\bar{d}(\gamma, \gamma') = \hat{d}(x^\gamma, x^{\gamma'})$ . In this metric,  $\Gamma(q, N)$  is a compact metric space. Note also that in these metrics

the mapping that maps  $\gamma$  to  $x^\gamma$  is injective and continuous as is the mapping which maps  $x$  to its related measure  $\rho$ .

Now we construct the representing measure in the theorem: Let  $CS(q, N; k)$  be the set of  $x \in CS(q, N)$  for which their representing measures  $\rho$  have support only on the grid  $g_k = \{p \mid \exists n \in N, \text{ s.t. } \forall m \neq n, 2^k p_m \in Z\}$  (this is a grid and not a lattice, as illustrated in Figure 7). Note that  $g_k \subset g_{k+1}$  and therefore  $CS(q, N; k) \subset CS(q, N; 2k)$  for all  $k \geq 1$ . Similarly define  $\mathcal{C}(q, N; k)$  which is the set of paths which have support on  $g_k$ , for which  $\mathcal{C}(q, N; k) \subset \mathcal{C}(q, N; 2k)$  for all  $k \geq 1$ . It is straightforward to show that  $CS(q, N; k)$  is the set of CSMs which only depend on the value of the cost function at the intersection points of the grid, i.e.,  $\hat{g}_k = \{p \mid \forall n \in N, 2^k p_n \in Z\}$ . This is the problem considered by Wang [23]. In that paper he proves the following using an elegant argument based on a result of Sprumont [21]:

**Proposition 2 (Wang (1998))** *The set ext  $CS(q, N; k)$  is equal to the set  $\{x^\gamma \mid \gamma \in \mathcal{C}(q, N; k)\}$ .*

Note that  $\mathcal{C}(q, N; k)$  is finite; thus for each  $x^k \in CS(q, N; k)$  there exists a vector probability measure  $\mu^k$  on  $\mathcal{C}(q, N; k)$  such that  $x^k = \int_{\gamma \in \Gamma(q, N; k)} x^\gamma d\mu^k(\gamma)$ . Now, by taking the “limit” as  $k \rightarrow \infty$  we get the desired result, after noting the following.

**Lemma 1** *Given an  $x \in CS(q, N)$  there exists a sequence  $x^k \in CS(q, N; k)$  such that  $\lim_{k \rightarrow \infty} x^k = x$ .*

Proof: Let  $\mathcal{C}^\dagger(q, N)$  be the subset of  $\mathcal{C}(q, N)$  which contains only functions for which all cross partial derivatives are continuous, i.e., the functions  $\partial_S C(q)$  is continuous for all  $S \subseteq N$ .



Note that  $\mathcal{C}^\dagger(q, N)$  is dense in  $\mathcal{C}(q, N)$  and also that any function in  $\mathcal{C}^\dagger(q, N)$  can be uniquely decomposed into a sum of functions  $C = \sum_{S \subseteq N} C^S$ , where  $C^S$  has support only on the interior of  $[0, q_S]$ .

Following [8] (Lemma 1) starting with the representation formula:

$$x_n(q; C) = \int_{[0, q]} \partial_n C(p) d\rho_n(q)(p)$$

we can integrate by parts multiple times to get

$$x_n(q; C) = \sum_{S \subseteq N} \int_{[0, q_S]} \partial_S^{|S|} C(p) \lambda_n^S(p) dp_S$$

where  $\partial_S C(p)$  is  $(\prod_{m \in S} \partial_m) C(p)$  and  $\lambda_n^S$  is an  $L^1$  density function on  $[0, q_S]$  satisfying  $\sum_{m \in S} \lambda_m^S(p) = 1$ . (Note that this is the continuous version of the discrete representation formula in [16].)

Now, given such a set of  $\lambda$ 's which represent the CSM  $x$ , construct a CSM on  $x^k$  as follows. Let  $H^S(p) \subset [0, q_S]$  be the hypercube in the grid  $g_k$  which contains  $p \in [0, q_S]$ , i.e., let  $p_-$  be the largest  $p$  in  $\hat{g}_k$  which is not larger (in the vector ordering) than  $p$  and  $p_+$  to be the smallest element of  $\hat{g}_k$  which is not smaller than  $p$ . Now define  $\lambda_m^S(p; k) = \int_{[p_-, p_+]} \lambda_m^S(p)$  and note that by construction the CSM induced by these  $\lambda_m^S(p; k)$ 's only depends on the values of  $C(p)$  on  $\hat{g}_k$  and thus defines an element of  $CS(q, N; k)$ . Also, note that since it only depends on the values of  $C(p)$  on  $\hat{g}_k$ , it is well defined on all of  $\mathcal{C}(q, N)$  and not just on  $\mathcal{C}^\infty(q, N)$ . Also, on all  $C \in \mathcal{C}^\dagger(q, N)$   $\lim_{k \rightarrow \infty} x^k(q; C) = x(q; C)$ . Since  $\mathcal{C}^\dagger$  is dense in  $\mathcal{C}$  this implies that  $\lim_{k \rightarrow \infty} d(x^k, x) = 0$ .  $\diamond$

**Proof of theorem:** Consider some  $x \in CS(q, N)$  and choose any sequence  $x^k \in CS(q, N; k)$  such that  $\lim_{k \rightarrow \infty} x^k = x$ . Now, using the above proposition, there exists a sequence of vector

probability measures  $\mu^k$  which represent the CSMs  $x^k$  and note that  $\mu^k$  induces a measure on  $(q, N)$  which we will also (with a slight abuse of notation) denote by  $\mu^k$ . Since  $(q, N)$  is a compact metric space, the space of probability measures on  $(q, N)$  must be compact [18], and therefore there exists a subsequence on which  $\mu^k$  converges to a measure on  $(q, N)$ . This measure is the desired representation of  $x$  as an integral over  $(q, N)$ , proving the theorem.  $\square$

## B Theorem 3

*The following are equivalent:*

- i)  $x \in DCON$ .
- ii) For all  $N \in \mathcal{N}$  there exists a probability measure  $\mu$  on  $_s(N)$  such that for any  $m \in N$ ,

$$x^N = \int_{\gamma \in \Gamma_s(N)} x^\gamma d\mu(\gamma)$$

and

$$x^{N \setminus m} = \int_{\gamma \in \Gamma_s(N)} \pi_m^N[x^\gamma] d\mu(\gamma).$$

**Proof:** It is easy to see that a CSM generated by a seperable path is dummy consistent and thus a convex combination of such methods is also dummy consistent; thus we concentrate on the converse, that (i) implies (ii).

Consider some dummy consistent CSM,  $x \in DCON$ . For any  $N \in \mathcal{N}$ , using Theorem 1, we can write the restriction of  $x$  to  $N$ ,  $x^N$ , as an integral over paths. Now for some  $m \in N$  consider the CSM  $x'$  on  $N \setminus m$ . Given  $C \in \mathcal{C}(N)$  for which agent  $m$  is a dummy we can

compute  $x'$  for  $n \in N \setminus m$  as

$$x^N = \int_{\{\gamma^\omega \mid \omega \in \Omega\}} x^\gamma d\mu(\gamma)$$

using dummy consistency, this is equal to

$$x^{N \setminus m} = \int_{\{\gamma^\omega \mid \omega \in \Omega\}} \pi_m^N[x^\gamma] d\mu(\gamma).$$

which must yield the same answer for any  $q_m$  between 0 and  $\bar{q}$  and in particular when  $q_m = 0$ .

Thus after reparameterizing (each path separately) in terms of  $t$ , the representation for  $N \setminus m$ , can be given by  $(\gamma_{-m}^\omega(t; q_{-m}), \gamma_n^\omega(t; q_m))$  for  $\omega \in \Omega$ . Repeating this process for all  $m \in N$  yields the desired set of paths  $\gamma_m^\omega(t; q_m)$ .  $\square$

## C Theorem 4

*Assume that for all  $N \in \mathcal{N}$  there exists a probability measure  $\mu$  on  $\Gamma_S(N)$  such that for any  $m \in N$ ,*

$$x^N = \int_{\gamma \in \Gamma_S(N)} x^\gamma d\mu(\gamma)$$

*and*

$$x^{N \setminus m} = \int_{\gamma \in \Gamma_S(N)} \pi_m^N[x^\gamma] d\mu(\gamma),$$

*then  $x \in CON$ .*

**Lemma 2** *If  $\gamma$  is associative then  $\gamma_n(t, q_n)$  is nondecreasing and right differentiable in  $q_n$ .*

*In addition*

$$\partial_s^+ \phi(t, s) = \partial_n^+ \gamma_n(t; \gamma_n(s; q_n)) \dot{\gamma}_n(s; q_n) / \dot{\gamma}_n(t; \gamma_n(s; q_n)),$$

*where  $\partial^+$  denotes right differentiation and  $\dot{\gamma}_n(t; q_n) = \partial_t^+ \gamma_n(t; q_n)$ ,*

Proof: Given  $q_n$  choose any  $q'_n > q_n$  and let  $s$  solve  $\gamma(s, q'_n) = q_n$  and  $s$  solve  $\gamma_n(s, q'_n) = q'_n$  and note that  $s' > s$ . Then by associativity  $\gamma_n(t; q_n) = \gamma_n(t, \gamma_n(s; q'_n)) = \gamma_n(\phi(t, s); q'_n)$  and similarly  $\gamma_n(t, q'_n) = \gamma_n(\phi(t, s'), q'_n)$ , but since  $\phi(t, s') \geq \phi(t, s)$  which implies that  $\gamma_n(t; q'_n) \geq \gamma_n(t; q_n)$  since  $\gamma_n(\cdot, \cdot)$  is nondecreasing in its first argument by definition.

Differentiating the identity  $\gamma_n(t; \gamma_n(s; q_n)) = \gamma_n(\phi(t, s); q_n)$  with respect to  $s$  yields

$$\partial_n \gamma_n(t; \gamma_n(s; q_n)) \dot{\gamma}_n(s; q_n) = \dot{\gamma}_n(t; \gamma_j(s; q_j)) \partial_s \phi(t, s),$$

thus since all the other right derivatives exist, the right derivative of  $\gamma_n(t, q_n)$  with respect to  $q_n$  must exist and solve the equation.  $\diamond$

**Lemma 3** *If  $\gamma$  is an associative path, then  $x^\gamma$  is strongly consistent.*

Proof: Note that since all the functions involved ( $\gamma$  and  $\phi$ ) are nondecreasing and right differentiable, all the functional compositions which we use in the following calculations are also nondecreasing and right differentiable. Thus, in the following we use right differentiation, denoted by  $\partial^+$ . Also, to simplify notation assume that  $\gamma(1, q) = q$ .

Assume that  $C$  is twice continuously differentiable and that  $n \neq m \in N$ . Now

$$x_n(q; C) = \int_0^1 dt \partial_n^+ C(\gamma(t; q)) \dot{\gamma}_n(t; q_n),$$

where we are using the separability of  $\gamma$ . Differentiating yields,

$$\partial_m x_n(q; C) = \int_0^1 dt \partial_{nm} C(\gamma(t; q)) \dot{\gamma}_n(t; q_n) \partial_m^+ \gamma_m(t; q_m). \quad (*)$$

Strong consistency implies that  $x_m(q; \hat{C}) = x_m(q_{-n}; \hat{C}) - x_m(q_{-n}; C')$ , where  $C'(p_{-n}) = x_n(p_{-n}, q_n; C)$ , where  $\hat{C}(p_{-i}) = C(p_{-i}, p_i)$ , and we ignore the fact that  $C'$  is not nondecreasing, as it does not affect the calculations. Now we compute the second term. Using  $(*)$  yields

$$x_m(q_{-m}; C') =$$

$$\int_0^1 ds \int_0^1 dt \partial_{nm} C(\gamma_{-n}(t; \gamma_{-n}(s; q_{-n})), \gamma_n(t; q_n)) \dot{\gamma}_n(t; q_n) \dot{\gamma}_m(t; q_m) \partial_m^+ \gamma_m(t; \gamma_m(s; q_m))$$

and since  $\gamma$  is associative

$$x_m(q_{-m}; C') = \int_0^1 ds \int_0^1 dt \partial_{nm} C(\gamma_{-n}(\phi(t, s); q_{-n})), \gamma_n(t; q_n)) \dot{\gamma}_n(t; q_n) \dot{\gamma}_m(t; q_m) \partial_m^+ \gamma_m(t; \gamma_m(s; q_m))$$

Changing variables to  $u = \phi(t, s)$  and  $v = t$ , noticing that the Jacobian of the transformation is  $\partial_s^+ \phi(t, s)$ , and using the identities,  $\phi(t, 0) = 0$  and  $\phi(t, 1) = t$  results in

$$x_m(q_{-n}; C') = \frac{\int_0^1 du \int_u^1 dv \partial_{nm} C(\gamma_{-n}(u; q_{-n}), \gamma_n(v; q_n)) \dot{\gamma}_n(v; q_n) \dot{\gamma}_m(s; q_m) \partial_m^+ \gamma_m(u; q_m)}{\partial_s^+ \phi(t, s)} (**).$$

Combining the formula from the preceding lemma (with  $m$  replacing  $n$  and  $v$  replacing  $t$ ) with  $(**)$  yields

$$x_m(q_{-n}; C') = \int_0^1 du \int_u^1 dv \partial_{nm} C(\gamma_{-n}(u; q_{-n}), \gamma_n(v; q_n)) \dot{\gamma}_n(v; q_n) \dot{\gamma}_m(u; q_m).$$

Integrating with respect to  $v$  yields

$$x_m(q_{-n}; C') = \int_0^1 du \partial_m [C(\gamma_{-n}(u; q_{-n}), q_n) - C(\gamma(u; q))] \dot{\gamma}_m(u; q_m),$$

which equals  $x_m(q; C) - x_m(q_{-n}; \hat{C}_{q_n})$ , proving strong consistency for twice differentiable  $C$ .

Since the set of twice differentiable functions is dense in  $\mathcal{C}(N)$  this suffices to prove the theorem.  $\square$

Noting that a convex combination of strongly consistent CSMs is strongly consistent, completes the proof of the Theorem.

## D Theorem 5

For any  $\bar{q} < \infty$  the following are equivalent:

i)  $x \in DM \cap DCON$ .

ii) For all  $N \in \mathcal{N}$  there exists a probability measure  $\mu$  on  $,_{DM,DCON}(N)$  such that for any  $m \in N$ ,

$$x^N = \int_{\gamma \in \Gamma_{DM,DCON}(N)} x^\gamma d\mu(\gamma)$$

and

$$x^{N \setminus m} = \int_{\gamma \in \Gamma_{DM,DCON}(N)} \pi_m^N[x^\gamma] d\mu(\gamma).$$

**Proof:** Consider any  $x \in DM \cap DCON$ . For any  $N \in \mathcal{N}$  let  $\mu(q)$  be the representing measure (over  $,_{DM,DCON}(N)$ ) via Theorem 1 of  $x(q; \cdot) \in CS(N)$ , for  $\bar{q} < \infty$ . Let  $\rho(q)$  be the representing vector measure (over  $[0, q]^N$ ) via Proposition 1. The following property of  $\rho$  is given in [8].

**Proposition 3 (Friedman and Moulin (1998))** *If  $x(q; \cdot) \in DM(q, N)$  and  $\rho(q)$  is the representing vector measure, then for all  $n \in N$  and  $q'_n > q_n$  then  $\rho_n(q) = \rho_n(q_{-n}, q'_n)$  on  $[0, q]$ .*

Now, consider some  $\gamma \in ,_{DM,DCON}$ . By dummy consistency we know that it can be written, for each  $n \in N$  as  $\gamma_n(t; q_n)$ ; however, by the above proposition we must have that  $\gamma_n(t; q_n) = \gamma_n(t, q'_n)$  for any  $q_n < q'_n < \bar{q}$  whenever  $\gamma_n(t, q'_n) \leq q_n$ . Thus, we can write  $\gamma_n(t, q_n) = \min[q_n, \gamma_n(t; \bar{q}_n)]$ .  $\square$

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