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Paths in Additive Cost Sharing

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Abstract

In this paper we develop a unified framework for the study of additive cost sharing methods. We show that any additive cost sharing method satisfying the dummy axiom can be generated by a (possibly infinite) convex combination of path generated methods. We also show that the set of scale invariant cost sharing methods can be generated by the set of scale invariant paths and the set of demand monotonic methods by the set of demand monotonic paths, both of which we construct.

We first apply these results to the study a strict version of marginality, and show that none of the standard methods satisfy this requirement. We construct two new methods, which are generated by infinite sums of paths, and show that these satisfy strict marginality.

We then note that the minimum of any concave functional over the set of cost sharing methods, either general, scale invariant, or demand monotonic, must be path generated, and therefore can be computed using techniques from the theory of optimal control. This allows us to provide a new characterization of the Random Order methods as the methods which minimize a lexicographic function of agents' payments according for supermodular cost functions. It may also lead to new characterizations of other interesting methods.

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1 Introduction

In this paper we provide a simple characterization of all additive cost sharing methods (CSMs) satisfying the dummy axiom.¹ Our characterization is in terms of path generated CSMs – CSMs which are constructed by integrating along a single path. Such path generated methods are quite common. Many of the standard CSMs – Aumann-Shapley (Billera and Heath 1982, Mirman and Tauman 1982), extremal Random Order² (Weber 1988), and Serial Cost (Moulin and Shenker 1992, Friedman and Moulin, 1995) – are path generated. Our main result is that *any* additive CSM can be expressed as a (possibly infinite) sum of path generated methods, since the path generated methods comprise the set of extreme points in the (convex) set of such CSMs. We construct similar characterizations for the CSMs satisfying either scale invariance or demand monotonicity. In these cases, the extreme CSMs are generated from sets of paths which we construct.

The view of CSMs as sums of path generated CSMs allows one to simplify the analysis of general CSMs and to more easily analyze various questions about CSMs in general.³ This is because the extremal CSMs are much easier to analyze analytically than general CSMs.⁴

¹For motivation and detailed references to such CSMs we refer the reader to Friedman and Moulin (1995).

²The Shapley-Shubik method (Shubik, 1962) is the average of all extremal Random Order methods.

³Friedman and Moulin (1997) use the path generated CSMs to simplify the analyses of extensions from homogeneous CSMs to general CSMs, and Friedman (1997) has applied these ideas to understand consistency relationships among CSMs.

⁴See, e.g. Athey (1995), who has recently applied this idea to stochastic decision problems.

Our first application of this theory is to the study of a strict version of marginality. We show that none of the standard methods satisfies this requirement; in fact no method which is constructed as a finite sum of paths can satisfy it. We then construct two new methods which satisfy strict marginality.

Our second application concerns the minimization of functionals over the set of CSMs. Since the path generated methods are the set of extreme points of a convex set, every concave functional on the set of CSMs is minimized by a path generated method. (For example, the CSM most preferred by a particular agent can be represented as the minimum of a linear functional.) This path can then be computed using standard techniques from the theory of optimal control.

This raises an interesting question: what concave (or linear) functionals do the standard CSMs minimize? We provide one such characterization for the extremal Random Order methods, as the minimization of a lexicographic function of agents' payments for supermodular cost functions. Such questions can also be asked about the Aumann-Shapley method and the Serial Cost method, and (perhaps) lead to interesting characterizations of these.

The paper is organized as follows. Section 2 presents the basic cost sharing problem and introduces the path generated methods. Section 3 the main representation theorem, and Section 4 some specializations of it. Section 5 concludes with applications and open questions. Several more technical proofs are in the appendix.

2 Additive Cost Sharing

Let $C(q)$ be the cost of supplying commodity $q \in \mathfrak{R}_+^n$ (the set of nonnegative vectors), where $C \in \Xi(n)$, the space of nondecreasing once continuously differentiable functions from $\mathfrak{R}_+^n \rightarrow \mathfrak{R}_+$. A cost sharing mechanism (CSM) is a (nonnegative) mapping $x : \mathfrak{R}_+^n \times \Xi(n) \rightarrow \mathfrak{R}_+^n$ satisfying “budget balance,” $\sum_{i=1}^n x_i(q; C) = C(q)$, where $x_i(q; C)$ is the costs allocated to agent i . We say that agent i is a dummy agent if $C(q_i, q_{-i})$ is independent of q_i . The following two standard assumptions are crucial to our results and will be assumed throughout.⁵

Assumption 1 (Dummy) *If agent i is a dummy for $C \in \Xi(n)$ then $x_i(q, C) = 0$.*

Assumption 2 (Additivity) *For all $C, C' \in \Xi(n)$, $x_i(q, C + C') = x_i(q, C) + x_i(q, C')$.*

Let $CS(q, n)$ be the set of CSMs satisfying dummy and additivity for $n > 0, q \in \mathfrak{R}_+^n$. The following definitions and notations will be used: $CS(n) = \bigcap_{q \in \mathfrak{R}_+^n} CS(q, n)$, $CS(q) = \bigcap_{0 < n < \infty} CS(q, n)$, and $CS = \bigcap_{n \in \mathbb{Z}_+} CS(n)$. Also we will denote the (multi-dimensional) intervals by $[0, q]$ which will be the set $\{p \mid \forall 1 \leq i \leq n \quad 0 \leq p_i \leq q_i\}$ where the dimension will be implicitly understood to be the dimension of q .

2.1 Path Generated Methods

A path is a natural way to generate a CSM. A mapping $\gamma : [0, 1] \times \mathfrak{R}_+ \rightarrow [0, q]$ is an admissible family of paths if for each q : i) $\gamma(0; q) = 0$, ii) $\gamma(1; q) = q$, and iii) $\gamma(t; q)$

⁵Dummy is a statement of basic equity, costless goods should not be charged, while additivity is necessary for decentralizability of the CSM.

is nondecreasing in t . Let Γ be the set of all admissible families of paths satisfying these properties and $\Gamma(q)$ be the specialization of Γ to a specific q .

Given a path $\gamma \in \Gamma(q)$, define the CSM $x_\gamma^i(q; C) \in CS(q, n)$ as the Riemann-Stieltjes integral

$$x_\gamma^i(q; C) = \int_0^1 \partial_i C(\gamma(t; q)) d\gamma_i(t; q),$$

where n is the dimension of q . Friedman and Moulin (1995) have proven that path generated methods are CSMs.

Three important examples are:⁶

- 1) Aumann-Shapley: $\gamma_i(t; q) = q_i t$.
- 2) Serial Cost: $\gamma_i(t; q) = \min[q_i, q_{\max} t]$, where $q_{\max} = \max_i q_i$.
- 3) Extremal Random Order method with order $\sigma \in \Sigma(n)$, where $\Sigma(n)$ is the set of all permutations of $\{1, 2, \dots, n\}$: $\gamma_j(t; q) = 0$ if $t \leq \sigma(j-1)/n$, $\gamma_j^j(t; q) = n(t - (\sigma(j) - 1)/n)q_j$ for $t \in ((\sigma(j) - 1)/n, \sigma(j)/n)$, and $\gamma_j^j(t; q) = q_j$ otherwise.

Recall that the Shapley-Shubik method is obtained by averaging the extremal Random Order methods over all orders, and a Random Order method is any convex combination of extremal Random Order methods. More generally, given a sequence of weights $\{w_i\}$ with $w_i > 0$, one can represent weighted versions of Aumann-Shapley method (McLean and Sharkey 1994) by $\gamma_i(t; q) = q_i t^{w_i}$, and the Serial Cost method by $\gamma_i(t; q) = \min[q_i, q_{\max} t^{w_i}]$.

⁶These are discussed in detail in Friedman and Moulin (1995).

Let $CP(q, n) = \{x^\gamma \mid \gamma \in \Gamma(q)\}$, the set of path generated CSMs.

3 A Representation for CS(q,n)

In this section we will describe and prove our key result, which says that all CSMs can be constructed as sums of path generated methods.

Friedman and Moulin (1995) have shown that any cost sharing method in $CS(q, n)$ can be (uniquely) represented as

$$x_i(q; C) = \int_{[0, q]} \partial_i C(p) d\mu_i(p; q),$$

where $\mu_i(\cdot; q)$ is a (nonnegative) measure such that its projection (marginal) $\hat{\mu}_i$ defined on $S \subseteq [0, q_i]$ by $\hat{\mu}_i(S; q) = \mu^i([0, q_{-i}] \times S; q)$ is the Lebesgue measure.⁷

Define convergence in $CS(q, n)$ by $\lim_{j \rightarrow \infty} x^j(q; \cdot) = x(q; \cdot)$ if and only if $\lim_{j \rightarrow \infty} x_i^j(q; C) = x_i(q; C)$ for all $C \in \Xi(n)$. This induces the topology of weak convergence on the set of measures which generate $CS(q, n)$. In particular, this space is metrizable.⁸ The following lemma follows immediately.

Lemma 1 *For all n, q , $CS(q, n)$ is a closed compact convex subset of a metrizable space.*

Proof: Closure follows immediately, since any convergent sequence of CSMs satisfying nonnegativity, budget balance, dummy and additivity clearly converges to a CSM.

⁷Friedman and Moulin provide a complete characterization which imposes constraints on a large number of higher dimensional projections.

⁸Given any countable basis C^k of $\Xi(n)$ define $d(x_i, y_i) = \sum_{k=1}^i nfty(|x_i(q; C^k) - y_i(q; C^k)|/2^k)$ and $d(x, y) = \sup_i d(x_i, y_i)$. See, e.g. Phelps (1980) or Billingsley (1968) for details.

Metrizability and compactness follow since there is a natural mapping from CSMs to a bounded and compact set of measures which is metrizable. \diamond

Now we examine the extreme points of $CS(q, n)$.

Definition 1 *A point $c \in A$ (where A is convex) is an extreme point of A if whenever $a, b \in A$, and $c = a/2 + b/2$, then $a = b$. Let $ext(A)$ be the set of extreme points of A .*

Our first result is for the case where there are two agents.

Lemma 2 $CP(q, 2) = ext(CS(q, 2))$.

Proof: In Appendix.

Now we extend this to an arbitrary number of agents. Define the projection $\hat{\pi}^{ij}$ for any $i, j \in N$ and $i \neq j$ on $S \subseteq [0, (q_i, q_j)]$ by $\hat{\pi}^{ij}(\mu_i)(S; q) = \mu_i(S \times [0, q_{ij}]; q)$. It is easy to see that $\hat{\pi}^{ij}$ is a linear operator.

Since any CSM has a unique measure μ associated with it we define the projection $\pi^{ij} : CS(q, n) \rightarrow CS((q_i, q_j), 2)$ by allowing $\hat{\pi}$ to act on the underlying measures. Note that π inherits the linearity of $\hat{\pi}$ and that π is onto $CS(q_{ij}, 2)$.

Lemma 3 *If $x \in ext(CS(q, n))$ then $\pi(x) \in ext(CS(q_{ij}, 2))$.*

Proof: Assume that $\pi(x) = a/2 + b/2$ where $a, b \in CS((q_{ij}, 2)$ and $a \neq b$. Choose any a', b' such that $\pi(a') = a$ and $\pi(b') = b$. Then by linearity $a'/2 + b'/2 \notin ext(CS(q, n))$. \diamond

Now we present our main result.

Theorem 1 $ext(CS(q, n)) = CP(q, n)$.

Proof: Let $x \in \text{ext}(CS(q, n))$. Then for all $i \neq j$ $\pi^{ij}(x)$ is extreme by the previous lemma and therefore generated by some path. Now it is straightforward to see that if a measure projects to paths for all i, j then it must itself be a path, as its support must be a path. (See Friedman and Moulin (1995) Lemma 6 for the formal argument.) \square

Theorem 2 *The following statements are equivalent:*

i) $x \in CS(q, n)$.

ii) *There exists a nonnegative measure ν on $CP(q, n)$ such that*

$$x = \int_{\gamma \in \Gamma(q)} x^\gamma d\nu(\gamma).$$

Proof: This is a direct application of Choquet's theorem. See, e.g. Phelps (1980). \square

Since the space $CP(q, n)$ is metrizable, the above integration is well behaved, and the following corollary is immediate.

Corollary 1 *For any $x \in CS(q, n)$ there exists a sequence of CSMs, each generated by a finite sum of path generated methods, which converge to x .*

4 CS(n), Scale Invariance, and Demand Monotonicity

When there are no restrictions on the behavior of a CSM with respect to q , the space of all CSMs, $CS(n)$, is quite easy to analyze. Define $\gamma(t; q)$ to be an admissible path from 0 to q for any q and let $\Gamma(n)$ be the set of all such paths in dimension n . Let $CP(n) = \{x^\gamma \mid \gamma \in \Gamma(n)\}$.

Corollary 2 *Let $x \in CS(n)$ then there exists a family of nonnegative measures ν_q for each q on $CP(q, n)$ such that*

$$x(q; C) = \int_{\gamma \in \Gamma(q)} x^\gamma d\nu_q(\gamma).$$

While a general extreme point is quite complex – a different path for every q , the extreme points under two important axioms are much simpler, as shown below.

One of the most common restrictions imposed on a CSM is that of scale invariance. Given $\lambda \in \mathfrak{R}_+^n$ define $\tau_\lambda(q)$ by $\tau_\lambda(q)_i = \lambda_i q_i$ and define $\tau_\lambda(C)$ by $\tau_\lambda(C)(q) = C(\tau_\lambda(q))$.

Definition 2 (Scale Invariance) *For all $\lambda \in \mathfrak{R}_+^n$ and all C a CSM is scale invariant if $x(\tau_\lambda(q); C) = x(q; \tau_\lambda(C))$.*

Let $SI(n)$ be the set of scale invariant CSMs in $CS(n)$, and note that it is a convex set. It is easy to characterize this set of scale invariant CSMs. Define $\Gamma_{SI}(n)$ to be the set of paths such that $\gamma(t; q) = \tau_q(\gamma(t; e))$, thus each component is $\gamma_i(t; q) = \gamma_i(t, e)q_i$, where e represents unit vector $(1, 1, \dots, 1)$.

Theorem 3 $ext(SI(n)) = \{x_\gamma \mid \gamma \in \Gamma_{SI}(n)\}$.

Proof: This follows upon noting that a scale invariant CSM is completely determined by its behavior on any particular q , such as $q = e$. Thus $SI(N)$ is isomorphic to $CS(e, N)$. \square

Another important axiom is demand monotonicity.

Definition 3 (Demand monotonicity) *For all q, q' such that $q_i \leq q'_i$ and $q_{-i} = q'_{-i}$ and all C , a CSM is demand monotonic if $x_i(q; C) \leq x_i(q'; C)$.*

There are some technical difficulties involved with characterizing demand monotonicity when the domain of q 's is unbounded, since the limiting behavior of the class of demand monotonic paths is not well behaved. Thus we will consider the class of CSMs which are defined only on a bounded domain: let $CS^{\hat{q}}(n)$ be the set of CSMs defined for $q \leq \hat{q}$.

Let $DM^{\hat{q}}(n)$ be the set of demand monotonic CSMs in $CS^{\hat{q}}(n)$. In this case the extreme CSMs are constructed from a single infinitely long path. Let $\omega(s)$ be a path defined on $t \in [0, 1)$ which is nondecreasing and $\omega(1) \geq \hat{q}$. Let $\Omega^{\hat{q}}(n)$ be the set of all such paths. Given $\omega \in \Omega^{\hat{q}}(n)$ define γ_ω by $\gamma_\omega(t)_i = \min[\omega(t), q_i]$.

Theorem 4 $ext(DM^{\hat{q}}(n)) = \{x^{\gamma_\omega} \mid \omega \in \Omega^{\hat{q}}(n)\}$.

Proof: In Appendix.

Using these characterizations, it is quite easy to show that any scale invariant or demand monotonic CSM is continuous in q , since one only needs to check continuity at the extreme points.

Corollary 3 *If x is scale invariant or demand monotonic then $x_i(q; C)$ is continuous in q for fixed C .*

Proof: We only need to show that for x which satisfies SI or DM, that the CSM generated by a single path is continuous. This is true because paths in $\Gamma_{SI}(n)$ and $\Gamma_{DM}(n)$ are continuous in q . \square

5 Applications

5.1 Strict Marginality

A CSM is said to satisfy marginality (Young 1985b) if agent i 's cost share depends (monotonically) only on her marginal costs, $\partial_i C(\cdot)$. For ordinary TU games, Young (1985a) has shown that marginality essentially characterizes the Shapley value. For general CSMs satisfying Additivity and Dummy no specific CSM is singled out⁹ since they all satisfy marginality, as was noted in Friedman and Moulin 1995, and is obvious from Theorem 1.

Definition 4 *A CSM is said to satisfy marginality if for all demands q and all cost functions $C, D \in \Xi(n)$ which satisfy $\partial_i C(p) \geq \partial_i D(p)$ for all $p \in [0, q]$ then $x_i(q; C) \geq x_i(q; D)$.*

Theorem 5 *A CSM satisfying Additivity and Dummy also satisfies marginality.*

Proof: This follows immediately from Theorem 1, since the cost share is computed by integrating the marginal costs with a nonnegative measure, $x_i(q; C) = \int_{p \in [0, q]} \partial_i C(p) d\mu_i(q)(p) \geq \int_{p \in [0, q]} \partial_i D(p) d\mu_i(q)(p) = x_i(q; D)$. \square

We now consider a stronger version of marginality, which requires that the cost share of a player strictly increase when her marginal costs increase.

Definition 5 *A CSM is said to satisfy strict marginality if for all demands q and all cost functions $C, D \in \Xi(n)$ which satisfy $\partial_i C(\cdot) \geq \partial_i D(\cdot)$ for all $p \in [0, q]$ and for*

⁹Young (1985b) uses a stronger requirement, symmetric marginality, to characterize the Aumann-Shapley method.

some $p \in [0, q]$ $\partial_i C(p) > \partial_i D(p)$ then $x_i(q; C) > x_i(q; D)$.

Somewhat surprisingly, none of the standard CSMs satisfy strict marginality.

Theorem 6 *None of the following methods satisfy strict marginality:*

Aumann-Shapley, Random Order, Serial Cost, and weighted versions of these.

Proof: This follows immediately from the observation that each of these methods is generated by a finite set of paths and the following lemma. \square

Lemma 4 *If $x \in CS(n)$ can be written as a finite sum $x = \sum_{a \in A} x_{\gamma_a}$ where $|A| < \infty$ and $x_a \in \Gamma(n)$, then x does not satisfy strict marginality.*

Proof: Given x as in the lemma, consider the measure which represents x in Theorem 1, $\mu_i(q)$. It is easy to see that the support of $\mu_i(q)$ is contained in the set p 's such that $p = \gamma(t; q)$ for some t . However, this set can not be dense in $[0, q]$ as it is constructed from a finite number of monotonic lines. Now consider $C, D \in \Xi(n)$ where $\partial_i C(p) = \partial_i D(p)$ except in some small neighborhood B which does not intersect the support of $\mu_i(q)$, and $\partial_i C(p) \geq \partial_i D(p)$ on B is strictly larger at some point of B . By Theorem 1 $x_i(q; C) = x_i(q; D)$ violating strict marginality. \diamond

Thus for a method to satisfy strict marginality, it must be generated by an infinite sum of paths which ‘cover’ the entire interval $[0, q]$. This is because any CSM satisfying strict marginality must use *all* the marginal cost information. We now present two interesting methods which satisfy strict marginality.

EXAMPLE: The “smoothed” weighted Aumann-Shapley methods.

Given $w \in \mathfrak{R}_{++}^n$, let x_w be the weighted Aumann-Shapley method corresponding to w . Define $x_{swas}(q; C) = \int_{w \in (0,1)} x_w(q; c) \rho(w) dw$ where $\rho(\cdot)$ is a nonnegative probability density on $w \in [0, e]$. Thus, x^{sw} is just an average over various weighted Aumann-Shapley methods.

EXAMPLE: The “almost flat” method.

Since the key behind strict marginality is the support of the measures $\mu_i(q)$, the simplest CSM satisfying strict marginality would be when $\mu_i(q)$ is “flat” on $[0, q]$, by which we mean that it is proportional to the Lebesgue measure. However, it is not possible to construct a CSM out of paths which is flat for all i , and thus such a CSM does not exist. Instead, we construct a CSM which for all i is flat on the interior of $[0, q]$.

Given an ordering $\sigma \in \Sigma(n)$ and an $r \in [0, q]$ define $\gamma_{\sigma,r}(\cdot; q)$ to be the path which is made up of $2n - 1$ straight lines: first from 0 to $(0_{-\sigma(1)}, r_{\sigma(1)})$ then to $(0_{-(\sigma(1), \sigma(2))}, r_{(\sigma(1), \sigma(2))})$ and continuing this process until it reaches r and then in reverse order to $(q_{\sigma(n)}, r_{-\sigma(n)})$ to $(q_{\sigma(n), \sigma(n-1)}, r_{-(\sigma(n), \sigma(n-1))})$ until it finally reaches q . Let

$$x^{af} = \frac{1}{n!} \sum_{\sigma \in \Sigma(n)} \int_{r \in [0, q]} x^{\gamma_{\sigma,r}} dr,$$

where the integration over $dq_{\sigma(n)}$ is formally unnecessary since $\gamma_{\sigma,r}(\cdot; q)$ does not depend on $r_{\sigma(n)}$.

We can evaluate this CSM as follows:

$$x_i^{af} = \frac{1}{n!} \sum_{\sigma \in \Sigma(n)} \int_{(r_i, r_{\sigma^{-1}(i)}) \in [0, (q_i, q_{\sigma^{-1}(i)})]} [C(r_i, r_{\sigma^{-1}(i)}, \mathbf{0}_{\sigma^+(i)}) - C(\mathbf{0}_i, r_{\sigma^{-1}(i)}, \mathbf{0}_{\sigma^+(i)}) \\ + C(q_i, r_{\sigma^+(n)}, q_{\sigma^+(n)}) - C(r_i, r_{\sigma^+(n)}, q_{\sigma^+(n)})],$$

where $\sigma^-(i) = \{j \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ and $\sigma^+(i) = \{j \mid \sigma^{-1}(j) > \sigma^{-1}(i)\}$. This formula can be further simplified into one with only 2^{n-1} terms (instead of $n!$) by considering partitions of $\{1, 2, \dots, n\}$. In particular, when $n = 2$,

$$x_1^{af} = \int_0^{q_2} dp_2 [C(q_1, p_2) - C(0, p_2)] + \int_0^{q_1} dp_1 [C(q_1, q_2) - C(p_1, q_2) + C(p_1, 0)].$$

Theorem 7 *Both x^{sw} and x^{af} satisfy strict marginality.*

Proof: In both cases the representing measure $\mu_i(q)$ has full support on $[0, q]$: This is obvious for the almost flat method by construction since $\gamma_{\sigma, r}$ passes through r , and easy to see for the “smoothed” AS method after noting that for any $p \in [0, q]$ it is possible to choose t, w such that for all i , $p_i = q_i t^{w_i}$. Thus if $C, D \in \Xi(n)$ satisfy $\partial_i C(p) \geq \partial_i D(p)$ for all $p \in [0, q]$ and for some $p \in [0, q]$ $\partial_i C(\cdot) > \partial_i D(\cdot)$ for some $p \in [0, q]$ we can compute $x(q; C) = \int_{p \in [0, q]} \partial_i C(p) d\mu_i(q)(p) > \int_{p \in [0, q]} \partial_i D(p) d\mu_i(q)(p) = x_i(q; D)$ where the inequality is strict because $\partial_i C(\cdot) > \partial_i D(\cdot)$ on a set of positive measure by the continuity of $D(\cdot)$. \square

5.2 Concave Minimization and a Characterization of Extremal Random Order Methods

One important result from the theory of convex sets, is that the minimum of a concave (or linear) functional on a convex set, always has extreme point solutions. Thus, any

concave minimization over CSMs yields a path generated method.

Lemma 5 *Let B be either $CS(n, q)$, $SI(n)$, or $DM^{\hat{q}}(n)$, for some \hat{q} . If $f : B \rightarrow \mathfrak{R}_+$ is concave and upper-semicontinuous then*

$$\min\{f(x) \mid x \in B\}$$

exists and the set of $x \in B$ which attains this minimum contains some $x \in \text{ext}(B)$.

Proof: This follows from Aubin Theorem ? (p.??) (1979) since all three spaces are nonempty compact convex subspaces of a locally convex Hausdorff space. \diamond

EXAMPLE: Consider the case where we want to choose a CSM that minimizes agent 1's expected payments over a set of possible cost functions Ξ , for q fixed. Thus $f(x) = -E[x_i(q; C)]$, and by linearity, $f(x) = -x_i(q; \hat{C})$ where $\hat{C} = E[C]$. By The above corollary this problem is equivalent to

$$\min_{\gamma} \int_0^1 \partial_1 C(\gamma(t)) d\gamma_1(t)$$

subject to $\gamma(t)$ is continuous and nondecreasing, $\gamma(0) = 0$ and $\gamma(1) = q$. This can be solved using the techniques of optimal control.

EXAMPLE: We can extend the previous example if there is also a distribution over the possible demand vectors q and we were interested in finding a scale invariant (resp. demand monotonic) method which minimizes player 1's expected cost. In this case $f(x) = -E[x_i(q; \hat{C})]$ where the expectation is taken over q and $\hat{C} = E[C]$. Once

again this problem can be reduced to the simpler one of optimal control using the characterization of $SI(n)$ (resp. $DM^{\hat{q}}(n)$) in Theorem 3 (resp. Theorem 4).

EXAMPLE: In Friedman and Moulin (1995), demand monotonicity is motivated as a weak incentive constraint. Using the above techniques we can compute the ‘most’ demand monotonic CSM by defining $f(x) = -E(\frac{\partial x_i(q;C)}{\partial q_1})$ where the expectation is taken over both a set of cost functions and a set of possible demands, and then minimizing over the set of general or demand monotonic CSMs.

EXAMPLE: A characterization of Extremal Random Order Methods

Now we show that the extremal Random Order methods minimize a lexicographic ordering for any set of cost functions which are supermodular.¹⁰ Recall that a function $C(\cdot)$ is supermodular if for all i , $\partial_i C(q)$ is nondecreasing in q_{-i} , and let $\Xi_s(n)$ be the subset of $\Xi(n)$ which are supermodular.

Given an ordering $\sigma(\cdot) \in \Sigma(n)$, a probability measure over $q \in [0, \hat{q}]$ and a set of cost functions $C_\alpha(\cdot) \in \Xi_s(n)$ for $\alpha \in A$ with a probability measure defined on A . define the σ -lexicographic ordering among CSMs $x, y \in CS(n, q)$ as follows: define $I(x, y) = \min_j \{j \mid E[x_j(C; q)] \neq E[y_j(C; q)]\}$ then we say that CSM x is less than CSM y with respect to σ , written $x \leq_\sigma y$, if $E[x_{\sigma(I(x,y))}(C; q)] \leq E[y_{\sigma(I(x,y))}(C; q)]$.

Theorem 8 *Given the above conditions, the CSM which is minimal with respect to the σ -lexicographic ordering over any of the sets $CS(q)$, $DM^{\hat{q}}(q)$ or $SI(q)$ is the extremal Random Order method with order σ .*

¹⁰Moulin (1989) has studied other properties of supermodular functions for cost allocation problems.

Proof: To simplify notation we consider the natural order, $\sigma(i) = i$ and define $\hat{C}(q) = E_{\alpha \in A}[C_\alpha(q)]$. We now argue by contradiction. Consider the extremal Random Order method x^σ generated by $\sigma(\cdot)$, and any other path generated method x^γ and assume that $x^\gamma <_\sigma x^\sigma$. Then

$$x_1^\gamma(q; \hat{C}) - x_1^\sigma(q; \hat{C}) = \int_0^1 [\partial_1 \hat{C}(\gamma(t)) - \partial_1 \hat{C}(\gamma_1(t), \mathbf{0}_{-1})] d\gamma_1(t)$$

and note that $[\partial_1 \hat{C}(\gamma(t)) - \partial_1 \hat{C}(\gamma_1(t), \mathbf{0}_{-1})] \geq 0$ by supermodularity (integrate \hat{C} from $(\gamma_1(t), \mathbf{0}_{-1})$ to $\gamma(t)$) this implies that $x_1^\gamma(q; \hat{C}) \geq x_1^\sigma(q; \hat{C})$. If this inequality is strict then we have a contradiction. If not, then consider

$$x_2^\gamma(q; \hat{C}) - x_2^\sigma(q; \hat{C}) = \int_0^1 [\partial_2 \hat{C}(\gamma(t)) - \partial_2 \hat{C}(\gamma_1(t), \mathbf{0}_{-1})] d\gamma_1(t)$$

and once again it is easy to see that $x_2^\gamma(q; \hat{C}) \geq x_2^\sigma(q; \hat{C})$. Repeating the argument (at most n times) completes the proof. \square

Thus, when $C(\cdot)$ is supermodular we can characterize an extremal Random Order method, as one that minimizes some lexicographic ordering of agents' payments.

We end this paper with an interesting (and unsolved) question: What linear functional is minimized by the various CSMs such as Aumann-Shapley or Serial Cost? One might expect that these would lead to useful characterizations.

A Appendices

A.1 Proof of Lemma 2

Given a set of measures $\{\mu^i\}$ define the semi-projected measures by $\hat{\mu}_{p_{-i}}^i(S_i) = \mu^i(S_i \times [0, p_{-i}])$, where S_i is a measurable subset of $[0, p_i]$. (These are essentially

the conditional probabilities.) Note that $\hat{\mu}_{q-i}^i$ is Lebesgue measure and thus $\hat{\mu}_{p-i}^i$ is dominated by Lebesgue measure and therefore has a well defined density (in the L^1 sense). Let $\phi_i(p)$ be this density which we will call the ‘potential’, and let Φ be the set of all pairs of potentials which generate a CSM.¹¹

Lemma 6 *If $(\phi_1, \phi_2) \in \Phi$ then $\phi_1(p) + \phi_2(p) = 1$ a.e.*

Proof: Assume that $\mu_i(p)$ has a density $\rho_i(p)$. Choose any arbitrary nonnegative continuous function $f(p)$ which is zero on the boundary of $[0, q]$ and let $C(p) = \int_{[0,p]} f(z)dz$. Then

$$x_i(q; C) = \int_{[0,q]} \partial_i C(p) \rho_i(p) dp,$$

which upon integration by parts yields

$$x_i(q; C) = C(q) - \int_{[0,q]} f(p) \phi_i(p) dp,$$

since in the differentiable case $\phi_i(p) = \int_0^{q-i} \rho_i(q_i, t) dt$.

Then budget balance requires that $2C(q) - \int_{[0,q]} f(p) [\phi_1(p) + \phi_2(p)] dp = C(q)$, or $\int_{[0,q]} f(p) [\phi_1(p) + \phi_2(p) - 1] dp = 0$. Since f is arbitrary this implies that $\phi_1(p) + \phi_2(p) = 1$ a.e.

Since ϕ is well defined, and any measure can be approximated in the weak topology by one with a density, a straightforward approximation argument completes the proof. \diamond

¹¹Superficially, it appears that our ‘potential’ could be related to those in Mclean, Pazgal, and Sharkey (1994) which are the generalizations of the potential found in Hart and Mas-Colell (1989) to cost sharing methods with divisible goods. However, our use of the term is based on the ‘potential theory’ of differential geometry, since the semi-projected measures and their natural generalizations to n dimensions are the closed n -forms which generate the measure under exterior differentiation. Using this idea it should be possible to construct a direct proof of Theorem 1 without using projections.

Now we note some properties of ϕ_1 .

Lemma 7

i) $\phi_1(p_1, 0) = 0$, *a.e.*

ii) $\phi_1(p_1, q_2) = 1$, *a.e.*

iii) $\phi_1(p)$ is nonincreasing in p_1 and nondecreasing in p_2 , *a.e.*

Proof: i) This is true by the definition of ϕ_1 .

ii) This is true since the projection of μ^1 is Lebesgue measure.

iii) That $\phi_1(p)$ is increasing in p_1 follows from its definition. It is decreasing in p_2 since $\phi_2(p)$ is increasing in p_2 and $\phi_1 + \phi_2 = 1$. \diamond

Since, $\phi(p)$ is defined as an element of L^1 and is always less than 1 (since it is dominated by Lebesgue measure), we can always choose a representation in which all three parts of the previous lemma hold everywhere; we will assume this throughout the remainder of the proof.

Given a path γ let ϕ_γ be the potential whose first component is 0 for any p below γ and 1 above and on γ , *a.e.* Clearly $\phi_\gamma \in \Phi$. Let $\hat{\Phi}$ be the set of such potentials.

Lemma 8 $ext(\Phi) = \hat{\Phi}$.

Proof: Assume that $\phi \notin \hat{\Phi}$. Then there must exist some $0 < \alpha < 1$ such that $\nu(\{p \in [0, q] \mid 0 < \phi(p) < \alpha\}) > 0$. Define $\phi^+(p) = (\max[\phi(p), \alpha] - \alpha)/(1 - \alpha)$ and $\phi^-(p) = (\min[\phi(p), \alpha])/\alpha$. Then $\phi^- \neq \phi$ (*a.e.*) and $\phi^-/2 + \phi^+/2 = \phi$ proving that ϕ is not an extreme point. The proof of the converse is straightforward. \diamond

Now we complete the proof of the theorem. Let Φ be the space of potentials. Let τ the mapping from $CS(2)$ to Φ . It is easy to see that τ is a linear bijection, thus $\tau(ext(\Phi)) = ext(CS(2))$. By the previous lemma one can show that if $\phi \in ext(\Phi)$ where there exists some γ such that $\phi = \phi_\gamma$ then $\tau^{-1}(\phi)$ is x_γ . \square

B Proof of Theorem 4

The proof of Corollary parallels that of Theorem 1. Once we show that the extreme CSMs in the set of two agent demand monotonic CSMs are path generated, a projection argument analogous to that in the proof of Theorem 1 completes the proof. We present the first part below, and leave the second part to the reader.

Lemma 9 *Let $DM^{\hat{q}}(2)$ be the set of two agent demand monotonic CSMs defined for $q \leq \hat{q}$. Then $ext(DM^{\hat{q}}(2)) = \{x_{\gamma_\omega} \mid \omega \in \Omega^{\hat{q}}(2)\}$.*

Proof: Combining the above definitions with Theorem 1 shows that

$$x(\hat{q}) = \int_{\omega \in \Omega^{\hat{q}}(2)} x_{\gamma_\omega} d\nu^{\hat{q}}(\omega)$$

and this induces a pair of measures on $[0, \hat{q}]$ which we denote by $\mu(\hat{q})$. Friedman and Moulin (1995), have shown that for two player demand monotonic methods when $q \leq \hat{q}$, $\mu^i(\hat{q})$ and $\mu^i(q)$ are equivalent measures on $[0, q]$. Denoting the restriction of a path γ_ω to $[0, q]$ by $\pi_q(\gamma_\omega)$ we get the following,

$$x(q) = \int_{\omega \in \Omega^{\hat{q}}(2)} x_{\pi_q(\gamma_\omega)} d\nu^{\hat{q}}(\omega),$$

proving the lemma. \diamond .

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