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# **ARMA-GARCH MODELS: BAYES ESTIMATION VERSUS MLE, AND BAYES NON-STATIONARITY TEST**

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## **Abstract**

We compare small-sample properties of Bayes estimation and Maximum likelihood estimation (MLE) of ARMA-GARCH models. Our Monte Carlo experiments indicate that in small sample, the Bayes estimator beats the MLE. We also develop a Bayes method of testing strict stationarity and ergodicity of the conditional variance in the GARCH(1,1) process, near epoch dependence (NED), and finiteness of unconditional moments of the GARCH(1,1) process by using a Markov chain Monte Carlo (MCMC) method. We apply this method to test these properties in the ARMA-GARCH models of weekly foreign exchange rates.

**KEY WORDS:** Generalized Autoregressive Conditional Heteroskedasticity (GARCH), Markov Chain Monte Carlo (MCMC), Near Epoch Dependence (NED).

## Introduction

One of purposes of our study is comparison between Bayes estimation and maximum likelihood estimation (MLE) of a linear regression model with an ARMA error whose conditional variance follows a generalized autoregressive conditional heteroskedasticity (GARCH) process. We call this model an ARMA-GARCH model. GARCH by Bollerslev (1986), its predecessor autoregressive conditional heteroskedasticity (ARCH) by Engle (1982), and other GARCH variants have been widely studied and applied in economics and finance. While many of the previous studies of GARCH models have been made in the classical or frequentist framework, some researchers (Geweke (1989a, 1989b), Kleinbergen and Van Dijk (1993), Müller and Pole (1995), Nakatsuma (1996), among others) studied the GARCH models in the Bayesian framework. It will be interesting to examine which estimation, Bayes or MLE, is better. To answer this question, we attempt to compare estimation results of the ARMA-GARCH model between the Bayes estimation and the MLE in various scenarios by Monte Carlo experiments. As the measurement of accuracy of estimation, we calculate root relative mean square errors (RRMSE) of the posterior mean (Bayes estimation) and the MLE of parameters in the ARMA-GARCH model replicated in the Monte Carlo experiments, and see which estimation produces smaller RRMSE's. The results of our Monte Carlo experiments indicate that the posterior mean is a better estimator than the MLE.

Another purpose of our study is to test stationarity and other properties in the GARCH(1,1) process. Nelson (1990) derived a condition for strict stationarity and ergodicity of the conditional variance in the GARCH(1,1) process as well as conditions for finiteness of

unconditional moments of the GARCH(1,1) process. Hansen (1991) derived a condition for near epoch dependence (NED) of the GARCH(1,1) process. These conditions in Nelson (1990) and Hansen (1991) are expressed as inequalities of parameters in the GARCH(1,1) model. In our study, we test a) NED, b) finite unconditional variance, c) finite unconditional standard deviation, and d) strict stationary and ergodicity in the GARCH(1,1) process by testing whether the corresponding inequalities hold. In the frequentist framework, it is difficult to test whether those inequalities hold or not. In the Bayesian framework, on the other hand, we can test the inequalities by estimating their posterior probabilities. To do so, however, we need to calculate multiple integrals of the posterior density with respect to nuisance parameters, and it may be a difficult task. Kleinbergen and Van Dijk (1993) used an importance sampling method to estimate the posterior probabilities of these inequalities. In this paper, we apply a Markov chain Monte Carlo (MCMC) method to estimate the posterior probabilities, and we test whether the GARCH(1,1) process has these properties. As an example, we test these properties in the ARMA-GARCH model of weekly foreign exchange rates of five major currencies against U.S. dollar: British pound, Canadian dollar, Deutsche mark, Japanese yen, and Swiss franc.

Organization of this paper is as follows. In Section 1, we explain the ARMA-GARCH model and its estimation methods. In Section 2, we present results of our Monte Carlo experiments to compare the Bayes estimation with the MLE. In Section 3, we present conditions for stationarity and other properties of the GARCH(1,1) process and show how to test these properties. In Section 4, we estimate ARMA-GARCH models of weekly foreign exchange rates, and test these properties of the GARCH(1,1) process. In Section 5, conclusion remarks are given.

# 1. ARMA-GARCH Model and Estimation Methods

We consider the following linear regression model with an ARMA(p,q)-GARCH(r,s) error, or simply an ARMA-GARCH model:

$$\begin{aligned}
 y_t &= x_t' \gamma + u_t, \quad (t = 1, \dots, T), \\
 u_t &= \sum_{j=1}^p \phi_j u_{t-j} + \epsilon_t + \sum_{j=1}^q \theta_j \epsilon_{t-j}, \\
 \sigma_t^2 &= \omega + \sum_{j=1}^r \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2,
 \end{aligned} \tag{1}$$

where  $x_t$  and  $\gamma$  are  $k \times 1$  vectors, and  $\epsilon_t$  follows  $N(0, \sigma_t^2)$ . We call  $\{u_t\}$  an ARMA(p,q)-GARCH(r,s) process, and  $\{\epsilon_t\}$  a GARCH(r,s) process. Let  $y \equiv [y_1, \dots, y_T]'$ ,  $x \equiv [x_1, \dots, x_T]'$ ,  $\delta \equiv (\gamma, \phi, \theta, \omega, \alpha, \beta)$ .

In the maximum likelihood estimation (MLE) of the ARMA-GARCH model, the estimator of  $\delta$  is obtained by solving

$$\hat{\delta}_{MLE} \equiv \max_{\delta \in S_\delta} f(\delta | y, x). \tag{2}$$

where  $f(y|x, \delta)$  is the likelihood function of the ARMA-GARCH Model:

$$f(y|x, \delta) = \prod_{t=1}^T \frac{1}{\sqrt{2\pi} \sigma_t} \exp\left(-\frac{\hat{\epsilon}_t^2}{2\sigma_t^2}\right), \tag{3}$$

and  $S_\delta$  is the parameter space of  $\delta$ . In most practices,  $S_\delta$  is a subset of  $\mathbb{R}^{k+p+q+r+s+1}$ . Thus we need to solve a constrained maximum likelihood problem to obtain the the estimator of  $\delta$ . In the Bayes estimation, we consider the posterior density:

$$\pi(\boldsymbol{\delta} | y, x) \propto \pi(\boldsymbol{\delta}) \cdot f(y | x, \boldsymbol{\delta}), \quad (4)$$

where  $\pi(\boldsymbol{\delta})$  is the prior density. The posterior mean of  $\delta_1$ , which is one of parameters in  $\boldsymbol{\delta}$ , is obtained by

$$\begin{aligned} E(\delta_1 | y, x) &= \int_{S_{\delta_1}} \delta_1 \pi(\boldsymbol{\delta}_1 | y, x) \mathbf{d}\boldsymbol{\delta}_1, \\ &= \int_{S_{\delta_1}} \delta_1 \left\{ \int_{S_{\delta_{-1}}} \pi(\boldsymbol{\delta} | y, x) \mathbf{d}\boldsymbol{\delta}_{-1} \right\} \mathbf{d}\boldsymbol{\delta}_1, \end{aligned} \quad (5)$$

where  $\pi(\delta_1 | y, x)$  is the marginal posterior density of  $\delta_1$ ,  $\boldsymbol{\delta}_{-1}$  is parameters except  $\delta_1$ ,  $S_{\delta_1}$  and  $S_{\delta_{-1}}$  are the parameter spaces of  $\delta_1$  and  $\boldsymbol{\delta}_{-1}$  respectively, and  $\int(\cdot) \mathbf{d}\boldsymbol{\delta}_{-1}$  is the multiple integral with respect to  $\boldsymbol{\delta}_{-1}$ .

## 2. Monte Carlo Experiments: Bayes vs. MLE

In this section, we compare the Bayes estimation and the MLE of the ARMA-GARCH model by Monte Carlo experiments. In the Monte Carlo experiments, we use the following data generating process:

$$\begin{aligned} y_t &= \gamma_0 + u_t, \\ u_t &= \phi_1 u_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1}, \\ \sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \end{aligned} \quad (6)$$

We set  $\gamma_0 = 1$ ,  $\theta_1 = .1$ , and  $\omega = 1$ . Values of the other parameters are listed in Table 1. The

sample size is 200 in Case 1-6, and 400 in Case 7. For each case, we generate  $\{y_t\}$  from the data generating process (6) given values of parameters shown above.

Let us explain properties of the ARMA(1,1)-GARCH(1,1) process in each case in Table 1. In Case 1, 2, 3, and 7, the GARCH(1,1) process is integrated, *i.e.*, it is an IGARCH(1,1) process. (See Engle and Bollerslev (1986).) The values of  $\alpha_1$  and  $\beta_1$  are ones used in Lumsdaine (1995). In Case 3,  $\phi_1$  is .99 and this is a near unit root situation. In Case 4, the conditional variance is strictly stationary and ergodic but unconditional moments of the GARCH(1,1) process are all infinite. (See Figure 2.) In Case 5, the conditional variance is non-stationary. (See also Figure 2.) Finally, in Case 6, the conditional variance is stationary and all unconditional moments of the GARCH(1,1) process is finite.

By using data simulated from the data generating process, we estimate the same model as the data generating process. Thus there is no misspecification. We impose the following constraints on parameters:

$$-1 < \theta_1 < 1, \quad \omega > 0, \quad \alpha_1 > 0, \quad \beta_1 > 0, \quad (7)$$

Under constraints in (7),  $\{y_t\}$  is invertible, and the conditional variance is positive for all t. In the MLE, we solve the constrained maximum likelihood problem (2) by a quasi-Newton method. In the Bayes estimation, we use a Gaussian quadrature method to calculate posterior means. For dimensions of 6 or less, a Gaussian quadrature formula with Simpson's rule yields good results. We calculate root relative mean squared errors (RRMSE) of posterior means and MLE's to compare the Bayes estimation and the MLE. The obtained RRMSE's are listed in Table 2.

For all cases except  $\alpha_1$  and  $\beta_1$  in Case 7, the Bayesian estimation produces smaller RRMSE than the MLE. In Case 3 in which  $\phi_1$  is .99, and in Case 5 in which the conditional

variance in the GARCH(1,1) process is non-stationary, the RRMSE's of the MLE of  $\gamma_0$  and  $\omega$  are much larger than those of the Bayesian estimation. Interestingly, the magnitude of the RRMSE's in Case 4, in which the conditional variance is strictly stationary and ergodic but the GARCH(1,1) process has no finite unconditional moments, is comparable to that in Case 6 in which all unconditional moments are finite. In Case 7 in which the sample size is twice as many but the values of parameters are the same as in Case 1, the RRMSE's become smaller than in Case 1 as we expect from the consistency of the MLE.

In figure 1, we show kernel-smoothed densities of replicated posterior means and MLE's of  $\beta_1$  for Case 1. The mode of the MLE is nearer to the true value than that of the posterior mean, but the spread of the distribution is wider in the MLE than in the posterior mean. Thus the RRMSE of the posterior mean is smaller than that of the MLE as shown in Table 2. This is a typical result of a “shrinkage” estimator.

In summary, the Bayes estimation is better than the MLE in small sample in terms of smaller mean square errors. In both Bayes estimation and MLE, IGARCH(1,1) process and GARCH(1,1) process with the non-stationary conditional variance tend to yield instable estimates when  $\beta_1$  is large and near 1. Infinite unconditional moments in the GARCH(1,1) process, however, do not seem to affect the stability of estimates when the conditional variance is strictly stationary and ergodic, and  $\alpha_1$ , instead of  $\beta_1$ , is large.



### 3. Testing Stationarity and Other Properties of the GARCH(1,1) Process

In this section, we present some important properties of the GARCH(1,1) process and how to test these properties. Hansen (1991) showed that the GARCH(1,1) process is  $L^2$ -near epoch dependent ( $L^2$ -NED) if

$$\beta_1^2 + 2\alpha_1\beta_1 + 3\alpha_1^2 < 1. \quad (8)$$

when  $\varepsilon_t/\sigma_t (\equiv z_t)$  is i.i.d. normal. From Corollary in Nelson (1990), the unconditional variance of the GARCH(1,1) process is finite if

$$E(\beta_1 + \alpha_1 z_t^2) < 1, \quad \text{or} \quad \beta_1 + \alpha_1 < 1, \quad (9)$$

if  $z_t$  is i.i.d. normal, and the unconditional standard deviation of the GARCH(1,1) process is finite if

$$E(\beta_1 + \alpha_1 z_t^2)^{\frac{1}{2}} < 1. \quad (10)$$

Nelson (1990) showed that the conditional variance of the GARCH(1,1) process is strictly stationary and ergodic if

$$E[\ln(\beta_1 + \alpha_1 z_t^2)] < 0 \quad (11)$$

The regions of  $(\alpha_1, \beta_1)$  satisfying (8)-(11) are shown in Figure 2.

We test these properties by estimating the posterior probabilities of inequalities (8)-(11).

We estimate the posterior probabilities by a Monte Carlo method with a kernel estimation. For

example, for the NED condition (8), we calculate

$$\begin{aligned} & \text{Prob} \{ \beta_1^2 + 2 \alpha_1 \beta_1 + 3 \alpha_1^2 < 1 \} \\ & \cong \frac{1}{M} \sum_{j=1}^M \text{K} \left( \frac{1 - \{ (\beta_1^{(j)})^2 + 2 \alpha_1^{(j)} \beta_1^{(j)} + 3 (\alpha_1^{(j)})^2 \}}{h_j} \right), \end{aligned} \quad (12)$$

where  $\alpha_1^{(j)}$  and  $\beta_1^{(j)}$  ( $j=1, \dots, M$ ) are realized values in a Monte Carlo simulation,  $\text{K}(\cdot)$  is a kernel for the distribution function, and  $h_j$  is the window width. We choose the standard normal distribution function as  $\text{K}(\cdot)$ , and  $h_j$  is decided by a local smoothing method. For the condition of finite unconditional variance, we also use the same method. However, for conditions of finite unconditional standard deviation and of strict stationarity and ergodicity, we need to calculate the expectation of non-linear functions. To do so, we approximate the expectation by a Monte Carlo method. For example,  $E[\ln(\beta_1^{(j)} + \alpha_1^{(j)} z^2)]$  is approximated by

$$\begin{aligned} & E[\ln(\beta_1^{(j)} + \alpha_1^{(j)} z^2)] \\ & \cong \frac{1}{N} \sum_{h=1}^N \ln(\beta_1^{(j)} + \alpha_1^{(j)} (z^{(h)})^2), \end{aligned} \quad (13)$$

where  $z^{(h)}$  is generated from  $N(0,1)$ . Once we have the expectation corresponding to each pair of  $(\alpha_1^{(j)}, \beta_1^{(j)})$ , we can apply the same kernel estimation method as in (12) to estimate the posterior probabilities of finite standard deviation and of strict stationarity and ergodicity.

Finally, we need the Monte Carlo sample of  $\alpha_1$  and  $\beta_1$ . Kleinbergen and Van Dijk (1993) used an importance sampling method to estimate the posterior probabilities of the stationarity condition of the conditional variance and the finite unconditional variance condition. In our study, we apply a Markov chain sampling method to generate  $\alpha_1$  and  $\beta_1$  to estimate the posterior

probabilities. The Markov chain sampling scheme we use in this paper is explained in Nakatsuma (1996). To construct the Markov chain sampling scheme for the ARMA-GARCH model, we consider the following auxiliary ARMA models:

*AM1: regression model with an ARMA(p,q) error:*

$$y_t = X_t\gamma + \sum_{j=1}^p \phi_j(y_{t-j} - X_{t-j}\gamma) + \epsilon_t + \sum_{j=1}^q \theta_j \epsilon_{t-j}, \quad \epsilon_t \sim N(0, \sigma_t^2), \quad (14)$$

*AM2: ARMA(l,s) model of the squared errors  $\epsilon_t^2$ :*

$$\epsilon_t^2 = \omega + \sum_{j=1}^l (\alpha_j + \beta_j) \epsilon_{t-j}^2 + w_t - \sum_{j=1}^s \beta_j w_{t-j}, \quad w_t \sim N(0, 2\sigma_t^4), \quad (15)$$

where  $l \equiv \max\{r, s\}$ ,  $\alpha_j = 0$  for  $j > r$ , and  $\beta_j = 0$  for  $j > s$ . The outline of our Markov chain sampling scheme is as follows:

- a) Generate  $(\gamma, \phi, \theta)$  from AM1 given  $\{\sigma_t^2\}$  and the rest of parameters.
- b) Generate  $(\omega, \alpha, \beta)$  from AM2 given  $\{\epsilon_t^2\}$ ,  $\{\sigma_t^2\}$ , and the rest of parameters.
- c) Apply the Metropolis-Hastings algorithm after each parameter is generated. The Metropolis-Hastings algorithm is given as follows:

I) Generate a candidate of  $\delta$  from the proposal distribution given  $\delta^{(i)}$ .

II) Accept or reject this candidate by

$$\delta^{(j+1)} = \begin{cases} \delta & \text{with probability } a, \\ \delta^{(j)} & \text{with probability } 1-a, \end{cases} \quad (16)$$

where

$$a = \min \left\{ \frac{\pi(\delta)}{\pi(\delta^{(j)})} \frac{g(\delta, \delta^{(j)})}{g(\delta^{(j)}, \delta)}, 1 \right\} \quad (17)$$

and  $\pi(\delta)$  is the density of the target distribution and  $g(\delta^{(j)}, \delta)$  is the density of the proposal distribution.

- d) Repeat a) - c) until the sequences become stable.

In this sampling scheme, we update  $\{\varepsilon_t^2\}$  and  $\{\sigma_t^2\}$  at every time corresponding parameters are updated. See Chib and Greenberg (1994), Müller and Pole (1995) and Nakatsuma (1996) for more details.

#### 4. Examples: Weekly Foreign Exchange Rates

In this section, we estimate ARMA-GARCH models of weekly foreign exchange rates of five currencies: British pound, Canadian dollar, Deutsche mark, Japanese yen, and Swiss franc against U.S. dollar, and we test stationarity and other properties of the GARCH(1,1) process in the estimated ARMA-GARCH models of the exchange rates.

The sample period is from the week of June 4th in 1974 to the week of May 18th in

1987. The sample size is 729. We take the natural logarithm of the exchange rates, and subtract the first period's value from the current values. Thus the first period's value of the log exchange rate series is always 0. Using weekly exchange rates data, We estimate the same ARMA(1,1)-GARCH(1,1) model specification as in (6) except omitting the constant term  $\gamma_0$ . We also impose the same constraints as in (7) on parameters. The Bayes estimation of the ARMA-GARCH model is obtained by an MCMC method explained in Section 3. The number of iterations of the Markov chain sampling is 11000, and we discard first 1000 as burn-in. Thus the size of the Monte Carlo sample is 10000. Posterior means and standard deviations of parameters are listed in Table 3. Posterior probabilities of near epoch dependence (NED), finite unconditional standard deviation and variance, and strict stationarity and ergodicity of the GARCH(1,1) process are estimated by a kernel smoothing method with the Gaussian kernel as explained in Section 3. To estimate posterior probabilities of (10) and (11), we need to generate  $z \sim N(0,1)$ . The number of replications of  $z$  is 10000. Estimated posterior probabilities are listed in Table 4. Estimated marginal posterior densities of (8) - (11) for Swiss franc are plotted in Figure 3 and 4.

For Canadian dollar, all posterior probabilities in Table 4 is more than 99%. British pound and Japanese yen also show a similar pattern to Canadian dollar, but their posterior probabilities are a few points less than those of Canadian dollar. In particular, the probability of NED of British pound and Japanese yen is about 90%. Deutsche mark and Swiss franc show a similar pattern but different from the other three currencies. The posterior probability of NED is low for both currencies, and it is unlikely that the NED condition holds. The posterior probabilities of finite variance and standard deviation are less than 90%. Thus the GARCH(1,1) processes in Deutsche mark and Swiss franc may not have finite unconditional moments. Finally, the posterior

probability of strict stationarity and ergodicity is more than 90% but less than 95% for both currencies. Hence there is a reasonable probability that the conditional variance in the GARCH(1,1) processes in Deutsche mark and Swiss franc are not strictly stationary or ergodic.

We note that even though the posterior means and standard deviations of  $\alpha_1$  and  $\beta_1$  seem similar for British pound, Deutsche mark, and Swiss franc, the posterior probabilities of stationarity and other properties of the GARCH(1,1) process for British pound are strikingly different from Deutsche mark and Swiss franc. However, this may not be surprising. In Figure 2 for low  $\alpha_1$  and high  $\beta_1$ , the boundaries of NED, finite unconditional moments, and strict stationarity and ergodicity are so close to each other that only a slight change in  $\alpha_1$  and/or  $\beta_1$  may change the properties of the GARCH(1,1) process.

## **5. Conclusion Remarks**

In this paper, we compare small-sample properties of the Bayes estimation and MLE of the ARMA-GARCH model, and we find that the Bayes estimation is a better method than the MLE in terms of smaller mean square errors in particular when the sample size is relatively small. We also develop a method to test strict stationarity and ergodicity of the conditional variance, near epoch dependence (NED), and finite unconditional variance and standard deviation in the GARCH(1,1) process. We apply this method to weekly foreign exchange rates data, and we find that even if the posterior means and standard deviations are similar, the posterior probabilities of properties of the GARCH(1,1) process may not.

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TABLE 1. VALUES OF PARAMETERS IN MONTE CARLO EXPERIMENTS

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
$\phi_1$	.8	.8	.99	.8	.8	.8	.8
$\alpha_1$	.15	.05	.15	1.6	.25	.1	.15
$\beta_1$	.85	.95	.85	.1	.85	.5	.85

TABLE 2. RRMSE IN MONTE CARLO EXPERIMENTS

		$\gamma_0$	$\phi_1$	$\theta_1$	$\omega$	$\alpha_1$	$\beta_1$
Case 1	MLE	5.3811	.0799	.8879	12.5389	.5167	.1391
	Bayes	1.4410	.0620	.5669	2.8547	.5003	.1105
Case 2	MLE	5.3591	.0796	.8897	12.6988	2.6501	.1954
	Bayes	1.4420	.0617	.5596	3.1396	.5266	.1123
Case 3	MLE	66.9227	0.230	.7593	16.095	.5643	.1922
	Bayes	1.0449	.0170	.5567	2.8569	.5613	.1096
Case 4	MLE	.8048	.0310	.7453	.6476	.2297	.8308
	Bayes	.6123	.0214	.5830	.4132	.1498	.3081
Case 5	MLE	28.7341	.1050	1.2289	97.7718	3.1006	.4241
	Bayes	1.2753	.0856	.6396	3.7631	.5176	.3216
Case 6	MLE	.7892	.0779	.9055	.6964	.9181	.6146
	Bayes	.4669	.0602	.5515	.4418	.8123	.3827
Case 7	MLE	2.2573	.0522	.6561	3.4636	.3333	.0628
	Bayes	1.5657	.0417	.5012	.3966	.6725	.0992

Notes: the sample size is 200 in Case 1-6, and 400 in Case 7. The number of replications is 100 for the Bayes estimation in Case 7. Otherwise, it is 500.

TABLE 3.

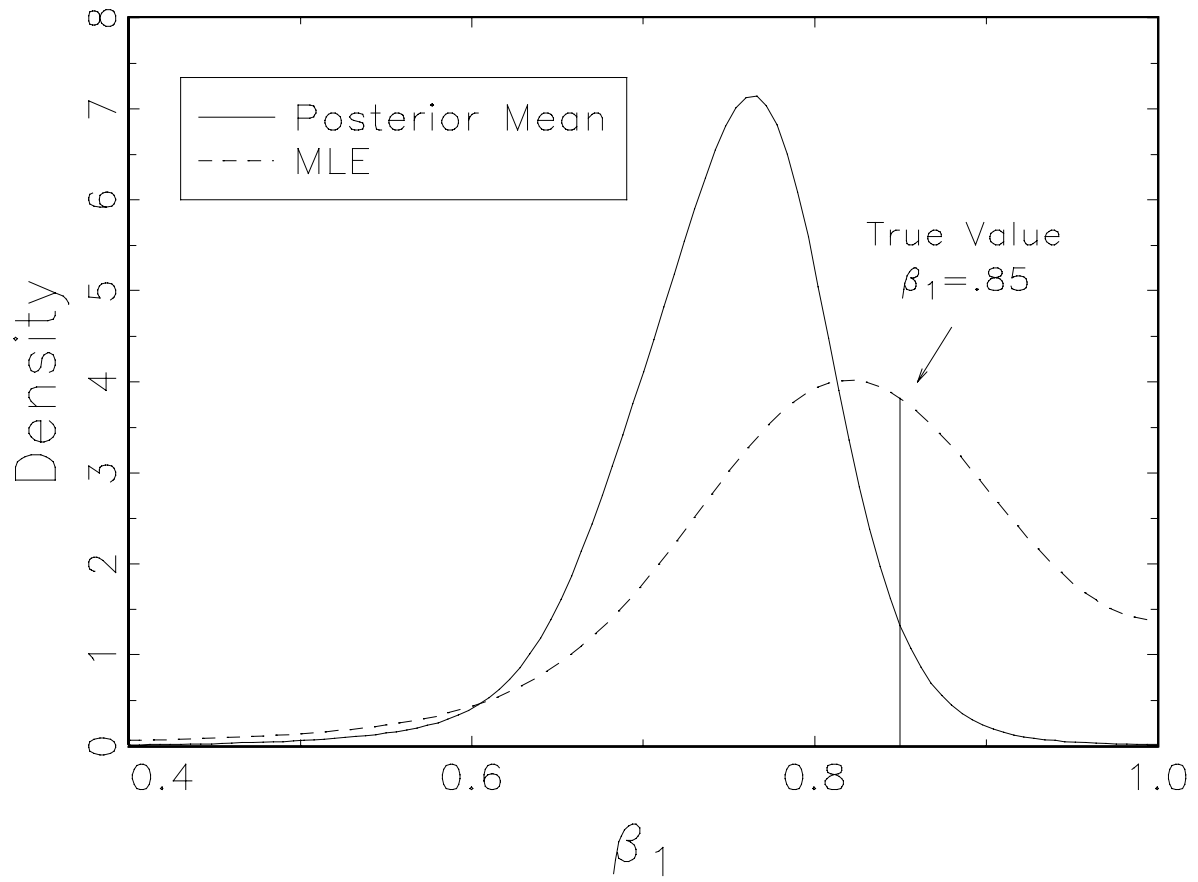
## POSTERIOR MEANS AND STANDARD DEVIATIONS OF PARAMETERS

	British Pound	Canadian Dollar	Deutsche Mark	Japanese Yen	Swiss Franc
$\phi_1$	1.001 (0.00132)	1.001 (0.00105)	1.001 (0.00232)	1.003 (0.00269)	1.001 (0.00145)
$\theta_1$	0.05827 (0.0395)	0.08501 (0.0405)	0.08702 (0.0367)	0.08806 (0.0383)	0.07520 (0.0378)
$\omega$	$1.065 \times 10^{-5}$ ( $3.78 \times 10^{-6}$ )	$5.769 \times 10^{-6}$ ( $2.20 \times 10^{-6}$ )	$5.797 \times 10^{-6}$ ( $3.54 \times 10^{-6}$ )	$5.012 \times 10^{-6}$ ( $1.41 \times 10^{-6}$ )	$5.139 \times 10^{-6}$ ( $2.93 \times 10^{-6}$ )
$\alpha_1$	0.1279 (0.0314)	0.2290 (0.0447)	0.1322 (0.0329)	0.08792 (0.0177)	0.1371 (0.0246)
$\beta_1$	0.8231 (0.0406)	0.5887 (0.104)	0.8455 (0.0418)	0.8914 (0.0211)	0.8518 (0.0269)

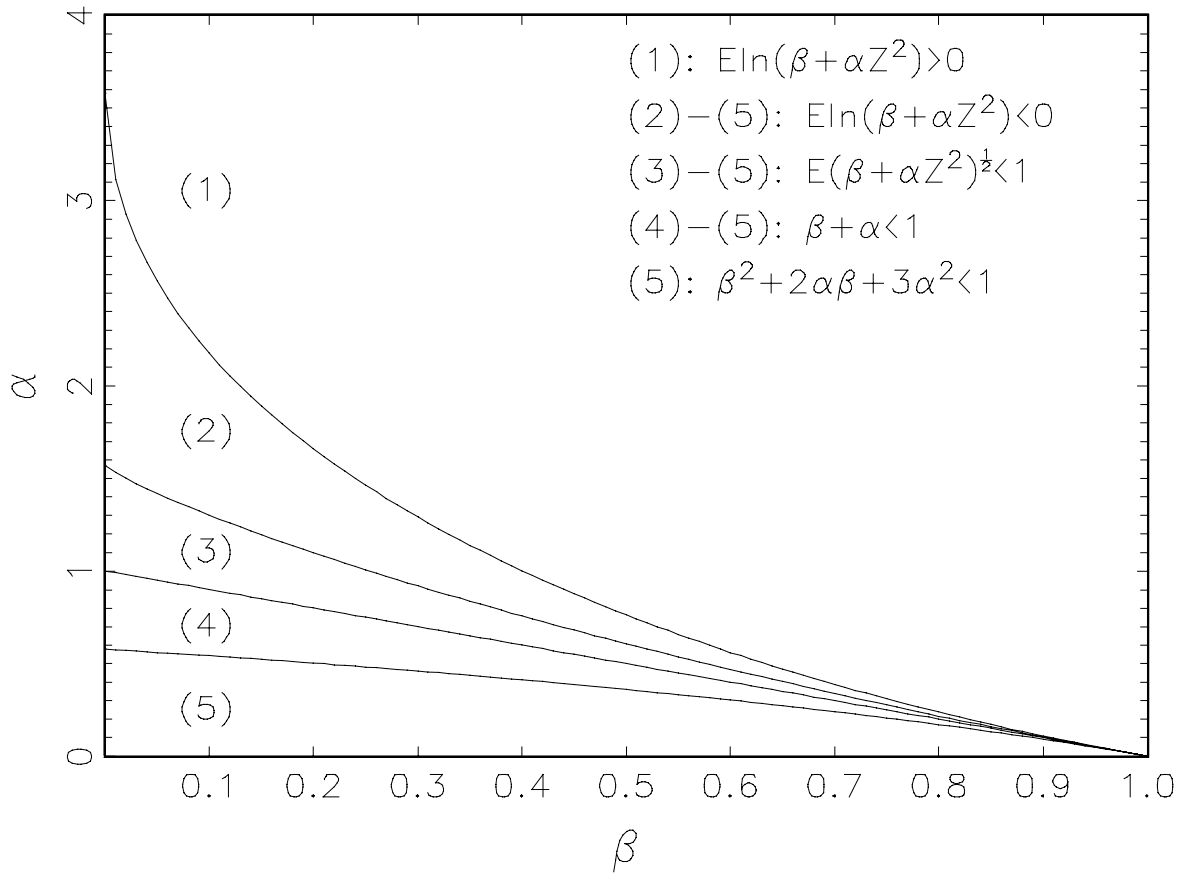
TABLE 4.

POSTERIOR PROBABILITIES OF NED, FINITENESS OF UNCONDITIONAL MOMENTS,  
AND STRICT STATIONARITY AND ERGODICITY

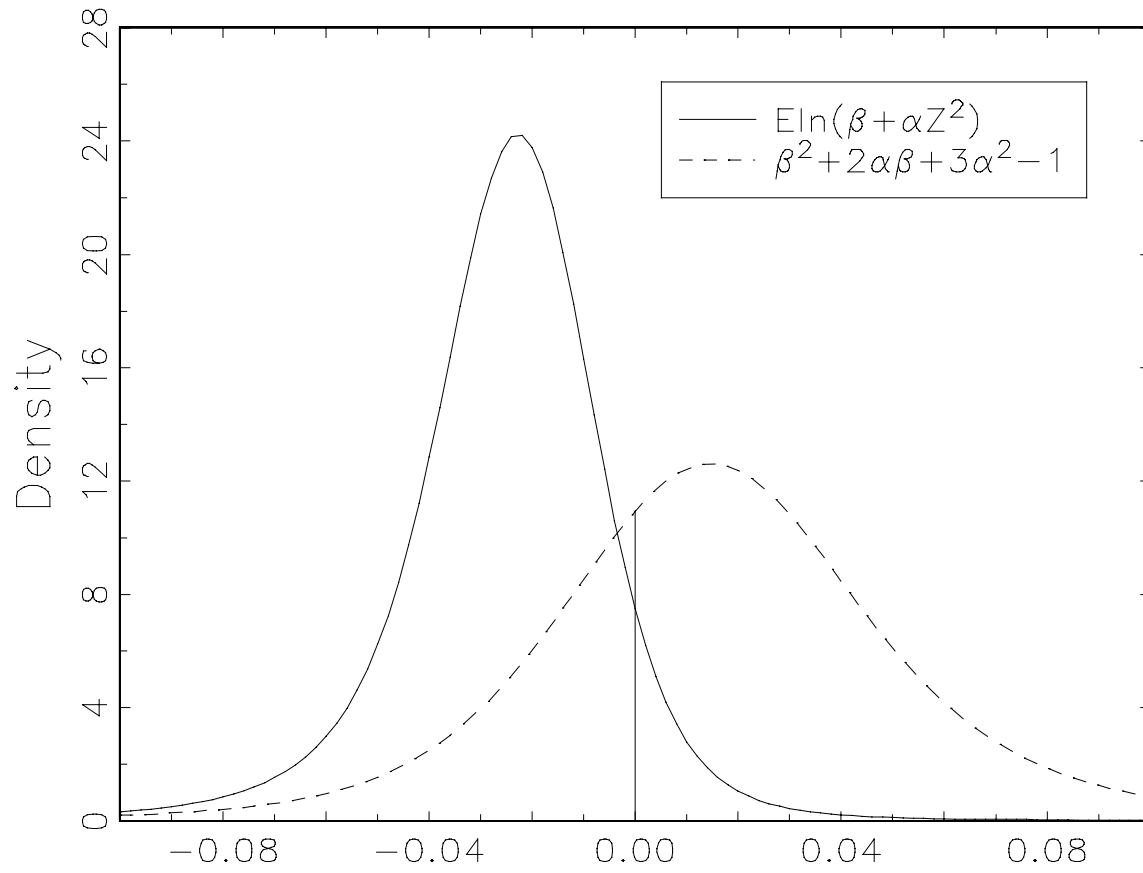
	British Pound	Canadian Dollar	Deutsche Mark	Japanese Yen	Swiss Franc
NED	90.1%	99.0%	57.9%	89.3%	31.0%
$\beta_1^2 + 2\alpha_1\beta_1 + 3\alpha_1^2 < 1$					
Finite Variance	95.8%	99.7%	86.1%	94.4%	74.0%
$\beta_1 + \alpha_1 < 1$					
Finite Std. Dev.	95.3%	99.4%	85.8%	91.4%	79.9%
$E[(\beta_1 + \alpha_1 z^2)^{1/2}] < 1$					
Strict Stationarity	97.9%	99.8%	94.0%	97.1%	92.0%
$E[\ln(\beta_1 + \alpha_1 z^2)] < 0$					



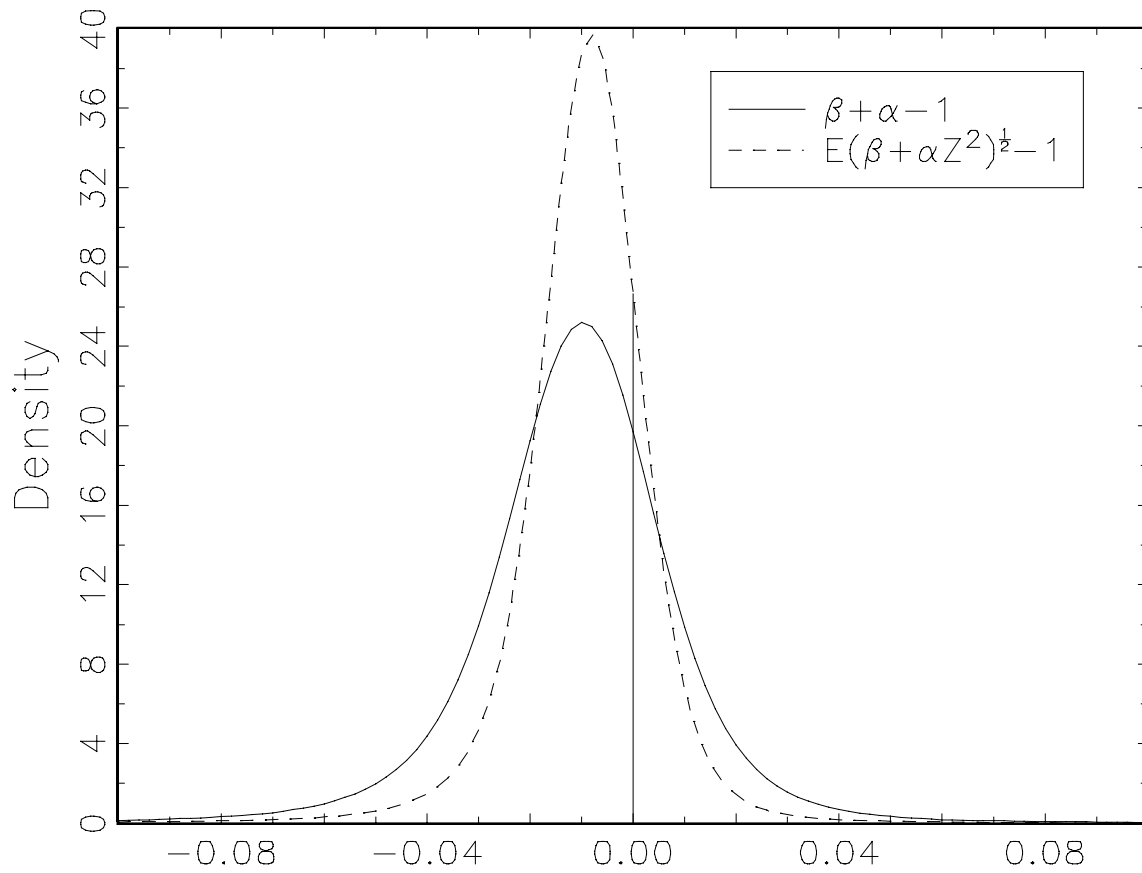
**Figure 1** Bayes Estimation versus MLE



**Figure 2** Regions of  $(\alpha, \beta)$



**Figure 3** Marginal Posterior Densities of Stationarity and NED Conditions (Swiss Franc)



**Figure 4** Marginal Posterior Densities of Finite Moments Conditions (Swiss Franc)