

Ma, Jinpeng

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# ENGLISH AUCTIONS AND WALRASIAN EQUILIBRIA WITH MULTIPLE OBJECTS: a Dynamic Approach\*

Jinpeng Ma<sup>†</sup>

Rutgers University

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## Abstract

This paper studies the English (progressive) auction for an exchange economy with multiple objects. The English auction is a *tâtonnement* process and lasts multiple rounds. It is modeled as a sequence of round games. Each round game is a normal form game in which an agent's strategies are his bids and his payoff is his trading profits of his winning bundle at that round. Among these normal form games, all intermediary round games are in fact the "virtual" games because the payoffs to agents are not finalized unless the auction closes. We show that any ascending price sequence obtained from the English auction converges to a Walrasian equilibrium (if any) within finite rounds when agents submit their bids that consist of a Nash equilibrium in each round game.

We also provide a sufficient condition for the English auction to converge to a Walrasian equilibrium in finite rounds. But this condition is weaker than the Nash equilibrium. This shows that the Nash equilibrium is not necessary (though sufficient) for the English auction to converge to a Walrasian equilibrium.

**Keywords:** English Auction, Walrasian Equilibrium, Nash Equilibrium, Multiple Object Auctions

**JEL classification** numbers: D41, D44, D50

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<sup>†</sup>Department of Economics, Rutgers University, Camden, NJ 08102. jinpeng@crab.rutgers.edu

# 1 Introduction

The English auction and a variety of auction forms have been extensively studied in the auction literature since the pioneering works of Vickrey (1961) and Wilson (1969). The analysis of an auction form is treated as a noncooperative game with the application of the Bayesian Nash equilibrium. Milgrom and Weber (1982) provided a general framework for analyzing a number of auction forms with symmetric Bayesian Nash equilibrium. The auction literature has been well surveyed in McAfee and McMillan (1987), Milgrom (1985, 1987) and Wilson (1992).

An absolute English auction (Bulow and Klemperer (1996)) raises the price continuously until one and only one bidder remains and this last bidder wins the object and pays his winning bid. This English auction as an allocation mechanism has been widely used in practice for the sale of a single object. The existing auction literature is mainly concerned with the sale of a single object and the analysis of an auction form focuses on one normal form game. This paper studies an analogy of the English auction for the sale of multiple objects and presents an analysis of the English auction by a sequence of normal form games.<sup>1</sup> The English auction in this paper is a tâtonnement process and it captures some merits of the spectrum auction. The auction form works as follows. At the beginning of each round agents submit their bid vectors and the price of an object at this round equals the highest bid(s). Then the price vector is publicly announced and agents begin to conduct their trade of objects under the announced prices.<sup>2</sup> The auction stops whenever each object is demanded by one and only one agent under the announced prices. Otherwise all objects become open again for the next round in which each agent submits his new bids. These new bids determine a new price vector which should be no less than the prior one and at least one object receives a higher price bid. As the auction continues, it generates an ascending sequence of price vectors. The problem is that not all ascending price sequences generated by the English auction converge to a Walrasian equilibrium within finite rounds. This motivates a natural question: what are the conditions on the ascending price sequences such that every ascending price sequence generated by the English auction converges to the Walrasian equilibria within finite rounds.

To answer the question we formulate the English auction with a sequence of multiple round normal form games. In each round game the strategies and payoffs for an agent are respectively his bids and the “trading” profit of his winning bundle at that round. An agent’s winning bundle at a round is those objects for each of which he has submitted the highest bid. There is no tie-breaking device used in this definition of the winning bundles and these winning bundles may overlap. Note

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<sup>1</sup>The study of auctions with multiple objects is largely motivated by the novel spectrum auction adopted by the Federal Communication Commission (FCC) to sell a few thousands of spectrum licenses. The spectrum auction is a sequential auction form designed by Professors Paul Milgrom and Robert Wilson of Stanford University and Preston McAfee of the University of Texas (Cramton (1995)). Cramton (1995) and McMillan (1994) provided a detailed introduction of this auction.

<sup>2</sup>This trade can be implemented by the auctioneer if agents truthfully report their demand bundles at the announced prices.

that all games but the one at the close of the auction are in fact the “virtual” games since the assigned payoffs in the intermediary rounds are just the “virtual” payoffs and they are not finalized unless the auction closes. This framework is based on the idea that agents in a multiple round auction do not know exactly when an auction closes. Therefore they may indeed play their bids in each intermediary round as if that round were the close one. This feature that only the payoffs at the close of the auction matter is similar to the noncooperative bargaining model as in Rubinstein (1982). But the sequence of normal form games is modeled as in an infinitely repeated game.

We show that any ascending price sequence obtained in the English auction converges to a Walrasian equilibrium within finite rounds when players submit their bids that consist of a Nash equilibrium in each round game. Moreover, the auction can start with arbitrary price vectors below Walrasian equilibrium price ones and the convergence result still holds. An implication of this is that agents can submit arbitrary bids (say, switch bids, jump bids; see Cramton (1995)) at the very beginning of the auction, the auction still converges to a Walrasian equilibrium within finite rounds once agents’ bidding behavior conforms with the Nash equilibrium bids starting from some round and thereafter.

In the above result we assume that agents are myopic and submit Nash equilibrium bids at each round game. But this myopic Nash equilibrium (MNE) assumption is not necessary for the result. We find one additional sufficient condition (C2) for the auction to converge to a Walrasian equilibrium within finite rounds. Intuitively C2 states that each agent is at least willing to buy his winning bundle at each round if the auction stops at that round, though he may be willing to buy those he is not winning. Alternatively this condition means that agents are not willing to withdraw their winning bids after winning.

In the discussion section we introduce the second additional condition (C3). We say that an agent “wins” an object if his bid for that object is no less than its prior price. Hence each agent can determine in advance in each round which bundle he is going to “win” since the prior price vector is public information. C3 says that each agent is at least willing to buy his “winning” bundle at each round if the auction stops at that round. C3 is a special case of C2. A corollary of the above result is that any ascending price sequence obtained from the English auction converges to a Walrasian equilibrium in finite rounds under C3.

We show the convergence result under condition C2. Because the MNE assumption satisfies C2, the result under MNE is a corollary of the one under C2. The major differences among MNE, C2 and C3 are the facts that bids under C2 and C3 may not be the Nash equilibrium bids. An implication of this is that Nash equilibrium is not a necessary (though sufficient) condition (even at the close of the auction) for the English auction to achieve a Walrasian equilibrium within finite rounds. C3 is different from C2 by the fact that bids under C3 depend on the prior prices only while bids under C2 may depend on the historic prices beyond the last round.

When agents' utility functions satisfy the gross substitutes condition in Kelso and Crawford (1982), an agent can ignore how and what the others may bid in the auction; as long as each agent is willing to buy his "winning" bundle and pay his "winning" bids at each round, any ascending price sequence obtained from the English auction converges to a Walrasian equilibrium within finite rounds.

## Related Literature

We consider an exchange economy in which there are a finite number of agents and objects. Agents may consume as many objects as they wish. Therefore each agent's utility function is described by a set function over all commodity bundles. This economy has been studied by Bikhchandani and Mamer (1994) and Gül and Stacchetti (1996b). A related version of it in which objects are personalized has been studied in Ma (1996). It is analogous to the job-matching market in Kelso and Crawford (1982) by identifying agents and objects in this paper with firms and workers respectively in theirs. The major issue in these papers is the existence of a Walrasian equilibrium. In what follows we assume that a Walrasian equilibrium always exists.

There has been a growing interest of the studies of auction forms with multiple objects. A list of lately studies of a variety of auction forms with multiple objects may be found in Bikhchandani (1996). Our paper is perhaps mostly related to Demange, Gale and Sotomayor (1986), Gül and Stacchetti (1996a) and Kelso and Crawford (1982). Demange, Gale and Sotomayor (1986) designed an auction form for the assignment problem (Shapley and Shubik (1972)) by the Hall Theorem. Their auction form starts with zero bids. In each round, given the prior price vector, agents report their demand objects to the auctioneer. The auctioneer stops the auction if each object demanded by one and only one agent. Otherwise the Hall Theorem (Gale (1961) and Hall (1935)) is used to find out the minimum overdemanded bundle. The prices of those objects in the minimal overdemanded bundle are each increased by a unit and the auction moves to the next round. They showed that their auction converges to the minimum (discrete) Walrasian price vector within finite rounds. The unit of increments can be chosen such that their auction converges to the minimum Walrasian price vector with any degree of approximation.

Kelso and Crawford (1982) generalized the job-matching market in Crawford and Knoer (1981) to allow a firm to hire as many workers as he wishes and to have more complex utility functions. As in Crawford and Knoer (1981), they generalized the Gale and Shapley (1962) deferred proposal algorithm to their job-matching market. In their algorithm each firm starts with zero salary offers. In each intermediary round each firm makes offers to a demand set of workers and this set of workers includes those who accept its prior offers; Each worker rejects all but the most preferred offer, which he then temporarily accepts; The salaries of each rejected offer is raised by a unit and the algorithm

continues until each worker has one and only one offer.<sup>3</sup> Under their gross substitutes condition, they showed that their algorithm must stop within finite rounds with a (discrete) core outcome (thus a discrete Walrasian equilibrium) and the unit of increments of salaries can be chosen such that the algorithm converges to a (continuous) core outcome with any degree of approximation. Their algorithm is in fact a version of the English auction.

Gül and Stacchetti (1996a) studied the exchange economy with multiple objects and designed an English auction by a generalization of the Hall Theorem. Their auction starts with zero vector. Given the prior price vector, agents report their demand bundles to the auctioneer. The auctioneer stops the auction if no minimal overdemanded bundle<sup>4</sup> can be found. Otherwise the minimal overdemanded bundle can be found and prices are equally increased for those objects in the minimal overdemanded bundle and these prices are increased beyond such a degree that the minimal overdemanded bundles in two adjacent rounds are the same. They showed that under the gross substitutes condition<sup>5</sup> their auction converges to the minimum Walrasian price vector within finite rounds.

There are several features that distinguish our paper from these papers. First, the major concern in these papers is to design one Walrasian equilibrium price path while our paper focuses on the conditions under which every ascending price sequence generated by the English auction is Walrasian. Therefore, there may exist infinitely many Walrasian price paths that satisfy our conditions for an economy. Second, the tie-breaking device often plays an important role in the design and the study of an auction form. For example, the minimum overdemanded bundles in Demange, Gale and Sotomayor (1986) and Gül and Stacchetti (1996a) both play in part a role of tie-breaking device. We bypass the tie-breaking problem by allowing the overlapping of the winning bundles. Third, since we formulate the English auction by a sequence of normal form games, the bid increments at each round become strategic variables and are endogenously determined by agents. The auctioneer does not play a role in how agents may submit their bids. Fourth, we only assume the existence of a Walrasian equilibrium and this assumption is minimal in the sense that if a situation has no Walrasian equilibria then no auction form can achieve a Walrasian equilibrium.

Finally, Bikhchandani (1996) studied the exchange economy and the implementation of the Walrasian allocations by a (static) English auction. He showed that pure strategy Nash equilibria exist if and only if Walrasian equilibria exist. Moreover, the set of pure strategy Nash equilibrium allocations coincides with the set of Walrasian allocations. These conclusions may not apply to the

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<sup>3</sup>Their algorithm is asymmetric in the sense that only firms make offers. But Roth (1984) found that this algorithm can be made symmetric and its core has a surprising polarization property.

<sup>4</sup>Their definition is quite different from the one in Demange, Gale and Sotomayor (1986).

<sup>5</sup>Gül and Stacchetti (1996b) provided two new sufficient conditions, the no complementarities and the single improvement property, for the existence of a Walrasian equilibrium. They showed that the set of Walrasian prices forms a lattice under the common interests of sellers and the gross substitutes condition is equivalent to their two new conditions.

mixed Nash equilibria though. Bikhchandani's approach is static and it is quite different from ours.

## Outline of the Paper

The outline of this paper is as follows. Section 2 introduces the exchange economy. Section 3 introduces the English auction. Section 4 formulates the auction form by a sequence of normal form games and introduces the first convergence result under the MNE assumption. Section 5 introduces the main convergence theorem under the condition C2. Section 6 introduces one additional condition C3 and an application of the results when agents' utility functions satisfy the gross substitutes condition in Kelso and Crawford (1982). We also show what may be the outcomes when agents make their "mistakes". Section 7 provides the proofs of the two main results under condition C2 and MNE.

## 2 Exchange Economies

We use some notation and definitions from Gül and Stacchetti (1996a,b). Let  $N = \{1, 2, \dots, n\}$  denote the set of agents. Denote  $\Omega$  the set of all commodities in the economy and  $2^\Omega$  the set of all commodity bundles. For an agent  $i \in N$ , his utility function  $u_i$  is a set function  $u_i : 2^\Omega \rightarrow R$  satisfying  $u_i(\emptyset) = 0$ . We assume that  $u_i$  is weakly monotone, i.e., for all  $A, B \subseteq \Omega$  such that  $A \subseteq B$ ,  $u_i(A) \leq u_i(B)$ . Free disposal is a sufficient condition for this assumption.

A feasible allocation  $X$  is a partition  $(X(0), X(1), \dots, X(n))$  of  $\Omega$ , where  $X(i)$  is agent  $i$ 's commodity bundle and  $X(0)$  is the unallocated bundle. An allocation  $X$  is optimal if  $\sum_{i \in N} u_i(X(i)) \geq \sum_{i \in N} u_i(Y(i))$  for all allocations  $Y$ . Define  $V(N) = \sum_{i \in N} u_i(X(i))$  for an optimal allocation  $X$ .

Given a price vector  $p \in R_+^{|\Omega|}$ , agent  $i$ 's trading profit function  $v_i : 2^\Omega \times R_+^{|\Omega|} \rightarrow R$  is defined by

$$v_i(A, p) = u_i(A) - \sum_{a \in A} p_a$$

and his demand correspondence  $D_i : R_+^{|\Omega|} \rightarrow 2^\Omega$  is defined by

$$D_i(p) = \{A \subseteq \Omega : v_i(A, p) \geq v_i(B, p), \forall B \subseteq \Omega\}.$$

A pair  $(X, p)$  of an allocation  $X$  and a price vector  $p$  is a Walrasian (competitive) equilibrium if  $p_a = 0$  for all  $a \in X(0)$  and  $X(i) \in D_i(p)$  for all  $i \in N$ .

## 3 The English Auction

We now present the English auction with multiple objects. Let  $t = 0, 1, 2, \dots$  be the infinite time horizon and each  $t$  represents a round.

Round  $t = 0$ . The auction starts with any price vector below the equilibrium price vectors. For example, the auction may start with the zero price vector.

Round  $t$ . Each agent submits  $b^i(t) \in R_+^m$ . The price vector  $p(b(t))$  at  $t$  is determined by

$$p_a(b(t)) = \max_{i \in N} b_a^i(t) \quad \forall a \in \Omega.$$

An agent  $i$ 's winning bundle is defined by

$$W_i(b(t)) = \{a \in \Omega : b_a^i(t) \geq b_a^{i'}(t) \quad \forall i' \in N\}.$$

$p(b(t))$  is publicly announced. Under  $p(b(t))$  agents conduct trade of objects. The auction will stop if there is a trade such that each object is demanded by one and only one agent under  $p(b(t))$ . Otherwise, all objects become open again and the auction goes to the next round,  $t + 1$ .

This English auction is a tâtonnement process and it is designed to obtain a Walrasian price vector since as long as the price vector is Walrasian the trade conducted by agents will be Walrasian. The stopping rule of the above auction is an analogy to the all-open-until-all-close rule in the spectrum auction. But there is one significant difference. In the spectrum auction each object is awarded to its winner at the close. Since the winning bundles may overlap in this paper, the awarded bundles at the close may be quite different from the winning bundles in the above auction form.

Is it possible for the auction to stop with a non-Walrasian price vector for the stopping rule designed this way? The following result derived from the extension of the Hall Theorem provides a negative answer to the question. For any non-Walrasian price vector with no object oversupplied, there exists at least one object demanded by two agents.

**Theorem 0:** Let  $p \in R_+^m$  be a price vector. Suppose that  $p$  is not Walrasian. Then for any collection  $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$  such that  $C_i \in D_i(p)$  for all  $i \in N$  and  $\cup_{i \in N} C_i = \Omega$ , there exists  $S \subseteq N$  such that

$$|\cup_{i \in S} C_i| < \sum_{i \in S} |C_i|.$$

**Proof:** Define the biparte graph  $G(p) = (N, V_2, E)$  such that  $V_2 = \{C_i : C_i \text{ is an element in } \mathcal{C}\}$ .  $(i, C) \in E$  if and only if  $C = C_i$ . Since  $p$  is not Walrasian, there does not exist a perfect match of the graph  $G(p)$ . Therefore Theorem A in the appendix applies to the graph  $G(p)$  and provides the desired result. *Q.E.D.*

Hence some commodity bundles must be overdemanded for a non-Walrasian price vector. Note that Theorem 0 is true for any collection of commodity bundles in agents' demand correspondences.



Therefore the English auction stops if and only if it stops with a Walrasian price vector. Because of this, the English auction above may not work because not all price sequences generated by it converge to a Walrasian equilibrium. Our task is to find the conditions such that every price sequence generated by the English auction converges to a Walrasian equilibrium within finite rounds. In other words we are looking for the environments such that the English auction designed this way works. This is the main task in the next two sections.

Before we leave this section, we would like to introduce the ascending bid condition. Let  $\{p(b(t))\}_{t=0,1,2,\dots}$  be a price sequence obtained from the English auction and define  $A_{t+1} = \{a \in \Omega : p_a(b(t+1)) > p_a(b(t))\}$ . The ascending bid condition is as follows.

**C1:**  $p(b(t+1)) \geq p(b(t))$  and  $A_{t+1} \neq \emptyset$  for each  $t$ .

Clearly the condition only applies to those rounds in which the auction does not stop. Since C1 is conditioned on the price vector not on agents' bids, agents' bids may not be ascending. C1 should be embodied as an auction rule in the design of the English auction.

## 4 Nash Equilibrium Bids

We first formulate the English auction by a sequence of normal form games. We now define the round games,  $\Gamma(t)$ . In each  $\Gamma(t)$ , we assume that the strategies for an agent are his bids in  $R_+^m$ . Given a profile of bids  $b(t)$ ,  $b(t) = (b^1(t), b^2(t), \dots, b^n(t)) \in R_+^{mn}$ , the payoff for an agent  $i$  in the game  $\Gamma(t)$  is the "trading" profit of his winning bundle defined by

$$v_i(W_i(b(t)), p(b(t))).$$

Equivalently,  $v_i(W_i(b(t)), p(b(t))) = v_i(W_i(b(t)), b^i(t))$ .

Hence the English auction is considered as a sequence of normal form games  $\Gamma(t)$ . We say that  $b(t) = (b^1(t), b^2(t), \dots, b^n(t))$  is a *Nash equilibrium* of the game  $\Gamma(t)$  if for all agents  $i \in N$

$$v_i(W_i(b(t)), p(b(t))) \geq v_i(W_i(\hat{b}^i, b^{-i}(t)), p(\hat{b}^i, b^{-i}(t)))$$

for all  $\hat{b}^i(t) \in R_+^m$ .

We assume that agents are myopic: Each agent  $i \in N$  submits his bids  $b^i(t)$  at each round  $t$  such that  $b^i(t)$  is a best response to the others' bids,  $b^{-i}(t)$ , at that round. Precisely, we make the following assumption on agents' bidding behavior:

**Myopic Nash Equilibrium (MNE):**  $\{p(b(t))\}_{t=0,1,2,\dots}$  satisfies the MNE assumption if for each round  $t$  each profile of bids  $b(t) = (b^1(t), b^2(t), \dots, b^n(t))$  is a *Nash equilibrium* of each round

game  $\Gamma(t)$ .

We often use  $p(t)$  for  $p(b(t))$  and  $W_i(t)$  for  $W_i(b(t))$ .

**Theorem 1:** Let  $\{p(t)\}_{t=0,1,2,\dots}$  be any price sequence obtained from the English auction that satisfies C1 and MNE. Then there exists a finite  $T$  such that  $p(T)$  is Walrasian.

We will show Theorem 1 in Section 7 after Theorem 2. Before we leave this section two remarks are in order. Although the English auction does not award the winning bundles to each agent at the close, the trading profit received by each agent at the close is equivalent to his trading profit under the rule such that he is awarded with his winning bundle under the MNE assumption. To see this, note that MNE implies that  $W_i(T) \in D_i(p(T))$  for all  $i \in N$ ; see the proof of Theorem 1. At the close  $T$ ,  $p(T)$  is Walrasian. Therefore we know that each agent is awarded with a bundle of a Walrasian allocation  $X$ , say. Since  $X(i) \in D_i(p(T))$ , it follows that

$$v_i(W_i(T), p(T)) = v_i(X(i), p(T)), \forall i \in N.$$

Even so, one may still wonder why not to award agents with their winning bundles at the close as in the single object situation. The answer is that the single object is a very special situation in which the awarded object accidentally coincides with the Walrasian allocation (i.e., the highest valuation bidder obtains the object with a price equal to the second highest valuation) and the final bid by the winner turns out to be a Nash equilibrium bid. In the multiple objects situation, these nice properties with the single object are no longer true. In fact as long as agents submit their Nash equilibrium bids at the close, the winning bundles must overlap. Indeed one can easily show that each object must receive at least two winning bids at the Nash equilibrium bids. Otherwise a bidder can always lower his bid for an object but still wins it; With the new bids he gets strictly better off and this disputes the Nash equilibrium assumption. In fact the overlapping situation happens with the single object in which the highest valuation and the second highest valuation bidders bid the same. Unfortunately the overlapping winning bundles no longer coincide with Walrasian allocation bundles in the multiple object situation.

An implication of this argument is that once agents are awarded with their winning bundles their winning bids must not be Nash equilibrium bids. In order to conform the Nash equilibrium with such an auction rule that awards agents with their winning bundles, complicated tie-breaking rules must be imposed. The tie-breaking approach in Bikhchandani (1996) may provide a way how to do this. However, we prefer not to use any tie-breaking device in this paper.

## 5 Non-Nash Equilibrium Bids

This section presents a weaker assumption on agents' bidding behavior than the MNE one. The reason for such a search comes from the fact that agents may not behave so rationally as the Nash equilibrium predicts. Nonetheless this does not mean that agents behave in an arbitrary manner in their bidding. For example, agents would at least like to “buy” what they win when they submit their bids. We will show that this simple condition together with the ascending bid one in C1 provides a sufficient condition for the English auction to achieve a Walrasian equilibrium in finite rounds.

We start with an example and deliver some intuition behind. The following example is due to Professor Preston McAfee and taken from McMillan (1994).

Suppose there are two licenses, 1 and 2. Hence  $\Omega = \{1, 2\}$ . There are three bidders  $N = \{e, w, nw\}$ , East, West and Nationwide. Bidders' utility functions or reservation values (in billions) are as follows

$$\begin{aligned} u_e(\{1\}) &= 2, & u_e(\{2\}) &= 1, & u_e(\{1, 2\}) &= 3 \\ u_w(\{1\}) &= 1, & u_w(\{2\}) &= 2, & u_w(\{1, 2\}) &= 3 \\ u_{nw}(\{1\}) &= 1.6, & u_{nw}(\{2\}) &= 1.6, & u_{nw}(\{1, 2\}) &= 3.3 \end{aligned}$$

These utility functions are clearly interdependent. We first find out all Walrasian equilibrium prices. Since the optimal allocation of licenses is such that  $X(e) = \{1\}$ ,  $X(w) = \{2\}$  and  $X(nw) = \emptyset$ , we must have  $\{1\} \in D_e(p)$ ,  $\{2\} \in D_w(p)$  and  $\emptyset \in D_n(p)$  for any Walrasian price vector  $p$ . Hence we obtain a system of linear inequalities

$$\begin{aligned} 1.6 &\leq p_1 \leq 2, & 1.6 &\leq p_2 \leq 2 \\ -1 &\leq p_1 - p_2 \leq 1, & p_1 + p_2 &\geq 3.3. \end{aligned}$$

This system has a solution and therefore a Walrasian equilibrium exists. For examples,  $(1.6, 1.7)$  and  $(1.7, 1.6)$  are Walrasian. Note that  $(1.6, 1.6)$  is not. This shows that not all exchange economies with multiple objects have the minimum Walrasian equilibrium.

McMillan (1994) provided a detailed analysis of this example and noted that (arbitrary) open auctions (that also allow bidders to submit combination bids) may not allocate the licenses efficiently. Therefore, open auctions that also allow combination bids may not converge to a Walrasian equilibrium. Nevertheless, we now use the English auction in Section 3 to find a price sequence that does converge to a Walrasian equilibrium. Because the English auction provides infinitely many Walrasian price sequences. The analysis given below provides only one such price sequence.

Round 0.

$i$	$e$	$w$	$nw$
$b^i(0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
$W_i(0)$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$
$D_i(p(0))$	$\{\{1, 2\}\}$	$\{\{1, 2\}\}$	$\{\{1, 2\}\}$

Round 1.

$i$	$e$	$w$	$nw$
$b^i(1)$	$(1, 1)$	$(1, 1)$	$(1, 1)$
$W_i(b(1))$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$
$D_i(p(1))$	$\{\{1\}, \{1, 2\}\}$	$\{\{2\}, \{1, 2\}\}$	$\{\{1, 2\}\}$

Round 2.

$i$	$e$	$w$	$nw$
$b^i(2)$	$(1.7, 1)$	$(1, 1)$	$(1.6, 1)$
$W_i(b(2))$	$\{1, 2\}$	$\{2\}$	$\{2\}$
$D_i(p(2))$	$\{\{1\}, \{1, 2\}\}$	$\{\{2\}\}$	$\{\{1, 2\}\}$

Round 3.

$i$	$e$	$w$	$nw$
$b^i(3)$	$(1.7, 1)$	$(1, 1.6)$	$(1.6, 1.6)$
$W_i(b(3))$	$\{1\}$	$\{2\}$	$\{2\}$
$D_i(p(3))$	$\{\{1\}\}$	$\{\{2\}\}$	$\{\{\emptyset\}, \{2\}, \{1, 2\}\}$

The price sequence is  $(0, 0), (1, 1), (1.7, 1)$  and  $(1.7, 1.6)$ . At  $(1.7, 1.6)$ , there is an allocation  $X$  such that  $X(e) = \{1\}, X(w) = \{2\}$  and  $X(nw) = \emptyset$ . Therefore, the auction closes with  $(1.7, 1.6)$ .

The importance of this price sequence is not the sequence itself because we know the set of Walrasian equilibrium prices. One can design many such price sequences. What is important is that this price sequence satisfies the following condition:

**C2:** At each round  $t$  and  $i \in N \exists A_i(t) \in D_i(p(t))$  such that  $W_i(b(t)) \subseteq A_i(t)$ .

That is the winning bundle at each round is included in one demand bundle for each agent in each round. The intuition of this condition is as follows. Imagine that there is a clock setting up for the auction. Suppose that the clock suddenly stops. Then the condition says that each agent is at least willing to buy his winning bundle when the clock stops although he may want to buy those objects that he is not winning. Equivalently this means that an agent does not want to withdraw his winning bids after winning.

**Theorem 2:** Let  $\{p(t)\}_{t=0,1,2,\dots}$  be a price sequence obtained from the English auction that satisfies C1 and C2. Then there always exists a finite  $T$  such that  $p(T)$  is Walrasian.

Note that C2 is different from MNE because a profile of bids satisfying C2 may not be a Nash equilibrium. For example, the bid profile  $((1, 7, 1), (1, 1.6), (1.6, 1.6))$  does not satisfy MNE at round 3. A MNE profile at round 3 could be  $((1.7, 1), (1, 1.6), (1.7, 1.6))$  or  $((1.7, 1), (1, 1.5), (1.7, 1.5))$ .

Therefore as long as agents' bidding behavior conforms with the condition C2 the English auction in Section 3 will work. It will close within finite rounds with a Walrasian equilibrium.

We will show Theorem 2 in Section 7. In what follows we develop some intuition behind the proof. We first introduce some useful notation and two preliminary results. Let  $p \in R_+^m$ . Since  $v_i(A, p) = v_i(B, p)$  for all  $A, B \in D_i(p)$ , we define  $v_i : R_+^m \rightarrow R_+$  by  $v_i(p) = v_i(C, p)$  for  $C \in D_i(p)$  and denote  $v(p)$  for  $(v_1(p), \dots, v_n(p))$ .

**Lemma 1:** Let  $p \in R_+^m$ . Then  $\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) \geq V(N)$ .

Suppose  $p$  is Walrasian. Then there exists a Walrasian allocation  $X$  such that

$$\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) = \sum_{i \in N} u_i(X(i)) = V(N)$$

since a Walrasian allocation is optimal. What follows says that the above equality is in fact a necessary and sufficient condition for a price vector to be Walrasian. This result is built on the necessary and sufficient condition for the existence of Walrasian equilibrium in Bikhchandani and Mamer (1994).

**Theorem 3:** Let  $p \in R_+^m$ .  $p$  is Walrasian if and only if

$$\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) = V(N).$$

Now define

$$\mathcal{G}(t) = \sum_{a \in \Omega} p_a(t) + \sum_{i \in N} v_i(p(t)).$$

Lemma 1 shows that  $\mathcal{G}(t) \geq V(N)$ . Theorem 3 shows that if  $p(t)$  is a price vector such that  $\mathcal{G}(t) = V(N)$  then  $p(t)$  is Walrasian. Therefore we need to find conditions such that  $\{\mathcal{G}(t)\}_{t=0,1,2,\dots}$  is decreasing since it is bounded below by  $V(N)$ . Because  $\{p(t)\}_{t=0,1,2,\dots}$  is ascending, the first term in  $\mathcal{G}(t)$  is strictly increasing. Lemma 4 in Section 7 shows that  $\mathcal{G}(t+1) \leq \mathcal{G}(t)$ , i.e., the second term in  $\mathcal{G}(t)$  must be strictly decreasing. However this does not mean that the decrease in the second term in  $\mathcal{G}(t)$  strictly dominates the increase in the first term for every  $t$ , i.e.,  $\mathcal{G}(t+1) > \mathcal{G}(t)$  for every  $t$ , because it may be the case that  $\mathcal{G}(t) = \mathcal{G}(t+1)$  for some  $t$ . What we have to make sure is that a state  $t$  such that  $\mathcal{G}(t) = \mathcal{G}(t+1)$  will not last infinitely many rounds, i.e., absorbing.

This is shown by Lemma 5 in Section 7. Therefore, we show that starting with any  $t$  such that the auction is not closed there always exists a finite  $T$  such that  $\mathcal{G}(t+T) < \mathcal{G}(t)$ , i.e., the decrease in the second term eventually strictly dominates the increase in the first term. This implies that there must exist a finite round such that  $\{\mathcal{G}(t)\}_{t=0,1,2,\dots}$  reaches its bound  $V(N)$ . Then Theorem 3 provides the result.

Without C1 Theorems 1 and 2 are not true. For example, each agent can submit  $0 \in R_+^m$  bids for each round and this satisfies both C2 and MNE. But the price vector 0 is not Walrasian for some exchange economies, e.g., the McAfee example. Without C2 Theorem 2 is not true either because it is possible that the increase in the first term dominates the decrease in the second term in  $\mathcal{G}(t)$ . For example, consider such situation in which each agent  $i$  bids  $0 \in R_+^m$  in round 0 and  $(B^i, \dots, B^i) \in R_+^m$  in round 1, where  $B^i > u_i(\Omega)$ . Then  $\mathcal{G}(0) = \sum_{i \in N} u_i(\Omega)$  but  $\mathcal{G}(1) = \sum_{a \in \Omega} \max_{i \in N} B_a^i$ , with  $\mathcal{G}(0) < \mathcal{G}(1)$ . By C1 it follows that  $\mathcal{G}(t) > \mathcal{G}(1)$  for every  $t = 2, 3, \dots$ . The auction never ends up with a Walrasian equilibrium, by Theorem 3.

## 6 Discussions

### Markovian Bidding

The winning bundles  $W_i(t)$  in C2 depend on how the other agents submit their bids. What follows is to define the “winning” bundles according to the prior price vector  $p(t-1)$ . The advantage for such a definition is the fact that bidders can determine their “winning” bundles in advance since the price vector  $p(t-1)$  is public information. One may consider such bidding behavior as a version of the “Markovian” bidding, i.e., agents play their bids dependent on the prior prices only. The bids under C2 may be depend on the prior prices and beyond. Define

$$\tilde{W}_i(t) = \{a \in \Omega : b_a^i(t) \geq p_a(t-1)\} \quad \forall i \in N.$$

**C3:** For each  $t$  and  $i \in N$   $\exists A_i(t) \in D_i(p(t))$  such that  $\tilde{W}_i(t) \subseteq A_i(t)$ .

Since  $p_a(t) = \max_{i \in N} b_a^i(t)$  and  $p(t) > p(t-1)$ , it follows that  $W_i(t) \subseteq \tilde{W}_i(t)$ . By C3 there exists  $A_i(t) \in D_i(p(t))$  such that  $\tilde{W}_i(t) \subseteq A_i(t)$ . Therefore, C3 satisfies C2.

**Corollary 1:** Let  $\{p(t)\}_{t=0,1,2,\dots}$  be a price sequence obtained from the English auction that satisfies C1 and C3. Then there always exists a finite  $T$  such that  $p(T)$  is Walrasian.

## The Gross Substitutes Condition

Kelso and Crawford (1982) studied a job-matching market in which firms may hire as many workers as they wish and have interdependent utility functions over workers. The exchange economy studied in this paper is analogous to theirs. But there is a significant difference. A worker's utility function in Kelso and Crawford (1982) depends on not only the salaries but also the firm who hired him. Therefore, their job-matching model is more complicated in the aspects of sellers' utility functions. In the commodity markets we assume that objects (sellers) do not care about the names of the buyers.

Given a utility function  $u$  and any two price vectors  $p, q \in R_+^m$ , define  $T : 2^\Omega \rightarrow 2^\Omega$  by

$$T(A) = \{a \in A : A \in D(p) \text{ and } p_a = q_a\}.$$

**Gross Substitutes (GS):** For every  $i \in N$ , if  $A \in D_i(p)$  and  $q \geq p$ , then there exists  $\bar{A} \in D_i(q)$  such that  $T_i(A) \subseteq \bar{A}$ .

Intuitively, an agent demands an object at a price vector  $p$  and he still demands it if some other objects' prices are increased. Under the gross substitutes, a Walrasian equilibrium always exists in the current model. This follows from the fact that the core is nonempty (Kelso and Crawford (1982)) and the equivalence theorem.

Gül and Stacchetti (1996b) introduced a new condition, *no complementarities* (NC), and showed that their NC condition is equivalent to GS.

**No Complementarities (NC):** A utility function  $u$  satisfies the NC condition if for each price vector  $p$ , and all bundles  $A, B \in D(p)$  and  $X \subseteq A$ , there exists a bundle  $Y \subseteq B$  such that  $[A \setminus X] \cup Y \in D(p)$ .

Under the GS or NC condition agents have a simple way to bid.

**C4:** For each  $t$  and  $i \in N \exists A_i(t) \in D_i(b^i(t))$  such that  $\tilde{W}_i(t) \subseteq A_i(t)$ .

C4 states that agents submit bids on those objects that he demands at his bids, given the price vector  $p(t-1)$ . This condition depends on the information disclosed after each prior round. Hence, under the gross substitutes condition, an agent can ignore how the others may submit their bids. Indeed, what he needs to do is to decide which bundle he wants to win relative to the prior price

vector. And the winning bundle is a bundle he really demands according to his own submitted bids. He needs to know nothing about the others.

Suppose that  $u_i$  satisfies the gross substitutes condition for all  $i \in N$ . We show that C4 satisfies C2. To see this, note that  $W_i(t) \subseteq \tilde{W}_i(t)$ . C4 requires that there exist  $A \in D_i(b^i(t))$  such that  $W_i(t) \subseteq \tilde{W}_i(t) \subseteq A$ . On the other hand, for each  $a \in W_i(t)$ ,  $p_a(t) = b_a^i(t)$ . Since  $p(t) \geq b^i(t)$  by definition, the gross substitutes condition requires that there exist  $B \in D_i(p(t))$  such that  $W_i(t) \subseteq B$ . This is exactly C2. Thus, we obtain a useful corollary.

**Corollary 2:** Suppose that  $u_i$  satisfies the GS or NC condition for each  $i \in N$ . Let  $\{p(t)\}_{t=0,1,2,\dots}$  be any price sequence obtained in the English auction that satisfies C1 and C4. Then there always exists a finite  $T$  such that  $p(T)$  is Walrasian.

Corollary 2 is also true if C4 takes the following form:

**C4':** For each  $t$  and  $i \in N \exists A_i(t) \in D_i(b^i(t))$  such that  $W_i(t) \subseteq A_i(t)$ .

### “Mistakes”

In the close of this section we will discuss what will happen if agents do not conform with the MNE assumption or the condition C2 for sometimes. Does the English auction still work by allowing agents to make these “mistakes” occasionally? As long as agents do not make such mistakes for infinitely many times, the English auction will still end up within finite rounds with a Walrasian equilibrium. This follows from the following two versions of Theorems 1 and 2.

**Theorem 1':** Let  $\{p(t)\}_{t=0,1,2,\dots}$  be any price sequence obtained from the English auction and  $\{q(s)\}_{s=0,1,2,\dots}$  be any subsequence of  $\{p(t)\}_{t=0,1,2,\dots}$  that satisfies C1 and MNE. Then there always exists a finite  $T$  such that  $q(T)$  is Walrasian.

**Theorem 2':** Let  $\{p(t)\}_{t=0,1,2,\dots}$  be any price sequence obtained from the English auction and  $\{q(s)\}_{s=0,1,2,\dots}$  be any subsequence of  $\{p(t)\}_{t=0,1,2,\dots}$  that satisfies C1 and C2. Then there always exists a finite  $T$  such that  $q(T)$  is Walrasian.

The proofs of Theorems 1' and 2' directly follow from the proofs of Theorems 1 and 2. These results show that “mistakes” are forgettable as long as agents do not make such “mistakes” so often. This is especially useful at the very early stage of the auction.



## 7 Proofs of Theorems 1 and 2

### 7.1 Preliminary Results

We first prove the two preliminary results, Lemma 1 and Theorem 3.

**Lemma 1:** Let  $p \in R_+^m$ . Then  $\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) \geq V(N)$ .

**Proof:**  $v_i(p) \geq u_i(A) - \sum_{a \in A} p_a$  for any  $A \subseteq \Omega$ . Hence

$$\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) \geq \sum_{a \in \Omega \setminus Y(0)} p_a + \sum_{i \in N} v_i(Y(i), p)$$

for any allocation  $Y$ . It follows that

$$\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) \geq \sum_{a \in \Omega \setminus X(0)} p_a + \sum_{i \in N} v_i(X(i), p) = V(N)$$

for an optimal allocation  $X$ .

*Q.E.D.*

**Theorem 3:** Let  $p \in R_+^m$ .  $p$  is Walrasian if and only if

$$\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) = V(N).$$

**Proof:** It is sufficient to show the only if part. The value  $V(N)$  is computed according to the following integer linear problem, as in Bikhchandani and Mamer (1994).

$$\begin{aligned}
 (1) \quad V(N) &= \max_{x_{il}} \sum_{i=1}^n \sum_{l=1}^{2^{|\Omega|-1}} u_i(S_l) x_{il} \\
 (2) \quad & \text{s.t.} \quad \sum_{l=1}^{2^{|\Omega|-1}} a_{sl} \sum_{i=1}^n x_{il} \leq 1, \forall s = 1, 2, \dots, |\Omega| \\
 (3) \quad & \sum_{l=1}^{2^{|\Omega|-1}} x_{il} \leq 1, \forall i = 1, 2, \dots, n \\
 & x_{il} = 0 \text{ or } 1, \forall i, l
 \end{aligned}$$

where  $S_l \subseteq \Omega$ ,  $a_{sl}$  is the elements in the matrix  $A_{|\Omega| \times (2^{|\Omega|-1})}$  such that  $a_{sl} = 1$  if  $s \in S_l$  or 0 otherwise.

Consider the following linear programming without the integer constrains.

$$\begin{aligned} \bar{V}(N) &= \max_{x_{il}} \sum_{i=1}^n \sum_{l=1}^{2^{|\Omega|-1}} u_i(S_l) x_{il} \\ (4) \quad & \text{s.t.} \quad \sum_{l=1}^{2^{|\Omega|-1}} a_{sl} \sum_{i=1}^n x_{il} \leq 1, \forall s = 1, 2, \dots, |\Omega| \\ (5) \quad & \sum_{l=1}^{2^{|\Omega|-1}} x_{il} \leq 1, \forall i = 1, 2, \dots, n \\ (6) \quad & x_{il} \geq 0, \forall i, l \end{aligned}$$

It follows from the dual theorem that

$$\begin{aligned} (7) \quad \bar{V}(N) &\equiv \min_{x_s, v_i} \sum_{s=1}^{|\Omega|} x_s + \sum_{i=1}^n v_i \\ (8) \quad & \text{s.t.} \quad \sum_{s=1}^{|\Omega|} a_{sl} x_s + v_i \geq u_i(S_l), \forall l \\ (9) \quad & x_s \geq 0, v_i \geq 0, \forall s, i \end{aligned}$$

Now eq. (8) is equivalent to

$$(10) \quad v_i \geq u_i(S_l) - \sum_{s \in S_l} x_s, \forall l.$$

Therefore, it is binding for all  $l^*$  such that  $S_{l^*} \in D_i(x)$  and  $v_i(x) = u_i(S_{l^*}) - \sum_{a \in S_{l^*}} x_a$ . Substitute this into  $\bar{V}(N)$ , we obtain that

$$\begin{aligned} \bar{V}(N) &= \min_{x \in R_+^{|\Omega|}} \sum_{s=1}^{|\Omega|} x_s + \sum_{i=1}^n v_i(x) \\ &= \min_{(x, v(x))} \sum_{a \in \Omega} x_a + \sum_{i \in N} v_i(x). \end{aligned}$$

Clearly  $V(N) \leq \bar{V}(N)$ . Thus, by the assumption that

$$\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) = V(N),$$

it follows that  $V(N) = \bar{V}(N)$ . Now Lemma 6 in Bikhchandani and Mamer (1994) showed that there exists an allocation  $X$  such that  $(p, X)$  is Walrasian. *Q.E.D.*

The next two lemmas are useful for their own sakes.

**Lemma 2:**  $v(p)$  is convex, continuous and monotonely decreasing in  $p$ .

**Proof :** (a).  $v$  is convex. Let  $p$  and  $q$  be two price vectors. Then  $v_i(\lambda p + (1 - \lambda)q) \leq \lambda v_i(p) + (1 - \lambda)v_i(q)$  for all  $i \in N$  since

$$\max_{A \subseteq \Omega} u_i(A) - \sum_{a \in A} [\lambda p_a + (1 - \lambda)q_a] \leq \lambda [\max_{A \subseteq \Omega} u_i(A) - \sum_{a \in A} p_a] + (1 - \lambda) [\max_{A \subseteq \Omega} u_i(A) - \sum_{a \in A} q_a].$$

(b).  $v$  is continuous.  $v(p)$  is continuous because  $v_i(A, p) = u_i(A) - \sum_{a \in A} p_a$  is continuous in  $p$ .

(c).  $v$  is monotonely decreasing. Suppose that  $p \geq q$ . Let  $i \in N$  and  $A \in D_i(p)$  and  $B \in D_i(q)$ .

Then

$$\begin{aligned} v_i(p) - v_i(q) &= [u_i(A) - \sum_{a \in A} p_a] - [u_i(B) - \sum_{b \in B} q_b] \\ &\leq [u_i(A) - \sum_{a \in A} p_a] - [u_i(A) - \sum_{a \in A} q_a] \\ &= \sum_{a \in A} q_a - \sum_{a \in A} p_a \\ &\leq 0. \end{aligned}$$

*Q.E.D.*

Define  $\mathcal{F} = \{(p, v(p)) : p \text{ is Walrasian}\}$ .

**Lemma 3:**  $\mathcal{F}$  is a convex set.

**Proof:** Let  $(p, v(p))$  and  $(q, v(q))$  in  $\mathcal{F}$ . Then Theorem 3 shows that

$$\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p) = V(N)$$

and

$$\sum_{a \in \Omega} q_a + \sum_{i \in N} v_i(q) = V(N).$$

Therefore, by the convexity of  $v$  in Lemma 2,

$$\begin{aligned} \sum_{a \in \Omega} [\lambda p_a + (1 - \lambda)q_a] + \sum_{i \in N} v_i(\lambda p + (1 - \lambda)q) \\ \leq \lambda [\sum_{a \in \Omega} p_a + \sum_{i \in N} v_i(p)] + (1 - \lambda) [\sum_{a \in \Omega} q_a + \sum_{i \in N} v_i(q)] \\ = V(N). \end{aligned}$$

It follows from Lemma 1 that

$$V(N) = \sum_{a \in \Omega} [\lambda p_a + (1 - \lambda)q_a] + \sum_{i \in N} v_i(\lambda p + (1 - \lambda)q).$$

Therefore,  $(\lambda p + (1 - \lambda)q, v(\lambda p + (1 - \lambda)q)) \in \mathcal{F}$ , by Theorem 3.

*Q.E.D.*

## 7.2 Proofs of Theorems 1 and 2

We now turn to the proofs of Theorems 1 and 2. We need Lemmas 4 and 5. Lemma 4 shows that  $\mathcal{G}(t)$  is not increasing. Lemma 5 shows that those rounds  $t$  such that  $\mathcal{G}(t+1) = \mathcal{G}(t)$  are not absorbing.

**Lemma 4:** For any ascending price sequence  $\{p(t)\}_{t=0,1,2,\dots}$  obtained from the English auction that satisfies C2, we have that  $\mathcal{G}(t+1) \leq \mathcal{G}(t)$ .

**Proof:** Let  $A_{t+1} = \{a \in \Omega : p_a(t+1) > p_a(t)\}$  and  $\delta_a = p_a(t+1) - p_a(t)$ . Let  $\{A_1(t+1), \dots, A_n(t+1)\}$  be a collection such that  $A_i(t+1) \in D_i(p(t+1))$  and  $W_i(t+1) \subseteq A_i(t+1)$  for all  $i \in N$ . Denote  $I = \{i \in N : W_i(t+1) \cap A_{t+1} \neq \emptyset\}$ . Therefore  $\cup_{i \in I} W_i(t+1) \cap A_{t+1} = A_{t+1}$  and then  $\cup_{i \in I} A_i(t+1) \cap A_{t+1} = A_{t+1}$ . It follows that

$$\sum_{a \in A_{t+1}} \delta_a - \sum_{i \in N} \sum_{a \in A_i(t+1)} \delta_a = \sum_{a \in A_{t+1}} \delta_a - \sum_{i \in I} \sum_{a \in A_i(t+1) \cap A_{t+1}} \delta_a \leq 0.$$

Let  $\{A_1(t), \dots, A_n(t)\}$  be any collection such that  $A_i(t) \in D_i(p(t))$ . Then  $\sum_{i \in N} [v_i(A_i(t+1), p(t+1)) - v_i(A_i(t), p(t))] \leq 0$  and

$$\begin{aligned} \mathcal{G}(t+1) - \mathcal{G}(t) &= \sum_{a \in \Omega} [p_a(t+1) - p_a(t)] + \sum_{i \in N} [v_i(p(t+1)) - v_i(p(t))] \\ &= \sum_{a \in A_{t+1}} \delta_a + \sum_{i \in N} [v_i(A_i(t+1), p(t+1)) - v_i(A_i(t), p(t))] \\ &= \sum_{a \in A_{t+1}} \delta_a + \sum_{i \in N} [v_i(A_i(t+1), p(t)) - v_i(A_i(t), p(t))] - \sum_{i \in N} \sum_{a \in A_i(t+1)} \delta_a \\ &\leq \sum_{a \in A_{t+1}} \delta_a - \sum_{i \in N} \sum_{a \in A_i(t+1)} \delta_a \\ &\leq 0. \end{aligned}$$

*Q.E.D.*

**Lemma 5:** Suppose that  $\{p(t)\}_{t=0,1,2,\dots}$  is an ascending price sequence obtained from the English auction that satisfies C2. If  $p(t)$  is not Walrasian, then there exists a finite  $T$  such that  $D_i(p(t+T)) \not\subseteq D_i(p(t))$  for some  $i \in N$ .

**Proof:** Suppose on the contrary that for all  $s \in \{t, t+1, \dots\}$  we have either (a)  $D_i(p(s+1)) = D_i(p(s))$  for all  $i \in N$  or (b)  $D_i(p(s+1)) \subseteq D_i(p(s))$  for all  $i \in N$  and  $D_i(p(s+1)) \subset D_i(p(s))$  for some  $i$ . Since for any such round  $s$  the fact that  $p(s+1)$  is Walrasian implies that  $p(s)$  is Walrasian as well under both cases (a) and (b),  $p(s)$  is not Walrasian for any  $s \in \{t, t+1, \dots\}$ . The auction must continue forever. Let  $M = \max_{i \in N} u_i(\Omega)$ . Clearly  $M$  is finite. Because  $p(t)$  is ascending, it

follows that there exist at least one object  $a$  and  $s \in \{t, t+1, \dots\}$  such that  $p_a(s) > M$ . Therefore,  $a$  is not demanded by any agent with price  $p_a(s)$ . But by C2 there exist  $i \in N$  and  $A \in D_i(p(s))$  such that  $a \in W_i(s) \subseteq A$ . This is a contradiction. *Q.E.D.*

**Theorem 2:** Let  $\{p(t)\}_{t=0,1,2,\dots}$  be a price sequence obtained from the English auction that satisfies C1 and C2. Then there always exists a finite  $T$  such that  $p(T)$  is Walrasian.

**Proof:** Suppose the auction does not stop at  $t$ . We show that there exists a finite  $T(t)$  and a positive  $\epsilon(t) > 0$  such that

$$\mathcal{G}(t + T(t)) + \epsilon(t) \leq \mathcal{G}(t + T(t) - 1) \leq \dots \leq \mathcal{G}(t + 1) \leq \mathcal{G}(t).$$

Since  $\epsilon(t)$  is bounded away from zero and  $T(t)$  is finite, there exists a finite  $T$  such that  $\mathcal{G}(T) = V(N)$  because  $V(N)$  is the lower bound of  $\mathcal{G}(t)$  by Lemma 1 and Theorem 3. Therefore, there must exist a finite  $T$  such that

$$\sum_{a \in \Omega} p_a(T) + \sum_{i \in N} v_i(p(T)) = V(N).$$

Theorem 3 shows that  $p(T)$  must be Walrasian. The auction must stop at  $T$ .

Since  $p(t)$  is not Walrasian, Lemma 5 shows that there exists a finite  $T(t)$  such that  $D_i(p(t + T(t))) \not\subseteq D_i(p(t + T(t) - 1))$  for some  $i$ . We now show that there exists  $\epsilon(t) > 0$  such that

$$\mathcal{G}(t + T(t)) + \epsilon(t) \leq \mathcal{G}(t + T(t) - 1).$$

Without loss of generality, suppose  $T(t) = 1$ . For each  $A \in D_i(p(t))$  and  $B \notin D_i(p(t))$ , there exists  $\epsilon > 0$  such that  $v_i(A, p(t)) \geq \epsilon + v_i(B, p(t))$ . The fact that  $D_i(p(t + 1)) \not\subseteq D_i(p(t))$  for some  $i$  implies that there exists  $A$  such that  $A \in D_i(p(t + 1))$  and  $A \notin D_i(p(t))$ . Let  $\{A_1(t + 1), A_2(t + 1), \dots, A_n(t + 1)\}$  be the collection such that  $A_i(t + 1) \in D_i(p(t + 1))$ ,  $W_i(t + 1) \subseteq A_i(t + 1)$  for all  $i \in N$  and  $A_i(t + 1) \notin D_i(p(t))$  for some  $i$ . Also let  $\{B_1(t + 1), \dots, B_n(t + 1)\}$  and  $\{A_1(t), \dots, A_n(t)\}$  be any collections such that  $B_i(t + 1) \in D_i(p(t + 1))$  and  $A_i(t) \in D_i(p(t))$  for all  $i \in N$ . Then there exists  $\epsilon(t) > 0$  such that

$$\begin{aligned} \mathcal{G}(t + 1) &= \sum_{a \in \Omega} p_a(t + 1) + \sum_{i \in N} v_i(B_i(t + 1), p(t + 1)) \\ &= \sum_{a \in \Omega} p_a(t + 1) + \sum_{i \in N} v_i(A_i(t + 1), p(t + 1)) \\ &= \sum_{a \in \Omega} p_a(t) + \sum_{a \in A_{t+1}} \delta_a + \sum_{i \in N} [v_i(A_i(t + 1), p(t)) - \sum_{a \in A_i(t+1)} \delta_a] \\ &\leq -\epsilon(t) + \sum_{a \in \Omega} p_a(t) + \sum_{a \in A_{t+1}} \delta_a + \sum_{i \in N} [v_i(A_i(t), p(t)) - \sum_{a \in A_i(t+1)} \delta_a] \\ &= -\epsilon(t) + \mathcal{G}(t) + \sum_{a \in A_{t+1}} \delta_a - \sum_{i \in N} \sum_{a \in A_i(t+1) \cap A_{t+1}} \delta_a \\ &\leq -\epsilon(t) + \mathcal{G}(t). \end{aligned}$$

In the first inequality above we use the fact that  $D_i(p(t+1)) \not\subseteq D_i(p(t))$  for some  $i$ . The last inequality follows from the fact that

$$\sum_{a \in A_{t+1}} \delta_a - \sum_{i \in N} \sum_{a \in A_i(t+1) \cap A_{t+1}} \delta_a \leq 0,$$

as shown in the proof of Lemma 4.

Hence it follows from Lemma 4 that there exists a finite  $T(t)$  and a positive  $\epsilon(t) > 0$  such that

$$\mathcal{G}(t + T(t)) + \epsilon(t) \leq \mathcal{G}(t + T(t) - 1) \leq \dots \leq \mathcal{G}(t + 1) \leq \mathcal{G}(t).$$

*Q.E.D.*

**Theorem 1:** Let  $\{p(t)\}_{t=0,1,2,\dots}$  be any price sequence obtained from the English auction that satisfies C1 and MNE. Then there exists a finite  $T$  such that  $p(T)$  is Walrasian.

**Proof:** We show that MNE satisfies condition C2. Then Theorem 1 follows from Theorem 2. We need to show that  $W_i(b(t)) \in D_i(p(b(t)))$  for all  $i \in N$  whenever  $b(t)$  is a Nash equilibrium of the round game  $\Gamma(t)$ . Suppose on the contrary that  $W_i(b(t)) \notin D_i(p(b(t)))$  for some  $i \in N$ . Then

$$v_i(A_i(t), p(b(t))) > v_i(W_i(b(t)), p(b(t))), \quad \forall A_i(t) \in D_i(p(b(t))).$$

Given  $A_i(t) \in D_i(p(b(t)))$ , we construct a profile of bids  $\bar{b}^i(t) \in R_+^m$  as follows:

$$\bar{b}_a^i(t) = \begin{cases} b_a^i(t) & \forall a \in A_i(t) \cap W_i(b(t)) \\ p_a(b(t)) & \forall a \in A_i(t) \setminus W_i(b(t)) \\ 0 & \forall a \notin A_i(t) \text{ and } p_a(b(t)) = 0 \\ p_a(b(t)) - \epsilon & \forall a \notin A_i(t) \text{ and } p_a(b(t)) > 0 \end{cases}$$

where  $\epsilon > 0$ .

Denote  $\mathcal{O} = \{a \in \Omega : a \notin A_i(t) \text{ and } p_a(b(t)) = 0\}$ . Hence,

$$W_i((b^{-i}, \bar{b}^i)(t)) = A_i(t) \cup \mathcal{O}.$$

Since  $b(t)$  is a Nash equilibrium of the game  $\Gamma(t)$ , there are at least two different agents  $j$  and  $k$  such that  $a \in W_j(b(t)) \cap W_k(b(t))$  for any  $a \in \Omega$ . It follows that there exists  $j \in N \setminus \{i\}$  such that

$$p_a((b^{-i}, \bar{b}^i)(t)) = b_a^j(t) = p_a(b(t))$$

for any  $a \notin A_i(t)$  such that  $a \in W_i(b(t))$  and  $p_a(b(t)) > 0$ . Therefore, we have that  $p((b^{-i}, \bar{b}^i)(t)) = p(b(t))$ .

Since  $u_i$  is weakly monotone, it follows that

$$\begin{aligned} v_i(W_i((b^{-i}, \bar{b}^i)(t)), p((b^{-i}, \bar{b}^i)(t))) &\geq v_i(A_i(t), p((b^{-i}, \bar{b}^i)(t))) \\ &= v_i(A_i(t), p(b(t))) \\ &> v_i(W_i(b(t)), p(b(t))). \end{aligned}$$

This is a contradiction to the assumption that  $b(t)$  is a Nash equilibrium.

*Q.E.D.*

## Appendix

In this appendix we introduce the extension of the Hall Theorem. Let  $G = (V_1, V_2, E)$  be a bipartite graph with vertex classes  $V_1 = \{x_1, x_2, \dots, x_n\}$  and  $V_2 = \{y_1, y_2, \dots, y_m\}$  and edges  $E \subseteq V_1 \times V_2$ . Let  $\Gamma(S)$  denote the set of vertices adjacent to  $S$  for  $S \subseteq V_1$  and  $d(x) = |\Gamma(x)|$ . The next definition is standard and a matching is often called a perfect match in the graph theory. A subgraph  $H$  of  $G$  is called a many-to-one (one-to-one) *matching* if  $H : V_1 \cup V_2 \rightarrow 2^{V_1 \cup V_2}$  such that:

1. For all  $x_i \in V_1$ , ( $|\Gamma_H(x_i)| \leq 1$  and)  $\Gamma_H(x_i) \subseteq V_2$ ;
2. For all  $y_j \in V_2$ ,  $|\Gamma_H(y_j)| \leq 1$  and  $\Gamma_H(y_j) \subseteq V_1$ ;
3. For all  $x_i \in V_1$  and  $y_j \in V_2$ ,  $\Gamma_H(y_j) = x_i$  if and only if  $y_j \in \Gamma_H(x_i)$ .

The Hall theorem and its extension are as follows. For a proof of Theorem A, see Bollobás (1979). Also see Gale (1960) and Hall (1935).

**Theorem A** (The Hall Theorem and its Extension): Let  $G$  be a bipartite graph with vertex classes  $V_1 = \{x_1, x_2, \dots, x_m\}$  and  $V_2 = \{y_1, \dots, y_n\}$ . Then  $G$  contains a matching  $H$  of  $G$  such that  $d_H(x_i) = d_i$  and  $0 \leq d_H(y_j) \leq 1$  if and only if

$$|\Gamma(S)| \geq \sum_{x_i \in S} d_i$$

for every  $S \subseteq V_1$ , where  $d_H(x)$  denotes the number of edges connected with  $x$  in the matching  $H$ .

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