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# The Perils of Taylor Rules

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## Abstract

Since John Taylor's (1993) seminal paper, a large literature has argued that active interest rate feedback rules, that is, rules that respond to increases in inflation with a more than one-for-one increase in the nominal interest rate, are stabilizing. In this paper, we argue that once the zero bound on nominal interest rates is taken into account, active interest-rate feedback rules can easily lead to unexpected consequences. Specifically, in the context of a sticky-price model, we show that even if the steady state at which monetary policy is active is locally the unique equilibrium, typically there exists an infinite number of equilibrium trajectories originating arbitrarily close to that steady state that converge either to another steady state at which monetary policy is passive or to a stable limit cycle around the active steady state. We conclude that the use of local techniques for monetary policy evaluation might lead to spurious policy recommendations.

*JEL Classification* Numbers: E52, E31, E63.

**Keywords:** Interest rate feedback rules, zero bound on nominal rates, liquidity traps, multiple equilibria.

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# 1 Introduction

Since John Taylor's (1993) seminal paper describing Federal Reserve policy, there has been a resurgence of interest in monetary policies that target the nominal rate. Much of the literature has explored the efficiency and dynamic effects of such policies, with particular attention to their stabilization properties. A central policy recommendation that has emerged from this body of research is that "active monetary policy," that is, a policy that strongly responds to the rate of inflation in setting the nominal interest rate, is stabilizing.<sup>1</sup> In an earlier paper (Benhabib, Schmitt-Grohé and Uribe, 1998) we argue that such result depends very much on the specification of the model, and that indeed often active monetary feedback policies lead to multiple equilibria under standard specifications, assumptions and calibrations, including models with sticky prices, Taylor rules that allow for leads or lags, and Ricardian and non-Ricardian monetary/fiscal regimes. In this paper, we take an even stronger position and argue that active monetary policy generally leads to indeterminacy and multiple equilibria, and that pursuing such a policy can easily lead to unexpected consequences even in the simplest and most innocuous monetary models, using the simplest and most standard assumptions.

Our method of analysis departs from the conventional local approach to study multiple equilibria that proceeds by linearizing around a steady state. The reason for this departure stems from the observation that the nominal rate must be constrained to be non-negative, since negative nominal rates are impossible. It immediately follows from this observation, as we illustrate below, that if there is a steady state with an active monetary policy, there must necessarily exist another steady state with a passive policy. As a result, local analysis is inadequate because paths of the economy diverging from one steady state can converge to the other steady state or to another attracting set, thus qualifying as equilibrium trajectories. We show these results in the context of a sticky-price model both theoretically and through simulations of calibrated economies.

To intuitively illustrate the source of multiplicity, consider the Fisher equation

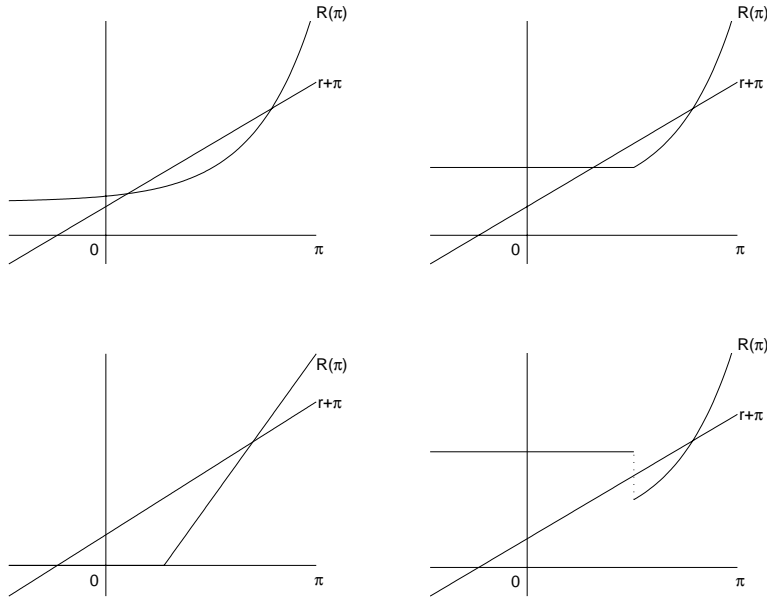
$$R(\pi) = r + \pi,$$

where  $R(\pi)$  is a simplified Taylor rule that sets the nominal rate,  $R(\pi)$ , as a non-decreasing function of inflation,  $r$  is the discount rate, and  $\pi$  is the inflation rate. This steady-state relation is common to a wide range of monetary models with representative agents and an infinite horizon and holds irrespective of whether prices are flexible or sticky or of whether money enters the model through the utility function, the production function or a cash-in-advance constraint. If there exists a steady state with active monetary policy, that is, a value of  $\pi$

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<sup>1</sup>For papers arriving at this conclusion in the context of non-optimizing models, see Levin, Wieland, and Williams (1998) and Taylor (1998a, 1998b); for optimizing models with flexible prices, see Leeper (1991); and for optimizing models with nominal frictions see Rotemberg and Woodford (1997, 1998), Christiano and Gust (1998), and Clarida, Gali, and Gertler (1997).

Figure 1: Taylor Rules, zero-bound on nominal rates, and multiple steady states



that solves the Fisher equation and satisfies  $R'(\pi) > 1$ , then if  $R(\pi) \geq 0$  and continuous, there must exist another steady state with  $R'(\pi) < 1$  (figure 1).<sup>2</sup>

Since the inflation rate  $\pi$  is a jump variable whose initial value is not exogenously given, the existence of a steady state with active policies immediately establishes the possibility of the existence of at least two steady state equilibria. Clearly, the existence of multiple solutions to the steady-state Fisher equation need not imply the existence of multiple steady-state equilibria, for in general the equilibrium conditions will involve other equations. However, in this paper we show, in the context of a standard model with nominal rigidities, that in general the presence of a steady state equilibrium at which monetary policy is active gives rise to at least one other steady-state equilibrium at which monetary policy is passive.

But it would be naive to conclude that active interest-rate rules are destabilizing solely because they give rise to multiple steady-state equilibria. After all, the observed inflation dynamics do not seem to be consistent with an economy in which inflation constantly jumps from a level at which monetary policy

<sup>2</sup>Note that for the existence of two solutions to the steady-state Fisher equation it is not crucial that the Taylor rule be continuous. It is sufficient that the Taylor rule is non-negative, non-decreasing and that one solution occurs at a value of  $\pi$  for which  $R'(\pi) > 1$ . The bottom right panel of figure 1 displays a case in which there is a unique solution to the Fisher equation even though at that solution  $R'(\pi) > 1$ . The absence of a second solution results not because the Taylor rule is discontinuous but because it is non-monotonic. We will not explore the macroeconomic consequences of Taylor rules of this type because we believe that they are irrelevant for it is implausible that the central bank will implement a discrete increase in the nominal interest rate in the context of declining inflation.

is active to another at which monetary policy is passive. The main result of this paper is that active Taylor rules are destabilizing because the multiplicity of steady-equilibria that they induce opens the door to a much larger class of equilibria. Specifically, we show that in general there exists an infinite number of equilibrium trajectories originating in the vicinity of the active steady state that converge either to the steady state at which monetary policy is passive (a saddle connection) or to a stable limit cycle around the active steady state. Interestingly, along both the saddle connection and the limit cycle, the inflation rate fluctuates for long periods of time around the steady state at which monetary policy is active. Thus, an econometrician using data generated from such equilibria to estimate the slope of the Taylor rule may very well conclude that monetary policy has always been active.

Simulations of calibrated versions of the model indicate that saddle connections from the active steady state to the passive steady state exist for empirically plausible parameterizations and are indeed the most typical pattern as they are robust to wide parameter perturbations. This type of equilibrium is of particular interest because it sheds light on the precise way in which economies may fall into liquidity traps. The results suggest that central banks that maintain an active monetary policy stance near a given inflation target are more likely to lead the economy into a deflationary spiral—like the one currently observed in Japan and, as some may argue, in the United States—than central banks that maintain a globally passive monetary stance such as an interest- or exchange-rate peg.

## 2 A Sticky-Price Model

The economy is assumed to be populated by a continuum of household–firm units indexed by  $j$  each of which produces a differentiated good  $Y^j$ . Firms have market power and set prices so as to maximize profits. The demand faced by firm  $j$  is given by  $Y^d d(P^j/P)$ , where  $Y^d$  denotes the level of aggregate demand,  $P^j$  the price firm  $j$  charges for the good it produces, and  $P$  the aggregate price level. Such a demand function can be derived by assuming that households have preferences over a composite good that is produced from differentiated intermediate goods via a Dixit-Stiglitz production function. The function  $d(\cdot)$  is assumed to satisfy  $d(1) = 1$  and  $d'(1) < -1$ . The restriction imposed on  $d'(1)$  is necessary for the firm’s problem to be well defined in a symmetric equilibrium. The production of good  $j$  uses labor  $h^j$  provided by the household  $j$  as the only input:

$$Y^j = y(h^j)$$

where  $y(\cdot)$  is positive, strictly increasing, strictly concave and satisfies the Inada conditions.

We introduce nominal price rigidity, following Rotemberg (1982), by assuming that households face convex costs of adjusting prices. Specifically, the

household's lifetime utility function is assumed to be of the form

$$U^j = \int_0^\infty e^{-rt} \left[ u(c^j, m^j) - z(h^j) - \frac{\gamma}{2} \left( \frac{\dot{P}^j}{P^j} - \pi^* \right)^2 \right] dt \quad (1)$$

where  $c^j$  denotes consumption of the composite good by household  $j$ ,  $m^j \equiv M^j/P$  denotes real money balances held by household  $j$ , and  $M^j$  denotes nominal money balances. The utility function  $u(\cdot, \cdot)$  is assumed to be strictly increasing and strictly concave, and satisfies  $u_{cc} - u_{cm}u_c/u_m < 0$  and  $u_{mm} - u_{cm}u_m/u_c < 0$ , which ensures that both real balances and consumption are normal goods. The function  $z(\cdot)$  measures the disutility of labor and is assumed to be convex. The parameter  $\gamma$  measures the degree to which household-firm units dislike to deviate in their price-setting behavior from the constant rate of inflation  $\pi^* > -r$ .

The household's instant budget constraint and no-Ponzi-game restriction are

$$\dot{a}^j = (R - \pi)a^j - Rm^j + \frac{P^j}{P}y(h^j) - c^j - \tau \quad (2)$$

and

$$\lim_{t \rightarrow \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a^j(t) \geq 0, \quad (3)$$

where  $a$  denotes the real value of the household's financial assets which consist of money holdings and government bonds,  $R$  denotes the nominal interest rate,  $\pi$  the rate of change in the aggregate price level, and  $\tau$  denotes real lump-sum taxes. In addition, firms are subject to the constraint that given the price they charge, their sales are demand-determined

$$y(h^j) = Y^d d \left( \frac{P^j}{P} \right). \quad (4)$$

The household chooses sequences for  $c^j$ ,  $m^j$ ,  $P^j \geq 0$ , and  $a^j$  so as to maximize (1) subject to (2)–(4) taking as given  $a^j(0)$ ,  $P^j(0)$ , and the time paths of  $\tau$ ,  $R$ ,  $Y^d$ , and  $P$ . The first-order conditions associated with the household's optimization problem are

$$u_c(c^j, m^j) = \lambda^j \quad (5)$$

$$u_m(c^j, m^j) = \lambda^j R \quad (6)$$

$$z'(h^j) = \lambda^j \frac{P^j}{P} y'(h^j) - \mu^j y'(h^j) \quad (7)$$

$$\dot{\lambda}^j = \lambda^j (r + \pi - R) \quad (8)$$

$$\lambda^j \frac{P^j}{P} y(h^j) + \mu^j \frac{P^j}{P} Y^d d' \left( \frac{P^j}{P} \right) = \gamma r (\pi^j - \pi^*) - \gamma \dot{\pi}^j \quad (9)$$

$$\lim_{t \rightarrow \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a^j(t) = 0 \quad (10)$$

where  $\pi^j \equiv \dot{P}^j / P^j$ .

## Monetary and Fiscal Policy

The monetary authority is assumed to set the nominal interest rate as an increasing function of the inflation rate. Specifically, it conducts open market operations so as to ensure that

$$R = R(\pi) \equiv R^* e^{\frac{A}{R^*}(\pi - \pi^*)} \quad (11)$$

where  $R^*$ ,  $A$ , and  $\pi^*$  are positive constants.<sup>3</sup> This specification of the feedback rule implies that the nominal interest rate is strictly positive and strictly increasing in the inflation rate. We will refer to monetary policy as active (passive) if the monetary authority raises the nominal interest rate by more (less) than one-for-one in response to an increase in the inflation rate, that is, if  $R'(\pi) > (<)1$ .

The instant budget constraint of the government is given by

$$\dot{a} = (R - \pi)a - Rm - \tau, \quad (12)$$

where  $a$  denotes the real value of aggregate per capita government liabilities, which consist of real balances and bonds. The monetary-fiscal regime is assumed to be Ricardian in the sense of Benhabib, Schmitt-Grohé and Uribe (1998). That is, the monetary-fiscal regime ensures that total government liabilities converge to zero in present discounted value for all (equilibrium or off-equilibrium) paths of the price level or other endogenous variables:<sup>4</sup>

$$\lim_{t \rightarrow \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a(t) = 0. \quad (13)$$

## Equilibrium

In a symmetric equilibrium all household–firm units choose identical sequences for consumption, asset holdings, and prices. Thus, we can drop the superscript  $j$ . In addition, the goods market must clear, that is,

$$c = y(h). \quad (14)$$

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<sup>3</sup>Note that we assume that the constant  $\pi^*$  appearing in the interest rate feedback rule is the same constant that plays a role in the household's cost of adjusting prices. We make this assumption for analytical convenience.

<sup>4</sup>As discussed in Benhabib, Schmitt-Grohé and Uribe (1998), an example of a Ricardian monetary-fiscal regime is an interest-rate feedback rule like (11) in combination with the fiscal rule  $\tau + Rm = \alpha a$ ;  $\alpha > 0$ . In the case in which  $\alpha = R$ , this fiscal rule corresponds to a balanced-budget requirement.

Combining equations (5) and (6) yields a liquidity preference function of the form

$$m = m(c, R). \quad (15)$$

Given our maintained assumption about the normality of consumption and real balances, the demand for money is increasing in consumption and decreasing in the nominal interest rate. Using (11), (14), and (15) to eliminate  $R$ ,  $c$ , and  $m$  from (5) yields the following expression for  $h$ :

$$h = h(\lambda, \pi), \quad (16)$$

where  $h_\lambda < 0$ ,  $h_\pi u_{cm} < 0$  if  $u_{cm} \neq 0$ , and  $h_\pi = 0$  if  $u_{cm} = 0$ .<sup>5</sup>

Let  $\eta \equiv d'(1) < -1$  denote the equilibrium price elasticity of the demand function faced by the individual firm. Using (7), (11), and (16) to eliminate  $\mu$ ,  $R$ , and  $h$  from equations (8) and (9) yields

$$\dot{\lambda} = \lambda [r + \pi - R(\pi)] \quad (17)$$

$$\dot{\pi} = r(\pi - \pi^*) - \frac{y(h(\lambda, \pi))\lambda}{\gamma} \left[ 1 + \eta - \frac{\eta z'(h(\lambda, \pi))}{\lambda y'(h(\lambda, \pi))} \right] \quad (18)$$

A perfect-foresight equilibrium is a pair of sequences  $\{\lambda, \pi\}$  satisfying (17) and (18). Given the equilibrium sequences  $\{\lambda, \pi\}$ , the corresponding equilibrium sequences  $\{h, c, R, m\}$  are uniquely determined by (16), (14), (11), and (15), respectively. The assumed Ricardian nature of the monetary-fiscal regime requires that the fiscal authority sets taxes in such a way that, given paths for  $R$ ,  $\pi$ , and  $m$  and an initial condition  $a(0)$ , the path for  $a$  implied by equation (12) satisfies the transversality condition (13).

### 3 Steady-state equilibria

A steady-state equilibrium is defined as a pair of constant sequences  $\{\lambda, \pi\}$  satisfying equations (17) and (18); that is,

$$0 = r + \pi - R^* e^{\frac{A}{R^*}(\pi - \pi^*)} \quad (19)$$

$$0 = r(\pi - \pi^*) - \frac{\lambda y(h(\lambda, \pi))}{\gamma} \left( 1 + \eta - \eta \frac{z'(h(\lambda, \pi))}{\lambda y'(h(\lambda, \pi))} \right) \quad (20)$$

Recalling that  $R^* = r + \pi^*$ , it is clear from (19) that in general there exist two steady-state levels of inflation,  $\pi^*$  and  $\bar{\pi}$ , with  $\bar{\pi} < (>) \pi^*$  if  $A > (<) 1$ . If  $A = 1$ , then  $\pi^*$  is the unique steady-state level of inflation. Note that if  $A > 1$ , then monetary policy is active at  $\pi^*$  and passive at  $\bar{\pi}$ . Conversely, if  $A < 1$ , monetary policy is passive at  $\pi^*$  and active at  $\bar{\pi}$ .

<sup>5</sup>To see this, note that  $h_\lambda = [u_{mm} - (u_m/u_c)u_{cm}]/[y'(u_{cc}u_{mm} - u_{cm}^2)]$ . The assumed concavity of the instant utility function and normality of consumption imply, respectively, that the denominator of this expression is positive and that the numerator is negative. Also,  $h_\pi = -h_\lambda u_{cm} m_R R'(\pi)$ , which is of the opposite sign of  $u_{cm}$ .



The steady-state level of  $\lambda$  associated with  $\pi^*$ ,  $\lambda^*$ , is given by the solution to

$$\frac{1 + \eta}{\eta} \lambda = \frac{z'(h(\lambda, \pi^*))}{y'(h(\lambda, \pi^*))}$$

Because the right-hand side of this expression is positive and decreasing in  $\lambda$ ,  $\lambda^*$  exists and is unique. The steady-state value of  $\lambda$  associated with  $\bar{\pi}$  is the solution to

$$\frac{1 + \eta}{\eta} \lambda = \frac{z'(h(\lambda, \bar{\pi}))}{y'(h(\lambda, \bar{\pi}))} - \frac{r\gamma}{\eta} \frac{(\pi^* - \bar{\pi})}{y(h(\lambda, \bar{\pi}))}.$$

If  $A < 1$ , then  $\pi^* - \bar{\pi} < 0$  and hence the right-hand side of this expression is decreasing in  $\lambda$ . Therefore, if a steady-state value of  $\lambda$  exists, it is unique. On the other hand, if  $A > 1$ , then  $\pi^* - \bar{\pi} > 0$  and the right-hand side of the above expression may not be monotone in  $\lambda$ . Thus, multiple steady-state values of  $\lambda$  may exist.

## 4 Local equilibria

We now consider perfect-foresight equilibria in which  $\lambda$  and  $\pi$  remain bounded in a small neighborhood around the steady state  $(\lambda^*, \pi^*)$  and converge asymptotically to it.

Linearizing equations (17) and (18) around  $(\lambda^*, \pi^*)$ , we obtain the system:

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\pi} \end{pmatrix} = J \begin{pmatrix} \lambda - \lambda^* \\ \pi - \pi^* \end{pmatrix} \quad (21)$$

where

$$J = \begin{bmatrix} 0 & u_c(1 - A) \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{21} = \frac{y\eta}{y'\gamma} \left[ \left( z'' - \frac{z'y''}{y'} \right) h_\lambda - \frac{z'}{\lambda} \right] > 0$$

$$J_{22} = r + \frac{y\eta}{y'\gamma} \left( z'' - \frac{z'y''}{y'} \right) h_\pi$$

The sign of the coefficient  $J_{22}$  depends on the sign of  $h_\pi$ , which in turn depends on whether consumption and real balances are Edgeworth substitutes or complements. Specifically,  $J_{22}$  is positive if  $u_{cm} \geq 0$ , and may be negative if  $u_{cm} < 0$ .<sup>6</sup>

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<sup>6</sup> An aggregate supply schedule like the one given by the second row of (21) also arises in the context of a staggered price setting model with optimizing firms like Yun's (1996) extension of Calvo (1983). In Calvo's original model, firms change prices according to a rule of thumb that results in an aggregate supply function in which  $\dot{\pi}$  is only a function of aggregate demand ( $J_{22} = 0$ ).

If monetary policy is active at  $\pi^*$  ( $A > 1$ ), then the determinant of  $J$  is positive, so that the real part of the roots of  $J$  have the same sign. Since both  $\lambda$  and  $\pi$  are jump variables, the equilibrium is locally determinate if and only if the trace of  $J$  is positive. It follows that if  $u_{cm} \geq 0$ , the equilibrium is locally determinate. If  $u_{cm} < 0$ , the equilibrium may be locally determinate or indeterminate.<sup>7</sup> If monetary policy is passive at  $\pi^*$ , ( $A < 1$ ), then the determinant of  $J$  is negative, so that the real part of the roots of  $J$  are of opposite sign. In this case, the equilibrium is locally indeterminate.

One may be tempted to conclude from the above analysis that if  $u_{cm} \geq 0$ , there is no indeterminacy problem under active monetary policy. This conclusion is incorrect, however, because globally the picture may be quite different.

## 5 Global equilibria

In order to characterize global equilibrium dynamics, in this section, we assume particular functional forms for preferences and technology. We assume that the instant utility function displays constant relative risk aversion in a composite good, which in turn is produced with consumption goods and real balances via a CES aggregator. Formally,

$$u(c, m) - z(h) = \frac{\left[ (xc^q + (1-x)m^q)^{\frac{1}{q}} \right]^w}{w} - \frac{h^{1+v}}{1+v}; \quad q, w \leq 1, v > 0 \quad (22)$$

The restrictions imposed on  $q$  and  $w$  ensure that  $u(\cdot, \cdot)$  is concave,  $c$  and  $m$  are normal goods, and the interest elasticity of money demand is strictly negative.<sup>8</sup> The production function takes the form

$$y(h) = h^\alpha; \quad 0 < \alpha < 1$$

In the recent related literature on determinacy of equilibrium under alternative specifications of Taylor rules, it is typically assumed that preferences are separable in consumption and real balances (e.g., Woodford, 1996; Bernanke and Woodford, 1997; Clarida, Galí, and Gertler, 1997). We therefore characterize the equilibrium under this preference specification first, before turning to the more general case.

### 5.1 Separable preferences

The case of separable preferences arises when the intra- and intertemporal elasticities of substitution take the same value, that is, when  $q = w$ . In this case

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<sup>7</sup> Benhabib, Schmitt-Grohé and Uribe (1998) show that the economy with  $u_{cm} < 0$  is similar to one without money in the utility function but money entering the production function.

<sup>8</sup> Note that the sign of  $u_{cm}$  equals the sign of  $w - q$ .

the equilibrium conditions (17) and (18) become

$$\dot{\lambda} = \lambda \left[ r + \pi - R^* e^{\frac{A}{R^*}(\pi - \pi^*)} \right] \quad (23)$$

$$\dot{\pi} = r(\pi - \pi^*) - \frac{1 + \eta}{\gamma} \lambda^\omega x^{\alpha\theta} + \frac{\eta}{\alpha\gamma} \lambda^\beta x^{(1+v)\theta} \quad (24)$$

where  $\beta \equiv (1 + v)/(\alpha(w - 1)) < 0$ ,  $\omega \equiv w/(w - 1)$ , and  $\theta \equiv 1/(\alpha(1 - w))$ . Throughout this subsection we assume that

$$R^* = r + \pi^*.$$

This expression implies that  $\pi^*$  solves (23) when  $\dot{\lambda} = 0$ . Evaluating (24) at  $\dot{\pi} = 0$  and  $\pi = \pi^*$ , it follows that  $\lambda^*$  must satisfy

$$\frac{1 + \eta}{\gamma} \lambda^{*\omega} x^{\alpha\theta} = \frac{\eta}{\alpha\gamma} \lambda^{*\beta} x^{(1+v)\theta} \equiv M < 0.$$

Evaluating (24) at  $\pi = \bar{\pi}$  and setting  $\dot{\pi} = 0$  yields the following expression defining the steady-state value of  $\lambda$ ,  $\bar{\lambda}$ , associated with  $\bar{\pi}$

$$r(\bar{\pi} - \pi^*) = M(\bar{\lambda}^\omega - \bar{\lambda}^\beta).$$

Because  $\bar{\pi} < \pi^*$  for  $A > 1$  and  $\omega \geq 0$  for  $w \leq 0$ , it follows from this expression that if  $A > 1$  and  $w \leq 0$ , then  $\bar{\lambda}$  exists and is unique.<sup>9</sup> Thus, in this subsection we assume that  $w \leq 0$ ; that is, the intertemporal elasticity of substitution does not exceed unity.

The main result of this subsection is that in the economy described above there exists an infinite number of equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to the steady state at which monetary policy is passive or to a limit cycle. In section 4, we showed that if one restricts the analysis to equilibria in which  $\pi$  and  $\lambda$  remain forever bounded in an arbitrarily small neighborhood of the active steady state, then the unique perfect-foresight equilibrium is the steady state itself. Thus, the picture that arises from a local analysis might wrongly lead one to conclude that active monetary policy is stabilizing when in fact it is not. The following proposition formalizes this result.

**Proposition 1 (Global indeterminacy under active monetary policy and separable preferences)** *Suppose preferences are separable in consumption and real balances ( $q = w$ ). Then, for  $r$  and  $A - 1$  positive and sufficiently small, the equilibrium exhibits indeterminacy as follows: trajectories originating in the neighborhood of the steady state  $(\lambda, \pi) = (\lambda^*, \pi^*)$ , at which monetary policy is active, converge either to the other steady state,  $(\bar{\lambda}, \bar{\pi})$ , at which monetary policy is passive or to a limit cycle around  $(\lambda^*, \pi^*)$ . In the first case, there exists a saddle connection and the dimension of indeterminacy is one, whereas in the latter case the dimension of indeterminacy is two.*

<sup>9</sup>If  $w = 0$ ,  $\bar{\lambda}$  may not exist for all parameterizations of the model. For  $r$  sufficiently close to zero or  $A$  sufficiently close to one (or both),  $\bar{\lambda}$  always exists.

**Proof:** See the Appendix.■

This result is likely to also arise in models with alternative specifications of the source of nominal rigidities. For example, in a model with staggered price setting like Yun's (1996) extension of Calvo (1983), the aggregate supply schedule takes a form that is qualitatively similar to (24). Thus, we conjecture that the Calvo-Yun model exhibits global indeterminacy of the kind described in Proposition 1 as well.<sup>10</sup>

Figure 2 illustrates the existence of a saddle connection from the steady state at which monetary policy is active to the steady state at which monetary policy is passive. In computing the equilibrium dynamics of  $\pi$  and  $\lambda$ , the assumed time unit is a quarter. The parameters  $R^*$ ,  $\pi^*$ , and  $r$  were set at .06/4, .042/4, and .018/4, respectively. The parameter  $A$  was set at 1.5, so that at the active steady state the Taylor rule has the slope suggested by Taylor (1993). These parameter values imply that at the active steady state the nominal interest rate is 6 percent per year, which equals the average three-months Treasury Bill rate in the period 1960:1-1998:9, the inflation rate is 4.2 percent per year, which is consistent with the average US inflation rate over the period 1960:Q1-1998:Q3 as measured by the GDP deflator, and the real discount rate is 1.8 percent per year. In addition, we set  $w = q = -1$  so that the instant utility function is separable in consumption and real balances, and the intertemporal elasticity of substitution equals 1/2. The parameter  $x$  was set at a value consistent with an annual consumption velocity of money of 3. The labor share,  $\alpha$ , was set at .7, and the labor supply elasticity at 1. The value of  $\eta$  was chosen so that the implied markup of prices over marginal cost at the active steady state is 5 percent, which is consistent with the evidence presented by Basu and Fernald (1997). Finally, following Sbordone (1998), we set  $\gamma$ , the parameter governing the disutility of deviating from the inflation target, at  $-17.5(1 + \eta)$ . Table 1 summarizes the calibration. The inflation rate at the passive steady state is 0.7 percent per year, and the

Table 1: Calibration

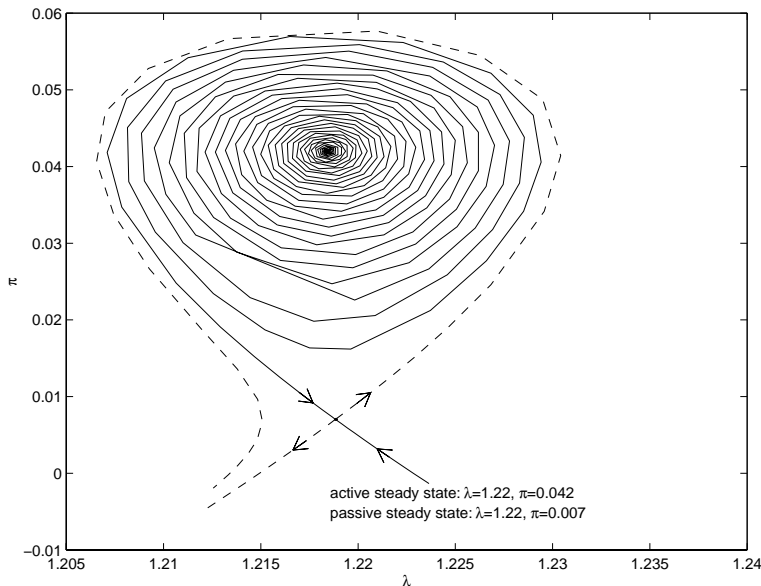
$R^*$	$\pi^*$	$r$	$A$	$w$	$q$	$c/m$	$\alpha$	$v$	$\frac{\eta}{1+\eta}$	$\gamma$
.06/4	.042/4	.018/4	1.5	-1	-1	3/4	.7	1	1.05	350

Notes: (1) The time unit is one quarter. (2)  $x/(1 - x) = (c/m)^{1-q}/R^*$ .

sensitivity of the Taylor rule with respect to inflation is .63. The active steady

<sup>10</sup>In Calvo's (1983) original sticky-price model, the aggregate supply function takes the form  $\pi = f(\lambda)$ . In this case, the equilibrium conditions (23)-(24) form a conservative Hamiltonian system whose Jacobian has a zero trace and a positive determinant under active monetary policy. Such a system gives rise to a continuum of cycles surrounding the active steady state. These cycles are enclosed by a homoclinic orbit formed by the connection of the stable and unstable manifolds of the passive steady state. The period of the cycles approaches infinity as the cycles get closer to the homoclinic orbit.

Figure 2: Separable preferences: Saddle connection from the active to the passive steady state

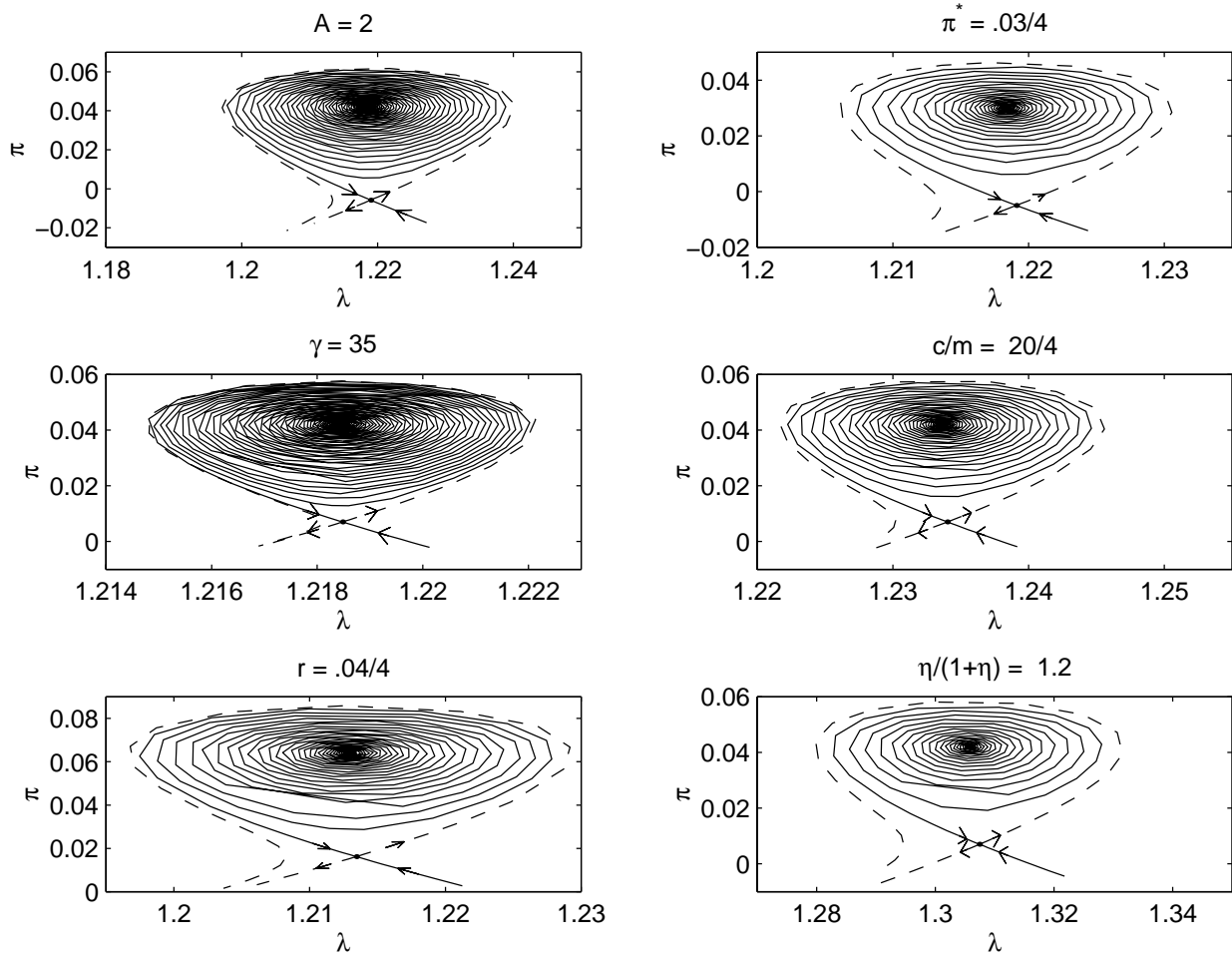


state is a source and the passive steady state is a saddle. Thus, the active steady state is locally the unique rational expectations equilibrium whereas around the passive steady state the equilibrium is indeterminate. The solid line in figure 2 displays the saddle path converging to the passive steady state. The dashed line corresponds to the unstable manifold diverging from the passive steady state.

Three features of figure 2 are noteworthy. First, the indeterminacy result established in Proposition 1 seems to hold not only for pairs  $(r, A)$  close to  $(0, 1)$  but also for empirically relevant values. Second, the saddle connection is not inconsistent with the observation that the inflation rate fluctuates for long periods of time in a region in which monetary policy is active, as has been argued is the case of the U.S. economy since the Volker era (see Clarida, et al., 1997; and Rotemberg and Woodford, 1997). In our calibrated economy monetary policy is active for all inflation rates exceeding 2.6 percent per year. Third, one argument for restricting attention to local dynamics is that observed inflation fluctuations at business-cycle frequencies are relatively small. The global dynamics illustrated in figure 2 suggest that the short-term fluctuations in the inflation rate along the saddle connection are empirically plausible, with a maximum annual inflation rate of 5.7 percent and a minimum of 0.7 percent.

The dynamics are robust to wide variations in parameter values. Figure 3 illustrates that the saddle path connecting the steady state at which monetary policy is active with the steady state at which policy is passive does not disappear if: (a)  $A$ , the slope of the Taylor rule at  $\pi = \pi^*$ , is increased from the baseline value of 1.5 to a value of 2, which, as some authors may argue,

Figure 3: Separable preferences: Sensitivity analysis



reflects more closely the stance of U.S. monetary policy in the post-Volker era (see, again, Clarida, et al., 1997; and Rotemberg and Woodford, 1997); (b)  $\pi^*$ , the inflation rate associated with the active steady state, is set at 3 percent per year. This case illustrates that the global indeterminacy result does not hinge in any important way on the inflation rate being high at the active steady state. (Note that the inflation rate at the passive steady state is negative.); (c)  $\gamma$ , the parameter governing the cost of deviating from the inflation target, is reduced from its baseline value of 350 to 35. Although not noticeable in the figure, for such a low value of  $\gamma$ , the economy converges from the vicinity of the active steady state to the passive steady state at a much higher speed than under the baseline calibration; (d) the annual consumption velocity of money is increased from 3 to 20. This result is of particular interest in light of the view that as a result of financial innovation agents are increasingly able to perform transactions without money; (e) The discount rate,  $r$ , takes a value of 4, a value commonly used in the real-business-cycle literature (Prescott, 1986); and (f) a markup of prices over marginal cost of 20 percent is assumed. This number reflects the upper range of available empirical estimates (Basu and Fernald, 1997).

## 5.2 Non-separable preferences

In this subsection, we consider preference specifications for which the intra- and intertemporal elasticities of substitution are different from each other ( $q \neq w$ ). In this case, the equilibrium conditions (17) and (18) can be written as

$$\dot{\lambda} = \lambda \left[ r + \pi - R^* e^{\frac{A}{R^*}(\pi - \pi^*)} \right] \quad (25)$$

$$\begin{aligned} \dot{\pi} = & r(\pi - \pi^*) - \frac{1 + \eta}{\gamma} x^{\alpha\theta} \lambda^\omega \left[ (R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi} e^{\frac{A}{R^*}\chi(\pi - \pi^*)} + 1 \right]^{\alpha\xi} \\ & + \frac{\eta}{\alpha\gamma} x^{(1+v)\theta} \lambda^\beta \left[ (R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi} e^{\frac{A}{R^*}\chi(\pi - \pi^*)} + 1 \right]^{(1+v)\xi}, \end{aligned} \quad (26)$$

where  $\beta$  and  $\omega$  are defined as in the previous subsection and  $\chi \equiv q/(q-1)$ ,  $\xi \equiv (w-q)/[\alpha q(1-w)] \neq 0$ , and  $\theta \equiv w/[\alpha q(1-w)]$ . Let  $\lambda^*$  be the steady-state value of  $\lambda$  associated with  $\pi = \pi^*$ ,  $A = 1$ ,  $r = r^c$ , and  $R^* = \pi^* + r^c$  (with  $r^c$  to be determined below). Then, by equation (26)  $\lambda^*$  is implicitly defined by

$$\begin{aligned} \frac{1 + \eta}{\gamma} x^{\alpha\theta} (\lambda^*)^\omega \left( (R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi} + 1 \right)^{\alpha\xi} &= \\ \frac{\eta}{\alpha\gamma} x^{(1+v)\theta} (\lambda^*)^\beta \left( (R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi} + 1 \right)^{\xi(1+v)} &\equiv M. \end{aligned}$$

The parameter  $r^c$  is defined as the value of  $r$  that makes the trace of the Jacobian of the system (25)-(26) equal to zero for  $A = 1$ . That is,  $r^c$  is implicitly given

by

$$r^c = -B\xi \left( \frac{1}{r^c + \pi^*} \right) \chi M (1 + v - \alpha), \quad (27)$$

where

$$B \equiv \frac{(R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi}}{1 + (R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi}}.$$

Inspection of (27) reveals that the existence of a positive  $r^c$  depends on parameter values. For example, one can show that a positive  $r^c$  always exists if  $q \in (0, 1)$  and  $q - w, \pi^* > 0$ . Throughout this subsection, we assume that  $R^*$  is fixed and equal to  $r^c + \pi^*$ . When  $(r, A) = (r^c, 1)$ , the point  $(\lambda, \pi) = (\lambda^*, \pi^*)$  is the unique steady state of the system (25)-(26). At that point, monetary policy is neither active nor passive ( $R'(\pi^*) = 1$ ). For parameter configurations in which  $(r, A) \neq (r^c, 1)$ , the economy displays in general either none or two steady-state values of  $\pi$ . When two steady-state values of  $\pi$  exist, the larger of them corresponds to an active monetary policy stance and the smaller one to a passive stance. In addition, each steady-state value of  $\pi$  is associated with one or two steady-state values of  $\lambda$ . The following lemma shows that under the assumption that the intertemporal elasticity of substitution is less than one ( $w < 0$ ), each steady-state value of  $\pi$  is associated with a unique steady-state value of  $\lambda$ . For this reason and because it is clearly the case of greatest empirical relevance, in what follows we assume that  $w < 0$ . The lemma also shows that the steady state at which monetary policy is active is either a sink or a source, while the steady state at which monetary policy is passive is always a saddle.

**Lemma 1** *Suppose  $w < 0$ . Then, the steady states of the system (25) and (26) satisfy: (i) for each steady-state value of  $\pi$  there exists a unique steady-state value of  $\lambda$ ; and (ii) the steady state at which monetary policy is active is either a sink or a source and the steady state at which monetary policy is passive is always a saddle.*

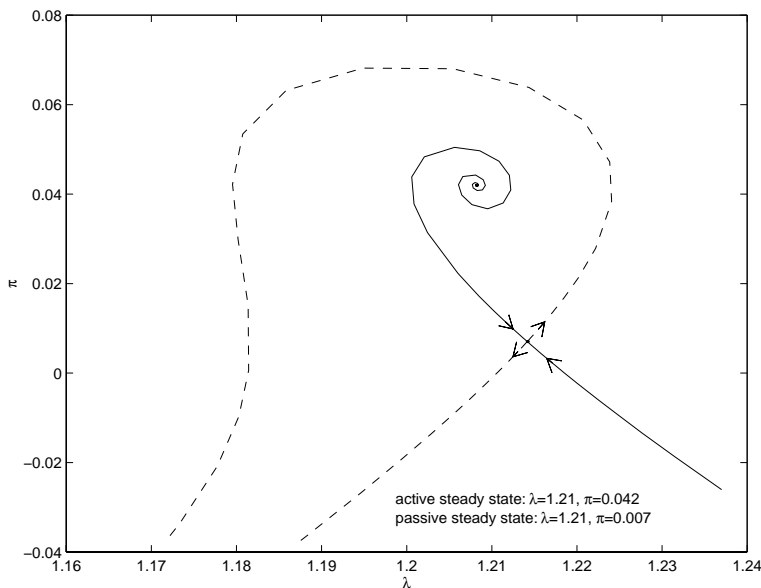
**Proof:** See the appendix. ■

The next proposition contains the main result of this subsection. Namely, that if the steady state at which monetary policy is active is locally the unique equilibrium (i.e., the steady state is a source), then the equilibrium is globally indeterminate. Specifically, there exist equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to a limit cycle or to the other steady state, at which monetary policy is passive.

**Proposition 2 (Global indeterminacy under active monetary policy and non-separable preferences)** *For parameter specifications  $(r, A)$  sufficiently close to  $(r^c, 1)$ , the economy with non-separable preferences exhibits indeterminacy as follows: There always exist an infinite number of equilibrium*



Figure 4: Non-separable preferences,  $w > q$ : Saddle connection from the active to the passive steady state



trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to: (i) that steady state; (ii) a limit cycle; or (iii) the other steady state at which monetary policy is passive. In cases (i) and (ii) the dimension of indeterminacy is two, whereas in case (iii) it is one.

**Proof:** See the appendix. ■

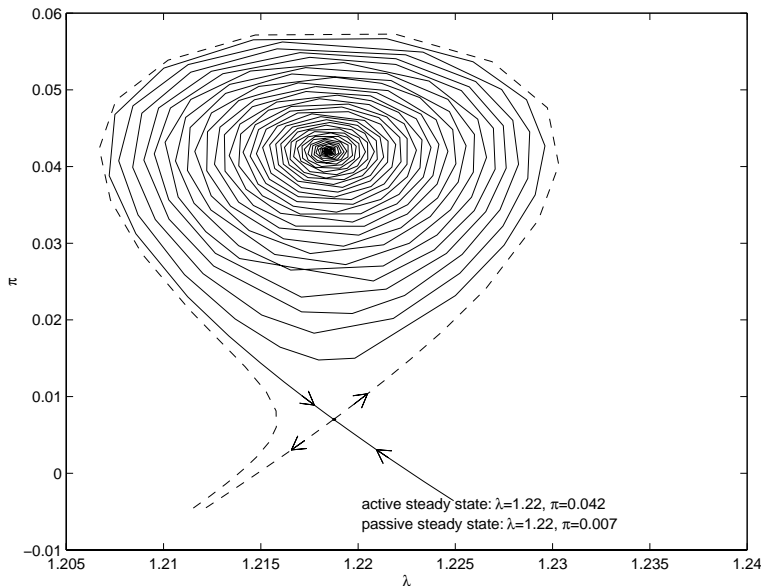
The following corollary establishes parameter restrictions under which attracting limit cycles exist around the steady state at which monetary policy is active.

**Corollary 1 (Periodic equilibria)** *If  $-\frac{1-B}{B(1+v+\alpha)} < \xi < 0$ , then there exists a region in the neighborhood of  $(r, A) = (r^c, 1)$  for which the active steady state is a source surrounded by a stable limit cycle. On the other hand, if  $\xi > 0$  or  $\xi < -\frac{1-B}{B(1+v+\alpha)}$ , then stable limit cycles do not exist.*

**Proof:** See the appendix. ■

It is important to recall that the equilibrium remains globally indeterminate even if limit cycles do not exist. This is because in that case there always exists an equilibrium trajectory connecting the active steady state with the passive one. In fact, as shown in figures 4 and 5, a saddle connection is the typical pattern that arises under plausible parameterizations of the model with non-separable preferences. In both figures, the calibration is the same as the one

Figure 5: Non-separable preferences,  $w < q$ : Saddle connection from the active to the passive steady state



used in the economy with separable preferences, summarized in table 1, except, of course, that now the intratemporal elasticity of substitution between consumption and real balances,  $1/(1 - q)$ , is assumed to be different from the intertemporal elasticity of substitution,  $1/(1 - w)$ . In both figures, the intertemporal elasticity of substitution takes its baseline value of .5. In figure 4,  $q$  is set at  $-9$ , a value consistent with an interest elasticity of money demand of  $-1$ . In this case,  $w > q$ , which implies that consumption and real balances are Edgeworth complements ( $U_{cm} > 0$ ).<sup>11</sup> In figure 5,  $q$  is set at  $-0.975$ , which corresponds to a log-log interest elasticity of money demand of  $1/2$ . In this case,  $w < q$ , so consumption and money are Edgeworth substitutes ( $U_{cm} < 0$ ). Under both parameterizations, the active steady state is locally the unique perfect-foresight equilibrium (i.e., the active steady state is a source). However, as the figures suggest, from a global perspective it is clear that an infinite number of trajectories originating arbitrarily close to the active steady state and on the saddle connection can be supported as equilibrium outcomes because they converge to the passive steady state.

The pattern illustrated in figures 4 and 5 is unchanged for values of  $q$  between  $-0.975$  and  $-9$ , the two values assumed in the figures. As  $q$  is increased above  $-0.975$ , the active steady state becomes a sink and thus the equilibrium is locally indeterminate. If  $w > q$  ( $U_{cm} > 0$ ), the simulation results are, as in the case

<sup>11</sup>When  $U_{cm} > 0$ , the economy is equivalent to a cash-in-advance economy with cash and credit goods like the one developed by Lucas and Stokey (1987).

of separable preferences, robust to wide variations in other parameter values. In particular, a saddle path connecting the active steady state with the passive one continues to exist for more aggressive Taylor rules ( $A \geq 2$ ) and lower costs of price adjustment (for example,  $\gamma = 35$ ). In the case that  $q > w$  ( $U_{cm} < 0$ ), parameter variations may or may not eliminate the saddle connection. However, when the saddle connection disappears, it is typically replaced by a situation in which the active steady state is a sink, which is locally a more severe case of indeterminacy.

## 6 Final Remarks

This paper shows that in a sticky-price model when a global analysis is undertaken, the existence of a steady state with active monetary policy generically leads to global indeterminacy. Although the prepositions above are proven for specific functional forms to facilitate checking for non-degeneracies, it is clear from the general structure of the equilibrium conditions that generically alternative specifications for smooth preferences and the interest rate feedback rule will give rise to similar results, as long as the feedback rule assures the existence of a steady state with an active monetary policy. The main results of the paper also obtain in flexible-price versions of the model (Schmitt-Grohé and Uribe, 1998). In this case, equilibrium dynamics are described by a scalar system where again a continuous feedback rule generating a steady state with active monetary policy implies the existence of a passive steady state, with all the implications for global indeterminacy, quite independently of the structure of preferences and production.

# Appendix

## Proof of Proposition 1

### Preliminaries

To prove Proposition 1 we apply the following theorem due to Kopell and Howard (1975):

**Theorem (Kopell and Howard, 1975, Theorem 7.1):** *Let  $\dot{X} = F_{\mu,\nu}(X)$  be a two-parameter family of ordinary differential equations on  $R^2$ ,  $F$  smooth in all of its four arguments, such that  $F_{\mu,\nu}(0) = 0$ . Also assume:*

1.  $dF_{0,0}(0) \equiv A$  has a double zero eigenvalue and a single eigenvector  $e$ .
2. The mapping  $(\mu, \nu) \rightarrow (\det dF_{\mu,\nu}(0), \text{tr } dF_{\mu,\nu}(0))$  has a nonzero Jacobian at  $(\mu, \nu) = (0, 0)$ .
3. Let  $Q(X, X)$  be the  $2 \times 1$  vector containing the terms quadratic in the  $x_i$  and independent of  $(\mu, \nu)$  in a Taylor series expansion of  $F_{\mu,\nu}(X)$  around 0. Then  $[dF_{(0,0)}(0), Q(e, e)]$  has rank 2.

*Then: There is a curve  $f(\mu, \nu) = 0$  such that if  $f(\mu_0, \nu_0) = 0$ , then  $\dot{X} = F_{\mu_0, \nu_0}(X)$  has a homoclinic orbit. This one-parameter family of homoclinic orbits (in  $(X, \mu, \nu)$  space) is on the boundary of a two-parameter family of periodic solutions. For all  $|\mu|, |\nu|$  sufficiently small, if  $\dot{X} = F_{\mu,\nu}(X)$  has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.*

To apply this theorem, we must first perform several changes of variables and a Taylor series expansion of the equilibrium conditions around the steady state. Let  $p \equiv \pi - \pi^*$  and  $n \equiv \ln(\lambda/\lambda^*)$ . Then the equilibrium conditions (23) and (24) can be expressed as

$$\dot{n} = R^* + p - R^* e^{\left(\frac{A}{R^*}\right)p} \quad (28)$$

$$\begin{aligned} \dot{p} = & rp - \gamma^{-1}(1 + \eta)(\lambda^*)^\omega e^{\omega n} x^{\alpha\theta} \\ & + \alpha^{-1}\gamma^{-1}\eta(\lambda^*)^\beta e^{\beta n} x^{\theta(1+v)} \end{aligned} \quad (29)$$

Defining  $y = p/[M(\beta - \omega)]$  and  $s = R^*n$ , we have

$$\begin{aligned} \dot{s} &= R^{*2} + M(\beta - \omega)R^*y - R^{*2}e^{\frac{A}{R^*}M(\beta - \omega)y} \\ \dot{y} &= ry + \left(e^{\frac{\beta}{R^*}s} - e^{\frac{\omega}{R^*}s}\right)/(\beta - \omega) \end{aligned}$$

We now take a Taylor series expansion of these two equations around  $(s, y) = (0, 0)$ , which yields

$$\dot{s} = R^*(1 - A)M(\beta - \omega)y - \frac{[AM(\beta - \omega)]^2}{2}y^2 - \frac{[AM(\beta - \omega)]^3}{3!R^*}y^3 - \dots \quad (30)$$

$$\dot{y} = ry + \frac{s}{R^*} + \frac{1}{\beta - \omega} \left( \frac{(\beta^2 - \omega^2)}{2R^{*2}}s^2 + \frac{(\beta^3 - \omega^3)}{3!R^{*3}}s^3 + \dots \right) \quad (31)$$

with Jacobian

$$J = \begin{bmatrix} 0 & R^*(1-A)M(\beta-\omega) \\ 1/R^* & r \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 0 & 0 \\ 1/R^* & 0 \end{bmatrix}$$

when  $r = (1-A) = 0$ . We are now ready to prove Proposition 1.

### Proof of Proposition 1:

We prove the proposition by showing that for  $(r, 1-A)$  small enough, the system of differential equations (30)-(31) satisfies the hypotheses of the Kopell-Howard Theorem stated above. Let  $\mu \equiv r$ ,  $\nu \equiv 1-A$ , and  $X \equiv [s; y]$ . Then, the system (30)-(31) can be expressed as  $\dot{X} = F_{\mu,\nu}(X)$ . We have that  $dF_{0,0}(0) = [0 \ 0; 1/R^* \ 0]$ . Clearly,  $dF_{0,0}$  has a double zero eigenvalue and a single eigenvector  $e = [0; 1]$ . The Jacobian of the mapping  $(\mu, \nu) \rightarrow (\det dF_{\mu,\nu}(0), \text{tr } dF_{\mu,\nu}(0))$  at  $(\mu, \nu) = (0, 0)$  is given by  $[0 \ -M(\beta-\omega); 1 \ 0]$  and is different from zero. Note that neither  $\lambda^*$  nor  $M$  depend on  $\mu$  or  $\nu$ . The vector  $Q(e, e)$  is given by  $[-M^2(\beta-\omega)^2/2; 0]$ . It follows that  $[dF_{0,0}(0) \ Q(e, e)]$  has rank 2. The proposition follows from the facts that the active steady state is a source, the passive steady state is a saddle, and both  $s$  and  $y$  are jump variables. ■

### Proof of Lemma 1

(i)  $w < 0$  implies that  $\omega > 0$ . Given a steady-state value  $\bar{\pi}$  the uniqueness of the associated steady-state value of  $\lambda$  follows directly from evaluating (26) at  $\hat{\pi} = 0$  and  $\pi = \bar{\pi}$  and recalling that  $\beta, 1 + \eta < 0$ . (ii) By definition, monetary policy is active (passive) at a given steady state  $(\bar{\lambda}, \bar{\pi})$  if and only if  $Ae^{(\frac{A}{R^*})\bar{\pi} - \bar{\pi}^*} > (<) 1$ . Let  $J$  denote the Jacobian of (25)-(26). Then  $J_{11} = 0$ . Therefore, the determinant of  $J$  is given by  $-J_{21}J_{12}$ . The element  $J_{12}$  equals  $\bar{\lambda} \left[ 1 - Ae^{\frac{A}{R^*}(\bar{\pi} - \bar{\pi}^*)} \right]$  which is negative (positive) if monetary policy is active (passive). The element  $J_{21}$  is given by

$$J_{21} = -\omega \frac{1+\eta}{\gamma} x^{\alpha\theta} \bar{\lambda}^{\omega-1} \left[ (R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi} e^{\frac{A}{R^*}\chi(\bar{\pi} - \bar{\pi}^*)} + 1 \right]^{\alpha\xi} + \beta \frac{\eta}{\alpha\gamma} x^{(1+v)\theta} \bar{\lambda}^{\beta-1} \left[ (R^*)^\chi \left( \frac{1-x}{x} \right)^{1-\chi} e^{\frac{A}{R^*}\chi(\bar{\pi} - \bar{\pi}^*)} + 1 \right]^{(1+v)\xi},$$

which is clearly positive. Therefore, the determinant of  $J$  is positive (negative) if and only if monetary policy is active (passive). ■

## Proof of Proposition 2

We prove the proposition by applying a theorem and a lemma from Kuznetsov (1995) that together allow us to transform the system of equilibrium conditions into a simpler, topologically equivalent planar system of differential equations with known bifurcation diagram. Technically, we show that the system (25)-(26) exhibits a Bogdanov-Takens (double-zero) bifurcation at  $(r, A) = (r^c, 1)$ .

### Preliminaries

Let  $n \equiv \ln(\lambda/\lambda^*)$  and  $y \equiv (\pi - \pi^*)/[M(\beta - \omega)]$ . Then, equilibrium conditions (25) and (26) can be written as

$$\dot{y} = ry - \left(\frac{e^{\omega n}}{\beta - \omega}\right) \left(Be^{\frac{A}{R^*}\chi M(\beta - \omega)y} + 1 - B\right)^{\alpha\xi} \quad (32)$$

$$+ \left(\frac{e^{\beta n}}{\beta - \omega}\right) \left(Be^{\frac{A}{R^*}\chi M(\beta - \omega)y} + 1 - B\right)^{(1+v)\xi}$$

$$\dot{n} = r + \pi^* + M(\beta - \omega)y - R^*e^{\frac{A}{R^*}M(\beta - \omega)y}. \quad (33)$$

Taking a Taylor series expansion of the right-hand side of this system around  $(y, n) = (0, 0)$  yields

$$\dot{y} = \left[r + B\xi\frac{A}{R^*}\chi M(1 + v - \alpha)\right] y + n + \quad (34)$$

$$\frac{1}{2}B\xi(\beta - \omega) \left(\frac{A}{R^*}\chi M\right)^2 [(1 + v - \alpha)(1 - B) + B\xi((1 + v)^2 - \alpha^2)] y^2 +$$

$$B\xi\frac{A}{R^*}\chi M[\beta(1 + v) - \omega\alpha]yn + \frac{1}{2}(\beta + \omega)n^2 + \dots$$

$$\dot{n} = (r + \pi^* - R^*) + M(\beta - \omega)(1 - A)y -$$

$$\frac{1}{2}R^* \left(\frac{A}{R^*}M(\beta - \omega)\right)^2 y^2 - \dots \quad (35)$$

The Jacobian of this system is

$$\begin{bmatrix} r + B\xi\frac{A}{R^*}\chi M(1 + v - \alpha) & 1 \\ M(\beta - \omega)(1 - A) & 0 \end{bmatrix},$$

At  $(r, A) = (r^c, 1)$  this Jacobian collapses to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which has two zero eigenvalues (the Bogdanov-Takens condition). We now state the aforementioned theorem and lemma from Kuznetsov (1995).

**Theorem (Normal form representation [Kuznetsov, 1995, Theorem 8.4]):** *Suppose that a planar system*

$$\dot{x} = f(x, \alpha), \quad x \in R^2, \quad \alpha \in R^2,$$

with smooth  $f$ , has at  $\alpha = 0$  the equilibrium  $x = 0$  with a double zero eigenvalue. Via a Taylor series expansion around  $x = 0$  and transformation of variables, this system can be expressed as:

$$\begin{aligned} \dot{y}_1 &= y_2 + a_{00}(\alpha) + a_{10}(\alpha) y_1 + a_{01}(\alpha) y_2 \\ &\quad + \frac{1}{2} a_{20}(\alpha) y_1^2 + a_{11}(\alpha) y_1 y_2 + \frac{1}{2} a_{02}(\alpha) y_2^2 + P_1(y, \alpha) \\ \dot{y}_2 &= b_{00}(\alpha) + b_{10}(\alpha) y_1 + b_{01}(\alpha) y_2 \\ &\quad + \frac{1}{2} b_{20}(\alpha) y_1^2 + b_{11}(\alpha) y_1 y_2 + \frac{1}{2} b_{02}(\alpha) y_2^2 + P_2(y, \alpha), \end{aligned}$$

where  $a_{lk}(\alpha)$ ,  $b_{lk}(\alpha)$ , and  $P_{1,2}(y, \alpha) = O(\|y\|^3)$  are smooth functions of their arguments. Assume that

$$a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0$$

and that the following nondegeneracy conditions are satisfied:

$$(BT.0) \text{ the Jacobian matrix } A(0) = \frac{\partial f}{\partial x}(0, 0) \neq 0;$$

$$(BT.1) \ a_{20}(0) + b_{11}(0) \neq 0;$$

$$(BT.2) \ b_{20}(0) \neq 0;$$

(BT.3) the map

$$(x, \alpha) \mapsto \left( f(x, \alpha), \operatorname{tr} \left( \frac{\partial f(x, \alpha)}{\partial x} \right), \det \left( \frac{\partial f(x, \alpha)}{\partial x} \right) \right)$$

is regular at point  $(x, \alpha) = (0, 0)$ .

Then there exist smooth invertible variable transformations smoothly depending on the parameters, a direction-preserving time reparameterization, and smooth invertible parameter changes, which together reduce the system to

$$\begin{cases} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2 \eta_1 + \eta_1^2 + s \eta_1 \eta_2 + O(\|\eta\|^3) \end{cases},$$

where  $s = \operatorname{sign}[b_{20}(0)(a_{20}(0) + b_{11}(0))] = \pm 1$ .

The explicit steps of the transformation of variables is given in Kuznetsov (1995). We note that  $\beta_1$  and  $\beta_2$  are functions of  $\alpha$  satisfying  $\beta_1(\alpha) = \beta_2(\alpha) = 0$  for  $\alpha = 0$ .

**Lemma (Effect of higher-order terms [Kuznetsov, 1995, Lemma 8.8]):**

The system

$$\begin{cases} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2 \eta_1 + \eta_1^2 \pm \eta_1 \eta_2 + O(\|\eta\|^3) \end{cases}$$

is locally topologically equivalent near the origin to the system

$$\begin{cases} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2 \eta_1 + \eta_1^2 \pm \eta_1 \eta_2 \end{cases}.$$

We are now ready to prove Proposition 2.

**Proof of proposition 2:**

We first show that the system (25)-(26) of equilibrium conditions of the economy with non-separable preferences is in general locally (i.e., near  $(r, A) = (r^c, 1)$ ) topologically equivalent near the steady state  $(\lambda, \pi) = (\lambda^*, \pi^*)$  to the system

$$\begin{cases} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2. \end{cases} \quad (36)$$

The first step is to show that the conditions of Theorem 8.4 of Kuznetsov (1995) are satisfied by the transformation of (25)-(26) given by (32)-(33). Let  $x \equiv (y, n)$  and  $\alpha \equiv (1 - A, r - r^c)$ . Then the system (32)-(33) can be expressed as  $\dot{x} = f(x, \alpha)$ . We have shown above that (32)-(33) has at  $\alpha = 0$  the equilibrium  $x = 0$  with a non-zero Jacobian. Thus, BT.0 is satisfied. We have also shown that at  $(x, \alpha) = (0, 0)$  the Jacobian has a double zero eigenvalue. It is clear from (34)-(35) that

$$a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0.$$

Also,  $a_{20}(0) = B\xi(\beta - \omega) \left(\frac{1}{R^*}\chi M\right)^2 [(1 + v - \alpha)(1 - B) + B\xi((1 + v)^2 - \alpha^2)]$  and  $b_{20}(0) = -R^*[1/R^*M(\beta - \omega)]^2$  are in general non-zero while  $b_{11}(0) = 0$ . Therefore, BT.1 and BT.2 are satisfied. The Jacobian of the mapping  $(x, \alpha) \mapsto \left(f(x, \alpha), \text{tr}\left(\frac{\partial f(x, \alpha)}{\partial x}\right), \det\left(\frac{\partial f(x, \alpha)}{\partial x}\right)\right)$  at  $(x, \alpha) = (0, 0)$  is given by:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{20}(0) & a_{11}(0) & r^c & 1 \\ -b_{20}(0) & 0 & -1 & 0 \end{bmatrix},$$

where  $a_{11}(0) = B\xi\chi MR^{*-1}[\beta(1 + v) - \omega\alpha]$ . The determinant of this Jacobian is equal to  $-b_{20}(0)r^c$ , which is in general different from zero, so that the map is regular at  $(x, \alpha) = (0, 0)$  and condition BT.3 is satisfied. The claim that the equilibrium conditions have the normal form representation given by (36) follows from the theorem and the lemma stated above. Proposition 2 then follows directly from Lemma 1 and the properties of the bifurcation diagram of (36) (see Kuznetsov, 1995, section 8.4.2 for the case in which the coefficient on  $\eta_1\eta_2$  is  $-1$  and Guckenheimer and Holmes, 1983, section 7.3 for the case in which the coefficient on  $\eta_1\eta_2$  is  $+1$ ). ■

**Proof of Corollary 1**

The existence of stable limit cycles depends on the sign of the coefficient of  $\eta_1\eta_2$  in (36), which is equal to the sign of the parameter  $s$  defined in Theorem 8.4 of Kuznetsov (1995) stated above. As shown in Kuznetsov (1995), if  $s$  is negative there exists a region in the vicinity of  $(r, A) = (r^c, 1)$  for which stable limit cycles emerge. If  $s$  is positive, then stable limit cycles do not exist. In



the economy with non-separable preferences,  $s = -\text{sign}(a_{20}(0))$ , where  $a_{20}(0)$  is given in terms of the structural parameters of the economy in the proof of Proposition 2. ■

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