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# Weak and Strong Consistency in Additive Cost Sharing

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## Abstract

In this paper we study consistency in the context of additive cost sharing mechanisms. We contrast an extremely weak notion of consistency with the standard definition, which we denote strong consistency. First we show that many well known CSMs are consistent in both senses: Aumann-Shapley, random order methods, Shapley-Shubik, Serial cost, and also the weighted versions of these. We also provide general conditions which characterize the different types of consistency— all methods generated by separable paths are weakly consistent, while those generated by associative paths are strongly consistent. Using this characterization, we show that any weakly consistent method which is demand monotonic is also strongly consistent.

Next, we analyze the conditions under which a cost sharing method (CSM) for an arbitrary number of agents is uniquely defined by its behavior in the two agent case. We show that under weak (resp. strong) consistency all CSMs generated by a single separable (resp. associative) path are uniquely defined by their behavior on two agent problems. These include Aumann-Shapley, random order methods, Serial cost, and also the weighted versions of these. Shapley-Shubik, which is generated by multiple paths has a unique symmetric extensions, but also other nonsymmetric extensions as do many CSMs which are not generated by a single path.

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# 1 Introduction

In much of the literature on cost sharing with divisible idiosyncratic goods the analysis of cost sharing methods (CSMs) with two agents is quite similar to those with many agents. For example, the axiomatic characterization of the Aumann-Shapley method for two agents<sup>1</sup> is virtually identical to that for many agents (Mirman and Tauman 1982, Billera and Heath 1982). This is also true for the Shapley-Shubik method (Shubik 1962), the Serial methods (Moulin and Shenker, 1992, Friedman and Moulin, 1995), and weighted versions of these (e.g., Mclean, Pazgal, and Sharkey, 1994, Hart and Monderer 1994). In this paper we attempt to understand when and why this observation holds.

For TU games, results in this spirit were demonstrated by Hart and Mas-Colell (1989), who showed that the Shapley value is uniquely determined by its behavior on two agent problems, and a (strong) notion of consistency. Mclean, Pazgal, and Sharkey (1994) generalize these ideas to CSMs and show that Aumann-Shapley and its weighted versions are uniquely determined by their behavior on two agent problems and a natural generalization of consistency.

In this paper we consider additive CSMs and introduce an extremely weak notion of consistency. While (strong) consistency requires that the removal of any agent (and a subsequent modification of the cost function) should not affect cost shares, we require only that the removal of “dummies” (agents whose demands do not affect

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<sup>1</sup>In this paper we will always assume that each agent consumes a single good, and thus we refer to agents and goods interchangeably

the cost function) not affect the cost shares. While this consistency requirement is quite weak, it is strong enough to characterize many interesting CSMs in terms of their properties on two agent problems. In fact, for many interesting methods, both notions of consistency coincide, in the additive framework.

The following section reviews additive cost sharing, while Section 3 reviews strong consistency and introduces weak consistency. Section 4 review the path generated methods which is then applied in Section 5 to characterize the set of consistent (both weak and strong) CSMs. We conclude with Section 6 then develops conditions under which consistent extensions exist and are unique. Several technical proofs are in the appendix.

## 2 Additive Cost Sharing

We consider the problem in which each agent consumes an idiosyncratic good. Let  $q_i \in \mathbb{R}_+$  represent an agent's demand of that agent's good. Let  $\mathbb{R}_+^N$  represent a bundle of goods, where  $N$  is a finite subset of  $Z^+$ , which represents the set of all possible agents.

(Note that whenever we write  $N \subset Z^+$ , we will implicitly assume that  $N$  is finite.)

For any  $q \in \mathbb{R}_+^N$  let  $C(q)$  be the cost of producing those goods, where  $C \in \Xi(N)$ , the space of nondecreasing and nonnegative (once) continuously differentiable functions on  $\mathbb{R}^N$ . A cost sharing mechanism  $x$  is a nonnegative mapping such that  $x^i(q; C)$  is the cost allocated to agent  $i \in N$ , for  $q \in \mathbb{R}_+^N$ ,  $C \in \Xi(N)$ , where  $\sum_{i \in N} x^i(q; C) = C(q)$ .

We say that agent  $i$  is a dummy agent if  $C(q_{-i}, q_i)$  is independent of  $q_i$ . The following two standard assumptions will be assumed throughout.

**Assumption 1 (Dummy)** *If agent  $i \in N$  is a dummy agent for  $C \in \Xi(N)$  then  $x^i(q, C) = 0$ , for all  $q \in \mathbb{R}_+^N$ .*

**Assumption 2 (Additivity)** *If  $C, C' \in \Xi(N)$  then  $x^i(q, C + C') = x^i(q, C) + x^i(q, C')$ , for all  $q \in \mathbb{R}_+^N$ .*

Dummy is a basic equity axiom requiring that the consumption of goods which do not affect the cost function are free. Additivity has been justified on many grounds, of which perhaps the most relevant is that of decentralizability, which states that if the cost is being allocated by different agencies then the allocations do not depend on this division.

Let  $CS$  be the set of CSMs satisfying dummy and additivity, and let  $CS(N)$  be the restriction of  $CS$  to  $N$ . The following representation theorem will provide the basis for our analysis.

**Theorem 1 (Friedman and Moulin, 1996)** *For any  $x \in CS(N)$  and each  $i \in N$  there exists a (unique) nonnegative measure  $\rho_i^N(\cdot; q)$  on  $[0, q]$  such that*

$$x_i(q; C) = \int_{[0, q]} \partial_i C(p) d\rho_i^N(p; q)$$

*for all  $C(\cdot) \in \Xi(N)$ , where  $\rho$ 's projection (marginal)  $\hat{\rho}_i^N(\cdot; q)$ , defined for  $S \subseteq [0, q_i]$  by  $\hat{\rho}_i^N(S; q) = \rho([0, q_{-i}] \times S; q)$ , is Lebesgue measure.*

Thus, we can reduce the study of CSMs to that of a vector of unnormalized probability measures.

## 2.1 Scale invariance and demand monotonicity

So far we have imposed no restrictions on how the CSM can depend on the demands, i.e. how  $x_i(q; C)$  varies with  $q$ . We now consider two such restrictions which will be useful.

One of the most common restrictions imposed on a CSM is that of scale invariance. Given  $\lambda \in \mathfrak{R}_+^n$  define  $\tau_\lambda(q)$  by  $\tau_\lambda(q)_i = \lambda_i q_i$  and define  $\tau_\lambda(C)$  by  $\tau_\lambda(C)(q) = C(\tau_\lambda(q))$ .

**Definition 1 (Scale Invariance)** *For all  $\lambda \in \mathfrak{R}_+^N$  and all  $C \in \Xi(N)$  a CSM in  $CS(N)$  is scale invariant if  $x(\tau_\lambda(q); C) = x(q; \tau_\lambda(C))$ .*

Scale invariance is the statement that different goods are not comparable, e.g. the units used to measure  $q_i$  can not be meaningfully compared with that used to measure  $q_j$  when  $j \neq i$ . Another important axiom is demand monotonicity.

**Definition 2 (Demand Monotonicity)** *A CSM in  $CS(N)$  is demand monotonic if  $x_i(q; C) \leq x_i(q'; C)$  for all  $q, q' \in \mathfrak{R}_+^N$  such that  $q_i \leq q'_i$  and  $q_{-i} = q'_{-i}$  and all  $C \in \Xi(N)$ .*

Demand monotonicity can be viewed as an extremely weak incentive constraint, since if a CSM is not demand monotonic then there are incentives for agents to overstate their demands.

In order to avoid technical complications about the measurability of function spaces, it is often useful to consider only CSMs defined for bounded demands, i.e.

there exists some  $\hat{q} > 0$  for which we only consider demands  $q \leq \hat{q}$ . We denote this space by  $CS^{\hat{q}}(N)$ .

### 3 Weak and Strong Consistency

Mclean, Pazgal, and Sharkey (1994) define a notion of consistency which is based on that defined by Hart and Mas-Colell (1989) for TU games. Their definition is analogous to many versions of consistency found in the literature.<sup>2</sup> However this definition is quite strong. We will denote it by strong consistency as compared with weak consistency to be defined below.

In order to discuss strong consistency we need some further regularity assumptions. Let  $\Xi^2(n)$  be the subset of  $\Xi(N)$  which has continuous cross-partial derivatives, i.e. for any  $C \in \Xi^2(N)$  and  $i, j \in N$  with  $i \neq j$  then  $\frac{\partial^2 C(q)}{\partial q_i \partial q_j}$  is continuous. Also let  $CS^2$  be the subset of  $CS$  which for all  $i \neq j$  we have that  $\frac{\partial x_i(C; q)}{\partial q_j}$  is integrable (as an element of  $L^1$ ) when  $C \in \Xi^2(N)$ .<sup>3</sup>

**Definition 3 (Strong Consistency)** *A CSM  $x \in CS^2$  is strongly consistent if for any  $N$ ,  $C \in \Xi^2(N)$  and any player  $i \in N$ , and all  $j \neq i$ ,*

$$x_j(q, C) = x_j(q_{-i}, R_i^{q_i}(C))$$

where  $R_i^{q_i}(C)(p_{-i}) = C(p_{-i}, q_i) - x_i(p_{-i}, q_i; C)$ , which is an element of  $\Xi(N - i)$ .

Thus, any player and her cost share can be removed from the cost function without affecting the cost shares of the remaining agents. We will denote the subset of  $CS$

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<sup>2</sup>See, e.g. Thomson (1996) for a survey of consistency in economics.

<sup>3</sup>For example, any continuous and monotone function has derivatives which are in  $L^1$ .

which is strongly consistent by  $CS_{strong}$ . By contrast, weak consistency only requires that dummies can be removed from a cost sharing problem without affecting the cost shares, a much milder requirement.

**Definition 4 (Weak Consistency)** *A CSM  $x \in CS(q)$  is weakly consistent if for all  $C \in \Xi(N)$  such that player  $i \in N$  is a dummy agent,  $x_j(q; C) = x_j(q_{-i}, R_i^{q_i}(C))$  for all  $j \neq i$ .*

Note that when  $i$  is a dummy agent, then  $R_i^{q_i}(C)(q_{-i}) = C(q_{-i}, 0_i)$ , and thus we need not impose any differentiability conditions on  $x(\cdot; C)$ . Let  $CS_{weak}$  denote the subset of  $CS$  which is weakly consistent.

Weak consistency seems (to us) to be an extremely natural requirement for CSMs, while strong consistency, which may be desirable, is not be as fundamental.

## 4 Path Generated Methods

In this section we review path generated methods as they will be central to our analysis.<sup>4</sup> A path is a natural way to generate a CSM. Fixing  $N$ , a mapping  $\gamma : [0, \infty] \times \mathfrak{R}_+^N \rightarrow \mathfrak{R}_+^N$  is an (admissible) path if: i)  $\gamma(0; q) = 0$ , ii)  $\gamma(\infty; q) = q$ , and iii)  $\gamma(t; q)$  is nondecreasing in  $t$ . Let  $\Gamma(N)$  be the set of all admissible paths and  $\Gamma = \bigcup_{N \subset \mathbb{Z}^+} \Gamma(N)$ .

Given a path  $\gamma$ , define the CSM,  $x \in CS(N)$  as the Riemann-Stieltjes integral

$$x_\gamma^i(q; C) = \int_0^\infty \partial_i C(\gamma(t; q)) d\gamma_i(t).$$

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<sup>4</sup>For a detailed analysis of path generated methods see Friedman (1996).



Friedman and Moulin (1995) have shown that this construction leads to a valid CSM.

Three of the best known CSM's are path generated.

- 1) Aumann Shapley:  $\gamma^i(t; q) = \min[t, 1]q_i$ .
- 2) Serial Cost:  $\gamma^i(t; q) = \min[t, q_i]$ .
- 3) Random order value (Weber, 1988) with order  $i_1, i_2, \dots, i_n$ :  $\gamma^{i_j}(t; q) = \min[1, \phi_{i_j}(t)]q_{i_j}$ , where  $\phi_k(t) = 0$  if  $t \leq (k - 1)$  and 1 if  $t \geq k$  and  $(t - k + 1)$  otherwise.<sup>5</sup>

Mclean and Sharkey (1994) define weighted Aumann-Shapley mechanisms. In fact, such weights can also be included into Serial cost. Given a sequence of weights,  $\{w_i\}$  such that for all  $i \in Z^+$ ,  $0 < w_i < \infty$ , the weighted Aumann-Shapley mechanism is generated by  $\gamma^i(t; q) = \min[1, t^{w_i}]q_i$ , while the weighted Serial costs mechanisms are generated by  $\gamma^i(t; q) = \min[t^{w_i}, q_i]$ .

Note that the CSMs generated by paths do not depend on the specific parameterizations of the paths. For example, Serial cost can be equivalently defined by  $\gamma^i(t; q) = \min[q_i, \phi(t)]$  where  $\phi(t)$  is any nondecreasing continuous function of  $t$  such that  $\phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ . Similarly, if a pair of weight sequences  $\{w_i\}$  and  $\{w'_i\}$  satisfy  $w'_i = \alpha w_i$  for some  $\alpha > 0$  and all  $i \in Z^+$ , then they lead to the same weighted Aumann-Shapley or Serial cost Mechanism.

**Definition 5** *Let  $CP(N)$  be the set of path generated CSMs in  $CS(N)$ : if  $x \in CP(N)$  then there exists a path  $\gamma \in \Gamma(N)$  such that  $x = x_\gamma$ .*

As shown in Friedman (1997), all CSMs can be generated as a infinite convex

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<sup>5</sup>The Shapley-Shubik method is obtained by averaging the random order values over all orderings.

combination of path generated methods.<sup>6</sup> The following theorem may be found there.

**Theorem 2** *The following are equivalent:*

i)  $x \in CS(N)$ .

ii) *There exists a non-negative probability measure  $\mu$  on  $\Gamma(N)$  such that*

$$x = \int_{\gamma \in \Gamma(N)} x_\gamma d\mu(\gamma).$$

Proof: See Friedman (1996).

Note that the paths in the above theorem need not be well behaved in  $q$  – there need be no relationship between  $\gamma(\cdot; q)$  and  $\gamma(\cdot; q')$  when  $q \neq q'$ . In particular  $\gamma(\cdot; q)$  need not be continuous in  $q$ . This is because we have made no assumptions about the dependence of  $x(q; C)$  on  $q$ . Both SI and DM impose such conditions and are characterized in Friedman (1997). In particular, either of these imply that  $\gamma(\cdot; q)$  is continuous in  $q$ .

## 5 Representations under Consistency

In this section, we provide representation theorems for both strongly and weakly consistent CSMs. These allow us to easily check whether many CSMs are consistent.

**Definition 6** *A path  $\gamma \in \Gamma(N)$ , for  $N \subset Z_+$  is separable if for each  $i \in N$ ,  $\gamma_i(t; q)$  is independent of  $q_{-i}$  which we write as (with a slight abuse of notation)  $\gamma_i(t; q_i)$ .*

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<sup>6</sup>The set of equivalence classes defined by the mapping from  $\gamma \in \Gamma(N)$  to  $x_\gamma CP(N)$  is measurable because the space of probability measures on a compact subset of  $\mathfrak{R}_+^N$  is metrizable in the topology of weak convergence. See Phelps (1980) for details, and Friedman (1997) for a discussion in the context of CSMs.

Note that a set of functions  $\{\gamma_i(\cdot; \cdot)\}$  for all  $i \in Z^+$  defines a path for any  $N \subset Z^+$ .

Let  $\Gamma_S$  be the set of such separable paths.

**Theorem 3** *For any  $\hat{q} > 0$  the following are equivalent:*

i)  $x \in CS_{weak}^{\hat{q}}$ .

ii) *There exists a probability measure  $\mu$  on  $\Gamma_{separable}^{\hat{q}}$  such that*

$$x = \int_{\gamma \in \Gamma_{separable}^{\hat{q}}} x_{\gamma} d\mu(\gamma),$$

where  $CS_{weak}^{\hat{q}}$  (resp.  $\Gamma_{separable}^{\hat{q}}$ ) is the restriction of  $CS_{weak}$  (resp.  $\Gamma_{separable}$ ) to the problem with demands bounded by  $\hat{q}$ .

Proof: In appendix.

Thus, the separable paths form a basis for  $CS_{weak}$ . As the next result demonstrates, all of the previously mentioned CSMs are generated by separable paths and therefore are weakly consistent.<sup>7</sup>

**Corollary 1** *The following CSMs are weakly consistent:*

- 1) *Aumann Shapley and weighted Aumann-Shapley.*
- 2) *Serial Cost and weighted Serial cost.*
- 3) *Random order methods and Shapley-Shubik.*

However, it is easy to show that not all CSMs are weakly consistent. Consider the CSM  $x_{\gamma}$  where  $\gamma(t; q) = (\min[q_1, t], \min[q_2, e^{tq_3}], \min[q_3, t])$ , which is not

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<sup>7</sup>Note that the boundedness of  $q$  in the above theorem does not affect these results, since we need check at each finite value of  $q$ .

separable, and the cost function  $C(q_1, q_2)$  which does not depend on  $q_3$ . Then  $x_\gamma(q; C) = \int_0^\infty dt \partial_2 C(\min[q_1, t], \min[q_2, e^{tq_3}]) q_3 e^{tq_3}$  which clearly depends on  $q_3$  even though agent 3 is a dummy agent. Thus it is not weakly consistent.

Strong consistency imposes a more stringent requirement on the paths. We now demonstrate a weakly consistent CSM which is not strongly consistent. Consider the CSM  $x_\gamma$  where  $\gamma(t; q) = (\min[1, t]q_1, \min[1, (t+t^2)/2]q_2, \min[1, t]q_3)$ , which is separable and scale invariant. Let  $C(q_1, q_2, q_3) = q_1 q_2 q_3$ . Then  $x_\gamma^1(q; C) = \int_0^\infty dt \min[q_2, q_2(t+t^2)/2] \min[q_3, tq_3] q_1 = 17q_1 q_2 q_3 / 24$ , while  $R_3^{q_3}(C)(p_1, p_2) = 17p_1 p_2 / 24$ . If  $x_\gamma$  was strongly consistent the cost share to agent 1 could be computed by removing agent 3 which yields  $\int_0^\infty dt 17(\min[q_2, q_2(t+t^2)/2]) / 24 q_3 q_1 = 85q_1 q_2 q_3 / 288$  which differs from the cost share which was computed directly, e.g. when  $q = (1, 1, 1)$  the direct computation shows that  $x_\gamma^1(q; C) = 17/24$  while the computation applying strong consistency computes a value of  $85/288$  which are unequal. Thus  $x_\gamma$  is not strongly consistent. This is because the path which defines this CSM is not associative.

**Definition 7** A path  $\gamma \in \Gamma(N)$ , for  $N \subset Z_+$  is associative if it is separable and there exists a continuous function  $\phi(t, s)$  (whose derivatives are in  $L^1$ ) such that,  $\gamma_i(t; \gamma_i(s; q_i)) = \gamma_i(\phi(t, s); q_i)$  for all  $i \in Z^+$ .

Let  $\Gamma_{\text{associative}}$  be the subset of paths in  $\Gamma_{\text{separable}}$  which are associative.

**Theorem 4** Let  $\mu$  be a probability measure on  $\Gamma_{\text{associative}}$  and

$$x = \int_{\gamma \in \Gamma_{\text{associative}}} x_\gamma d\mu(\gamma),$$

then  $x \in CS_{strong}$ .

Proof: In appendix.

Note that above characterization is only partial. We conjecture the complete characterization is true when  $q$  is bounded: for all  $\hat{q} > 0$ , all  $x \in CS_{strong}^{\hat{q}}$  can be written as a combination of CSMs generated by associative paths.<sup>8</sup> As we now show, all demand monotonic methods which are weakly consistent are also strongly consistent, since the paths associated with demand monotonic methods are associative. Thus it is easy to see that in this case strong consistency reduces to weak consistency, which is typically much easier to check.

**Corollary 2** *Let  $x \in CS$  be weakly consistent and demand monotonic, then it is strongly consistent.*

Proof: In Friedman (1996) it is shown that (for bounded  $q$ ) all demand monotonic CSMs are generated by paths of the form  $\gamma^i(t; q) = \min[\beta_i(t), q_i]$ , where  $\beta_i(t)$  is non-decreasing. Thus, demand monotonic paths are associative with  $\phi(t, s) = \min[t, s]$ . Since demand monotonic methods are generated by demand monotonic paths, and when applying strong consistency to  $x_i(q; C)$  only demands less than  $q$  need be considered, this completes the proof.  $\square$

We now show that all of the previously discussed CSMs are strongly consistent.

**Theorem 5** *The following CSMs are strongly consistent:*

1) *Aumann Shapley and weighted Aumann-Shapley.*

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<sup>8</sup>When  $q$  is not bounded, the set of associative paths is not closed in the appropriate topology.

2) *Serial Cost and weighted Serial cost.*

3) *Random order methods and Shapley-Shubik.*

Proof: Strong consistency of Serial cost and Random order methods follow immediately from Theorem 2 and the fact that they are demand monotonic. For the weighted Aumann-Shapley Mechanism it follows upon noting that the path which generates the mechanism is associative with  $\phi(t, s) = ts$ , since  $t^{w_i} s^{w_i} q_i = (ts)^{w_i} q_i$ .  $\square$

## 6 Consistent Extensions

Our goal in this section is to understand when a CSM defined for two agents extends to a (weakly or strongly) consistent CSM for an arbitrary number of players. Let  $CS(2) = \bigcap_{N \subset \mathbb{Z}^+, |N|=2} CS(N)$ , the set of all two player methods. Consider some  $x \in CS(2)$ . From the previous section we know that if  $x$  can not be generated by separable (resp. associative) paths there are no weakly (resp. strongly) consistent extensions of  $x$  to  $CS$ . Thus, if we can formulate  $x \in CS(2)$  as a sum of separable (resp. associative) path generated methods, then we know that there exists a weakly (resp. strongly) consistent extension.

**Theorem 6** *Let  $x \in CS(2)$ . Then the following statements are equivalent:*

- 1) *The CSM  $x$  can be written as a convex combination of CSMs,  $x_\gamma$ , where each  $\gamma$  is separable (resp. associative).*
- 2) *There exists at least one  $\hat{x} \in CS_{weak}$  (resp.  $\hat{x} \in CS_{strong}$ ) such that  $x = \hat{x}$  on  $CS(2)$ . (i.e.  $\hat{x}$  is a consistent extension of  $x$ .)*

Proof: This follows immediately from the representation theorems.  $\square$

Note that this extension is not necessarily unique. (We give an example of a method with multiple extensions in the following section.) However, it is easy to see that if  $x$  is generated by a single separable (resp. associative) path then  $x$  has a unique weakly (resp. strongly) consistent extension.

**Theorem 7** *Let  $x \in CS(2)$  be generated by a single separable (resp. associative) path, e.g. for all  $i \neq j$ ,  $x^{\{i,j\}}$  is generated by  $(\gamma_i(t; q_i), \gamma_j(t; q_j))$ . Then there exists a unique  $\hat{x} \in CS$  such that  $x = \hat{x}$  on  $CS(2)$  and is weakly (resp. strongly) consistent.*

Proof: It is easy to see that for any  $N \subset Z^+$ , the general method defined by  $(\gamma_i(t; q_i))_{i \in N}$  is well defined and weakly (resp. strongly) consistent. It is unique, since the projection of a general method onto a 2-agent CSM must be unique when the CSM is generated by a single path. Geometrically, all 2-dimensional coordinate views of a monotone curve are sufficient to reconstruct the curve. The formal details of this argument are in Friedman and Moulin (1995) Lemma 6.  $\square$

This theorem is applicable to almost all of the previously discussed methods.

**Corollary 3** *The following methods defined on  $CS(2)$  extend uniquely to  $CS_{weak}$  (resp.  $CS_{strong}$ ): Random order methods, Aumann-Shapley, Serial cost, and their weighted versions.*

Note that the above theorem does not include Shapley-Shubik, since it is a sum of multiple path generated methods. As we show in the next section Shapley-Shubik has many (nonsymmetric) extensions; however, it has only one symmetric extension.

**Theorem 8** *The Shapley-Shubik CSM defined on  $CS(2)$  has a unique weakly (resp. strongly) consistent symmetric extension.*

Proof: Fix  $N = \{1, 2, \dots, n\} \subset Z^+$  such that  $n \geq 3$ . Let  $\omega : N \rightarrow N$  be a permutation of  $N$ , i.e. a bijection, and let  $\Omega$  be the set of all permutations on  $N$ , and note that  $|\Omega| = n!$ . For simplicity assume that  $t \in [0, n]$ . The Shapley-Shubik CSM for the subset of  $CS(2)$  contained in  $N$  can be written as  $\sum_{\sigma \in \Sigma} x_{\gamma^\sigma} / n!$ , where for  $i \in N$ ,  $\gamma_i^\sigma(t; q) = \phi_{\sigma(i)}(t)q_{\sigma(i)}$ . It is easy to see that when viewed as a CSM over  $N$ , these are the only paths (up to equivalence of reparameterization) which “project” for all  $i, j \in N, i \neq j$ , to a two player random order value. Thus any extension to  $N$  can be written as  $\sum_{\sigma \in \Sigma} \lambda(\sigma)x_{\gamma^\sigma}$  where  $\sum_{\sigma \in \Sigma} \lambda(\sigma) = 1$  and for all  $\sigma \in \Sigma, \lambda(\sigma) \geq 0$ . However, unless all the  $\lambda$ 's are equal the method is not be symmetric.  $\square$

## 6.1 Nonsymmetric extensions

Somewhat surprisingly, a nonsymmetric CSM may be symmetric when “projected” onto  $CS(2)$ . Thus, as we now demonstrate, the Shapley-Shubik Mechanism has (many) nonsymmetric extensions.

Consider the case when  $N = \{1, 2, 3\}$ . We construct a CSM which is a convex combination of random order values with the following orderings:  $\sigma^1 = (1, 2, 3)$  and  $\sigma^2 = (3, 2, 1)$ . Define  $x_i$  to be the random order value for order  $\sigma^i$ . Then consider the CSM defined by  $(x_1 + x_3)/2$ . It is straightforward to check that all two agent methods induced by this method coincide with the Shapley-Shubik CSM. Thus, this provides an example of a nonsymmetric extension of the Shapley-Shubik CSM. (Agents 1 and



3 are interchangeable, but not agent 2 with either 1 or 3.) Note that there are other nonsymmetric extension which are combinations of various random order values.

In general there are even more complex examples of CSMs with non-unique consistent extensions. Note that generically, if  $x \in CS(2)$  is generated by a finite number of paths then the extension will be unique, since the paths can not be ‘mixed’ together; however, in particular cases they can have multiple extensions as shown here. Consider the following asymmetric CSM in  $CS(2)$  which is generated by the average of the following associative and scale invariant paths:

$$\gamma_a(t; q) = (tq_1, tq_2, t^2q_3, tq_4, tq_5, \dots),$$

$$\gamma_b(t; q) = (tq_1, t^{1.5}q_2, t^6q_3, tq_4, tq_5, \dots),$$

$$\gamma_c(t; q) = (tq_1, t^{0.5}q_2, t^3q_3, tq_4, tq_5, \dots),$$

where for simplicity we require that  $t \in [0, 1]$ . By this we mean that for  $N = \{1, 2\}$  the CSM is the average of the methods generated by the three paths,  $(tq_1, tq_2)$ ,  $(tq_1, t^{1.5}q_2)$ , and  $(tq_1, t^{0.5}q_2)$ .

Clearly, this element of  $CS(2)$  has the “natural” extension  $[x_{\gamma_a} + x_{\gamma_b} + x_{\gamma_c}]/3$ . However, it also has another extension which is given by  $[x_{\gamma_d} + x_{\gamma_e} + x_{\gamma_f}]/3$ , where

$$\gamma_d(t; q) = (tq_1, tq_2, t^6q_3, tq_4, tq_5, \dots),$$

$$\gamma_e(t; q) = (tq_1, t^{1.5}q_2, t^3q_3, tq_4, tq_5, \dots),$$

$$\gamma_f(t; q) = (tq_1, t^{0.5}q_2, t^2q_3, tq_4, tq_5, \dots).$$

Finally, any convex combination of these two methods is weakly consistent. Thus, there are infinitely many “nontrivial” extensions.

In general, extensions of arbitrary CSMs can be quite complex.

## A Appendix: Proofs

### A.1 Proof of Theorem 3

First we show the effect of weak consistency on the representing measure.

**Lemma 1** *Let the restriction of  $x \in CS_{weak}^{\hat{q}}$  to  $CS^{\hat{q}}(N)$  be represented by  $\rho^N(p; q)$  as in the Theorem 1. For any  $j \in N \setminus i$  define  $\pi_i[\rho_j^N(p; q)]$  to be the measure on  $[0, q_{-i}]$  defined such that for any measurable set  $S_{-i} \subseteq [0, q_{-i}]$ ,  $\pi_i[\rho_j^N(p; q)](S_{-i}) = \rho_j^N(p; q)(S_{-i} \times [0, q_i])$ .*

*Then for any  $i \in N$  the restriction of  $x$  to  $CS^{\hat{q}}(N \setminus i)$  is represented by the measure  $\rho^{N \setminus i}(p; q)$  which is equal to  $\pi_i[\rho_{-i}^N](p; q)$  a.e., and  $\pi_i[\rho_{-i}^N](p; q)$  is independent of  $q_i$ .*

**Proof:** Consider any  $C \in \Xi(N)$  such that agent  $i$  is a dummy agent. By weak consistency  $x$  can be computed either directly, or by applying weak consistency and removing agent  $i$ . Computing directly,

$$x_j(q; C) = \int_{[0, q]} \partial_j C(p) d\rho_j^N(p; q)$$

and integrating with respect to  $p_i$  gives

$$x_j(q; C) = \int_{[0, q_{-i}]} \partial_j C(p_{-i}, 0_i) d\pi_i[\rho_{-i}^N](p; q),$$

since  $\partial_j C(q)$  does not depend on  $p_i$  by assumption. Applying weak consistency yields,

$$x_j(q; C) = \int_{[0, q_{-i}]} \partial_j R_i(C)(p) d\rho^{N \setminus i}(p; q_{-i}).$$

Thus, if  $x$  is weakly consistent then these two methods of computing it must give the same result which implies that

$$0 = \int_{[0, q_{-i}]} \partial_j R_i(C)(p) \left\{ d\pi_i[\rho_{-i}^N](p; q) - d\rho^{N \setminus i}(p; q_{-i}) \right\}.$$

Since this holds for any  $C \in \Xi(N)$  the measures must be equal. Noting that  $\rho^{N \setminus i}(p; q_{-i})$  is independent of  $q_i$  completes the proof.  $\square$

Let  $\mathfrak{R}_+^\infty$  be the set of non-negative sequences  $\{q_i\}$  for  $i \in Z^+$ , for which  $|\{i \mid q_i > 0\}| < \infty$ . By the above lemma, any weakly consistent CSM is represented by  $\rho(p; q)$  where  $p, q \in \mathfrak{R}_+^\infty$ . Thus using Theorem 2 we can represent the CSM as a sum of paths  $\gamma(t; \cdot) : \mathfrak{R}_+^\infty \rightarrow \mathfrak{R}_+^\infty$ :

$$x = \int_{\gamma \in \Gamma(\infty)} x_\gamma d\mu(\gamma),$$

where we denote the set of such paths as  $\Gamma(\infty)$ . Given a path  $\gamma(t; q)$  let  $\phi(\gamma)$  be given by  $\phi(\gamma(t; q))_i = \gamma_i(t; q_i, 0_{-i})$ .

**Lemma 2** *If  $x$  is weakly consistent and is represented by  $\mu$  then  $x = x^\phi$  where  $x = \int_{\gamma \in \Gamma(\infty)} x_\gamma d\mu(\gamma)$  and  $x^\phi = \int_{\gamma \in \Gamma(\infty)} x_{\phi(\gamma)} d\mu(\gamma)$ .*

Proof: Consider  $i, j \in Z^+$  such that  $i \neq j$ . For any  $N \subset Z^+$  with  $i, j \in N$ , note that when  $q^j = 0$  the restriction of  $x$  and  $x^\phi$  to  $N \setminus j$  are equivalent under the weak topology. However, by weak consistency this implies that they are equal for all  $q_j \geq 0$ .

Repeatedly applying this argument proves the lemma.  $\square$

Noting that  $\phi(\gamma)$  is a separable path completes the proof of Theorem 3.

## A.2 Proof of Theorem 4

**Lemma 3** *If  $\gamma$  is an associative path, then  $x_\gamma$  is strongly consistent.*

Proof: Assume that  $C$  is twice continuously differentiable and that  $\gamma(t; q)$  is an associative and differentiable path (in the  $L^1$  sense). Now

$$x_i(q; C) = \int_0^1 dt \partial_1 C(\gamma(t; q)) \dot{\gamma}_i(t; q_i),$$

where  $\dot{\gamma}_i(t; q_i) = \frac{\partial \gamma_i(t; q_i)}{\partial t}$ , and we are using the separability of  $\gamma$ . For  $i \neq j$  differentiating yields,

$$\frac{\partial x_i(q; C)}{\partial q_j} = \int_0^1 dt \partial_{ij} C(\gamma(t; q)) \dot{\gamma}_i(t; q_i) \partial_j \gamma_j(t; q_j). \quad (*)$$

Strong consistency implies that  $x_j(q_{-i}; \hat{C}) = x_j(q_{-i}; \hat{C}) - x_j(q_{-i}; x_i(\cdot, q_i; C))$  and we compute the second term. Using (\*) yields  $x_j(q_{-j}; x_i(\cdot, q_i; C)) =$

$$\int_0^1 ds \int_0^1 dt \partial_{ij} C(\gamma_{-i}(t; \gamma_{-i}(s; q_{-i})), \gamma_i(t; q_i)) \dot{\gamma}_i(t; q_i) \dot{\gamma}_j(t; q_j) \partial_j \gamma_j(t; \gamma_j(s; q_j))$$

and since  $\gamma$  is associative

$$x_j(q_{-i}; x_i(\cdot, q_i; C)) = \int_0^1 ds \int_0^1 dt \partial_{ij} C(\gamma_{-i}(\phi(t, s); q_{-i}), \gamma_i(t; q_i)) \dot{\gamma}_i(t; q_i) \dot{\gamma}_j(t; q_j) \partial_j \gamma_j(\phi(t, s); q_j).$$

Changing variables to  $u = \phi(t, s)$  and  $v = \gamma_1(t; q_1)$  and noticing that the Jacobian of the transformation is  $\dot{\gamma}_i(t; q_i) \partial_j \phi(t, s)$  results in

$$x_j(q_{-i}; x_i(\cdot, q_i; C)) = \frac{\int_0^1 du \int_{\gamma_i(u; q_i)}^1 dv \partial_{ij} C(\gamma_{-i}(u; q_{-i}), v) \partial_j \gamma_j(u; q_j)}{\partial_j \phi(t, s)} (**).$$

Differentiating the identity  $\gamma_j(t; \gamma_j(s; q_j)) = \gamma_j(\phi(t, s); q_j)$  with respect to  $s$  yields

$$\partial_j \gamma_j(t; \gamma_j(s; q_j)) \partial_i \gamma_j(s; q_j) = \partial_i \gamma_j(t; \gamma_j(s; q_j)) \partial_j \phi(t, s).$$

Combining this with (\*\*) yields

$$x_j(q_{-i}; x_i(\cdot, q_i; C)) = \int_0^1 du \int_{\gamma_i(u; q_i}^{q_i} dv \partial_{ij} C(\gamma_{-i}(u; q_{-i}), v) \dot{\gamma}_j(u; q_j).$$

Integrating with respect to  $v$  yields

$$x_j(q_{-i}; x_i(\cdot, q_i; C)) = \int_0^1 du \partial_j [C(\gamma_{-i}(u; q_{-i}), q_i) - C(\gamma(u; q))] \dot{\gamma}_j(u; q_j),$$

which equals  $x_j(q; C) - x_j(q_{-i}; \hat{C}_{q_i})$ , proving strong consistency.  $\square$

Noting that a convex combination of strongly consistent CSMs is strongly consistent, completes the proof of the Theorem.

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