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**Working Paper**

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Working Paper, No. 2012-07

**Provided in Cooperation with:**

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*Suggested Citation:* Carbonell-Nicolau, Oriol; McLean, Richard P. (2012) : On equilibrium refinement in supermodular games, Working Paper, No. 2012-07, Rutgers University, Department of Economics, New Brunswick, NJ

This Version is available at:

<https://hdl.handle.net/10419/94233>

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# On Equilibrium Refinement in Supermodular Games

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February 2012

## Abstract

We show that supermodular games satisfying sequential better-reply security possess a pure strategy perfect equilibrium and a strategically stable set of pure strategies. We illustrate that, in continuous supermodular games, perfect equilibria may contain weakly dominated actions. Moreover, in discontinuous supermodular games, perfect equilibria may involve play of actions in the interior of the set of weakly dominated actions. We prove that when the set of undominated actions is a sublattice, supermodular games satisfying sequential better-reply security possess pure strategy perfect equilibria that can be obtained as limits of undominated actions.

*Keywords:* supermodular game, weakly dominated strategy, trembling-hand perfect equilibrium, strategically stable set.

*JEL classification:* C72.

## 1 Introduction

As pointed out in Carbonell-Nicolau [3], the existence of a perfect equilibrium in a normal form game depends crucially on the existence of Nash equilibria

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in the game's Selten perturbations (i.e., perturbations in which all players choose a completely mixed strategy with small but positive probability). In this paper, we show that the properties of supermodular games (cf. Topkis [16], Vives [17], Milgrom and Roberts [11], and Milgrom and Shannon [12]) are inherited by their Selten perturbations. Using this fact, and assuming *sequential better-reply security* (Carbonell-Nicolau and McLean [5]), a property of a game that guarantees that the Nash equilibrium correspondence of perturbed games has a closed graph, we prove the existence of a pure strategy perfect equilibrium (Theorem 1). A similar argument can be used to establish the existence of strategically stable sets (Kohlberg and Mertens [8]) of pure action profiles in supermodular games (Theorem 2). The set of pure strategy perfect equilibria (resp. any stable set) is shown to be compact in the set of Nash equilibria (resp. in the set of pure strategy perfect equilibria).

The fact that infinite-action, continuous games may exhibit unique Nash and perfect equilibria in weakly dominated actions (e.g., Simon and Stinchcombe [14], Example 2.1) fails to be true if one confines attention to the class of supermodular games. In fact, Kultti and Salonen [9] show that the set of Nash equilibria (and hence the set of perfect equilibria) in supermodular games cannot consist only of weakly dominated strategy profiles: at least one Nash equilibrium must be weakly undominated. However, pure strategy perfect equilibria in continuous, supermodular games may involve play of weakly dominated actions (Example 1). Moreover, in discontinuous supermodular games, perfect equilibria may select pure strategy profiles in the interior of the set of weakly dominated actions for some player (Example 2). When the set of weakly undominated actions is a sublattice, supermodular games satisfying sequential better-reply security have pure strategy perfect equilibria that do not involve play of actions in the interior of the set of weakly dominated actions (Theorem 3).

## 2 Preliminaries

### 2.1 Supermodular games

A **normal form game** (or simply a **game**) is a collection  $G = (X_i, u_i)_{i=1}^N$ , where  $N$  is a finite number of players,  $X_i$  is a nonempty set of actions for player  $i$ , and  $u_i \in B(X)$ , where  $B(X)$  denotes the space of bounded, real-valued functions defined on  $X := \times_{i=1}^N X_i$ . We view  $B(X)$  as a metric space

with associated metric defined by

$$\rho(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$

Let  $U(X)$  denote cartesian product of  $N$  copies of  $B(X)$ . We also view  $U(X)$  as a metric space, and denote, by a slight abuse of notation, the associated metric again by  $\rho$ , i.e.,

$$\rho((f_1, \dots, f_N), (g_1, \dots, g_N)) := \max_{i \in \{1, \dots, N\}} \left[ \sup_{x \in X} |f_i(x) - g_i(x)| \right].$$

Consequently, a net  $(u^\alpha)$  in  $U(X)$  is convergent with limit  $u$  if and only if for each  $i$ , the net  $(u_i^\alpha)$  is uniformly convergent with limit  $u_i$ .

A **lattice** is a pair  $(A, \leq)$ , where  $A$  is a nonempty set and  $\leq$  is a partial order (i.e., a reflexive, antisymmetric, and transitive binary relation) in  $A \times A$  such that for every  $\{a, b\} \subseteq A$ , the infimum of  $\{a, b\}$  (denoted  $\inf\{a, b\}$ ) and the supremum of  $\{a, b\}$  (denoted  $\sup\{a, b\}$ ) exist in  $A$ . A lattice  $(A, \leq)$  is **lattice complete** if  $\inf B \in A$  and  $\sup B \in A$  for every  $\emptyset \neq B \subseteq A$ . If  $G = (X_i, u_i)_{i=1}^N$  is a game and if each  $X_i$  is partially ordered by  $\leq_i$ , so that  $(X_i, \leq_i)$  is a lattice, then  $G$  is a **lattice game**.

If  $G = (X_i, u_i)_{i=1}^N$  is a game and  $Y_i \subseteq X_i$  for each  $i$ , we will write  $(Y_i, u_i|_{Y_i})_{i=1}^N$  simply as  $(Y_i, u_i)_{i=1}^N$ . If each  $X_i$  is a nonempty metric (resp. compact) space,  $G$  is said to be a **metric** (resp. **compact**) **game**. If  $G$  is a metric game and if  $u_i$  is Borel measurable for each  $i$ , we say that  $G$  is a **metric, Borel game**.

A **compact, metric, lattice, Borel game** is a lattice game  $(X_i, u_i)_{i=1}^N$  such that each  $X_i$  is a compact, metric lattice (cf. Birkhoff [2], Chapter X) and  $u_i$  is Borel measurable for each  $i$ .

Given a compact, metric, lattice, Borel game  $G = (X_i, u_i)_{i=1}^N$ , the pair  $(X, \leq)$  is a lattice, where  $\leq$  is the relation in  $X \times X$  defined as follows:

$$x \leq y \Leftrightarrow [x_j \leq_j y_j, \text{ for all } j \in \{1, \dots, N\}].$$

In addition,  $(X, \leq)$ , endowed with the product topology, is a compact, metric lattice.

Given  $i$ , the product lattice  $(X_{-i}, \leq_{-i})$  is defined similarly.

The following lemma will be useful.

**Lemma 1** (Ellis [7]). *Every compact, metric lattice is lattice complete.*

**Definition 1.** A compact, metric, lattice, Borel game  $(X_i, u_i)_{i=1}^N$  is **super-modular** if the following is satisfied:

- For each  $i$  and every  $x_{-i} \in X_{-i}$ ,  $u_i(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$ .
- For each  $i$  and every  $x_{-i} \in X_{-i}$ ,  $u_i(\cdot, x_{-i})$  is *supermodular*, i.e., for each  $i$ ,  $x_{-i} \in X_{-i}$ , and  $\{x_i, y_i\} \subseteq X_i$ ,

$$u_i(\sup\{x_i, y_i\}, x_{-i}) + u_i(\inf\{x_i, y_i\}, x_{-i}) \geq u_i(x_i, x_{-i}) + u_i(y_i, x_{-i}).$$

- For each  $i$ ,  $u_i$  exhibits *increasing differences in  $X_i$  and  $X_{-i}$* , i.e.,  $u_i(x_i, y_{-i}) - u_i(x_i, x_{-i})$  is increasing in  $x_i$  for all  $y_{-i} \geq_{-i} x_{-i}$ .

**Definition 2.** A strategy profile  $x = (x_i, x_{-i})$  in  $X$  is a **Nash equilibrium** of  $G = (X_i, u_i)_{i=1}^N$  if  $u_i(y_i, x_{-i}) \leq u_i(x_i, x_{-i})$  for each  $y_i \in X_i$  and every  $i$ .

Given a game  $G = (X_i, u_i)_{i=1}^N$ , the vector  $(u_1(x), \dots, u_N(x))$  is denoted as  $u(x)$ . Given  $\varepsilon \geq 0$ , define a correspondence  $\mathcal{N}_X^\varepsilon : U(X) \rightrightarrows X$  that assigns to each profile  $u = (u_1, \dots, u_N) \in U(X)$  the set of  $\varepsilon$ -Nash equilibria  $\mathcal{N}_X^\varepsilon(u)$  of  $(X_i, u_i)$ . The set of Nash equilibria  $(X_i, u_i)$  (i.e.,  $\varepsilon = 0$ ) will be denoted simply as  $\mathcal{N}_X(u)$ .

If  $(S^\nu)$  is a sequence of subsets of a topological space  $S$ , define the (Kuratowski-Painlevé) **topological limit superior of  $(S^\nu)$** , denoted  $\text{Ls}(S^\nu)$ , to be the set of  $y \in S$  such that there exist a subsequence  $(S^n)$  and a sequence  $(y^n)$  satisfying  $y^n \in S^n$  for each  $n$  and  $y^n \rightarrow y$ .

## 2.2 Sequential better-reply security

The following definitions are taken from Carbonell-Nicolau and McLean [5].

**Definition 3.** Let  $G = (X_i, u_i)_{i=1}^N$  be a metric game. Suppose that  $(u_i^\nu)$  is a sequence in  $U(X)$ . The game  $G$  satisfies **sequential better-reply security with respect to  $(u_i^\nu)$**  if the following condition is satisfied for each  $i$ : if  $(u_i^\nu)$  is a subsequence of  $(u_i^\nu)$ , if  $(x^n, u^n(x^n)) \in X \times \mathbb{R}^N$  is a convergent sequence with limit  $(x, \gamma) \in X \times \mathbb{R}^N$  satisfying  $x^n \in X$  for each  $n$ , and if  $x$  is not a Nash equilibrium of  $G$ , then there exist an  $i$ , an  $\eta > \gamma_i$ , a subsequence  $(x^k)$  of  $(x^n)$ , and a sequence  $(y_i^k)$  such that for each  $k$ ,  $y_i^k \in X_i$  and  $u_i^k(y_i^k, x_{-i}^k) \geq \eta$ .

**Definition 4.** A metric game  $G = (X_i, u_i)_{i=1}^N$  is **sequentially better-reply secure** if the following condition is satisfied: if  $(x^n, u(x^n)) \in X \times \mathbb{R}^N$  is a convergent sequence with limit  $(x, \gamma) \in X \times \mathbb{R}^N$  and if  $x$  is not a Nash equilibrium of  $G$ , then there exist an  $i$ , an  $\eta > \gamma_i$ , a subsequence  $(x^k)$  of  $(x^n)$ , and a sequence  $(y_i^k)$  such that for each  $k$ ,  $y_i^k \in X_i$  and  $u_i(y_i^k, x_{-i}^k) \geq \eta$ .

**Remark 1.** Within the class of compact, metric, lattice, Borel games, the class of sequentially better-reply secure, supermodular games subsumes the class of supermodular games as defined in Milgrom and Roberts [11].

**Remark 2.** Sequential better-reply security is weaker than several conditions introduced in the literature on the existence of Nash equilibrium. See Carbonell-Nicolau and McLean [6].

We next state a result characterizing sequential better-reply security with respect to a sequence  $(u_i^\nu)$ .

**Proposition 1** (Carbonell-Nicolau and McLean [5], Theorem 2). *Suppose that  $G = (X_i, u_i)_{i=1}^N$  is a metric game and suppose that  $(u^\nu)$  is convergent in  $U(X)$  with limit  $u$ . The following are equivalent:*

- (1)  $G$  satisfies sequential better-reply security with respect to  $(u_i^\nu)$ .
- (2) If  $\varepsilon^\nu \rightarrow 0$ , then  $\text{Ls}(\mathcal{N}_X^{\varepsilon^\nu}(u^\nu)) \subseteq \mathcal{N}_X(u)$ .

The following lemma will be useful.

**Lemma 2.** *Suppose that  $G = (X_i, u_i)_{i=1}^N$  is a metric game satisfying sequential better-reply security. Furthermore, suppose that  $(u^\nu)$  is convergent in  $U(X)$  with limit  $u$ . Then  $G$  satisfies sequential better-reply security with respect to  $(u^\nu)$ .*

*Proof.* Suppose that  $(u_i^n)$  is a subsequence of  $(u_i^\nu)$ ,  $(x^n, u^n(x^n))$  is a convergent sequence with limit  $(x, \gamma) \in X \times \mathbb{R}^N$  satisfying  $x^n \in X$  for each  $n$ , and  $x$  is not a Nash equilibrium of  $G = (X_i, u_i)_{i=1}^N$ . Since  $G$  satisfies sequential better-reply security, there exist an  $i$ , an  $\eta' > \gamma_i$ , a subsequence  $(x^k)$  of  $(x^n)$  and a sequence  $(y_i^k)$  such that, for each  $k$ ,  $y_i^k \in X$  and  $u_i(y_i^k, x_{-i}^k) \geq \eta'$ .

Next, choose  $\varepsilon > 0$  satisfying  $0 < \varepsilon < \eta' - \gamma_i$ , and index  $k'$  so that for each  $k > k'$ , we have  $\rho(u_i(\cdot, x_{-i}^{k'}), u_i^{k'}(\cdot, x_{-i}^{k'})) < \varepsilon$ . Then  $k > k'$  implies that

$$\begin{aligned} u_i^{k'}(y_i^{k'}, x_{-i}^{k'}) &> u_i(y_i^{k'}, x_{-i}^{k'}) - \varepsilon \\ &\geq \eta' - \varepsilon \\ &> \gamma_i. \end{aligned}$$

Defining  $\eta = \eta' - \varepsilon$ , we conclude that there exist an  $i$ , an  $\eta > \gamma_i$ , a subsequence  $(x^k)$  of  $(x^n)$  and a net  $(y_i^k)$  such that, for each  $k$ ,  $y_i^k \in X_i$  and  $u_i^k(y_i^k, x_{-i}^k) \geq \eta$ . This shows that  $G$  satisfies sequential better-reply security with respect to  $(u^\nu)$ .  $\blacksquare$

**Remark 3.** From Proposition 1, we conclude that a metric game  $G = (X_i, u_i)_{i=1}^N$  satisfies sequential better-reply security if and only if  $\varepsilon^\nu \rightarrow 0$  implies that  $\text{Ls}(\mathcal{N}_X^{\varepsilon^\nu}(u)) \subseteq \mathcal{N}_X(u)$ . In particular,  $\mathcal{N}_X(u)$  is a closed set if  $G$  satisfies sequential better-reply security.

### 2.3 Perfect equilibria and stable sets

Given a compact, metric game  $G = (X_i, u_i)_{i=1}^N$ , the *mixed extension* of  $G$  is the game

$$\overline{G} = (\Delta(X_i), u_i)_{i=1}^N,$$

where each  $\Delta(X_i)$  represents the set of regular Borel probability measures on  $X_i$ , endowed with the weak\* topology, and, abusing notation, we let  $u_i : \Delta(X) \rightarrow \mathbb{R}$  be defined by

$$u_i(\mu) := \int_X u_i d\mu,$$

where (abusing notation)  $\Delta(X) := \times_{i=1}^N \Delta(X_i)$ .

As usual, a *mixed strategy Nash equilibrium* of  $G$  is a Nash equilibrium of  $\overline{G}$ . To distinguish equilibria in  $G$  from equilibria in  $\overline{G}$ , we sometimes refer to a Nash equilibrium of  $G$  as a *pure strategy Nash equilibrium* of  $G$ .

A probability measure  $\mu_i \in \Delta(X_i)$  is said to be *strictly positive* if  $\mu_i(O) > 0$  for every nonempty open set  $O$  in  $X_i$ .

For each  $i$ , let  $\widehat{\Delta}(X_i)$  denote the set of all strictly positive members of  $\Delta(X_i)$ . The set of regular Borel measures on  $X_i$  is denoted as  $M(X_i)$ , and the subset consisting of the members  $\eta_i$  of  $M(X_i)$  such that  $\eta_i(X_i) < 1$  and  $\eta_i(O) > 0$  for every nonempty open set  $O$  in  $X_i$  is represented as  $\widehat{M}(X_i)$ . Let

$$\widehat{\Delta}(X) := \times_{i=1}^N \widehat{\Delta}(X_i) \quad \text{and} \quad \widehat{M}(X) := \times_{i=1}^N \widehat{M}(X_i).$$

For  $(\eta_1, \dots, \eta_N) \in \widehat{M}(X)$ , let

$$\Delta(X_i, \eta_i) := \{\nu_i \in \Delta(X_i) : \nu_i \geq \eta_i\}$$

and define the **Selten perturbation** of  $G$  as

$$\overline{G}_\eta = (\Delta(X_i, \eta_i), u_i)_{i=1}^N.$$

For  $\mu = (\mu_1, \dots, \mu_N) \in \widehat{\Delta}(X)$  and  $\delta = (\delta_1, \dots, \delta_N) \in [0, 1]^N$ , define the special Selten perturbation  $\overline{G}_{\delta*\mu}$  as

$$\overline{G}_{\delta*\mu} = (\Delta(X_i, \delta_i \mu_i), u_i)_{i=1}^N.$$

We sometimes abuse notation and represent the Selten perturbation  $\overline{G}_{(\delta_1, \dots, \delta_N)*\mu}$  in which  $\delta_1 = \dots = \delta_N = \delta$  simply as  $\overline{G}_{\delta*\mu}$ .

**Definition 5.** A strategy profile  $\mu \in \Delta(X)$  is a **trembling-hand perfect profile** in  $G = (X_i, u_i)_{i=1}^N$  if there are sequences  $(\delta^n)$ ,  $(\nu^n)$ , and  $(\mu^n)$  such that  $\delta^n \in (0, 1)^N$  and  $\nu^n \in \widehat{\Delta}(X)$  for each  $n$ ,  $\delta^n \rightarrow 0$ ,  $\mu^n \rightarrow \mu$ , and each  $\mu^n$  is a Nash equilibrium of  $\overline{G}_{\delta^n*\nu^n}$ .

**Remark 4.** The following definition of trembling-hand perfection is equivalent to Definition 5 (see Carbonell-Nicolau and McLean [5]): A strategy profile  $\mu \in \Delta(X)$  is a *trembling-hand perfect profile* in  $G = (X_i, u_i)_{i=1}^N$  if there are sequences  $(\eta^n)$  and  $(\mu^n)$  such that  $\eta^n \in \widehat{M}(X)$  for each  $n$ ,  $\eta^n \rightarrow 0$ ,  $\mu^n \rightarrow \mu$ , and each  $\mu^n$  is a Nash equilibrium of  $\overline{G}_{\eta^n}$ .

The reader is referred to Carbonell-Nicolau [4] for additional, alternative formulations of trembling-hand perfection.

Given a compact, metric game  $G = (X_i, u_i)_{i=1}^N$ , we will endow  $\Delta(X)$  with the product topology induced by the Prokhorov metric on  $\Delta(X_i)$ .<sup>1</sup> If  $\varrho_i$  denotes the Prokhorov metric on  $\Delta(X_i)$ , then given  $\{\mu, \nu\} \subseteq \Delta(X_i)$ ,

$$\varrho_i(\mu, \nu) := \inf \{ \varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon, \text{ for all } B \},$$

where

$$B^\varepsilon := \{x \in X_i : d_i(x, y) < \varepsilon \text{ for some } y \in B\},$$

and  $d_i$  denotes the metric associated with  $X_i$ . The product metric induced by  $(\varrho_1, \dots, \varrho_N)$  on  $\Delta(X)$  is denoted by  $\varrho$ .

For  $\varepsilon > 0$  and  $\emptyset \neq E \subseteq \Delta(X)$ , a profile  $\mu \in \Delta(X)$  is said to be  $\varepsilon$ -close to  $E$  if

$$\varrho(\mu, E) := \inf \{ \varrho(\mu, \nu) : \nu \in E \} < \varepsilon.$$

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<sup>1</sup>For compact metric games, this product topology coincides with the product topology induced by the weak\* topology on  $\Delta(X_i)$ .



Let  $\mathcal{S}_G$  be the family of all nonempty closed sets  $E$  of Nash equilibria of  $\overline{G}$  satisfying the following: for each  $\varepsilon > 0$ , there exists  $\alpha \in (0, 1]$  such that for each  $\delta \in (0, \alpha)^N$  and every  $\nu \in \widehat{\Delta}(X)$  the perturbed game  $\overline{G}_{\delta*\nu}$  has a Nash equilibrium  $\varepsilon$ -close to  $E$ .

Given  $x_i \in X_i$ , let  $\theta_{x_i}$  represent the Dirac measure on  $X_i$  with support  $\{x_i\}$ . Similarly, for  $x \in X$ ,  $\theta_x$  denotes the Dirac measure on  $X$  with support  $\{x\}$ . The map  $x_i \mapsto \theta_{x_i}$  (resp.  $x \mapsto \theta_x$ ) is an embedding, so  $X_i$  (resp.  $X$ ) can be topologically identified with a subspace of  $\Delta(X_i)$  (resp.  $\Delta(X)$ ). We sometimes abuse notation and refer to  $\theta_{x_i} \in \Delta(X_i)$  (resp.  $\theta_x \in \Delta(X)$ ) simply as  $x_i$  (resp.  $x$ ).

**Definition 6.** A set of pure strategy profiles in  $X$  is a **stable set** of  $G$  if it is a minimal element of the set  $\mathcal{S}_G$  ordered by set inclusion.

Given  $(\delta, \mu) \in [0, 1]^N \times \widehat{\Delta}(X)$ , let  $G_{(\delta, \mu)}$  be a normal form game defined as

$$G_{(\delta, \mu)} := (X_i, u_i^{(\delta, \mu)})_{i=1}^N,$$

where  $u_i^{(\delta, \mu)} : X \rightarrow \mathbb{R}$  is given by

$$u_i^{(\delta, \mu)}(x) := u_i((1 - \delta_i)x_i + \delta_1\mu_1, \dots, (1 - \delta_N)x_N + \delta_N\mu_N).$$

Here,  $(1 - \delta_i)x_i + \delta_i\mu_i$  represents the measure  $\sigma_i$  in  $\Delta(X_i)$  such that

$$\sigma_i(B) = (1 - \delta_i)\theta_{x_i}(B) + \delta_i\mu_i(B).$$

By a slight abuse of notation, we sometimes represent the game  $G_{((\delta_1, \dots, \delta_N), \mu)}$  in which  $\delta_1 = \dots = \delta_N = \delta$  simply as  $G_{(\delta, \mu)}$ .

Finally, given  $\varepsilon > 0$  and a nonempty subset  $E$  of  $\Delta(X)$ , the  $\varepsilon$ -neighborhood of  $E$  is denoted by  $N_\varepsilon(E)$ :

$$N_\varepsilon(E) := \bigcup_{\mu \in E} \{\nu \in \Delta(X) : \varrho(\mu, \nu) < \varepsilon\} = \{\nu \in \Delta(X) : \varrho(\mu, E) < \varepsilon\}.$$

### 3 Equilibrium refinement in supermodular games

**Lemma 3.** *Suppose that  $G$  is a compact, metric Borel game satisfying sequential better-reply security. Then the set of pure strategy trembling-hand perfect equilibria of  $G$  is closed in the set of pure strategy Nash equilibria of  $G$ .*

*Proof.* Fix a compact, metric Borel game  $G = (X_i, u_i)_{i=1}^N$  satisfying sequential better-reply security. Let  $(x^n)$  be a sequence of pure strategy trembling-hand perfect equilibria of  $G$  such that  $x^n \rightarrow x$ . To see that  $x$  is a trembling-hand perfect profile, note that for each  $n$ , there exist  $\nu^n \in \widehat{\Delta}(X)$  and  $\delta^n \in (0, 1)^N$  such that  $\delta^n < \frac{1}{n}$  and the game  $\overline{G}_{\delta^n * \nu^n}$  has a Nash equilibrium  $\mu^n$  with  $\varrho(\mu^n, x^n) < \frac{1}{n}$ . Hence, since  $x^n \rightarrow x$  and for each  $n$ ,

$$\varrho(\mu^n, x) \leq \varrho(\mu^n, x^n) + \varrho(x^n, x),$$

$x$  is a trembling-hand perfect profile.

It remains to show that  $x$  is a Nash equilibrium of  $G$ . But this flows from the following observations. First, each  $x^n$  is a Nash equilibrium of  $G$ , and  $x^n \rightarrow x$ . Second, the set of Nash equilibria in  $G$  is closed as consequence of Remark 3.  $\blacksquare$

**Lemma 4.** *Suppose that  $G = (X_i, u_i)_{i=1}^N$  is a compact, metric, Borel game satisfying sequential better-reply security. Suppose that there are sequences  $(\delta^n)$  and  $(\mu^n)$  satisfying the following:  $\delta^n \rightarrow 0$ , and, for each  $n$ ,  $\delta^n \in (0, 1)^N$ ,  $\mu^n \in \widehat{\Delta}(X)$ , and  $G_{(\delta^n, \mu^n)}$  has a Nash equilibrium. Then  $G$  has a pure strategy trembling-hand perfect equilibrium, and the set of pure strategy trembling-hand perfect equilibria of  $G$  is a compact subset of the set of Nash equilibria of  $G$ .*

*Proof.* Suppose that there are sequences  $(\delta^n)$  and  $(\mu^n)$  satisfying the following:  $\delta^n \rightarrow 0$ , and, for each  $n$ ,  $\delta^n \in (0, 1)^N$ ,  $\mu^n \in \widehat{\Delta}(X)$ , and  $G_{(\delta^n, \mu^n)}$  has a Nash equilibrium  $x^n$ . Since  $x^n \in X$  for each  $n$ , and because  $X$  is sequentially compact, we may write (passing to a subsequence if necessary)  $x^n \rightarrow x$  for some  $x \in X$ . Because  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu^n)}$  for each  $n$ , for each  $n$  the strategy profile

$$(1 - \delta^n)x^n + \delta^n \mu^n := ((1 - \delta_1^n)x_1^n + \delta_1^n \mu_1^n, \dots, (1 - \delta_N^n)x_N^n + \delta_N^n \mu_N^n)$$

is a Nash equilibrium of  $\overline{G}_{\delta^n * \mu^n}$ . Hence, since  $x^n \rightarrow x$  and  $\delta^n \rightarrow 0$ , we have

$$(1 - \delta^n)x^n + \delta^n \mu^n \rightarrow x,$$

and it follows that  $x$  is a trembling-hand perfect profile. In addition, since  $u_i^{(\delta^n, \mu^n)} \rightarrow u_i$  for each  $i$ , and because  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu^n)}$  for each  $n$  and  $x^n \rightarrow x$ , it follows from Lemma 2 and Proposition 1 that  $x$  is a Nash equilibrium of  $G$ .

It remains to show that the set of pure strategy trembling-hand perfect equilibria of  $G$  is a compact subset of the set of Nash equilibria of  $G$ . Because  $G$  satisfies sequential better-reply security, the set of pure strategy Nash equilibria of  $G$  is closed (hence compact) in  $X$  as consequence of Remark 3. Furthermore, by Lemma 3, the set of pure strategy trembling-hand perfect equilibria of  $G$  is closed in the set of pure strategy Nash equilibria of  $G$ . Therefore, the set of pure strategy trembling-hand perfect equilibria of  $G$  is a compact subset of the set of Nash equilibria of  $G$ . ■

**Lemma 5.** *Suppose that  $G = (X_i, u_i)_{i=1}^N$  is a compact, metric, Borel game satisfying sequential better-reply security. Suppose that there exists  $\alpha \in (0, 1)$  such that for every  $(\delta, \mu) \in (0, \alpha]^N \times \hat{\Delta}(X)$ ,  $G_{(\delta, \mu)}$  has a Nash equilibrium. Then  $G$  has a stable set of pure strategy profiles. Furthermore, any stable set of  $G$  is a compact subset of the set of Nash equilibria of  $G$  containing only pure strategy trembling-hand perfect equilibria of  $G$ .*

*Proof.* We first show that the set of Nash equilibria in  $G$ ,  $E_G$ , belongs to  $\mathcal{S}_G$ . First, this set is nonempty (by Lemma 4) and closed (Remark 3). Since  $E_G$  is closed in the compact set  $X$ ,  $E_G$  is itself compact, and therefore  $E_G$  can be viewed as a closed subset of  $\Delta(X)$ . Next, we show that for each  $\varepsilon > 0$ , there exists  $\alpha \in (0, 1]$  such that for each  $\delta \in (0, \alpha)^N$  and every  $\mu \in \hat{\Delta}(X)$ ,  $\overline{G}_{\delta * \mu}$  has a Nash equilibrium  $\varepsilon$ -close to  $E_G$ . Suppose that this is not true, i.e., suppose that there exists  $\varepsilon^* > 0$  such that for every  $\alpha \in (0, 1]$ , there exist  $\delta \in (0, \alpha)^N$  and  $\mu \in \hat{\Delta}(X)$  such that no Nash equilibrium of  $\overline{G}_{\delta * \mu}$  is  $\varepsilon^*$ -close to  $E_G$ . Then there are sequences  $(\delta^n)$  and  $(\mu^n)$  satisfying the following: for each  $n$ ,  $\delta^n \in (0, \frac{1}{n})^N$ ,  $\mu^n \in \hat{\Delta}(X)$ , and

$$E_{\overline{G}_{\delta^n * \mu^n}} \cap N_{\varepsilon^*}(E_G) = \emptyset, \quad (1)$$

where  $E_{\overline{G}_{\delta^n * \mu^n}}$  represents the set of Nash equilibria in  $\overline{G}_{\delta^n * \mu^n}$ . For each  $n$ , let  $x^n$  be a Nash equilibrium of  $G_{(\delta^n, \mu^n)}$ . Because  $x^n \in X$  for each  $n$ , and since  $X$  is sequentially compact, we may write (passing to a subsequence if necessary)  $x^n \rightarrow x$  for some  $x \in X$ . Hence, since  $u_i^{(\delta^n, \mu^n)} \rightarrow u_i$  for each  $i$ , and because  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu^n)}$  for each  $n$ , it follows from Lemma 2 and Proposition 1 that  $x$  is a Nash equilibrium of  $G$ , i.e.,  $x \in E_G$ . In addition, because each  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu^n)}$ , for each  $n$  the strategy profile

$$(1 - \delta^n)x^n + \delta^n \mu^n := ((1 - \delta^n)x_1^n + \delta^n \mu_1^n, \dots, (1 - \delta^n)x_N^n + \delta^n \mu_N^n)$$

is a Nash equilibrium of  $\overline{G}_{\delta^n * \mu^n}$ . Consequently, because

$$(1 - \delta^{n_k})x^{n_k} + \delta^{n_k}\mu^{n_k} \rightarrow x \in E_G,$$

(1) cannot hold for every  $n$ . This gives the desired contradiction.

We conclude that  $E_G \subseteq \mathcal{S}_G$ . Next, consider the collection  $\mathcal{S}_G^*$  of  $E$  in  $\mathcal{S}_G$  such that  $E \subseteq E_G$ . The collection  $\mathcal{S}_G^*$  can be ordered by set inclusion. We show that every decreasing chain  $(E^\alpha)$  in  $\mathcal{S}_G^*$  (ordered by set inclusion) has a lower bound. Fix a decreasing chain  $(E^\alpha)$  in  $\mathcal{S}_G^*$ . Since the collection  $(E^\alpha)$  has the finite intersection property, and since  $\Delta(X)$  is compact, we have

$$E^* := \bigcap_{\alpha} E^\alpha \neq \emptyset.$$

In addition,  $E^*$  is a lower bound for  $(E^\alpha)$ . This follows from the fact that  $E^*$  is a member of  $(E^\alpha)$ . To see that  $E^*$  is a member of  $(E^\alpha)$ , note first that  $E^* \neq \emptyset$  and that  $E^*$  is closed, since each  $E^\alpha$  is closed. Moreover, for each  $\varepsilon > 0$ , there exists  $a \in (0, 1]$  such that for each  $\delta \in (0, a)^N$  and every  $\nu \in \widehat{\Delta}(X)$ ,  $\overline{G}_{\delta * \nu}$  has a Nash equilibrium  $\varepsilon$ -close to  $E^*$ . In fact, given  $\varepsilon > 0$ , there exists an element  $E^\beta$  of  $(E^\alpha)$  such that

$$E^\beta \subseteq N_{\frac{\varepsilon}{2}}(E^*). \quad (2)$$

If that were not the case,  $(E^\alpha \setminus N_{\frac{\varepsilon}{2}}(E^*))$  would be a collection of closed subsets of the compact set  $\Delta(X)$  with the finite intersection property, and one would have

$$E^* \setminus N_{\frac{\varepsilon}{2}}(E^*) \supseteq \bigcap_{\alpha} (E^\alpha \setminus N_{\frac{\varepsilon}{2}}(E^*)) \neq \emptyset,$$

a contradiction. Because  $E^\beta$  is a member of  $\mathcal{S}_G$ , there exists  $b \in (0, 1]$  such that for each  $\delta \in (0, b)^N$  and every  $\nu \in \widehat{\Delta}(X)$ ,  $\overline{G}_{\delta * \nu}$  has a Nash equilibrium  $\frac{\varepsilon}{2}$ -close to  $E^\beta$ . In light of (2), therefore, for each  $\delta \in (0, b)^N$  and every  $\nu \in \widehat{\Delta}(X)$ ,  $\overline{G}_{\delta * \nu}$  has a Nash equilibrium  $\varepsilon$ -close to  $E^*$ .

We conclude that  $E^*$  is a member of  $(E^\alpha)$ , and therefore  $E^*$  is a lower bound for  $(E^\alpha)$ . We have shown that every decreasing chain  $(E^\alpha)$  in  $\mathcal{S}_G^*$  has a lower bound. Consequently, since  $E_G \in \mathcal{S}_G$ , Zorn's lemma gives a minimal element of  $\mathcal{S}_G^*$ , i.e., a stable set of  $G$ .

Next, we show that each element of a stable set is a trembling-hand perfect equilibrium. Let  $y \in S$ , where  $S$  is a stable set of  $G$ . Since  $S \in \mathcal{S}_G$ ,

$S$  is by definition a set of Nash equilibria of  $G$ , so it suffices to show that  $y$  is a trembling-hand perfect profile. Pick any  $\epsilon > 0$  such that  $S \setminus N_\epsilon(y) \neq \emptyset$ . Since  $S$  is stable and  $S \setminus N_\epsilon(y)$  is nonempty and closed,  $S \setminus N_\epsilon(y)$  is not stable. Consequently, there exists  $\epsilon^* > 0$  such that for each  $n$ , there exist  $\delta^n \in (0, \frac{1}{n})^N$  and  $\mu^n \in \widehat{\Delta}(X)$  such that the set of Nash equilibria of  $\overline{G}_{\delta^n * \mu^n}$ ,  $E_{\overline{G}_{\delta^n * \mu^n}}$ , satisfies

$$E_{\overline{G}_{\delta^n * \mu^n}} \cap N_{\epsilon^*}(S \setminus N_\epsilon(y)) = \emptyset. \quad (3)$$

Since  $S \subseteq N_{\epsilon^*}(S \setminus N_\epsilon(y)) \cup N_\epsilon(y)$ , there exists  $\gamma > 0$  such that

$$S \subseteq N_\gamma(S) \subseteq N_{\epsilon^*}(S \setminus N_\epsilon(y)) \cup N_\epsilon(y). \quad (4)$$

Therefore, since  $S$  is stable, there exists  $\alpha^* \in (0, 1]$  such that for each  $\delta \in (0, \alpha^*)^N$  and every  $\mu \in \widehat{\Delta}(X)$ , the set of Nash equilibria of  $\overline{G}_{\delta * \mu}$ ,  $E_{\overline{G}_{\delta * \mu}}$ , satisfies

$$E_{\overline{G}_{\delta * \mu}} \cap N_\gamma(S) \neq \emptyset.$$

In particular, for every large enough  $n$ ,

$$E_{\overline{G}_{\delta^n * \mu^n}} \cap N_\gamma(S) \neq \emptyset.$$

This, combined with (4), gives

$$E_{\overline{G}_{\delta^n * \mu^n}} \cap (N_{\epsilon^*}(S \setminus N_\epsilon(y)) \cup N_\epsilon(y)) \neq \emptyset$$

for any sufficiently large  $n$ . In light of (3), therefore,  $E_{\overline{G}_{\delta^n * \mu^n}} \cap N_\epsilon(y) \neq \emptyset$  for any sufficiently large  $n$ .

We conclude that for any  $\epsilon > 0$  such that  $S \setminus N_\epsilon(y) \neq \emptyset$ , there exist  $\delta \in (0, \epsilon)^N$  and  $\mu \in \widehat{\Delta}(X)$  such that  $\overline{G}_{\delta * \mu}$  has a Nash equilibrium in  $N_\epsilon(y)$ . This implies that  $y$  is a trembling-hand perfect profile.

Finally, stable sets are by definition closed in the set of Nash equilibria of  $G$ , and the set of Nash equilibria of  $G$  is closed (hence compact) in  $X$  (Remark 3). ■

**Lemma 6.** *Let  $f : X \rightarrow \mathbb{R}$  be a bounded Borel measurable function on  $X = \times_{i=1}^N X_i$ , where each  $X_i$  is a compact metric space. Given  $i$ , if  $f(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$  for every  $x_{-i} \in X_{-i}$ , then for each  $\mu_{-i} \in \Delta(X_{-i})$ , the map*

$$\mu_i \mapsto \int_{X_{-i}} \int_{X_i} f(x_i, x_{-i}) \mu_i(dx_i) \mu_{-i}(dx_{-i})$$

*defined on  $\Delta(X_i)$  is upper semicontinuous.*

*Proof.* Fix  $i$ . Because  $f(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$  for every  $x_{-i} \in X_{-i}$ ,  $f(\cdot, x_{-i})$  is upper semicontinuous on  $\Delta(X_i)$  for every  $x_{-i} \in X_{-i}$  (e.g., Aliprantis and Border [1], Theorem 15.5). Therefore, given  $x_{-i} \in X_{-i}$ , if  $(\nu_i^n)$  is a sequence in  $\Delta(X_i)$  with  $\nu_i^n \rightarrow \nu_i$ , then

$$\limsup_{n \rightarrow \infty} \int_{X_i} f(x_i, x_{-i}) \nu_i^n(dx_i) \leq \int_{X_i} f(x_i, x_{-i}) \nu_i(dx_i).$$

Consequently, for every  $\nu_{-i} \in \Delta(X_{-i})$ ,

$$\begin{aligned} \int_{X_{-i}} \left[ \limsup_{n \rightarrow \infty} \int_{X_i} f(x_i, x_{-i}) \nu_i^n(dx_i) \right] \nu_{-i}(dx_{-i}) \\ \leq \int_{X_{-i}} \int_{X_i} f(x_i, x_{-i}) \nu_i(dx_i) \nu_{-i}(dx_{-i}). \end{aligned} \quad (5)$$

Fix a sequence  $(\nu_i^n)$  in  $\Delta(X_i)$  with  $\nu_i^n \rightarrow \nu_i$ . For each  $n$ , define  $\phi_i^n : X_{-i} \rightarrow \mathbb{R}$  by

$$\phi_i^n(x_{-i}) := \int_{X_i} f(x_i, x_{-i}) \nu_i^n(dx_i).$$

Given  $\nu_{-i} \in \Delta(X_{-i})$ , we have, by Fatou's lemma,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{X_i} \int_{X_{-i}} f(x_i, x_{-i}) \nu_{-i}(dx_{-i}) \nu_i^n(dx_i) \\ = \limsup_{n \rightarrow \infty} \int_{X_{-i}} \phi_i^n(x_{-i}) \nu_{-i}(dx_{-i}) \\ \leq \int_{X_{-i}} \left[ \limsup_{n \rightarrow \infty} \phi_i^n(x_{-i}) \right] \nu_{-i}(dx_{-i}). \end{aligned}$$

This, combined with (5), gives

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{X_i} \int_{X_{-i}} f(x_i, x_{-i}) \nu_{-i}(dx_{-i}) \nu_i^n(dx_i) \\ \leq \int_{X_{-i}} \int_{X_i} f(x_i, x_{-i}) \nu_i(dx_i) \nu_{-i}(dx_{-i}). \end{aligned}$$

This establishes the result. ■

**Lemma 7.** *If  $G$  is supermodular, then  $G_{(\delta, \mu)}$  is supermodular for every  $(\delta, \mu)$ .*

*Proof.* Suppose that  $G = (X_i, u_i)_{i=1}^N$  is supermodular. Fix

$$((\delta_1, \dots, \delta_N), \mu) \in [0, 1]^N \times \widehat{\Delta}(X).$$

Since  $G$  is a compact lattice game, it is clear that  $G_{(\delta, \mu)}$  is a compact lattice game. Because  $u_i(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$  for each  $i$  and every  $x_{-i} \in X_{-i}$ , Lemma 6 implies that  $u_i(\cdot, \nu_{-i})$  is upper semicontinuous on  $\Delta(X_i)$  for each  $i$  and every  $\nu_{-i} \in \Delta(X_{-i})$ . Consequently, for each  $i$  and every  $x_{-i} \in X_{-i}$ ,  $u_i^{(\delta, \mu)}(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$ .

Next, we show that  $u_i^{(\delta, \mu)}(\cdot, x_{-i})$  is supermodular for each  $i$  and every  $x_{-i} \in X_{-i}$ . Fix  $i$ , and  $\{x_i, y_i\} \subseteq X_i$ . Define

$$\nu_{-i} := \bigotimes_{j \neq i} [(1 - \delta_j)x_j + \delta_j \mu_j].$$

Since

$$u_i(\sup\{x_i, y_i\}, z_{-i}) + u_i(\inf\{x_i, y_i\}, z_{-i}) \geq u_i(x_i, z_{-i}) + u_i(y_i, z_{-i})$$

for each  $z_{-i} \in X_{-i}$ , it follows that

$$u_i(\sup\{x_i, y_i\}, \nu_{-i}) + u_i(\inf\{x_i, y_i\}, \nu_{-i}) \geq u_i(x_i, \nu_{-i}) + u_i(y_i, \nu_{-i}).$$

Therefore,

$$\begin{aligned} & u_i^{(\delta, \mu)}(\sup\{x_i, y_i\}, x_{-i}) + u_i^{(\delta, \mu)}(\inf\{x_i, y_i\}, x_{-i}) \\ &= (1 - \delta_i) [u_i(\sup\{x_i, y_i\}, \nu_{-i}) + u_i(\inf\{x_i, y_i\}, \nu_{-i})] \\ & \quad + \delta_i [u_i(\mu_i, \nu_{-i}) + u_i(\mu_i, \nu_{-i})] \\ &\geq (1 - \delta_i) [u_i(x_i, \nu_{-i}) + u_i(y_i, \nu_{-i})] + \delta_i [u_i(\mu_i, \nu_{-i}) + u_i(\mu_i, \nu_{-i})] \\ &= u_i^{(\delta, \mu)}(x_i, x_{-i}) + u_i^{(\delta, \mu)}(y_i, x_{-i}). \end{aligned}$$

Finally, we show that  $u_i^{(\delta, \mu)}$  exhibits increasing differences in  $X_i$  and  $X_{-i}$  for each  $i$ . Fix  $i$  and  $\{y_{-i}, x_{-i}\} \subseteq X_{-i}$  with  $y_{-i} \geq_{-i} x_{-i}$ . Choose  $\{x_i, y_i\} \subseteq X_i$  with  $y_i \geq_i x_i$ . We must show that

$$u_i^{(\delta, \mu)}(y_i, y_{-i}) - u_i^{(\delta, \mu)}(y_i, x_{-i}) \geq u_i^{(\delta, \mu)}(x_i, y_{-i}) - u_i^{(\delta, \mu)}(x_i, x_{-i}).$$

Given  $I \subseteq \{1, \dots, N\}$  and  $i$ , the action profile

$$(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$$

in  $X_{-i}$  such that

$$a_j = \begin{cases} x_j & \text{if } j \in I \setminus \{i\}, \\ z_j & \text{if } j \in \{1, \dots, N\} \setminus (I \cup \{i\}), \end{cases}$$

is denoted by  $(x_{I \setminus i}, z_{-I \cup i})$ . Similarly, The action profile

$$(\nu_1, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_N)$$

in  $\Delta(X_{-i})$  such that

$$\nu_j = \begin{cases} x_j & \text{if } j \in I \setminus \{i\}, \\ \mu_j & \text{if } j \in \{1, \dots, N\} \setminus (I \cup \{i\}), \end{cases}$$

is denoted by  $(x_{I \setminus i}, \mu_{-I \cup i})$ .

For each  $I \subseteq \{1, \dots, N\}$  and every  $z_{-I} \in X_{-I} := \prod_{j \in \{1, \dots, N\} \setminus I} X_j$ , we have

$$\begin{aligned} u_i(y_i, (y_{I \setminus i}, z_{-I \cup i})) - u_i(y_i, (x_{I \setminus i}, z_{-I \cup i})) \\ \geq u_i(x_i, (y_{I \setminus i}, z_{-I \cup i})) - u_i(x_i, (x_{I \setminus i}, z_{-I \cup i})). \end{aligned}$$

Therefore, for each  $I \subseteq \{1, \dots, N\}$  and every  $\rho_{-I} \in \Delta(X_{-I}) := \prod_{j \in \{1, \dots, N\} \setminus I} \Delta(X_j)$ ,

$$\begin{aligned} u_i(y_i, (y_{I \setminus i}, \rho_{-I \cup i})) - u_i(y_i, (x_{I \setminus i}, \rho_{-I \cup i})) \\ \geq u_i(x_i, (y_{I \setminus i}, \rho_{-I \cup i})) - u_i(x_i, (x_{I \setminus i}, \rho_{-I \cup i})). \end{aligned}$$

This implies that

$$\begin{aligned} & \left[ u_i^{(\delta, \mu)}(y_i, y_{-i}) - u_i^{(\delta, \mu)}(y_i, x_{-i}) \right] - \left[ u_i^{(\delta, \mu)}(x_i, y_{-i}) - u_i^{(\delta, \mu)}(x_i, x_{-i}) \right] \\ &= \sum_{\substack{\emptyset \neq I \subseteq \overline{N} \\ : \overline{N} \setminus I \neq \{i\}}} \left[ \prod_{j \in I} (1 - \delta_j) \prod_{j \in \overline{N} \setminus I} \delta_j \right] \left[ u_i(y_i, (y_{I \setminus i}, \mu_{-I \cup i})) - u_i(y_i, (x_{I \setminus i}, \mu_{-I \cup i})) \right] \\ & \quad - \left[ \sum_{\substack{\emptyset \neq I \subseteq \overline{N} \\ : \overline{N} \setminus I \neq \{i\}}} \left[ \prod_{j \in I} (1 - \delta_j) \prod_{j \in \overline{N} \setminus I} \delta_j \right] \left[ u_i(x_i, (y_{I \setminus i}, \mu_{-I \cup i})) - u_i(x_i, (x_{I \setminus i}, \mu_{-I \cup i})) \right] \right] \\ & \geq 0, \end{aligned}$$

where  $\overline{N} := \{1, \dots, N\}$ .

We conclude that  $G_{(\delta, \mu)}$  is supermodular. ■



**Lemma 8.** *If  $G$  is supermodular, then  $G_{(\delta, \mu)}$  has a Nash equilibrium for every  $(\delta, \mu)$ .*

*Proof.* If  $G = (X_i, u_i)_{i=1}^N$  is supermodular,  $G_{(\delta, \mu)}$  is supermodular for every  $(\delta, \mu)$  (Lemma 7). Moreover, since  $X_i$  is a compact metric lattice for each  $i$ ,  $X_i$  is lattice complete for each  $i$  (Lemma 1). Consequently, for each  $i$ , the topology induced by the metric on  $X_i$  is finer than the interval topology (e.g., Lawson [10], Proposition 4). It follows from Theorem 4.2 in Vives [17] that  $G_{(\delta, \mu)}$  has a pure strategy Nash equilibrium. ■

**Theorem 1.** *Any supermodular game  $G$  satisfying sequential better-reply security has a pure strategy trembling-hand perfect equilibrium, and the set of pure strategy trembling-hand perfect equilibria of  $G$  is a compact subset of the set of Nash equilibria of  $G$ .*

*Proof.* Fix a supermodular game  $G = (X_i, u_i)_{i=1}^N$  satisfying sequential better-reply security. Take sequences  $(\delta^n)$  and  $(\mu^n)$  such that for each  $n$ ,

$$\delta^n \in (0, 1)^N \quad \text{and} \quad \mu^n \in \widehat{\Delta}(X),$$

and  $\delta^n \rightarrow 0$ . Each  $G_{(\delta^n, \mu^n)}$  has a Nash equilibrium (Lemma 8). Apply Lemma 4. ■

**Theorem 2.** *Any supermodular game  $G$  satisfying sequential better-reply security has a stable set. Furthermore, any stable set of  $G$  is a compact subset of the set of Nash equilibria of  $G$  containing only pure strategy trembling-hand perfect equilibria of  $G$ .*

*Proof.* Fix a supermodular game  $G = (X_i, u_i)_{i=1}^N$  satisfying sequential better-reply security. Take any  $\delta \in (0, 1)^N$  and any  $\mu^n \in \widehat{\Delta}(X)$ . Then  $G_{(\delta, \mu^n)}$  has a Nash equilibrium (Lemma 8). Apply Lemma 5. ■

**Definition 7.** A strategy  $x_i \in X_i$  is **weakly dominated for  $i$**  if there exists a strategy  $\mu_i \in \Delta(X_i)$  such that  $u_i(\mu_i, x_{-i}) \geq u_i(x_i, x_{-i})$  for all  $x_{-i} \in X_{-i}$ , with strict inequality for some  $x_{-i}$ .

**Definition 8.** A strategy profile  $\mu \in \Delta(X)$  is **admissible** if  $\mu_i(D_i) = 0$  for all  $i$ , where  $D_i$  denotes the set of strategies weakly dominated for  $i$ .

**Definition 9.** A strategy profile  $\mu \in \Delta(X)$  is **limit admissible** if  $\mu_i(O_i) = 0$  for all  $i$ , where  $O_i$  denotes the interior of the set of strategies weakly dominated for  $i$ .

Example 2.1 in Simon and Stinchcombe [14] demonstrates there are continuous games whose unique Nash and trembling-hand perfect equilibrium is not admissible. While supermodular games must have at least one admissible Nash equilibrium (Kultti and Salonen [9], Theorem 1 and Proposition 1), the following example illustrates that continuous supermodular games may have trembling-hand perfect equilibria that are not admissible.

**Example 1.** Let  $(\alpha^n)$  be a sequence in  $(0, \frac{1}{2})$  with  $\alpha^n \nearrow \frac{1}{2}$ . Let  $(\gamma^n)$  be a sequence in  $(\frac{1}{4}, 1)$ , such that  $\gamma^n \searrow \frac{1}{4}$  and  $1 - 2\gamma^{2(n+1)} > 0$  for each  $n$ .

Consider the game

$$G := (X_i, u_i)_{i=1}^2 = ((\cup_n \alpha^n) \cup \{\frac{1}{2}, 1\}, \cup_n \gamma^n \cup \{\frac{1}{8}, \frac{1}{4}\}, u_1, u_2),$$

where  $u_2$  is identically zero and

$$u_1(x_1, x_2) := \begin{cases} -1 + \frac{x_2 - \gamma^{2n}}{(1 - 2\gamma^{2n})n} & \text{if } x_1 = \alpha^n, n = 1, \dots, \\ -1 & \text{if } x_1 = \frac{1}{2}, \\ -1 & \text{if } x_1 = 1 \text{ and } x_2 \in (\frac{1}{4}, 1], \\ -4x_2 & \text{if } x_1 = 1 \text{ and } x_2 \in [0, \frac{1}{4}]. \end{cases}$$

It is easy to see that  $G$  is continuous (i.e., each  $u_i$  is continuous). To see that  $G$  is supermodular, note that since  $G$  is continuous the first bullet point in Definition 1 is satisfied. Let  $\leq_1$  be the usual order on  $\mathbb{R}$ , and define  $\leq_2$  as follows:

$$a \leq_2 b \Leftrightarrow a \geq b.$$

If, for each  $i$ , player  $i$ 's action space is endowed with the order  $\leq_i$ , the resulting lattice game  $G$  clearly satisfies the second bullet point in Definition 1. In addition, it is routine to verify that  $u_1$  has increasing differences in  $X_1$  and  $X_2$  with respect to  $\leq_1$  and  $\leq_2$  ( $u_2$  clearly has increasing differences in  $X_2$  and  $X_1$  with respect to  $\leq_1$  and  $\leq_2$ ).

The strategy profile  $(\frac{1}{2}, \frac{1}{4})$  is a Nash equilibrium of  $G$ . This follows from the fact that  $u_2$  is identically zero and

$$u_1\left(\frac{1}{2}, \frac{1}{4}\right) = -1 = u_1\left(1, \frac{1}{4}\right) \geq u_1\left(\alpha^n, \frac{1}{4}\right), \quad \text{for all } n.$$

On the other hand,

$$u_1\left(\frac{1}{2}, x_2\right) \leq u_1\left(1, x_2\right), \quad \text{for all } x_2,$$

with strict inequality if  $x_2 = \frac{1}{8}$ , so the action  $\frac{1}{2}$  is weakly dominated for player 1.

We next show that  $(\frac{1}{2}, \frac{1}{4})$  is trembling-hand perfect. Given  $l \in \mathbb{N}$ , choose  $\beta_l > 1$  such that

$$\left(1 - \frac{1}{\beta_l}\right) A_l > \frac{1}{\beta_l} 2B, \quad (6)$$

where

$$A_l := \frac{\gamma^{2l+1} - \gamma^{2l+2}}{(1 - 2\gamma^{2(l+1)})(l+1)}$$

and  $|u_1(x_1, x_2)| \leq B$  for all  $(x_1, x_2) \in X_1 \times X_2$ . Take a sequence of trembles  $(\mu_2^l)$  in  $\widehat{\Delta}(X_2)$  satisfying the following for each  $l$ :

$$\begin{aligned} \mu_2^l(\{\tfrac{1}{4}\}) + \mu_2^l(\{\tfrac{1}{8}\}) + \sum_{k \neq 2l+1} \mu_2^l(\{\gamma^k\}) &= \frac{1}{\beta_l}, \\ \mu_2^l(\{\gamma^{2l+1}\}) &= 1 - \frac{1}{\beta_l}. \end{aligned}$$

For each  $l$ , we have

$$\begin{aligned} u_1(\alpha^{l+1}, \mu_2^l) - u_1(1, \mu_2^l) &= \mu_2^l(\{\tfrac{1}{8}\}) \left( -\tfrac{1}{2} + \frac{\frac{1}{8} - \gamma^{2(l+1)}}{(1 - 2\gamma^{2(l+1)})(l+1)} \right) \\ &\quad + \sum_{x_2 \in \{\frac{1}{4}\} \cup \bigcup_{k \neq 2l+1} \{\gamma^k\}} \mu_2^l(\{x_2\}) \left( \frac{x_2 - \gamma^{2(l+1)}}{(1 - 2\gamma^{2(l+1)})(l+1)} \right) \\ &\quad + \left(1 - \frac{1}{\beta_l}\right) \left( \frac{\gamma^{2l+1} - \gamma^{2(l+1)}}{(1 - 2\gamma^{2(l+1)})(l+1)} \right) \\ &\geq \left(1 - \frac{1}{\beta_l}\right) A_l - \frac{1}{\beta_l} 2B \\ &> 0, \end{aligned}$$

where the last inequality follows from (6). Therefore, given a sequence  $(\delta^l)$  in  $(0, 1)$  such that  $\delta^l \rightarrow 0$  and given  $\mu_1 \in \widehat{\Delta}(X_1)$ , we have, for each  $l$ ,

$$\begin{aligned} u_1((1 - \delta^l)\alpha^{l+1} + \delta^l\mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l\mu_2^l) \\ - u_1((1 - \delta^l)1 + \delta^l\mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l\mu_2^l) \\ = (1 - \delta^l) (u_1(\alpha^{l+1}, \mu_2^l) - u_1(1, \mu_2^l)) \\ > 0. \end{aligned}$$

Hence, if for each  $l$  and every  $n \in \{1, \dots, l\}$  we have

$$\begin{aligned} & u_1((1 - \delta^l)\alpha^{l+1} + \delta^l\mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l\mu_2^l) \\ & \quad - u_1((1 - \delta^l)\alpha^n + \delta^l\mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l\mu_2^l) \\ & \quad = (1 - \delta^l) (u_1(\alpha^{l+1}, \mu_2^l) - u_1(\alpha^n, \mu_2^l)) \\ & \quad > 0, \end{aligned} \tag{7}$$

then, for each  $l$ ,

$$\arg \max_{x_1} u_1^{(\delta^l, (\mu_1, \mu_2^l))}(x_1, \gamma^{2(l+1)}) \in \{\alpha^{l+1}, \alpha^{l+2}, \dots\} \cup \{\tfrac{1}{2}\}.$$

This means that, given  $l$ , if player 2 plays  $\gamma^{2(l+1)}$  in the game  $G_{(\delta^l, (\mu_1, \mu_2^l))}$ , player 1 best responds by choosing a member of the set

$$\{\alpha^{l+1}, \alpha^{l+2}, \dots\} \cup \{\tfrac{1}{2}\}.$$

Consequently, since  $\gamma^{2(l+1)}$  is always optimal for player 2 in  $G_{(\delta^l, (\mu_1, \mu_2^l))}$  (recall that  $u_2$  is identically zero), it follows that  $G_{(\delta^l, (\mu_1, \mu_2^l))}$  has an equilibrium  $(x_1^l, \gamma^{2(l+1)})$  for some

$$x_1^l \in \{\alpha^{l+1}, \alpha^{l+2}, \dots\} \cup \{\tfrac{1}{2}\}.$$

But then the sequence

$$((1 - \delta^l)x_1^l + \delta^l\mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l\mu_2^l)$$

converges to  $(\frac{1}{2}, \frac{1}{4})$ , and for each  $l$ ,

$$((1 - \delta^l)x_1^l + \delta^l\mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l\mu_2^l)$$

is a Nash equilibrium of  $\overline{G}_{\delta^l * (\mu_1, \mu_2^l)}$ , so  $(\frac{1}{2}, \frac{1}{4})$  is trembling-hand perfect.

We conclude that if (7) holds for each  $l$  and every  $n \in \{1, \dots, l\}$ , then  $(\frac{1}{2}, \frac{1}{4})$  is a trembling-hand perfect equilibrium of  $G$ . Hence, it suffices to show that  $u_1(\alpha^{l+1}, \mu_2^l) - u_1(\alpha^n, \mu_2^l) > 0$  for each  $l$  and every  $n \in \{1, \dots, l\}$ .

Fix  $l$  and  $n \in \{1, \dots, l\}$ . Then

$$\begin{aligned}
& u_1(\alpha^{l+1}, \mu_2^l) - u_1(\alpha^n, \mu_2^l) \\
&= \sum_{\substack{x_2 \in \{\frac{1}{4}, \frac{1}{8}\} \\ \cup \bigcup_{k \neq 2l+1} \{\gamma^k\}}} \mu_2^l(\{x_2\}) \left( \frac{x_2 - \gamma^{2(l+1)}}{(1-2\gamma^{2(l+1)})(l+1)} - \frac{x_2 - \gamma^{2n}}{(1-2\gamma^{2n})n} \right) \\
&\quad + \left(1 - \frac{1}{\beta_l}\right) \left( \frac{\gamma^{2l+1} - \gamma^{2(l+1)}}{(1-2\gamma^{2(l+1)})(l+1)} - \frac{\gamma^{2l+1} - \gamma^{2n}}{(1-2\gamma^{2n})n} \right) \\
&\geq \left(1 - \frac{1}{\beta_l}\right) A_l - \frac{1}{\beta_l} 2B \\
&> 0,
\end{aligned}$$

where the last inequality uses (6).

While continuous supermodular games may have trembling-hand perfect equilibria that fail admissibility, continuity ensures that perfect equilibria are always limit admissible. In fact, as shown in Carbonell-Nicolau [4], the following property suffices for a compact, metric normal form game  $G = (X_i, u_i)_{i=1}^N$  to have only limit admissible trembling-hand perfect equilibria: for each  $i$ , if  $x_i \in X_i$  is weakly dominated in  $G$  for player  $i$ , then for some  $\mu_i \in \Delta(X_i)$  that weakly dominates  $x_i$ , there exists  $y_{-i} \in X_{-i}$  with  $u_i(\mu_i, z_{-i}) > u_i(x_i, z_{-i})$  for all  $z_{-i}$  in some neighborhood  $V_{y_{-i}}$  of  $y_{-i}$ . If this condition is not fulfilled, trembling-hand perfect equilibria in supermodular games may fail limit admissibility. This is illustrated in the following example, which presents a supermodular, sequentially better-reply secure game with a trembling-hand perfect equilibrium that is not limit admissible.

**Example 2.** Consider the two-player game  $G := (X_1, X_2, u_1, u_2)$ , where  $X_1 = X_2 = [0, 1]$ ,

$$u_1(x_1, x_2) := \begin{cases} -1 & \text{if } x_1 \in [0, \frac{1}{2}) \text{ and } x_2 = 0, \\ x_2 & \text{elsewhere,} \end{cases}$$

and

$$u_2(x_1, x_2) := \begin{cases} 1 & \text{if } x_2 = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Let  $\leq_2$  be the usual order on  $\mathbb{R}$ , and define  $\leq_1$  as follows:

$$a \leq_1 b \Leftrightarrow a \geq b.$$

It is routine to verify that the game  $G$  endowed with  $\leq_1$  and  $\leq_2$  is supermodular.

We next show that  $G$  satisfies sequential better-reply security. Suppose that  $(x^n, u(x^n)) \in X \times \mathbb{R}^2$  is a convergent sequence with limit  $(x, \gamma) \in X \times \mathbb{R}^2$ . Suppose that  $x$  is not a Nash equilibrium of  $G$ . We must show that there exist an  $i$ , an  $\eta > \gamma_i$ , a subsequence  $(x^k)$  of  $(x^n)$ , and a sequence  $(y_i^k)$  such that for each  $k$ ,  $y_i^k \in X_i$  and  $u_i(y_i^k, x_{-i}^k) \geq \eta$ . This is clearly satisfied if  $x$  is a point of continuity of  $u$ . Suppose that  $x$  is a point of discontinuity of  $u$ . Then  $x$  must be a member of the set

$$Z := \{(z_1, z_2) \in X : z_2 = 0 \text{ and } z_1 \in [0, \tfrac{1}{2})\} \cup \{(z_1, z_2) \in X : z_2 = 1\}.$$

The set of pure strategy Nash equilibria of  $G$  in  $Z$  is  $\{(z_1, z_2) \in X : z_2 = 1\}$ . Consequently, since  $x$  is not a Nash equilibrium of  $G$ , and because  $x \in Z$ , we must have  $x_2 = 0$ . Since  $x_2 = 0$ , and because  $(x^n, u(x^n)) \rightarrow (x, \gamma)$ , the definition of  $u_2$  entails  $\gamma_2 = 0$ . But then, for each  $n$  we have

$$u_2(x_1^n, 1) = 1 > 0 = \gamma_2.$$

Next, observe that the strategy profile  $(0, 1)$  is not limit admissible. In fact, any action in  $[0, \frac{1}{2})$  is weakly dominated by any action in  $[\frac{1}{2}, 1]$  for player 1. In addition, the point  $(0, 1)$  is a trembling-hand perfect equilibrium. To see this, take  $\mu_2 \in \hat{\Delta}(X_2)$  with  $\mu_2(\{0\}) = 0$ . From the definition of  $u_2$  it is clear that for any  $(\delta, \mu_1) \in [0, 1) \times \hat{\Delta}(X_1)$ , 1 is a best response for player 2 to any  $x_1 \in X_1$  in the game  $G_{(\delta, (\mu_1, \mu_2))}$ . Moreover, since  $u_1(0, x_2) = u_1(x_1, x_2)$  for all  $(x_1, x_2) \in [0, 1] \times (0, 1]$  and  $\mu_2(\{0\}) = 0$ , we have

$$u_1((1 - \delta)0 + \delta\mu_1, (1 - \delta)1 + \delta\mu_2) \geq u_1((1 - \delta)x_1 + \delta\mu_1, (1 - \delta)1 + \delta\mu_2)$$

for all  $x_1 \in [0, 1]$ . Hence  $(0, 1)$  is a Nash equilibrium of  $G_{(\delta, \mu)}$ , and so  $(0, 1)$  is a trembling-hand perfect equilibrium of  $G$ .

Next, we show that for supermodular games satisfying sequential better-reply security, there are limit admissible, trembling-hand perfect equilibria. We first prove an intermediate result for a larger class of games.

**Lemma 9.** *Suppose that  $G = (X_i, u_i)_{i=1}^N$  is a compact, metric, Borel game satisfying sequential better-reply security. Suppose that for each  $\mu \in \hat{\Delta}(X)$ , there is a sequence  $(\delta^n)$  in  $(0, 1)^N$  with  $\delta^n \rightarrow 0$  such that the game*

$$\hat{G}_{(\delta^n, \mu)} := (X_i \setminus O_i, u_i^{(\delta^n, \mu)})_{i=1}^N$$

has a Nash equilibrium for each  $n$ . Then  $G$  has a limit admissible, pure strategy trembling-hand perfect equilibrium. Moreover, the set of limit admissible, pure strategy trembling-hand perfect equilibria of  $G$  is a compact subset of the set pure strategy trembling-hand perfect equilibria of  $G$ .

*Proof.* Since each  $X_i$  is separable, there exists, for each  $i$ , a countable dense subset  $Q_i$  of  $X_i$ . For each  $i$ , define  $\phi(i) : Q_i \cap O_i \rightarrow X_{-i}$  as follows:

$$\phi(i)(x_i) = (\phi_1(i)(x_i), \dots, \phi_{i-1}(i)(x_i), \phi_{i+1}(i)(x_i), \dots, \phi_N(i)(x_i)) = y_{-i},$$

where  $y_{-i}$  is such that, for some  $p_i \in \Delta(X_i)$ ,

$$u_i(p_i, x_{-i}) \geq u_i(x_i, x_{-i}), \quad \text{for all } x_{-i} \in X_{-i},$$

and  $u_i(p_i, y_{-i}) > u_i(x_i, y_{-i})$ . For each  $i$  and every  $j$ , let  $\rho_j(i)$  be a measure in  $\Delta(X_j)$  such that if  $j \neq i$ , then

$$\rho_j(i)(\{x_j\}) > 0, \quad \text{for all } x_j \in \phi_j(i)(O_i \cap Q_i).$$

For each  $i$ , let  $\rho(i)$  be the member of  $\Delta(X)$  defined by

$$\rho(i) := (\rho_1(i), \dots, \rho_N(i)).$$

Note that for each  $i$  we have

$$\rho_{-i}(i)(\{\phi(i)(x_i)\}) > 0, \quad \text{for all } x_i \in Q_i \cap O_i.$$

Define

$$\mu := \frac{1}{2}\sigma + \frac{1}{2} \sum_{i=1}^N \frac{1}{N} \rho(i), \tag{8}$$

where  $\sigma \in \widehat{\Delta}(X)$ . It is clear that  $\mu \in \widehat{\Delta}(X)$ .

By assumption, there is a sequence  $(\delta^n)$  in  $(0, 1)^N$  with  $\delta^n \rightarrow 0$  such that the game

$$\widehat{G}_{(\delta^n, \mu)} := (X_i \setminus O_i, u_i^{(\delta^n, \mu)})_{i=1}^N$$

has a Nash equilibrium  $x^n$  for each  $n$ . We show that  $x^n$  is also a Nash equilibrium of  $G_{(\delta^n, \mu)}$  for each  $n$ .

Fix  $n$ . We first note that for each  $i$ , the map  $u_i^{(\delta^n, \mu)}(\cdot, x_{-i}^n)$  is upper semi-continuous on  $X_i$ . Because  $u_i(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$  for every  $x_{-i} \in X_{-i}$ , it follows from Lemma 6 that  $u_i(\cdot, p_{-i})$  is upper semicontinuous

on  $\Delta(X_i)$  for every  $p_{-i} \in \Delta(X_{-i})$ . Therefore, the map  $u_i^{(\delta^n, \mu)}(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$  for every  $x_{-i} \in X_{-i}$ .

To see that  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu)}$ , suppose not. Then, there exist  $i$  and  $\varrho_i \in \Delta(X_i)$  such that

$$\int_X u_i^{(\delta^n, \mu)}(\cdot, x_{-i}^n) d\varrho_i > u_i^{(\delta^n, \mu)}(x^n). \quad (9)$$

Assume that

$$\varrho_i \in \arg \max_{\nu_i \in \Delta(X_i)} \int_{X_i} u_i^{(\delta^n, \mu)}(\cdot, x_{-i}^n) d\nu_i. \quad (10)$$

If that were not the case, one could always choose a member of

$$\arg \max_{\nu_i \in \Delta(X_i)} \int_{X_i} u_i^{(\delta^n, \mu)}(\cdot, x_{-i}^n) d\nu_i,$$

since this set is nonempty by compactness of  $\Delta(X_i)$  and upper semicontinuity of  $u_i^{(\delta^n, \mu)}(\cdot, x_{-i}^n)$  on  $X_i$ .

There is no loss of generality in assuming that no member of  $\Delta(X_i)$  weakly dominates  $\varrho_i$ . To see this, suppose that  $\nu_i \in \Delta(X_i)$  weakly dominates  $\varrho_i$ . Then, since  $u_i(\cdot, x_{-i})$  is upper semicontinuous on  $X_i$  for every  $x_{-i} \in X_{-i}$ , the map  $u_i(\cdot, p_{-i})$  is upper semicontinuous on  $\Delta(X_i)$  for every  $p_{-i} \in \Delta(X_{-i})$  (Lemma 6). Consequently, by Corollary 1 of Salonen [13], there exists a strategy  $\varrho_i^* \in \Delta(X_i)$  that is not weakly dominated by any member of  $\Delta(X_i)$  and that weakly dominates  $\varrho_i$ . By (10), it is clear that

$$\varrho_i^* \in \arg \max_{\nu_i \in \Delta(X_i)} \int_{X_i} u_i^{(\delta^n, \mu)}(\cdot, x_{-i}^n) d\nu_i.$$

If  $\varrho_i(O_i) = 0$ , (9) clearly contradicts the fact that  $x^n$  is a Nash equilibrium of  $\widehat{G}_{(\delta^n, \mu)}$ . Suppose that  $\varrho_i(O_i) > 0$ . It is convenient to analyze two subcases separately: (i) the case when  $\varrho_i$  has atoms in  $O_i$  (i.e.,  $\varrho_i(\{x_i\}) > 0$  for some  $x_i \in O_i$ ); (ii) and the case when  $\varrho_i$  has no atoms in  $O_i$ .

Suppose that  $\varrho_i(O_i) > 0$  and that  $\varrho_i(\{x_i\}) > 0$  for some  $x_i \in O_i$ . Because  $x_i \in O_i$ , there exists  $\nu_i \in \Delta(X_i)$  such that

$$u_i(\nu_i, x_{-i}) \geq u_i(x_i, x_{-i}), \quad \text{for all } x_{-i} \in X_{-i},$$

and  $u_i(\nu_i, y_{-i}) > u_i(x_i, y_{-i})$  for some  $y_{-i} \in X_{-i}$ . Define  $p_i \in \Delta(X_i)$  by

$$p_i(B) := \varrho_i(B \setminus \{x_i\}) + \nu_i(B)\varrho_i(\{x_i\}).$$



It is easy to see that  $p_i$  weakly dominates  $\varrho_i$ , contradicting that no member of  $\Delta(X_i)$  weakly dominates  $\varrho_i$ .

Suppose that  $\varrho_i(O_i) > 0$  and that  $\varrho_i$  has no atoms in  $O_i$ . Then, since  $Q_i \cap O_i$  is dense in  $O_i$ , there exists  $z_i \in Q_i \cap O_i \cap \text{supp}(\varrho_i)$ . To see this, define the measure  $\varrho_i^*$  on  $\text{cl}(O_i)$  by

$$\varrho_i^*(B) := \frac{\varrho_i(B \cap O_i)}{\varrho_i(O_i)}.$$

Since  $\varrho_i$  has no atoms in  $O_i$ ,  $\text{cl}(O_i)$  is uncountable and  $\varrho_i^*$  has no atoms in  $\text{cl}(O_i)$ . Therefore, there exists a one-to-one map  $\varphi : \text{cl}(O_i) \rightarrow [a, b]$  such that (i)  $\varrho_i^*(B) = m(\varphi(B))$  for every Borel set  $B \subseteq \text{cl}(O_i)$ , where  $m$  represents Lebesgue measure on  $[0, 1]$ , and (ii) the set of discontinuity points of  $\varphi$  (resp.  $\varphi^{-1}$ ) is contained in some Borel set  $B_\varphi \subseteq \text{cl}(O_i)$  (resp.  $B_{\varphi^{-1}} \subseteq [0, 1]$ ) such that  $\varrho_i^*(B_\varphi) = 0$  (resp.  $m(B_{\varphi^{-1}}) = 0$ ) (cf. Sun [15], Theorem 1). Choose  $z_i \in Q_i \cap O_i$ . Fix any  $\epsilon > 0$ . Let  $N_\epsilon(z_i)$  be the  $\epsilon$ -neighborhood of  $z_i$  in  $X_i$ . Note that

$$\varrho_i^*(N_\epsilon(z_i) \cap \text{cl}(O_i)) > 0 \tag{11}$$

implies that  $\varrho_i(N_\epsilon(z_i)) > 0$ , so to show that  $z_i \in \text{supp}(\varrho_i)$  it suffices to establish (11). Because  $\varphi^{-1}$  is continuous on  $[0, 1] \setminus B_{\varphi^{-1}}$ , there exists  $\beta > 0$  such that

$$\varphi^{-1}(((\varphi(z_i) - \beta, \varphi(z_i) + \beta) \cap [0, 1]) \setminus B_{\varphi^{-1}}) \subseteq N_\epsilon(z_i) \cap \text{cl}(O_i).$$

Consequently, since

$$\begin{aligned} m(((\varphi(z_i) - \beta, \varphi(z_i) + \beta) \cap [0, 1]) \setminus B_{\varphi^{-1}}) &= m((\varphi(z_i) - \beta, \varphi(z_i) + \beta) \cap [0, 1]) \\ &> 0, \end{aligned}$$

we have

$$\begin{aligned} \varrho_i^*(N_\epsilon(z_i) \cap \text{cl}(O_i)) &\geq \varrho_i^*(\varphi^{-1}(((\varphi(z_i) - \beta, \varphi(z_i) + \beta) \cap [0, 1]) \setminus B_{\varphi^{-1}})) \\ &= m(((\varphi(z_i) - \beta, \varphi(z_i) + \beta) \cap [0, 1]) \setminus B_{\varphi^{-1}}) \\ &> 0, \end{aligned}$$

as we sought.

We conclude that there exists  $z_i \in Q_i \cap O_i \cap \text{supp}(\varrho_i)$ . Since  $z_i \in Q_i \cap O_i$ , there exists  $\nu_i \in \Delta(X_i)$  such that

$$u_i(\nu_i, x_{-i}) \geq u_i(z_i, x_{-i}), \quad \text{for all } x_{-i} \in X_{-i}, \tag{12}$$

and

$$u_i(\nu_i, \phi(i)(z_i)) > u_i(z_i, \phi(i)(z_i)). \quad (13)$$

Let  $(\epsilon^k)$  be a sequence in  $(0, 1)$  with  $\epsilon^k \searrow 0$ . Define  $\varrho_i^k \in \Delta(X_i)$  by

$$\varrho_i^k(B_i) := \varrho_i(B_i \setminus N_{\epsilon^k}(z_i)) + \nu_i(B_i)\varrho_i(N_{\epsilon^k}(z_i))$$

(where  $N_{\epsilon^k}(z_i)$  denotes the  $\epsilon^k$ -neighborhood of  $z_i$  in  $X_i$ ). Observe that  $\varrho_i$  satisfies, for every Borel set  $B_i \subseteq X_i$ , and for each  $k$ ,

$$\varrho_i(B_i) = \varrho_i(B_i \setminus N_{\epsilon^k}(z_i)) + \nu_i^k(B_i)\varrho_i(N_{\epsilon^k}(z_i)), \quad (14)$$

where  $\nu_i^k \in \Delta(X_i)$  is defined as

$$\nu_i^k(B_i) := \frac{\varrho_i(B_i \cap N_{\epsilon^k}(z_i))}{\varrho_i(N_{\epsilon^k}(z_i))}.$$

Given (12) and (13), and since the construction of  $\mu$  in (8) entails  $\mu_{-i}(\{\phi(i)(z_i)\}) > 0$ , we have

$$u_i(\nu_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) > u_i(z_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}).$$

Therefore, since  $u_i(\cdot, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i})$  is upper semicontinuous at  $z_i$ , for large enough  $k$  we have

$$u_i(\nu_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) > \eta > u_i(w_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i})$$

for some  $\eta \in \mathbb{R}$  and all  $w_i \in N_{\epsilon^k}(z_i)$ . Consequently, for large enough  $k$  we have

$$u_i(\nu_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) > u_i(\nu_i^k, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}). \quad (15)$$

We are now ready to reach the desired contradiction. For large enough  $k$  we have

$$\begin{aligned} & u_i(\varrho_i^k, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) \\ &= \int_{X_i \setminus N_{\epsilon^k}(z_i)} u_i(\cdot, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) d\varrho_i \\ & \quad + \varrho_i(N_{\epsilon^k}(z_i)) u_i(\nu_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) \\ &> \int_{X_i \setminus N_{\epsilon^k}(z_i)} u_i(\cdot, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) d\varrho_i \\ & \quad + \varrho_i(N_{\epsilon^k}(z_i)) u_i(\nu_i^k, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) \end{aligned}$$

$$= u_i(\varrho_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}),$$

where the inequality uses (15) and the last equality follows from (14).

We have shown that for large enough  $k$ ,

$$u_i(\varrho_i^k, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}) > u_i(\varrho_i, (1 - \delta^n)x_{-i}^n + \delta^n \mu_{-i}).$$

But this contradicts (10).

We conclude that  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu)}$  for each  $n$ . Next, since  $(x^n)$  lies in  $X$  and  $X$  is sequentially compact, we may write (passing to a subsequence if necessary)  $x^n \rightarrow x$ . In addition, since each  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu)}$ , each  $(1 - \delta^n)x^n + \delta^n \mu$  is a Nash equilibrium of  $\overline{G}_{\delta\mu}$ , so, given that  $x^n \rightarrow x$ , we see that  $x$  is a pure strategy trembling-hand perfect profile of  $G$ . Since  $u_i^{(\delta^n, \mu)} \rightarrow u_i$  for each  $i$ , and because  $x^n$  is a Nash equilibrium of  $G_{(\delta^n, \mu)}$  for each  $n$  and  $x^n \rightarrow x$ , it follows from Lemma 2 and Proposition 1 that  $x$  is a Nash equilibrium of  $G$ . Furthermore, because  $x^n \in \times_{j=1}^N (X_j \setminus O_j)$  for each  $n$ , and since  $x^n \rightarrow x$  and  $\times_{j=1}^N (X_j \setminus O_j)$  is closed, we have  $x \in \times_{j=1}^N (X_j \setminus O_j)$ , so  $x$  is limit admissible.

Finally, the set of limit admissible, pure strategy trembling-hand perfect equilibria of  $G$  can be written as the intersection of the set of pure strategy trembling-hand perfect equilibria of  $G$  and the compact set  $\times_{j=1}^N (X_j \setminus O_j)$ . Hence, since set of pure strategy trembling-hand perfect equilibria of  $G$  is closed (hence compact) in the set of pure strategy Nash equilibria of  $G$  (Lemma 3), it follows that the set of limit admissible, pure strategy trembling-hand perfect equilibria of  $G$  is a compact subset of the set pure strategy trembling-hand perfect equilibria of  $G$ . ■

**Theorem 3.** *Suppose that  $G = (X_i, u_i)_{i=1}^N$  is a supermodular game satisfying sequential better-reply security. Suppose that for each  $i$  the set  $X_i \setminus O_i$  is a sublattice of  $X_i$ . Then  $G$  possesses a limit admissible, pure strategy trembling-hand perfect equilibrium. Moreover, the set of limit admissible, pure strategy trembling-hand perfect equilibria of  $G$  is a compact subset of the set pure strategy trembling-hand perfect equilibria of  $G$ .*

*Proof.* In light of Lemma 9, it suffices to show that for each  $\mu \in \widehat{\Delta}(X)$ , there is a sequence  $(\delta^n)$  in  $(0, 1)^N$  with  $\delta^n \rightarrow 0$  such that the game

$$\widehat{G}_{(\delta^n, \mu)} := (X_i \setminus O_i, u_i^{(\delta^n, \mu)})_{i=1}^N$$

has a Nash equilibrium for each  $n$ . Fix  $\mu \in \widehat{\Delta}(X)$  and a sequence  $(\delta^n)$  in  $(0, 1)^N$  with  $\delta^n \rightarrow 0$ . Because  $G$  is supermodular, the game  $G_{(\delta^n, \mu)}$  is

supermodular for each  $n$  (Lemma 7). Because for each  $i$  the set  $X_i \setminus O_i$  is a closed sublattice of  $X_i$ , and since  $G_{(\delta^n, \mu)}$  is supermodular for each  $n$ , the game  $\widehat{G}_{(\delta^n, \mu)}$  is supermodular for each  $n$ . Since  $X_i \setminus O_i$  is a compact metric lattice for each  $i$ ,  $X_i \setminus O_i$  is lattice complete for each  $i$  (Lemma 1). Consequently, for each  $i$ , the topology induced by the metric on  $X_i \setminus O_i$  is finer than the interval topology (e.g., Lawson [10], Proposition 4). Hence, since  $\widehat{G}_{(\delta^n, \mu)}$  is supermodular for each  $n$ , it follows from Theorem 4.2 in Vives [17] that  $\widehat{G}_{(\delta^n, \mu)}$  has a pure strategy Nash equilibrium for each  $n$ . ■

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