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Filling in the blanks: network structure and interbank contagion

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Non-technical summary

The pattern of financial linkages, or the financial network, is important in many areas of banking and finance, especially in such areas as financial stress testing of the entire system. Unfortunately, the interbank network often remains unobserved: as interbank loans are generally arranged over-the-counter, the bilateral exposures are known only by the immediate counterparties of each trade. The bilateral positions can sometimes be obtained from regulatory filings or credit registers. However, central banks and regulators do not usually observe the network because banks do not report their bilateral exposures. In those cases, the leading method is for researchers to fill in the blanks as evenly as possible, using the available information on each bank's total interbank lending. This approach, known as maximum entropy, effectively assumes that banks diversify their exposures by spreading their lending and borrowing across all other active banks. This paper proposes an efficient alternative that combines information-theoretic arguments with economic incentives to produce more realistic interbank networks that preserve important characteristics of the original interbank market. The maximum entropy (ME) approach for filling in the blanks can be misleading when the result is employed in network analysis or financial stress tests. The practical shortcoming is that applying ME tends to create complete networks which obscure the true structure of linkages in the original network. Key concepts in network analysis, such as degree (number of connections), become meaningless as a result of using ME. Furthermore, it has been demonstrated that when such estimated networks are used for purposes of stress testing, ME introduces a bias. The related conceptual shortcoming is that applying ME is optimal from an information-theoretic perspective only if nothing else is known about the network. But we do know that interbank networks are typically sparse, because interbank activity is based on relationships. Former work by the authors shows that smaller banks use a limited set of money center banks as intermediaries. This short paper proposes an alternative benchmark, one that minimizes the number of links necessary for distributing a given volume of loans on the interbank market. Our "minimum density" approach, in contrast to maximum entropy, identifies the most probable links and loads them with the largest possible exposures consistent with the total lending and borrowing observed for each bank. This link prediction method combines elements of information theory with economic rationale to determine a sparse network, without using any information from the (unobserved) bilateral positions of the original network. The second section confronts these two benchmarks with the true German interbank network that we observe for 1800 banks. We find that our minimum density solution preserves some of the network's structural features better than maximum entropy does. This makes the minimum density approach a reasonable benchmark for estimating missing counterparty exposures, one that does not wipe out the structure of linkages that is so central to network analysis. When used in a stress testing context, the minimum density approach performs better than maximum entropy and also permits more robust analysis. Using the two benchmarks side by side helps identify a range of possible systemic risk outcomes when the true pattern of counterparty exposures is unknown. Our approach may be of independent interest to finance and various other disciplines, where a network of interest is not fully observed, or has yet to be designed. Networks in transportation, financial markets or international trade are much sparser than maximum entropy would have us believe. Our minimum density approach may provide a meaningful alternative in these areas, guided by the simple economic rationale of minimizing cost.

Nicht-technische Zusammenfassung

Die Verflechtungen innerhalb des Finanzsektors, bzw. das Finanznetzwerk, spielt in vielen Bereichen des Bank- und Finanzwesens eine große Rolle, so z. B. bei Stresstests, die für das gesamte Finanzsystem durchgeführt werden. Allerdings bleibt das Interbankennetz dabei oft unbeobachtet, da Interbankenkredite meist außerbörslich direkt zwischen den Vertragspartnern verhandelt werden und die bilateralen Risikoengagements nur den Beteiligten bekannt sind. Mitunter lassen sich die bilateralen Positionen den bei Aufsichtsbehörden eingereichten Unterlagen oder Kreditregistern entnehmen. Jedoch wird das Interbankennetzwerk in der Regel nicht von den Zentralbanken und Aufsichtsbehörden beobachtet, da die Kreditinstitute ihre bilateralen Risiken nicht melden. Die gängige Herangehensweise der Forschung besteht darin, die Leerstellen mithilfe der verfügbaren Informationen zu den gesamten Interbankenkrediten einer Bank so gleichmäßig wie möglich zu füllen. Bei diesem als Maximum-Entropie-Methode (MEM) bezeichneten Ansatz wird letztendlich unterstellt, dass die Banken ihre Kreditrisiken diversifizieren, indem sie ihre aufgenommenen und ausgereichten Darlehen auf alle anderen aktiven Banken verteilen. In der vorliegenden Arbeit wird eine effiziente Alternative vorgeschlagen, die informationstheoretische Argumente mit ökonomischen Anreizen verknüpft, um zu einer realistischeren Darstellung von Interbankennetzwerken zu gelangen, in der wichtige Merkmale des ursprünglichen Interbankenmarkts erhalten bleiben. Das mit der MEM zum Auffüllen der Leerstellen ermittelte Ergebnis kann irreführend sein, wenn es zur Analyse des Netzwerks oder für Stresstests herangezogen wird. Aus praktischer Sicht hat die MEM den Nachteil, dass sie tendenziell vollständige Netzwerke hervorbringt, welche die tatsächliche Struktur der Verflechtungen innerhalb des ursprünglichen Netzwerks nicht richtig wiedergeben. Zentrale Begriffe der Netzwerkanalyse, wie etwa der Grad (Anzahl der Verbindungen), verlieren mit der Nutzung der MEM ihre Bedeutung. Zudem wurde gezeigt, dass die MEM zu Verzerrungen führt, wenn solche geschätzten Netzwerke in Stresstests verwendet werden. Das konzeptionelle Defizit besteht darin, dass die MEM aus informationstheoretischer Sicht nur dann optimal ist, wenn keine anderen Informationen zum untersuchten Netzwerk vorliegen. Wir wissen jedoch, dass Interbankennetze in der Regel eine geringe Dichte aufweisen, da der Aktivität am Interbankenmarkt Beziehungen zugrunde liegen. Frühere Arbeiten der Autoren zeigen, dass kleinere Banken lediglich eine begrenzte Zahl von großen Geschäftsbanken als Intermediäre in Anspruch nehmen. In diesem kurzen Beitrag wird eine alternative Messgröße vorgeschlagen, welche die zur Verteilung eines gegebenen Kreditvolumens am Interbankenmarkt erforderliche Anzahl von Verflechtungen minimiert. Anders als bei Verwendung der MEM lassen sich mit unserem Ansatz der „minimalen Dichte“ die Verbindungen mit der höchsten Wahrscheinlichkeit identifizieren. Diesen wird jeweils das größtmögliche Risiko entsprechend den insgesamt von einer Bank in Anspruch genommenen oder ausgereichten Krediten zugeordnet. Diese Methode zur Prognose von Verflechtungen kombiniert Elemente der Informationstheorie mit ökonomischen Argumenten, um ein Netzwerk von geringer Dichte zu bestimmen, und kommt dabei ohne Informationen aus (unbeobachteten) bilateralen Risikopositionen des ursprünglichen Netzwerks aus. Im zweiten Abschnitt werden die Ergebnisse der beiden Ansätze dem von uns beobachteten tatsächlichen Interbankennetz in Deutschland bestehend aus 1800 Kreditinstituten gegenübergestellt. Wir stellen fest, dass mit unserer Methode der minimalen Dichte einige strukturelle Merkmale des tatsächlichen Netzwerks

besser erhalten bleiben als bei Anwendung der MEM. Der Ansatz der minimalen Dichte führt daher zu einer angemessenen Schätzung unbeobachteter Positionen von Kontrahenten, wobei die Struktur der Verflechtungen, die in der Netzwerkanalyse eine zentrale Rolle spielt, sichtbar bleibt. Bei Stresstests liefert die auf der minimalen Dichte beruhende Methode bessere Ergebnisse als die MEM und ermöglicht zudem eine robustere Analyse. Die parallele Verwendung beider Messgrößen hilft bei der Ermittlung verschiedener Ergebnisse zu möglichen systemischen Risiken, wenn die tatsächliche Verteilung der Kontrahentenrisiken nicht bekannt ist. Unabhängig davon ist unser Ansatz auch für den Finanzsektor und verschiedene andere Disziplinen von Interesse, wenn es um ein Netzwerk geht, das nicht vollständig beobachtet wird oder noch zu modellieren ist. Netze in den Bereichen Verkehr, Finanzmärkte oder internationaler Handel haben bei Weitem nicht die hohe Dichte, die sich aus der Anwendung der maximalen Entropie ergibt. Unser Ansatz der minimalen Dichte bietet möglicherweise eine sinnvolle Alternative für diese Bereiche, folgt sie doch dem einfachen ökonomischen Kalkül der Kostenminimierung.

Filling in the Blanks: Network Structure and Interbank Contagion*

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Abstract

The pattern of financial linkages is important in many areas of banking and finance. Yet bilateral linkages are often unknown, and maximum entropy serves as the leading method for estimating unobserved counterparty exposures. This paper proposes an efficient alternative that combines information-theoretic arguments with economic incentives to produce more realistic interbank networks that preserve important characteristics of the original interbank market. The method loads the most probable links with the largest exposures consistent with the total lending and borrowing of each bank, leading to networks with *minimum density*. When used in a stress testing context, the minimum density approach performs better than maximum entropy and also permits more robust analysis. Using the two benchmarks side by side helps identify a range of possible systemic risk outcomes when the true pattern of counterparty exposures is unknown.

Keywords: Interbank markets, networks, entropy, intermediation, systemic risk

JEL classification: G21, L14, D85, C63.

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1 Introduction

Interbank contagion is a fundamental channel in many stress tests gauging systemic risk. Yet in practice the interbank network at the core of these simulations often remains unobserved: as interbank loans are generally arranged over-the-counter, the bilateral exposures are known only by the immediate counterparties of each trade. In some jurisdictions, bilateral positions can be obtained from regulatory filings or credit registers. More often, however, central banks and regulators do not observe the network because banks do not report their bilateral exposures. In those cases, the leading method is for researchers to fill in the blanks as evenly as possible, using the available information on each bank's total interbank lending (Upper 2011, Elsinger et al 2013). This approach, known as *maximum entropy*, effectively assumes that banks diversify their exposures by spreading their lending and borrowing across *all* other active banks.

The maximum entropy (ME) approach for filling in the blanks, however, can be misleading when the result is employed in network analysis or financial stress tests. The practical shortcoming is that applying ME tends to create *complete* networks which obscure the true structure of linkages in the original network. Key concepts in network analysis, such as degree (number of connections), become meaningless as a result of using ME. Furthermore, when such estimated networks are used for purposes of stress testing, ME introduces a bias (Mistrulli 2011, Markose et al 2012).

The related conceptual shortcoming is that applying ME is optimal from an information-theoretic perspective only if *nothing else* is known about the network. But we do know that interbank networks are typically sparse, because interbank activity is based on relationships (Cocco et al. 2009) and smaller banks use a limited set of money center banks as intermediaries (Craig and von Peter 2010). Indeed, most banks would find it prohibitively costly in terms of information processing and risk management to lend to *every* active bank in the system.

This short paper proposes an alternative benchmark, one that minimizes the number of links necessary for distributing a given volume of loans on the interbank market. Our “minimum density” approach, in contrast to maximum entropy, is based on the economic rationale that interbank linkages are costly to maintain, and determines an efficient pattern of linkages for allocating interbank positions. Intuitively, our approach identifies the most probable links and loads them with the largest possible exposures consistent with the total lending and borrowing observed for each bank. This link prediction method combines elements of information theory with economic rationale to determine a sparse network, without using any information from the (unobserved) bilateral positions of the original network.

The second section confronts these two benchmarks with the true German interbank network that we observe for 1800 banks. In providing an economically meaningful alternative, we find that our minimum density solution preserves some of the true network's structural features better than maximum entropy does. This makes the minimum density approach a reasonable benchmark for estimating missing counterparty exposures, one that does not wipe out the structure of linkages that is so central to network analysis.

The final section shows that systemic risk clearly depends on the pattern of interlinkages. We contrast the results from a standard stress test on the German banking system with those obtained when using the two alternative benchmarks instead. We find that

the maximum entropy approach overestimates contagion, whereas our minimum density solution underestimates it to a lesser extent. Using the two benchmarks in this context thus helps identify a range of possible stress test outcomes and also facilitates robust systemic risk analysis through repeated application. This makes the case for using both benchmarks when gauging how financial linkages affect systemic risk.

While these findings are relevant for central banks and regulators, our approach may be of independent interest. In finance and various other disciplines, situations arise where a network of interest is not fully observed, or has yet to be designed. Networks in transportation, financial markets or international trade are much sparser – and for good reason – than maximum entropy would have us believe. Our minimum density approach may thus provide a meaningful alternative in various areas, guided by the simple economic rationale of minimizing cost.

2 Minimum density networks

Consider a system of N banks engaged in interbank lending and borrowing. The matrix $\mathbb{X} \in [0, \infty)^{N \times N}$ represents gross interbank positions, where the typical element X_{ij} represents the amount bank i lends to bank j . Such networks are directed and valued. For each bank i , the row sum of \mathbb{X} shows total interbank assets, and the column sum tallies i 's interbank liabilities vis-à-vis all other banks,

$$\begin{cases} \text{Interbank assets :} & A_i = \sum_{j=1}^N X_{ij} \\ \text{Interbank liabilities :} & L_i = \sum_{j=1}^N X_{ji} \end{cases} \quad (1)$$

Matrix \mathbb{X} is the network of interest on which analysis would ideally be performed. In many situations, however, the true bilateral linkages are unknown. National authorities in many jurisdictions do not observe the full interbank network because banks either report no bilateral interbank positions at all, or only disclose their largest exposures. However, the authorities do generally have balance sheet information at their disposal, including the total interbank assets A_i and liabilities L_i of each bank i .

Suppose, for ease of exposition, that no bilateral positions are observed.¹ In this simple case, the authorities only know how much each bank borrows and lends on the interbank market overall (L_i and A_i), but not to whom (Figure 1 provides an example). Since information on counterparty exposures (X_{ij}) is essential for further analysis, it is necessary to resort to a method for filling in the interbank matrix, given knowledge of the “marginals” $\{A_i, L_i\}$.

The standard approach in the literature is to estimate the matrix \mathbb{X} by maximum entropy methods (Upper 2011, Elsinger et al 2013). This consists of maximizing the entropy function $-\sum_{i,j} X_{ij} \ln(X_{ij}/Q_{ij})$ subject to constraints (Equation 1), relative to prior information (Q_{ij}) on bilateral exposures, if available. As entropy is a measure of probabilistic uncertainty, this approach is optimal when selecting a probability distribution in the sense of using least information (Jaynes 1957, MacKay 2003). Entropy optimization is widely used across disciplines (Fang et al 1997), and can be implemented by a standard iterative

¹This is without loss of generality, as the method below can accommodate more information with exclusion restrictions on the prior distribution Q below.

algorithm (RAS), which can be generalized to handle additional constraints (Blien and Graef 1997, Elsinger et al 2013). In the simple case where only the marginals (1) are known, the maximum entropy (ME) solution takes the form of a gravity equation where the estimate of X_{ij} is approximately proportional to the product of marginals $A_i L_j$. To the extent that these marginals are positive, ME produces a complete network where each bank lends to all other active banks. A complete market structure is often thought to be more robust to contagion (Allen and Gale 2000); simulations on more realistic networks, however, show that greater diversification through more interlinkages may lower the probability of a crisis, but tends to raise its severity (Nier et al 2007).

Our approach for deriving an opposite benchmark starts from the premise that establishing and maintaining network linkages is costly and uses this information to estimate a different network from the “least information” entropy estimate. Banks do not spread their borrowing and lending as evenly as possible across the entire system, as the costs in terms of information processing, risk management and creditworthiness checks would be prohibitive for all but the largest banks. In reality, interbank activity occurs through relationships (Cocco et al. 2009), and interbank networks are *sparse* as a result, often with less than 1% of potential bilateral linkages in active use (see Bech and Atalay 2010 for the United States, and Craig and von Peter 2010 for Germany). These relationships are also *disassortative*, implying that less-connected banks are more likely to trade with well-connected banks than with other less-connected banks (Bech and Atalay 2010, Iori et al 2008). This reflects the economic rationale that smaller banks, rather than transacting with each other, typically use a small set of money center banks as intermediaries (Craig and von Peter 2010). Similar observations hold for other networks in financial markets (Li and Schürhoff 2013) or in international trade (Helpman et al 2008, Ahn et al 2011). More generally, minimally connected networks arise as the *efficient* solution in economic models where agents trade off the costs and rewards of forming links (Goyal 2007, Jackson 2008).

As an efficient alternative to maximum entropy, our approach minimizes the network’s density, the share of actual to potential bilateral links. It minimizes the total number of linkages necessary for allocating interbank positions, consistent with total lending and borrowing observed for each bank. Let c represent the fixed cost of establishing a link. Then the minimum density (MD) approach can be formulated as a constrained optimization problem,

$$\begin{aligned} \min_{\mathbf{X}} \quad & c \sum_{i=1}^N \sum_{j=1}^N \mathbf{1}_{[X_{ij}>0]} \quad \text{s.t.} \\ & \sum_{j=1}^N X_{ij} = A_i \quad \forall i = 1, 2, \dots, N \\ & \sum_{i=1}^N X_{ij} = L_j \quad \forall j = 1, 2, \dots, N \\ & X_{ij} \geq 0 \quad \forall i, j \end{aligned} \tag{2}$$

where the integer function $\mathbf{1}$ equals one only if bank i lends to bank j . The economic nature of this problem shares similarities with network design problems in transportation science and communication networks. Minimizing the cost of transporting a given volume of goods from origins to destinations appears analogous to moving money between banks. In our case, the capacities of transportation hubs (banks) are constrained by two marginals $\{A_i, L_i\}$, and the fixed cost of building new roads (credit relationships) must be considered

as well. Such network design problems have been studied for decades and are known to be NP-hard except in very special cases (Campbell and O’Kelly 2012 provide a survey).

Our approach specifies a link prediction method, combining elements of information theory and economic rationale. In problem (2), the *value* for any configuration of linkages \mathbb{X} can be expressed as follows. In our objective function which is used to assign links, we first soften the constraints by assigning penalties for *deviations* from the marginals,

$$\begin{aligned} AD_i &\equiv \left(A_i - \sum_j X_{ij} \right), \\ LD_i &\equiv \left(L_i - \sum_j X_{ji} \right). \end{aligned}$$

This makes the constraint part of a smooth optimization problem, so that the only non-smooth part lies in the cost of a link. When these deviations are introduced into the objective function, the problem maximizes

$$V(\mathbb{X}) = -c \sum_{i=1}^N \sum_{j=1}^N \mathbf{1}_{[X_{ij}>0]} - \sum_{i=1}^N [\alpha_i AD_i^2 + \delta_i LD_i^2]. \quad (3)$$

The solution would be straightforward if one could enumerate all possible network configurations \mathbb{X} , and rank them according to $V(\mathbb{X})$. But the computational complexity of such problems rises exponentially with N due to the number of possible subsets (2^N), even before allocating monetary values to each link. Since exhaustive search is impossible – certainly for our application with 1800 banks – we develop an algorithm with two main ingredients.

First, the probability distribution over possible networks, $P(\mathbb{X})$, is unknown. Drawing on the insights of Hansen and Sargent (2001), a distribution should be chosen according to *robust beliefs* which are derived from $\min_P \sum_{\mathbb{X}} P(\mathbb{X})V(\mathbb{X}) + \theta R(P \parallel Q)$, where $R(P \parallel Q)$ is the relative entropy between a prior distribution $Q(\mathbb{X})$ and the belief $P(\mathbb{X})$, and θ is a scaling parameter. Up to a constant, the entropy function $\sum_{\mathbb{X}} P(\mathbb{X}) \ln(P(\mathbb{X})/Q(\mathbb{X}))$ is in fact the only functional form that satisfies the decision-theoretic axioms relevant to this setting (Mattsson and Weibull 2002, Strzalecki 2011). The solution to this optimization problem can be obtained from the first order conditions as the multinomial logit choice function

$$P(\mathbb{X}) = \frac{Q(\mathbb{X}) e^{\theta V(\mathbb{X})}}{\sum_{\mathbb{X}'} Q(\mathbb{X}') e^{\theta V(\mathbb{X}')}}, \quad (4)$$

where \mathbb{X}' represents any other network structure.

However, exploring this distribution over the full space of network configurations is still a computationally complex task. We approximate this problem by assuming that the prior beliefs factorize over all links, $Q\left(X_{ij} | \{X\}_{kj \neq ij}\right) = Q(X_{ij}) = Q_{ij}$ which, as we describe below, allows us to treat individual links rather than entire matrices at a time.

The second important ingredient in our approach is to determine an economically meaningful prior distribution. Rather than using an uninformed flat prior, we build on the economic rationale that small banks seek to match their lending and borrowing needs through relationships with larger banks that are well placed to satisfy those needs.

Accordingly, the probability that i lends to j increases if either i is a large lender to a small borrower j , or i is a small lender to a large borrower j . This is summarized in the criterion

$$Q_{ij} \propto \max \left\{ \frac{AD_i}{LD_j}, \frac{LD_j}{AD_i} \right\} \quad \forall (i, j) \in \mu. \quad (5)$$

where μ is the set of “blank links” under consideration.

Armed with these beliefs, our MD algorithm proceeds as follows. At each iteration, a link (i, j) is selected with probability Q_{ij} and the link X_{ij} is loaded with the maximum value that this pair of banks can transact given their current asset and liability positions, which is $\min\{AD_i, LD_j\}$. If adding this link increases the value of the objective function $V(\mathbb{X}+X_{ij}) > V(\mathbb{X})$ in (3), the allocation is retained – otherwise, the link is rejected.² Finally, once positions have been updated we proceed to the next iteration until the total interbank market volume has been allocated. We also include a small probability of link deletion to allow the algorithm to cover the entire space of possible network configurations.³

Two useful aspects of the algorithm deserve special mention. First, we can parametrize how close the estimated network should be to the minimum density solution. Less aggressive solutions – with densities not quite as low as the minimum density – can be obtained by scaling the loadings for selected links to $\lambda \times \min\{AD_i, LD_j\}$, where $\lambda < 1$ forces the algorithm to select more links until the overall interbank volume is reached. Second, the algorithm lends itself to robust analysis through repeated applications. Since the link prediction method is stochastic, multiple solutions can be used as inputs for network analysis or stress tests. This allows, if needed, to produce confidence bands reflecting the uncertainty about the (unobserved) true network. The pseudo-code for our algorithm is provided in the Appendix.

We close this section with a simple example to illustrate the different approaches (Figure 1). The example network is represented by the top-left matrix; out of seven banks, two only lend (D,E), one only borrows (F), and the remaining four banks intermediate.⁴ Suppose the authorities observe only the marginals from individual bank balance sheets (bottom left-hand matrix), and estimate counterparty exposures by maximum entropy (top right-hand matrix in Figure 1). This preserves the marginals of each bank, and places zeros only on those rows and columns of banks that do not lend or do not borrow at all. However, the more banks are active, the less ME will preserve of the network structure. By contrast, our MD solution (with $\lambda = 1$) minimizes the number of linkages while also respecting all marginals and net positions.

The original network in this example has a density of 33%, with a third of the 42 potential bilateral links being used. The ME solution shows far higher density (62%), whereas our solution reaches the lowest density (21%) attainable with those marginals. While ME spreads interbank activity as evenly as possible, MD does the opposite by concentrating exposures on the smallest possible set of links. This illustrates that the two

²For reasons explained in the Appendix, we allow for a small probability of accepting such a link, $\exp[\theta (V(\mathbb{X}+X_{ij}) - V(\mathbb{X}))]$, which is derived from the ratio $P(\mathbb{X}+X_{ij})/P(\mathbb{X})$ of observing the network configurations.

³This ensures that the state space is ergodic (see Proposition 1 in Anand et al 2012).

⁴Banks A-C are core banks that play an central role in intermediating between periphery banks (Craig and von Peter 2010).

Actual Data									Estimated Networks								
True Network									Maximum Entropy Solution								
	A	B	C	D	E	F	G	A_i	A	B	C	D	E	F	G	A_i	
A	0	3	1	0	0	1	2	7	0	2.53	2.18	0	0	0.74	1.55	7	
B	2	0	2	0	0	0	1	5	1.72	0	1.6	0	0	0.54	1.14	5	
C	1	1	0	0	0	1	0	3	0.98	1.06	0	0	0	0.31	0.65	3	
D	1	0	0	0	0	0	0	1	0.25	0.27	0.23	0	0	0.08	0.17	1	
E	0	0	2	0	0	0	1	3	0.75	0.81	0.7	0	0	0.24	0.5	3	
F	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
G	0	1	0	0	0	0	0	1	0.3	0.32	0.28	0	0	0.09	0	1	
L_i	4	5	5	0	0	2	4	20	4	5	5	0	0	2	4	20	

Observable Interbank Market									Minimum Density Solution								
	A	B	C	D	E	F	G	A_i	A	B	C	D	E	F	G	A_i	
A								7	0	3	0	0	0	0	4	7	
B								5	3	0	2	0	0	0	0	5	
C								3	0	2	0	0	0	1	0	3	
D								1	0	0	0	0	0	1	0	1	
E								3	0	0	3	0	0	0	0	3	
F								0	0	0	0	0	0	0	0	0	
G								1	1	0	0	0	0	0	0	1	
L_i	4	5	5	0	0	2	4	20	4	5	5	0	0	2	4	20	

Figure 1: Illustrative example comparing the ME and MD solutions for a hypothetical interbank network.

benchmarks depart from the true network in opposite directions in how they trade off the number versus the size of interbank linkages.

3 Comparing network characteristics

This section evaluates the maximum entropy (ME) and minimum density (MD) solutions against a real-world network, the German interbank market. Our reference network \mathbb{X} represents the “true” bilateral interbank positions observed in the German banking system. The Deutsche Bundesbank compiles a set of comprehensive banking statistics on large loans and concentrated exposures (“*Gross- und Millionenkreditstatistik*”) comprising all positions between financial institutions in the amount of at least EUR 1.5 million or 10% of their liable capital. To obtain a consistent and self-contained network, we consolidate banks by ownership at the bank holding company level (“*Konzern*”) and exclude cross-border linkages (see Craig and von Peter 2010 for details). We use data from the second quarter of 2003; this choice avoids the crisis period and issues with backward data revisions.

The observed interbank network \mathbb{X} is a square matrix with 3.16 million cells containing the bilateral interbank exposures among 1779 active banks, amounting to EUR 855 billion in total value. The network is sparse, with a density of 0.62% (19643 active links). The bilateral counterparty exposures can also be estimated by maximum entropy, using only the marginals as explained. The ME solution (\mathbb{E}) yields a nearly complete network with density of 92.8%. Since 92% of banks are active both as lenders and borrowers, ME fails to place any zeros between these pairs. The original structure of linkages is essentially

lost as a result. The MD approach, at the other extreme, finds a solution (\mathbb{Z}) with density of 0.11% (3426 links). This solution allocates the full 100% of total interbank volume, using six times fewer links than in the original network.

The reason for this efficiency is that the MD algorithm identifies probable links and loads them with the largest possible exposures consistent with banks' total lending and borrowing in the interbank market. As a result, the largest links consistently account for more value than was the case in the original network (Figure 2). It is apparent that ME takes far longer to allocate total interbank volume, since it places positive value on virtually all pairs of banks. This illustrates how ME and MD differ in trading off the number versus the size of interbank linkages.

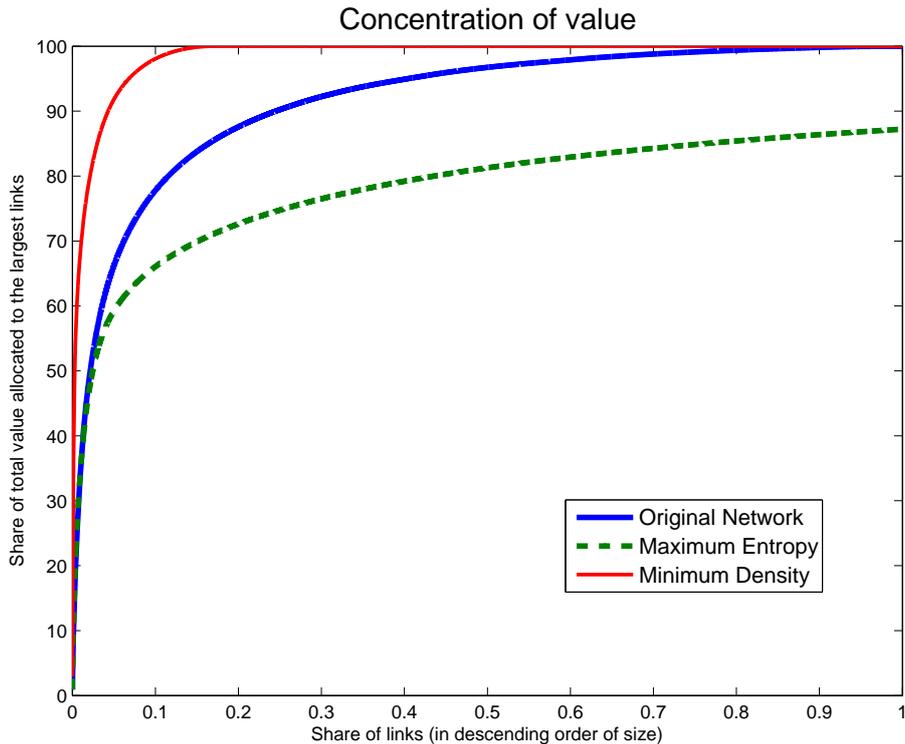


Figure 2: The figure shows the concentration of value on the largest links for the different networks. The x-axis lists bilateral linkages (in descending order of size) as a share of total number of links in original network. The y-axis shows the cumulative share of value allocated to the largest links relative to the total interbank volume in \mathbb{X} .

In so doing, MD preserves several network characteristics better than ME does (Table 1). The most fundamental statistic in network analysis, the *degree* (number of connections) of a node, typically becomes meaningless after using ME. In our application, the ME solution has nearly all banks lend to each other when banks in the true network have only 11 counterparties on each side of their balance sheet on average. The MD solution errs on the low side, but also assigns high degree to the most connected banks in the interbank market. As a result, its *degree distribution* broadly retains the shape of the original degree distribution (Figure 3) up to a factor reflecting its lower density. This is important since degree distributions are widely used as a diagnostic tool to characterize different types of networks (Erdős-Rényi random graphs, scale-free networks etc).

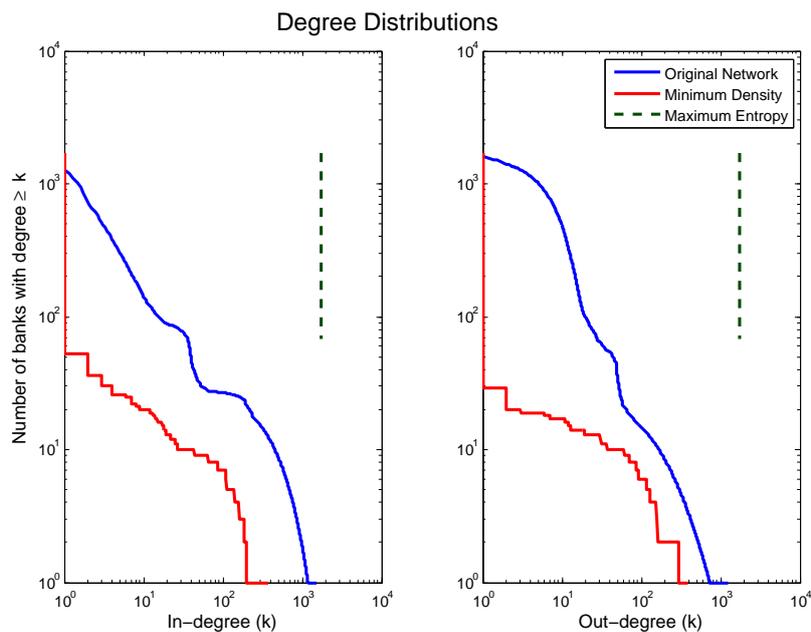


Figure 3: The figure displays the degree distribution in its cumulative form, showing the number of banks with degree greater than the value shown on the x-axis, on a double log scale. A straight line would indicate a Pareto cumulative indicative of a power law distribution. The original network has been smoothed to preserve the anonymity of individual bank data and represent averages at the end points.

The fact that interbank relations often involve smaller less-connected banks trading with larger well-connected banks gives rise to negative *assortativity*. A standard measure, the correlation between the node degrees across all linked pairs, is negative (-0.52) in the original network, as well as in the Italian interbank market and the Federal Funds market (Iori et al 2008, Bech and Atalay 2010). Disassortativity is even stronger under MD since this characteristic behavior was built into the link prediction model (5) of the algorithm. The focus of most banks on a few select counterparties comes out most clearly in the *dependence* measures listed next in Table 1. Under MD, the single largest link on average accounts for nearly 100% of a bank's total interbank borrowing. In the original network dependence on a single lender is also quite frequent, though less so for borrowers. The ME solution fails to preserve this relationship characteristic since it spreads links indiscriminately.

One network characteristic that appears to be lost under both ME and MD is *local clustering*, the propensity of nodes to form cliques. The local clustering coefficient averages the probabilities that two neighbors of a node are themselves connected (Jackson 2008). In matching big lenders with small borrowers and vice versa, MD tends to generate star-like networks where the extent of clustering is low. Indeed, it is part of the efficiency of

Network Characteristic	\mathbb{E} Max Entropy	\mathbb{X} True Network	\mathbb{Z} Min Density
Density, in %	92.8	0.6	0.1
Degree (average)	1649	11.0	1.9
Degree (median)	1710	6	1
Assortativity	0.00	-0.52	-0.66
Dependence on lender, %	12.2	87.0	99.3
Dependence on borrower, %	7.2	43.6	99.2
Clustering local avg, %	99.9	46.6	0.05
Core size, % banks	92.6	2.5	0.8
Error score, % links	21.8	12.2	12.5

Table 1: Comparing basic network characteristics

the MD solution that transitive relationships are replaced by a single link where possible, obviating the need for smaller banks to form additional local relationships for their lending or funding needs. At the other extreme, clustering under ME trivially equals 100% among active banks. As was the case for assortativity, ME provides no meaningful statistic since counterparty exposures are spread indiscriminately throughout entire system.

The final rows of Table 1 consider the extent to which the different networks exhibit *tiering*. The interbank market is tiered when few banks intermediate between other banks that do not transact directly with each other; how close an interbank market is to such a core-periphery structure can be measured using block modelling methods (Craig and von Peter 2010). The German interbank network is highly tiered, with only 12.2% of network links inconsistent with the tiering model (error score). Indeed, 98% of interbank volume has one of the 45 core banks (2.5% of banks in the system) on either side of the transaction. The MD network retains this structure, even if its core is only a third of the original size because exposures are concentrated on fewer links (Figure 2). Since core banks continue to be linked to each other, however, the interbank network remains a single market in the sense of a giant connected component. In the ME network, on the other hand, the core-periphery structure largely disappears: the best fit includes virtually all active banks in the core. This shows that ME fails to identify hierarchical networks structures.

We conclude that the minimum density approach preserves some of the network’s structural features better than the maximum entropy benchmark does. This makes the MD approach a promising alternative to ME for filling in missing counterparty exposures, especially if the estimated matrix is used in network analysis where the pattern of linkages is of central interest. An important application where both the pattern and size of linkages matter is explored next.

4 Performance in system stress tests

In this section, we contrast the results from a systemic stress test on the German banking system with the results obtained when using the two alternative benchmarks. We opt for

a standard simulation methodology, since the focus here is on variation in the simulation *inputs*, not on breaking new grounds on methodology. Three building blocks in any stress testing exercise are (i) the trigger event, (ii) the contagion mechanism, and (iii) data on bank balance sheets and counterparty exposures where we consider the estimated networks using maximum entropy and minimum density, alongside the “true” observed interbank network.

Regarding the trigger event, the majority of studies focus on the unanticipated failure of individual banks (Upper (2011), Mistrulli 2011), and we follow this approach here. Since common exposure to an aggregate shock is a somewhat more realistic crisis scenario, our second experiment combines a single failure with a decline in the regulatory capital ratio by 4 percentage points at all banks.

Regarding contagion, instead of employing the commonly used sequential default mechanism, we employ Eisenberg and Noe’s (2001) clearing vector methodology to find the fixed point at which contagion comes to a halt. The EN methodology is widely applied and discussed in Upper (2011) and Elsinger et al (2013). This approach has the advantage of delivering a unique solution that solves the simultaneity problem.⁵

We run three sets of simulations, one for each interbank network separately, and compare the results when using maximum entropy (\mathbb{E}) and minimum density (\mathbb{Z}) to those of the true interbank market (\mathbb{X}). Each set of simulations proceeds as follows: in each run, we let a single bank i fail exogenously to trigger the contagion process, and solve for the clearing vector to obtain (a) the number and identity of banks that default as a result of contagion (excluding the initial failure), and (b) the total assets and interbank liabilities of the defaulting banks. From these 1779 runs (one for each bank), we report the average number of contagious defaults and total bank assets affected by contagion (as in Mistrulli 2011).

In determining the system’s loss-given-default endogenously, the EN methodology assumes that all assets can be liquidated at book value to meet the liabilities of defaulting banks. The more contagion, the less tenable this assumption becomes. To capture possible distress selling and bankruptcy costs, we allow for an additional deadweight loss of β percent assessed on the liabilities of defaulting banks, and step up its value from 0% to 25% in separate simulations (where $\beta = 0$ leads to the original Eisenberg-Noe fixed point). This generalizes the results and allows to compute the overall liquidation and bankruptcy cost to the system as a whole. This is a true deadweight loss, net of all the redistributions that occur through the contagion process.

Figure 4 summarizes the results of the standard stress test with single bank failures. In the German interbank market (blue bold lines), the simulations suggest that about 6 banks default on average as a result of the failure of an arbitrary single bank, or 7 to 8 banks once bankruptcy costs exceed a few percent (left panel). In contrast with the average result, the worst case causes more than 1500 contagious defaults (85% of banks in the system). It is a well-known feature of complex networks with highly connected hubs that they are robust to failure but vulnerable to targeted attack (Albert et al 2000).

When the same stress test is run on the counterparty exposures estimated by maximum entropy, the results *overestimate* the extent of contagion (Figure 4, green lines). The number of contagious defaults, for instance, exceeds 25 banks on average. The MD

⁵Sequential default algorithms are internally inconsistent, ignoring that subsequent defaults induce further losses to banks that had already defaulted in earlier rounds of the process.

solution, on the other hand, *underestimates* the true extent of contagion (red lines). This remains the case when measuring contagion by the total assets of defaulting banks (Figure 4, center panel). The same ranking holds for the system-wide deadweight loss.

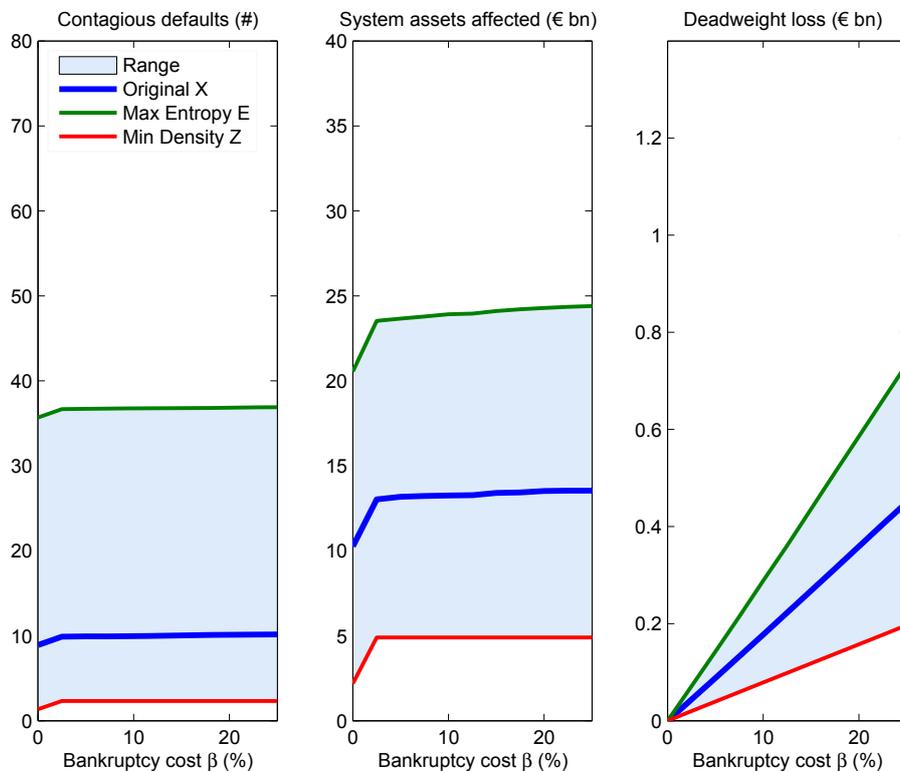


Figure 4: Figure 4 shows the results of the first stress test where contagion is triggered by individual bank failures. The three lines compare the different average results obtained by using maximum entropy (green), minimum density (red) and the “true” interbank network (blue). The stress tests are run separately for the different levels of bankruptcy costs shown on the x-axis. The left panel plots number of banks that default as a result of contagion (excluding the initial failure). The center panel expresses the extent of contagion by aggregating total assets of the banks that end up in default. The right panel displays the deadweight loss arising from distress selling and bankruptcy cost assessed on the liabilities of defaulting banks.

It is perhaps surprising to find that maximum entropy exaggerates contagion in the German interbank network when Mistrulli (2011) reported the opposite for Italy. In Mistrulli’s direct comparison, the ME solution *underestimated* systemic risk at low loss-given-default while overstating it at the higher end.⁶ We obtained a similar result when we employed the sequential default algorithm instead of the Eisenberg-Noe clearing vector. In those simulations (not reported here), ME underestimates the true extent of contagion far more than our MD does. This is perhaps because ME fails to place large exposures between the major banks in the system, even though such concentration is a realistic fea-

⁶The German interbank market is located at that higher end, given that interbank positions in the German banking system account for a greater share of total assets than is the case in Italy (Upper 2011, Figure 1).

ture of the interbank market (recall Figure 1). However, we hesitate to place much weight on this comparison, since the sequential default algorithm is not internally consistent. Moreover, the finding may not generalize to other networks where the structure and size of interbank linkages differ greatly from the interbank market considered here.

Our second experiment modifies these simulations by augmenting the single bank failures with an aggregate shock. There are many ways to design a stress test in which banks hold common exposure to a variety of factors, such as stock market declines, interest rate changes etc. Since all of these approaches cause losses to all banks simultaneously, we take a shortcut and directly hit their regulatory capital ratios by 4 percentage points. Figure 5 shows that this stress test produces more contagion for all three networks, without changing the qualitative character of the earlier results. In particular, ME overestimates substantially, and MD underestimates somewhat, the true extent of contagion.

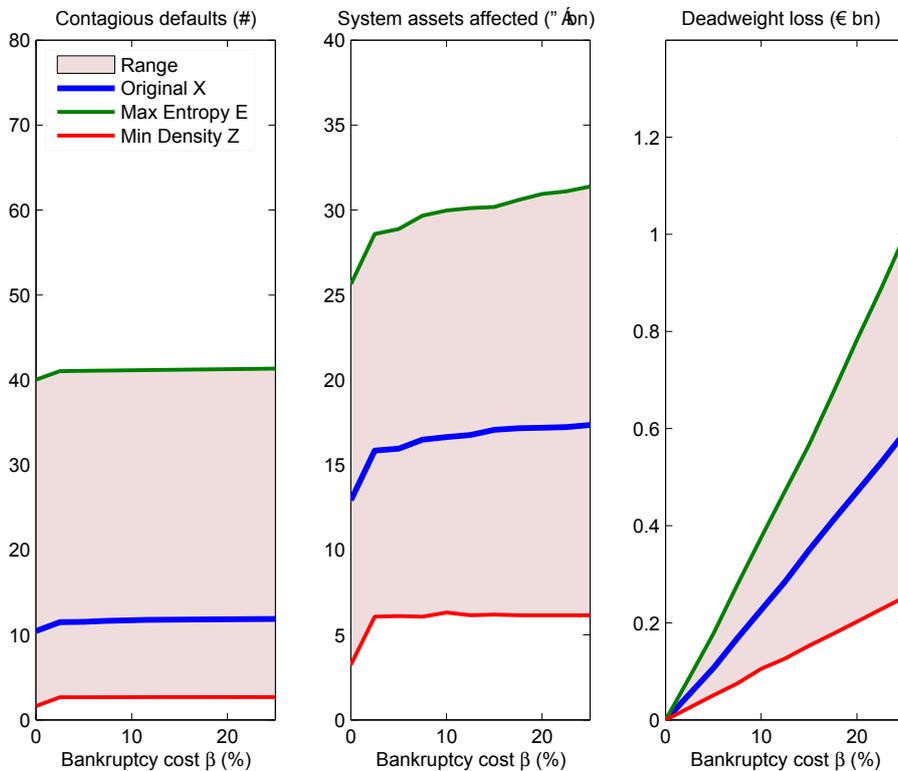


Figure 5: Figure 5 shows the results of the second experiment where individual bank failures are accompanied by a decline in the regulatory capital ratio by 4 percentage points at all banks. All other aspects as in Figure 4.

These findings suggest that the minimum density approach, while delivering an economically meaningful to maximum entropy, also leads to a reasonable estimate of *overall systemic risk* in stress tests. Using both benchmarks helps identify *a range* of possible stress test results when the true counterparty exposures are unknown (shaded areas in Figures 4-5). The alternatives can also be compared in terms of the rankings they predict on individual banks. Consider the 100 largest banks ranked by the damage their failure causes (as estimated from the true interbank network), in terms of contagious failures, total affected assets and other measures of contagion. The MD and ME results gener-

ally lead to similar rankings, with Kendall and Spearman rank correlations close to 80% relative to the “true” ranking. This suggests that both benchmarks deliver fairly reliable rankings of *systemically important banks*.

There are several reasons in our context why there is less contagion under minimum density than under maximum entropy. By reducing density, the MD network limits the scope of contagion since linkages are the conduits of the propagation of losses. Similar results are found in other sparse networks that are insufficiently connected (Nier et al 2007) or below a tipping point (Gai et al 2011). On the other hand, the concentration of exposures onto fewer links also means that the loss transmitted by a given link is more likely to exceed capital of the lending bank and thereby cause its default. The reason why this countervailing effect does not offset the sparsity effect in the German network has to do with negative assortativity in both MD and the true network. The failure of any small bank most likely hits a single large bank that can handle the loss, and this outcome dominates the averaged results in Figures 4-5. Indeed, we observe far more contagion when we set up the MD algorithm to enforce *positive* assortativity, by replacing the link prediction (5) by $Q_{ij} \propto \max \{AD_i, LD_j\}$.

5 Conclusion

The pattern and size of linkages are of central importance in most areas of network analysis and systemic risk. Yet the relevant linkages are often unknown, and maximum entropy serves as the leading method for estimating counterparty exposures. This paper proposes an economically meaningful alternative that combines known network features with information-theoretic arguments to produce more realistic interbank networks that better preserve characteristic features of the original interbank market. Moreover, in the context of the German banking system, the use of a minimum density network as input in systemic stress tests leads to more realistic outcomes than the use of the maximum entropy benchmark. Our minimum density approach does not necessarily produce better estimates of systemic risk in stress tests for other interbank markets. However, when used alongside maximum entropy, it does help to identify a range of possible outcomes when the true counterparty exposures are unknown. Indeed, the minimum density approach is more suitable for robustness analysis: whereas the maximum entropy network is unique, our link prediction method is stochastic and thus capable of generating many low-density networks for repeated application in stress tests. This is a useful feature in a context where networks are not fully observed yet the structure and size of linkages of great consequence for financial stability.

A Technical Appendix

The primitives of our algorithm are two Markov processes – one for adding new links and value to the interbank network (with probability $1 - \epsilon$), and the second to delete links, and correspondingly value, from the network (with probability ϵ). The intuition for this design is based on the notion of ergodicity, in that from any network structure, \mathbb{X} , through a finite series of link additions and deletions, we can obtain any other network \mathbb{X}' . As demonstrated in proposition 1 of Anand et al (2012), this ensures that the distribution

$P(\mathbb{X})$ over network configurations is stationary, and hence our Metropolis algorithm is well defined. As a consequence of being stochastic, our algorithm can generate multiple realizations of minimum density networks, which allows for robustness in conducting stress test exercises. Algorithm 1 below provides the pseudo-code for our procedure.

(??)

Algorithmus 1 Minimum Density algorithm to allocate interbank networks. The labels for the different lines are, (S) : update sets, (P) : update priors, (C) : update counter, (D) : draw random number, (L) : pick link, (A) : assign value, and (M) : update marginals.

begin

(S) : $\mu^{(0)} = \{(i, j)\}_{i, j=1}^N$ and $\nu^{(0)} = \{\emptyset\}$.

(P) : $Q_{ij}^{(0)} = \max \left\{ \frac{AD_i^{(0)}}{LD_j^{(0)}}, \frac{LD_j^{(0)}}{AD_i^{(0)}} \right\}$, $\forall (i, j) \in \mu^{(0)}$.

(C) : $\tau = 1$.

while

$$V(\mathbb{X}) < 0.999 \times \sum_{i=1}^N AD_i^{(0)}$$

do

(D) : $\rho \in [0, 1]$ at random.

if $\rho < \epsilon$ **then**

Remove link.

(L) : $(i, j) \in \nu^{(\tau-1)}$ with probability $1 / |\nu^{(\tau-1)}|$.

(M) : $AD_i^{(\tau)} = AD_i^{(\tau-1)} + X_{ij}$ and $LD_j^{(\tau)} = LD_j^{(\tau-1)} + X_{ij}$.

(A) : $X_{ij} = 0$.

(S) : $\mu^{(\tau)} = \mu^{(\tau-1)} \cup (i, j)$ and $\nu^{(\tau)} = \nu^{(\tau-1)} \setminus (i, j)$.

end

else

Add link.

(P) : $Q_{ij}^{(\tau-1)} = \max \left\{ \frac{AD_i^{(\tau-1)}}{LD_j^{(\tau-1)}}, \frac{LD_j^{(\tau-1)}}{AD_i^{(\tau-1)}} \right\}$, $\forall (i, j) \in \mu^{(\tau-1)}$.

(L) : (i, j) with probability $Q_{ij}^{(\tau-1)}$.

(A) : $X_{ij} \leftarrow \lambda \times \min \left\{ AD_i^{(\tau-1)}, LD_j^{(\tau-1)} \right\}$.

(A) : $\mathbb{X}' = \mathbb{X} + X_{ij}$.

(D) : $\psi \in [0, 1]$ at random.

if $V(\mathbb{X}') > V(\mathbb{X})$ **||** $\psi < \exp(\theta [V(\mathbb{X}') - V(\mathbb{X})])$ **then**

(A) : $\mathbb{X} = \mathbb{X}'$.

(M) : $AD_i^{(\tau)} = AD_i^{(\tau-1)} - X_{ij}$ and $LD_j^{(\tau)} = LD_j^{(\tau-1)} - X_{ij}$. (S) :
 $\mu^{(\tau)} = \mu^{(\tau-1)} \setminus (i, j)$ and $\nu^{(\tau)} = \nu^{(\tau-1)} \cup (i, j)$.

end

end

(P) : $Q_{ij}^{(\tau)} = \max \left\{ \frac{AD_i^{(\tau)}}{LD_j^{(\tau)}}, \frac{LD_j^{(\tau)}}{AD_i^{(\tau)}} \right\}$, $\forall (i, j) \in \mu^{(\tau)}$.

(C) : $\tau \leftarrow \tau + 1$.

end

end

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