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Discussion Paper 09/2013

Optimal Elimination Contest
by

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Optimal Elimination Contest

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Abstract

We consider multi-stage elimination contests, where agents' efforts at different stages generate some output for the organizers. Depending on the output function we characterize the optimal prize structure of the tournament and show that it is almost efficient. We have found that in some cases quite a strange structure turns out to be optimal, under which prizes for agents are smaller at the later stages than at the earlier ones. Sufficient conditions for optimality of such structures are provided for the case of a separable output function. Next we consider the modification, when the designer can specify a winning function. We provide sufficient conditions for optimality of a winning function and show that it can be found in the class of Tullock functions. This function does not depend on the output function. There is always an efficient equilibrium, under which the designer is able to extract the whole surplus from the agents and the corresponding optimal prize structure is always non-decreasing.

Keywords: Tullock contests; multiple-stage tournament; optimal structure; negative prizes.

JEL Classification Codes: C72, D86, J31, L2.

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1 Introduction

Many real life interactions among different agents, such as elections, implementation of innovations, promotion tournaments, or sports, can be very well described and analyzed by use of a contest model. Most contests involve multiple stages where the number of agents compete at successive stages until the winner is finally determined. The most prominent among the models which share this feature is the sequential elimination contest, which is commonly known from playoff rounds in sport competitions. It is also a good description of many corporate tournaments where employees from low hierarchy levels compete for promotions to higher hierarchy levels. This contest structure can also be encountered in politics, where candidates compete in localized contests and where the winners then often compete against each other on higher levels. R&D could be also described by such tournaments, where the firm with the most efficient technology wins the market.

In this paper we look for the optimal prize structure of an elimination contest, which maximizes the profit of the organizers. Our main assumption that differs this paper from the existing literature is that in the majority of papers the objective of the designer is to find a structure of prizes in the contest, which maximizes agents' efforts during the tournament (usually, either the average level, or the effort in the final). The prize pool is usually assumed to be a fixed amount of money, which can be distributed to the agents according to their performance. In our framework the designer does not have initially any fixed objective, such as, for example, maximization of the total effort of agents. Instead, agents' efforts at different stages produce the output according to some output function, which defines how exactly the organizers value various combinations of efforts at different stages. The prize pool is not fixed, rather the designers can pay to the contestants any amount of money they want. The designer's problem is to design the prize structure, which maximizes the organizers' profit, that is the difference between the value of the produced output and the value of the prizes, distributed to the agents. Competitor's performance incentives at any stage are set by the continuation value. Each player is guaranteed at least the loser's prize at each stage, but a win gives him the opportunity to continue participation in the contest. We find that depending on the output function several types of prize structures might be optimal. We give an example in which the output function depends only on one parameter and for various values of this parameter five types of prize structures might be optimal: 1) Increasing concave prize structure, 2) Increasing linear prize structure, 3) Increasing convex prize structure, 4) Winner-take-all structure, 5) Decreasing prize structure with the big prize to the winner. The last case with the negative prize differences at all stages, except the last one is the most striking result. It turns out that if the designer values efforts at each later stage much more

than at the previous stage, the difference between the prizes could be negative. We refer to this structure as a "trap structure". We are able to formulate sufficient conditions for prize structures to be increasing for a separable output function. We show that the optimal prize structure is also almost socially optimal, namely it provides the first best level of effort at all the stages except the first one.

Then we assume that the designer can optimize not only with respect to the prize structure, but also may choose the probability of winning function. It means that it can specify probability of winning depending on realized efforts in any way he wants. We provide sufficient conditions for the probability of winning function to be optimal. The important result here is that it is always possible to find such a function in the class of Tullock functions. It always generates the efficient efforts and the "trap structure" is never optimal here.

The paper is structured in the following way. In the next section we discuss the existing literature by this theme. Then, we describe our main model and find the optimal prize structure. Finally, we allow the designer to choose a probability of winning function. The last section concludes and summarizes the whole paper.

2 Related Literature

A number of literature exist on different tournaments, for example, on lottery contests, R&D, patents, innovation implementation. This literature is surveyed by Konrad (2007) in detail. Our attention is focused on the contributions, which are most relevant to our work.

The classic work which describes simple tournaments and their efficiency is Lazear and Rosen (1981). They compare rank order wage schemes to wages based on individual output and find that, for risk-neutral agents, both of them allocate resources in the efficient way. They consider the case of the simple simultaneous one stage tournament. In their setup the efficiency result requires only risk neutrality. In our paper the efficiency is generally lost because of the dynamic structure of the tournament.

The playoff or elimination tournaments considered in our paper were analyzed for the first time in Lazear and Rosen (1986). The tournament structure considered in this paper is similar to the those one. They have described double elimination tournament with multiple stages with a fixed prize pool. They have found the optimal payoff structure under the objective to maximize the same constant level of effort of all agents through the tournament. They have found that the optimal incentive scheme is such that the difference between prizes at all stages is the same. That is, the optimal prize growth is linear. It is true for all stages except the last one. Since there will be no continuation after this the increase in the prize between the first and the second places should have a significant jump compared with the

growth in prizes between semifinal and final, quarterfinal and semifinal, and so on. It is also never optimal to pay losers in the first round. It just lowers their incentives to provide an effort. The results of this work are the one particular case of our study when the output function is just a sum of all efforts.

There are also several papers that study the efficiency of contests. Chung (1996) discusses a rent seeking model where productive efforts increase the single rent which agents compete for. He considers a winner-take-all contests with linear costs. For this setup, he shows that the equilibrium efforts are always greater than socially optimal. It is not the case of our paper because in our model effort levels do not increase prizes directly. Our results suggest generally underprovision of efforts compared to the optimal level at the first stage.

There are several papers where simultaneous one-stage contests are considered. Moldovanu and Sela (2001) characterize the optimal prize structures in simultaneous tournaments. They analyze an exogenously given, fixed budget for prizes. They show that for convex cost functions it is optimal to give positive prizes not only to the winner, although for concave and linear cost functions winner-take-all structure is indeed optimal. In a similar framework, Shaffer (2006) compares payoffs and efforts arising from exogenously given prizes with those from effort dependent prizes. Cohen, Kaplan and Sela (2004) characterize the optimal effort dependent prize structure in the one-stage all-pay auction setup. Depending on the designer's objective, they find that the optimal reward may decrease or increase in the players' efforts.

Schweitzer and Segev (2012) analyze simultaneous Tullock contests. They give necessary and sufficient conditions under which there is an equilibrium under winner-take-all structure and show that if it exists, then it is unique. If it does not exist, then they construct a prize structure with several prizes, under which an equilibrium exists. In our paper an equilibrium is not necessary unique. Namely it turns out to be unique in the main model with the fixed Tullock function, but in the modification with a free choice of winning function there are two equilibria, with zero levels of efforts being applied in the second one. There is no contradiction between different results of the papers since the analyzed models are different. Schweitzer and Segev analyze prizes, awarded for different places in the one-stage tournament and we analyze prizes awarded to the winners at different stages. Thus, the optimization problems are substantially different. Essentially we optimize through a number of winner-take-all prizes in a sequence of tournaments. The equilibrium at each stage exists due to our assumptions (convex costs, a concave output function, the Tullock function of the form $P(x_i, \bar{x}_{-i}) = \frac{x_i}{x_i + \sum_{j \neq i} x_j}$)

Fu and Lu (2012) consider the question of the optimal design of the tournament. For a fixed prize pool, linear costs function they show that when the objective is to maximize

the total effort of all contestants the optimal structure is such that at each state only one contestant is eliminated until the finale and the single winner takes over the entire prize. However, in our paper we show the "winner-take-all" structure is almost never optimal if the cost function is not linear, but convex with the property $C'(0) = 0$.

There are also several papers, in which more general setting with heterogeneous agents is considered. See, for example, Gurtler and Krakel (2010), Parreiras and Rubinchik (2010), Ryvkin (2007).

3 Main Model

The tournament begins with m^N players and proceeds sequentially through N stages. At each stage all participants, who survived up to this stage are randomly selected into groups. In each group they compete and only the winner moves to the next stage. This situation is well known for $m = 2$. According to these schemes the playoffs of football cups are played or all tennis tournaments are conducted. Winners move to the next round and losers are eliminated from the subsequent play. In the next round other groups are randomly drawn, and again half of the participants are excluded from the further competition. In a general case there are m^{N+1-n} agents at the stage n , who are distributed to $m^{N+1-n}/m = m^{N-n}$ groups. The top prize W_w is awarded to the winner of the final match, who has won N matches overall. Other finalists, who are losers at the final stage are awarded the second place and get the prize W_N for having won $N - 1$ stages and lost the last one. At earlier stages all participants, eliminated at the particular stage, get equal prizes. We denote W_n as the prize for the losers at the stage n .

In our main model equal players are considered. It is a mechanism design problem with complete information. They can apply different levels of effort. Probability of moving to the next round $P(x_i, \bar{x}_{-i})$ is the function of agent's level of effort x_i and the vector of effort levels of his competitors \bar{x}_{-i} . It is assumed to be symmetric, increasing in x_i and decreasing in \bar{x}_{-i} . For this section the probability of winning a match at some particular stage is assumed to be the following Tullock function:

$$P(x_i, \bar{x}_{-i}) = \frac{x_i}{x_i + \sum_{j \neq i} x_j}$$

In real life it means that there are some observable characteristics which are connected with levels of efforts and show whose effort is higher, but not perfectly.

Here we come to the crucial part of our model, namely, the output function. Let $\Pi(\bar{x}_1, \dots, \bar{x}_N)$ be some concave, increasing in each component output function of efforts.

We especially emphasize that it is not necessary separable with respect to effort levels at different stages. In existing literature only specific examples are considered. For example, the designer wants to maximize the level of effort at the final or the average level of effort through the whole tournament. In this paper we characterize optimal prize structure for general form of output function. So, our results are generalization of the existing literature in this sense.

When the prize structure is announced agents make their decision about efforts, which obviously depends on announced prize structure. The profit of the designer if he has announced prize structure $\Omega = \{W_1, \dots, W_N, W_w\}$ is the difference between output produced by the agents and the total sum of prizes.

$$\Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega)) - \left(\sum_n (m^{N+1-n} - m^{N-n})W_n + W_w \right)$$

This expression can be written through differences between prizes as

$$\Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega)) - (m^N - m^{N-1})W_1 - \left(\sum_n m^{N-n} \Delta W_n \right)$$

where $\Delta W_n = W_{n+1} - W_n$.

At each stage every player who survived up to this moment decides which level of effort to apply. However, effort is not free. Applying the level of effort x is accompanied with the costs equal to $C(x) = x^\gamma$, $\gamma > 1$. A player's decision of how much effort to spend in any match depends on a cost—benefit analysis. Bigger efforts increase the probability of winning this match and moving to the next stage, but involves higher costs.

Suppose some prize structure Ω is announced. Let denote V_n to be the value of participation in the tournament for every player at the stage n . Since all players are assumed to be equal this would be the same amount for all players. This value consists of two components. The first is the prize earned if the match is lost and the player is eliminated. This event occurs with the probability $1 - P(x_i, \bar{x}_{-i})$. The other is the value of moving to the next stage if the match is won. The probability of this is $P(x_i, \bar{x}_{-i})$. Anyway, he also incurs costs of effort $C(x_i)$. So, we can write the agent's problem recursively as the following:

$$V_n = \max_{x_i} \left(1 - \frac{x_i}{x_i + \sum_{j \neq i} x_j} \right) W_n + \frac{x_i}{x_i + \sum_{j \neq i} x_j} V_{n+1} - C(x_i)$$

This finishes description of the model and we continue with the solution.

4 Solution and Main Results

4.1 Solution to agents' problem

As we have already said, agents take their levels of efforts after observing the proposed prize structure.

We start with the solution to the agents' problem:

$$V_n = \max_{x_i} \left(1 - \frac{x_i}{x_i + \sum_{j \neq i} x_j}\right) W_n + \frac{x_i}{x_i + \sum_{j \neq i} x_j} V_{n+1} - C(x_i)$$

Here we will give only a sketch of how we get our results. Complete proofs are delegated to the Appendix.

In the unique symmetric equilibrium we can find the equilibrium level of effort e_n at the stage n from the following equation:

$$e_n : \frac{m^2}{m-1} \gamma C(e_n) = V_{n+1} - W_n$$

Then the value function can be rewritten as the following:

$$V_n = W_n + \varkappa \Delta W_n + \varkappa^2 \Delta W_{n+1} + \dots + \varkappa^{N-n+1} \Delta W_N$$

where $\varkappa = \frac{(\gamma-1)m+1}{\gamma m^2}$

Then we can formulate the following Lemma:

Lemma 1 *The equilibrium level of effort at the stage n given the prize structure Ω :*

1. *Does not depend on the prize at that stage and at all earlier stages*
2. *Increases with an increase in the prize difference at that stage and at all later stages with decreasing weights:*

$$\frac{m^2}{m-1} \gamma C(e_n) = \Delta W_n + \varkappa \Delta W_{n+1} + \varkappa^2 \Delta W_{n+2} + \dots + \varkappa^{N-n} \Delta W_N$$

where $\varkappa = \frac{(\gamma-1)m+1}{\gamma m^2}$

Proof. see Appendix ■

We can see from Lemma 1 that the larger the difference between prizes is, the larger is the level of effort, which agents want to apply. At the same time the differences between prizes at the close stages weight more than those at the later stages.

Now this information can be used to find the optimal prize scheme.

4.2 Designer's problem

The organizers know how agents will choose their levels of efforts (from Lemma 1). So, organizers solve their profit maximization problem:

$$\Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega)) - (m^N - m^{N-1})W_1 - \left(\sum_n m^{N-n} \Delta W_n\right) \implies \max_{\Omega}$$

Solving this problem, we can formulate the following Proposition:

Proposition 1 *The optimal prize structure implements the efficient level of effort in the equilibrium at all the stages except the first one¹:*

$$\frac{\partial \Pi(\cdot)}{\partial e_n} = C'(e_n)$$

At the first stage:

$$\frac{\partial \Pi(\cdot)}{\partial e_1} = \frac{m}{m-1} \gamma C'(e_1)$$

Proof. see Appendix ■

This result states that the optimal prize structure would be the one, which implements the efficient level of efforts at all the stages, except the first one (if $\frac{m}{m-1} \gamma \neq 1$). The intuition is that the prize difference at the particular stage affects the level of effort not only at that stage, but also at all earlier stages. Since there are no stages in the tournament before the first stage, all inefficiency is concentrated there. We can also see that if the designer does not value efforts at some stage at all, then the equilibrium level of effort would be equal to zero at that stage.

Equations from Lemma 1 could also be rewritten as

$$\Delta W_n = \frac{\gamma m^2}{m-1} (C(e_n^*) - \alpha C(e_{n+1}^*)), \quad n \neq N$$

$$\Delta W_N = \frac{\gamma m^2}{m-1} C(e_N^*),$$

From Proposition 1 we know the levels of efforts, which would be implemented. So, we can formulate the following Proposition:

¹Here we use for simplicity the notation $\frac{\partial \Pi(\cdot)}{\partial e_n}$ to denote the derivative of the output function with respect to the effort of any agent i at the stage n , estimated in the equilibrium, that is $\frac{\partial \Pi(e_1 \dots e_1, e_2 \dots e_2, \dots, e_N \dots e_N)}{\partial x_{i,n}}$. The fact that the efforts of all the agents at the particular stage are the same in the equilibrium allows us to do that.

Proposition 2 *The optimal prize structure satisfies the following:*

$$\begin{aligned}\Delta W_n &= \frac{\gamma m^2}{m-1}(C(e_n^*) - \varkappa C(e_{n+1}^*)), \quad n \neq N \\ \Delta W_N &= \frac{\gamma m^2}{m-1}C(e_N^*),\end{aligned}$$

where $\{e_n^*\}$ satisfies:

$$\begin{aligned}\frac{\partial \Pi(\cdot)}{\partial e_n^*} &= C'(e_n^*), \quad n \neq 1 \\ \frac{\partial \Pi(\cdot)}{\partial e_1^*} &= \frac{m}{m-1}\gamma C'(e_1^*)\end{aligned}$$

Proof. see Appendix ■

We can make some important observations, based on this result.

First of all we can see that if the difference between equilibrium levels of efforts is not too large, then the prize structure is increasing because $\varkappa < 1$. Next, we can see that the difference between the prizes for the winner and for the other finalists is always nonnegative. However, and this is the most striking result, prize difference between other stages could be negative. At first view, this contradicts the normal sense. We could say that if an agent moves further in the tournament and his prize becomes smaller it seems a bit strange. But this is not completely true. The last prize difference would be necessary positive and the prize for the winner would be so high that an agent would benefit from going further and being closer to it. We can check this. The prize for the winner of the tournament can be rewritten as a sum of all increases in prizes:

$$\begin{aligned}\sum_n \Delta W_n &= \frac{\gamma m^2}{m-1} \left[C(e_N^*) + \sum_{n=1}^{N-1} (C(e_n^*) - \varkappa C(e_{n+1}^*)) \right] = \\ &= \frac{\gamma m^2}{m-1} \left[\sum_{n=2}^N (1 - \varkappa) C(e_n^*) + C(e_1^*) \right] > 0\end{aligned}$$

So, the winner of the tournament always gets the positive prize.

Moreover, we can calculate the sum of all prizes which is also positive:

$$\begin{aligned}
\sum_n m^{N-n} \Delta W_n &= \frac{\gamma m^2}{m-1} \left[C(e_N^*) + \sum_{n=1}^{N-1} m^{N-n} (C(e_n^*) - \varkappa C(e_{n+1}^*)) \right] = \\
&= \frac{\gamma m^2}{m-1} \left[\sum_{n=2}^N m^{N-n} (1 - m\varkappa) C(e_N^*) + m^{N-1} C(e_1^*) \right] > 0
\end{aligned}$$

since $\varkappa = \frac{(\gamma-1)m+1}{\gamma m^2} < \frac{\gamma m}{\gamma m^2} = \frac{1}{m}$

Let's finally verify that each agent indeed wants to participate in the tournament.

His value of participation is the following:

$$\begin{aligned}
V_1 &= W_1 + \sum_n \varkappa^n \Delta W_n = W_1 + \frac{\gamma m^2}{m-1} \left[\sum_{n=1}^{N-1} \varkappa^n (C(e_n^*) - \varkappa C(e_{n+1}^*)) + \varkappa^N C(e_N^*) \right] = \\
&= W_1 + \frac{\gamma m^2}{m-1} \varkappa C(e_1^*)
\end{aligned}$$

Suppose $W_1 = 0$. Since costs are always nonnegative, value of participation is always nonnegative. Choosing $W_1 = -\frac{\gamma m^2}{m-1} \varkappa C(e_1^*)$ will enable the designer to put agents on their participation constraint.

4.3 An Example

In this section we will provide a simple, but rich enough example to illustrate different optimal structures.

Suppose that the output function has the following form:

$$\Pi(\cdot) = \sum_n (a^n \sum_i x_{i,n} / m^{N+1-n}), a > 0.$$

Costs are quadratic:

$$C(x) = x^2$$

Thus, the designer cares about the average level of effort at different stages. The parameter a determines the weights the designer attaches to different stages.

Then we can apply Proposition 2 and get the following optimal prize structure:

$$\Delta W_n = \frac{m^2}{m-1} \frac{a^{2n}}{4} (1 - \varkappa a^2), n \neq 1, N$$

Now we consider several cases for the parameter a :

1. $a < 1$. In this case the designer values the later stages less than the earlier stages. The optimal prize structure is increasing ($1 - \varkappa a^2 > 0$) and concave (a^{2n} decreases with larger values on n , hence ΔW_n falls down).
2. $a = 1$. All stages are equally important. The optimal prize structure is linearly increasing ($\Delta W_n = const > 0$). This result is the case of (Rosen, 1986), where the designer maximizes the same average level of effort through the tournament.
3. $1 < a < \sqrt{\frac{1}{\varkappa}}$. The designer values the later stages more, but not drastically. The optimal prize structure is increasing ($1 - \varkappa a^2 > 0$) and convex (a^{2n} increases with larger values on n , hence ΔW_n increases).
4. $a = \sqrt{\frac{1}{\varkappa}}$. The optimal prize structure is winner-take-all (all $\Delta W_n = 0$ except the first and the last stages) . There are also several papers, where winner-take-all structure turns out to be optimal in other settings (Krishna, Morgan (1998), Moldovanu, Sela (2001)).
5. $a > \sqrt{\frac{1}{\varkappa}}$. The designer values the later stages drastically more than the earlier ones. $1 - \varkappa a^2 < 0$, hence $\Delta W_n < 0$ for all intermediate stages of the tournament. So, the optimal prize structure is decreasing, with large final prize being awarded for the winner. This is an example of a "trap structure". We do not know other papers, where such kinds of structures turn out to be optimal. Let us speak a bit more about this case because it seems to be the most interesting. When the organizer values each subsequent stage much more than the previous one, his valuation of the final is so high that he tries to make the gap between the prize for the winner and prizes for the other finalists as high, as possible. So, using negative prize differences and, hence, negative prizes he puts the agents in the situation where they are punished more if they go closer to the final and lose there. At the later stages stakes become extremely large, which enforces very high levels of efforts, as needed by the principal.

4.4 Sufficient conditions for nonnegativity of prizes

In the previous discussion we have shown that sometimes the optimal prize structure induces negative prize differences at some stages. Here we consider the case of separable output function and provide sufficient conditions, which guarantee that the prize differences at all stages are nonnegative.

We assume here that the output function is separable with respect to different stages, i.e. $\Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega)) = \sum_n \Pi_n(\bar{x}_n(\Omega))$.² If $\frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} \geq \frac{\partial \Pi_{n+1}(\bar{x}_{n+1}(\Omega))}{\partial e_{n+1}}$ then the equilibrium level of effort falls³ during the contest and obviously $\frac{\gamma m^2}{m-1}(C(e_n^*) - \varkappa C(e_{n+1}^*)) \geq 0$ and all the prize differences are nonnegative. Thus, we need to consider the case, when $\frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} < \frac{\partial \Pi_{n+1}(\bar{x}_{n+1}(\Omega))}{\partial e_{n+1}}$ (that is the organizers value the effort at later stages more than at earlier ones) and the equilibrium level of effort increases.

The main result here is that if the valuations of efforts do not increase too much from each stage to the next stage and at the same time the output function is concave enough, then the optimal prize structure is always nondecreasing. The inverted conditions together serve as sufficient conditions for "trap" structures. The exact statement is the following⁴:

Proposition 3 1. If $\frac{\partial \Pi_{n+1}(\bar{x}_{n+1}(\Omega))}{\partial e_{n+1}} \Big|_{\bar{x}_{n+1}(\Omega)=(e, \dots, e)} \leq \frac{1}{\varkappa} \frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} \Big|_{\bar{x}_n(\Omega)=(e, \dots, e)}$ for all $n > 1$ and $e \frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} \Big|_{\bar{x}_n(\Omega)=(e, \dots, e)}$ is nonincreasing for all e and $n > 1$, then the optimal prize structure is nondecreasing at all stages after the first one. If the inequality is strict and $e \frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} \Big|_{\bar{x}_n(\Omega)=(e, \dots, e)}$ is strictly increasing then the optimal prize structure is also strictly increasing.

2. If $\frac{\partial \Pi_{n+1}(\bar{x}_{n+1}(\Omega))}{\partial e_{n+1}} \Big|_{\bar{x}_{n+1}(\Omega)=(e, \dots, e)} \geq \frac{1}{\varkappa} \frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} \Big|_{\bar{x}_n(\Omega)=(e, \dots, e)}$ for all $n > 1$ and $e \frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} \Big|_{\bar{x}_n(\Omega)=(e, \dots, e)}$ is nondecreasing for all e and $n > 1$, then the optimal prize structure is nonincreasing at all stages, except the final. If the inequality is strict and $e \frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n} \Big|_{\bar{x}_n(\Omega)=(e, \dots, e)}$ is strictly decreasing then the optimal prize structure is also strictly decreasing.

However, these conditions are only sufficient, not necessary conditions.⁵

5 The optimal probability of winning function

5.1 Sufficient conditions for optimality of the winning function

Now we assume that the designer of the tournament is free to choose not only a prize structure, but also can implement any probability of winning at each stage that he wants.

²In many real life applications it is a reasonable assumption. For example, it is natural to assume that for sports events the revenues from selling tickets on semifinal matches do not depend on teams efforts in quarterfinals.

³As $\frac{\partial \Pi_n(\bar{x}_n(\Omega))}{\partial e_n^*} = C'(e_n^*)$, a decrease of the output function derivative will lead to a decrease of the equilibrium level of effort.

⁴Notations for derivatives are the same as earlier.

⁵See our previous example

In the previous part of the paper probability of winning function was assumed to be the following Tullock function: $P(x_i, \bar{x}_{-i}) = \frac{x_i}{x_i + \sum_{j \neq i} x_j}$. So, in this section we drop this assumption and ask the question, if the designer could choose any $P(x_i, \bar{x}_{-i})$, which one is better for him again in the sense of maximizing the profit. We also drop our previous assumption about the particular form of the cost function. Moreover, we even allow here the cost function to be concave.

The agent's problem is the following:

$$V_n = \max_{x_i} (1 - P(x_i, \bar{x}_{-i}))W_n + P(x_i, \bar{x}_{-i})V_{n+1} - C(x_i)$$

We can write F.O.C. assuming that maximum is interior. We will show later that the optimal winning function satisfies this.

$$P'_{x_i}(x_i, \bar{x}_{-i})(V_{n+1} - W_n) = C'(x_i)$$

Now we choose any probability function that satisfies the following condition:

$$\frac{P'_{x_i}(x_i, \bar{e}) |_{x_i=e}}{P(e, \bar{e})} = k \frac{C'(e)}{C(e)}, k \leq 1$$

Then in symmetric equilibrium we can write:

$$kP(e, \bar{e})(V_{n+1} - W_n) = C(e)$$

In our model from the previous chapter:

$$P(x_i, \bar{e}) = \frac{x_i}{x_i + (m-1)e}, \frac{C'(e)}{C(e)} = \frac{\gamma}{e}.$$

Hence,

$$\frac{P'_{x_i}(x_i, \bar{e}) |_{x_i=e}}{P(e, \bar{e})} = \frac{(m-1)e}{(x_i + (m-1)e)x_i} |_{x_i=e} = \frac{m-1}{me} = \frac{m-1}{m\gamma} \frac{\gamma}{e} = k \frac{C'(e)}{C(e)}$$

In our basic model $k = \frac{m-1}{m\gamma}$.

Then, using the similar arguments as in the previous section we can write the following system, which defines our optimal prize structure and equilibrium after choosing a winning function with the property $\frac{P'_{x_i}(x_i, \bar{e}) |_{x_i=e}}{P(e, \bar{e})} = k \frac{C'(e)}{C(e)}, k \leq 1$:

$$\frac{\partial \Pi(\cdot)}{\partial e_n^*} = C'(e_n^*), n \neq 1$$

$$\frac{\partial \Pi(\cdot)}{\partial e_1^*} = \frac{1}{k} C'(e_1^*)$$

$$\Delta W_n = \frac{m}{k} [C(e_n^*) - \frac{k-1}{km} C(e_{n+1}^*)], n \neq 1$$

$$\Delta W_N = \frac{m}{k} C(e_N^*)$$

Now let's put $k = 1$. Then this system has the following form:

$$\frac{\partial \Pi(\cdot)}{\partial e_n^*} = C'(e_n^*)$$

$$\Delta W_n = m C(e_n^*)$$

We can see that now all levels at all stages are efficient. The designer is still able to extract the whole surplus from the agents and, thus, obtains the maximal possible profit.

Hence, any probability function that satisfies $\frac{P'_{x_i}(x_i, \bar{e})|_{x_i=e}}{P(e, \bar{e})} = k \frac{C'(e)}{C(e)}$ with $k = 1$ would be the optimal winning function. Notice that if $k \neq 1$, then all the efforts except the effort at the first stage are efficient and, hence, "quasi-optimality" is achieved (see the previous section).

We can also notice that the optimal probability of winning function does not depend on the output function $\Pi(\cdot)$. So, no matter how the designer values efforts of agents, he should choose the same winning function at all stages.

5.2 The optimal Tullock function

In the previous subsection we have characterized sufficient conditions for the winning function to be optimal. But the existence of such probabilistic functions is not stated yet. In this subsection we will prove that the answer is affirmative. Namely, for any cost function $C(x)$ we can find a winning function from the class of Tullock functions $P(x_i, \bar{x}_{-i}) = \frac{f(x_i)}{f(x_i) + \sum_{j \neq i} f(x_j)}$, which satisfies sufficient conditions for optimality. Hence, it must be optimal.

Lemma 2 For any function $C(x)$ take $f(x) = C^{\frac{m}{m-1}}(x)$. Then $P(x_i, \bar{x}_{-i}) = \frac{f(x_i)}{f(x_i) + \sum_{j \neq i} f(x_j)}$

satisfies the following condition: $\frac{P'_{x_i}(x_i, \bar{e})|_{x_i=e}}{P(e, \bar{e})} = \frac{C'(e)}{C(e)}$

Proof. see Appendix' ■

This Lemma basically shows how the optimal winning function is constructed. Thus, the following must be true:

Proposition 4 *The following structure of the elimination tournament is optimal:*

1. $P(x_i, \overline{x_{-i}}) = \frac{f(x_i)}{f(x_i) + \sum_{j \neq i} f(x_j)}$, where $f(x) = C^{\frac{m}{m-1}}(x)$

2. $W_1 = 0, \Delta W_n = mC(e_n^*)$,

where $e_n^* : \frac{\partial \Pi(\cdot)}{\partial e_n^*} = C'(e_n^*)$.

Proof. see Appendix ■

This is our main result of this part of the paper. We can compare it to Proposition 2 in the previous section. We can see that if the designer is able to choose the winning function, then the optimal prize structure is always non-decreasing. Thus, a "trap structure" would never be optimal in this case. The intuition behind this fact is that the optimal winning function actually makes agents indifferent between participation at each stage and choosing zero level of effort. Hence, at each stage $V_n = W_n$ (compare to the previous section). The whole dynamic structure of the tournament is in fact destroyed because the value of participation at each stage equals exactly to the prize at that stage. Therefore, for the agents this tournament is equivalent to participation in a sequence of independent one-stage tournaments. Then there is no surprise that the prize difference must be nonnegative at each stage because this prize difference is equivalent to the prize in that particular one-stage tournament.

6 Conclusion

In this paper we consider elimination contests, in which the purpose of the organizers is to maximize their profits. Efforts of agents at different stages of a tournament generate some output for the organizers according to some output function. Depending on the output function we characterize the optimal prize structure in the tournament that gives the highest profit for the designer. We show that the optimal prize structure is also "quasi-efficient". It means that it generates the socially optimal level of efforts at all the stages of the tournament, except the first one. We provide a simple example where different prize structures can be optimal. Some of these structures are characterized as optimal in the existing literature, but we have found that under some conditions quite strange structures can be optimal, which we have called "trap structures". Under this prize structure prizes for agents are smaller at the later stages than at the earlier ones. If the prize at the first stage is negative, it means that all prizes at all the later stages would also be negative, except the prize for the winner of the tournament. We have provided sufficient conditions for optimality of such structures.

Then we add one more opportunity for the designer to change the tournament structure. Namely, we have found the optimal probability of winning function, which defines probability of moving to the next stage given levels of effort of this agent and his opponents. We have proved that the optimal winning function can be found in the class of Tullock functions. This function does not depend on the output function. So, no matter how the designer values levels of efforts at different stages, it is optimal to choose the same winning function. The optimal winning function and the optimal prize structure will generate the socially efficient equilibrium, in which the designer is able to extract the whole surplus from the agents. If probability of winning is chosen in the optimal way, then the corresponding optimal prize structure is always non-decreasing. It means that the "trap structure" would never be optimal in this case.

7 References

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8 Appendix

Proof of Lemma 1

The value function of each participant at the stage n is the following:

$$V_n = \max_{x_i} \left(1 - \frac{x_i}{x_i + \sum_{j \neq i} x_j}\right) W_n + \frac{x_i}{x_i + \sum_{j \neq i} x_j} V_{n+1} - C(x_i)$$

Differentiating with respect to x_i we get the first-order condition:

F.O.C.:

$$\frac{\sum_{j \neq i} x_j}{\left(x_i + \sum_{j \neq i} x_j\right)^2} (V_{n+1} - W_n) = C'(x_i)$$

This would be indeed solution to the problem because the problem is concave.

The solution to this system is unique and symmetric. We denote it by e_n . Pugging it into the system we get:

$$V_{n+1} - W_n = \frac{m^2}{m-1} e_n C'(e_n) = \frac{m^2}{m-1} \gamma C(e_n)$$

Taking $C(e_n)$ from the previous equation and plugging it into the value function we get the following difference equation for V_n :

$$V_n = \frac{m-1}{m}W_n + \frac{1}{m}V_{n+1} - \frac{m-1}{m^2} \frac{V_{n+1} - W_n}{\gamma}$$

Writing it recursively we get:

$$\begin{aligned} V_{n+1} - W_n &= \Delta W_n + \frac{(\gamma-1)m+1}{\gamma m^2} \Delta W_{n+1} + \left(\frac{(\gamma-1)m+1}{\gamma m^2} \right)^2 \Delta W_{n+2} + \\ &\dots + \left(\frac{(\gamma-1)m+1}{\gamma m^2} \right)^{N-n+1} \Delta W_N \end{aligned}$$

Then,

$$\begin{aligned} \frac{m^2}{m-1} \gamma C'(e_n) &= V_{n+1} - W_n = \Delta W_n + \frac{(\gamma-1)m+1}{\gamma m^2} \Delta W_{n+1} + \\ &+ \left(\frac{(\gamma-1)m+1}{\gamma m^2} \right)^2 \Delta W_{n+2} + \dots + \left(\frac{(\gamma-1)m+1}{\gamma m^2} \right)^{N-n+1} \Delta W_N \end{aligned}$$

Obviously, the level of effort does not depend on prizes at the previous stages.

Proof of Proposition 1

From Lemma 1 the designer knows, how agents will react to the proposed prize structure.

His problem is to maximize his profit, which is given by the following:

$$\Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega)) - (m^N - m^{N-1})W_1 - \left(\sum_n m^{N-n} \Delta W_n \right) \implies \max_{\Omega}$$

Firstly we note that from Lemma 1:

$$\frac{\partial e_n}{\partial \Delta W_n} = \frac{m-1}{m^2 \gamma C'(e_n)}$$

Since $C'(0) = \infty$ then $\frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n} \frac{\partial e_n}{\Delta W_n}$ would be infinitely large at $\Delta W_n = 0$ if $\frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n}$ is nonzero. Hence, the solution would be interior and we can use the first-order conditions to find it.

$$\frac{\partial}{\partial \Delta W_n} : m^{N+1-n} \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n} \frac{\partial e_n}{\partial \Delta W_n} + m^{N+1-(n-1)} \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_{n-1}} \frac{\partial e_{n-1}}{\partial \Delta W_n} +$$

$$\begin{aligned}
& + \dots + m^N \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_1} \frac{\partial e_1}{\partial \Delta W_n} = m^{N-n} \\
& \frac{\partial}{\partial \Delta W_{n-1}} : m^{N+1-(n-1)} \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_{n-1}} \frac{\partial e_{n-1}}{\partial \Delta W_{n-1}} + \\
& \dots + m^N \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_1} \frac{\partial e_1}{\partial \Delta W_{n-1}} = m^{N-(n-1)}
\end{aligned}$$

As we have already noticed

$$\frac{\partial e_{n-1}}{\partial \Delta W_{n-1}} = \frac{m-1}{m^2 \gamma C'(e_{n-1})}$$

Using Lemma 1 we can get the following:

$$\frac{\partial e_{n-1}}{\partial \Delta W_n} = \frac{(\gamma-1)m+1}{\gamma m^2} \frac{m-1}{m^2 \gamma C'(e_{n-1})} = \frac{(\gamma-1)m+1}{\gamma m^2} \frac{\partial e_{n-1}}{\partial \Delta W_{n-1}}$$

Substituting the last expression into the F.O.C. we get:

$$m^{N+1-n} \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n} \frac{m-1}{m^2 \gamma C'(e_n)} + \frac{(\gamma-1)m+1}{\gamma m^2} m^{N-(n-1)} = m^{N-n}$$

Hence:

$$\frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n} = C'(e_n)$$

This holds for every $n \neq 1$.

For $n = 1$ the following holds:

$$m^N \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_1} \frac{\partial e_1}{\partial \Delta W_1} = m^{N-1}$$

Hence,

$$\frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_1} = \frac{m}{m-1} \gamma C'(e_1)$$

Proof of Proposition 2

The expressions for the equilibrium levels of efforts follow from Proposition 1.

Then we can express ΔW_n using the expression for two consequent levels of efforts from Lemma 1:

$$\Delta W_n = \frac{\gamma m^2}{m-1} (C(e_n^*) - \varkappa C(e_{n+1}^*)), \quad n \neq N$$

$$\Delta W_N = \frac{\gamma m^2}{m-1} C(e_N^*)$$

Proof of Proposition 3

From Proposition 2 for $n \neq 1, N$ and a separable output function:

$$\Delta W_n = \frac{\gamma m^2}{m-1} (C(e_n^*) - \varkappa C(e_{n+1}^*)), \quad n \neq N$$

$$\frac{\partial \Pi_n(\cdot)}{\partial e_n^*} = C'(e_n^*)$$

The second equality can be equivalently rewritten as

$$e_n^* \frac{\partial \Pi_n(\cdot)}{\partial e_n^*} = \gamma C(e_n^*)$$

Assume that properties in 1) hold in nonstrict sense. Then we have:

$$\begin{aligned} C(e_n^*) &= e_n^* \frac{\partial \Pi_n(e_n^*, \dots, e_n^*)}{\partial e_n^*} / \gamma \geq \varkappa e_n^* \frac{\partial \Pi_{n+1}(e_n^*, \dots, e_n^*)}{\partial e_{n+1}^*} / \gamma \geq \\ &\geq \varkappa e_{n+1}^* \frac{\partial \Pi_{n+1}(e_{n+1}^*, \dots, e_{n+1}^*)}{\partial e_{n+1}^*} / \gamma = \varkappa C(e_{n+1}^*). \end{aligned}$$

Thus, $\Delta W_n \geq 0$ for $n \neq 1, N$.

In the final $\Delta W_N = \frac{\gamma m^2}{m-1} C(e_N^*)$, which is always nonzero.

The proof for the case with strict inequalities in the part 1) and the whole part 2) is similar.

Proof of Lemma 2

$$\begin{aligned}
\frac{P'_{x_i}(x_i, \bar{e})|_{x_i=e}}{P(e, \bar{e})} &= \frac{f'(e) \sum_{j \neq i} f(e)}{(f(e) + \sum_{j \neq i} f(e))^2} / \left(\frac{f(e)}{f(e) + \sum_{j \neq i} f(e)} \right) = \\
&= \frac{m-1}{m} \frac{f'(e)}{f(e)} = \frac{m-1}{m} \frac{m}{m-1} \frac{C^{\frac{m}{m-1}-1}(e)}{C^{\frac{m}{m-1}}(e)} C'(e) = \frac{C'(e)}{C(e)}
\end{aligned}$$

Proof of Proposition 4

The agent's problem is

$$V_n = \max_{x_i} (1 - P(x_i, \bar{x}_{-i}))W_n + P(x_i, \bar{x}_{-i})V_{n+1} - C(x_i)$$

Let's assume that the solution is interior and show later that this is indeed the case. Then:

$$P'_{x_i}(x_i, \bar{x}_{-i})(V_{n+1} - W_n) = C'(x_i)$$

Suppose that there is the equilibrium level of effort e_n . Then it satisfies the following:

$$P'_{x_i}(e, \bar{e})(V_{n+1} - W_n) = C'(e)$$

From Lemma 1 if $P(x_i, \bar{x}_{-i}) = \frac{f(x_i)}{f(x_i) + \sum_{j \neq i} f(x_j)}$, where $f(x) = C^{\frac{m}{m-1}}(x)$, then $\frac{P'_{x_i}(x_i, \bar{e})|_{x_i=e}}{P(e, \bar{e})} = \frac{C'(e)}{C(e)}$

Thus,

$$P(e_n, \bar{e}_n)(V_{n+1} - W_n) = C(e_n)$$

Then due to symmetry of the winning function,

$$C(e_n) = \frac{1}{m}(V_{n+1} - W_n)$$

Then plugging this in the value function we obtain:

$$V_n = (1 - \frac{1}{m})W_n + \frac{1}{m}V_{n+1} - \frac{1}{m}(V_{n+1} - W_n) = W_n$$

Thus, agents' valuations of the participation in each stage will be exactly equal to the prize at that stage. Applying this equilibrium nonzero level effort gives the same utility as

applying the zero level of effort. All other levels of efforts will give lower utility levels. Thus, this interior solution is indeed an equilibrium.

Then,

$$C(e_n) = \frac{1}{m}(W_{n+1} - W_n) = \frac{1}{m}\Delta W_n$$

Compare it with the result from Lemma 1. The level of effort at the particular stage depends only on the prize increase at that stage.

Now the designer can optimize with respect to the prize structure:

$$\Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega)) - (m^N - m^{N-1})W_1 - \left(\sum_n m^{N-n}\Delta W_n\right) \implies \max_{\Omega}$$

The similar procedure to the one from Proposition 1 is even easier now:

F.O.C.:

$$m^{N+1-n} \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n} \frac{\partial e_n}{\partial \Delta W_n} = m^{N-n}$$

The response of the effort to the change in prize is now:

$$\frac{\partial e_n}{\partial \Delta W_n} = \frac{1}{m} \frac{1}{C'(e_n)}$$

We plug in the last equation into the F.O.C.

$$m^{N+1-n} \frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n} \frac{1}{m} \frac{1}{C'(e_n)} = m^{N-n}$$

Hence,

$$\frac{\partial \Pi(\bar{x}_1(\Omega), \dots, \bar{x}_N(\Omega))}{\partial e_n} = C'(e_n)$$

Thus, the equilibrium level of effort is efficient. At the same time the designer obtains the whole surplus from the agents by putting $W_1 = 0$. In this case $V_1 = W_1 = 0$.

Hence, the proposed structure is indeed optimal.