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AN AXIOMATIC PROOF OF MIRRLEES' FORMULA*

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Mirrlees' optimal income tax formula has never been proven rigorously, and it is hard to understand it in economic terms. We prove an analogous formula for an economy with finitely many persons. This is easy and allows a simple economic interpretation. Thereafter, Mirrlees' original formula is derived by means of a limit theorem. The analysis also clarifies the discussion between Revesz and Saez published as "Communications" below in this issue of PF/FP. It shows which formula is correct and how the methodological problems, in particular the circularity problem, can be dealt with.

I. INTRODUCTION

Mirrlees' (1971) paper on the optimal non-linear income tax appears to be one of the most important and seminal contributions to public finance ever published. However, his celebrated tax formula has never been demonstrated rigorously, and its economic meaning is not entirely clear. A similar lack of clarity characterizes the exchange between Revesz (2003) and Saez (2003) published as "Communications" in this issue of PF/FP which leaves the reader somewhat perplexed. In a sense, both authors have shown that the optimal marginal income tax rate is inversely related to the compensated elasticity of labor supply. Their derivations rely partly on heuristic arguments and lead to two distinct formulae. As the discussion between Revesz and Saez makes clear, the crucial difference between the two formulae is an indirect effect following a change in the marginal tax rate, which is present in Saez' but not in Revesz' model.

Revesz believes that the incorporation of such an indirect effect "can be questioned on logical grounds, but cannot be disproved categorically". On the other hand, Saez points out that the inclusion of the indirect effect makes his result equivalent to Mirrlees' (1971) formula. Saez even confesses, in his footnote 1, that he had first neglected the indirect effect himself, but had then added it in order to reconcile his formula with that of Mirrlees. Hence, the equivalence between Saez' and Mirrlees' results is unsurprising, and Revesz rightly points out that "if there is a problem with the inclusion of second order effects by Saez, the same might also apply to the non-explicit formulae of Mirrlees". Worse yet,

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control theory and calculus of variations, as used by Mirrlees, Revesz and Saez, have never pretended to be logically air-tight branches of mathematical optimization; both are more or less heuristic methods, too.

The present paper takes up Revesz' final suggestion "that further research is needed to improve mathematical rigor and identify the best optimal income tax formula." In order to do so, we use an innovative approach which consists of two steps. Firstly, we derive the optimal income tax formula for a finite economy. Secondly, the corresponding formula for the continuum economy is established by means of a limit theorem. This approach, which also yields an elasticity formula as an immediate side-product, has several advantages over the traditional approach: It is very much easier to understand and requires far less calculations; it gives definite answers to the questions raised since it relies on axiomatic methods; and it facilitates a simple economic interpretation of the optimal income tax formula.

II. THE FINITE OPTIMAL TAX FORMULA

In the finite economy, gross wage rates (or skill levels) are given as $0 < n^0 < n^1 < \dots < n^H$ for some positive number H . The probability mass of people with wage rate n^h is $f^h > 0$. Every person with wage rate n^h (person h , for short) gets a pair (c^h, z^h) of consumption c^h and gross income z^h . The difference $z^h - c^h$ represents the tax payment. Individual utility $u(c^h, \ell^h)$ depends on consumption and working time $\ell^h = z^h/n^h$, and the utility function is twice continuously differentiable, strictly monotonic and strictly concave; leisure is non-inferior. Differentiating implicitly the equation $u(c^h, z^h/n) = \text{const.}$ one obtains the marginal rate of substitution of a person with an arbitrary wage rate n as:

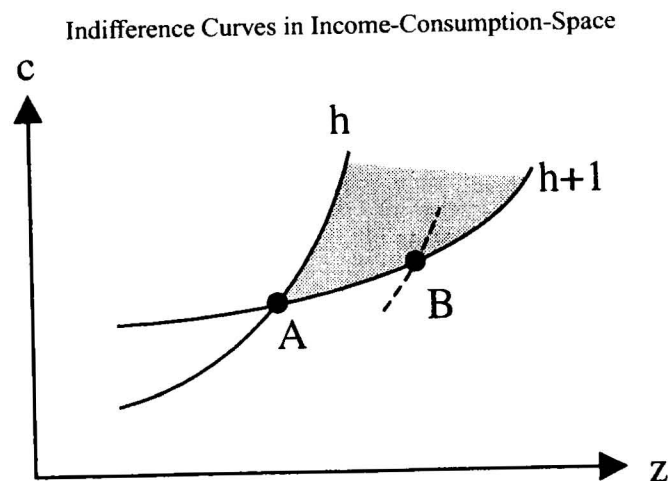
$$(1) \quad mrs^h(n) = - \frac{u_\ell(c^h, z^h/n)}{u_c(c^h, z^h/n)n}$$

The central premise of the theory of optimal income taxation, called agent monotonicity by Seade (1982), requires that the marginal rate of substitution at any given point in income-consumption-space decreases in n :

$$(2) \quad \Delta mrs^h = mrs^h(n^h) - mrs^h(n^{h+1}) > 0.$$

This is equivalent to assuming that the uncompensated elasticity of labor supply with respect to the net wage rate exceeds minus one. It is also equivalent to saying that whenever anyone is indifferent between two distinct pairs of consumption and income, each person with a lower wage rate strictly prefers the pair containing less consumption and income, and vice versa. In FIGURE 1, person $h + 1$ is indifferent between A and B, and h prefers A strictly.

FIGURE 1



The government maximizes expected utility of a person behind a "veil of ignorance", i.e. a person that does not know his own future wage rate. Total tax revenue must suffice to finance an exogenous amount R of public spending (resource constraint), and it must be impossible for each person k to make himself better off by mimicking some other person h (self-selection constraints). Hence, a second-best allocation solves

$$(3) \quad \begin{aligned} & \max_{(c^h, \ell^h)_{h=0 \dots H}} \sum_{h=0}^H u(c^h, \ell^h) f^h \\ & \text{s.t. } \quad \text{i) } \sum_{h=0}^H (z^h - c^h) f^h \geq R, \\ & \quad \quad \text{ii) } u(c^k, \ell^k) \geq u(c^h, z^h/n^k) \text{ for all } k \text{ and } h. \end{aligned}$$

Using an axiomatic approach it can be shown (Homburg 2001) that any solution to this problem has three important features: Firstly $z^{h+1} \geq z^h$ for all $h < H$. This is obvious from FIGURE 1: With point A representing the pair (c^h, z^h) , point B must lie in the shaded region because otherwise either h would mimic $h + 1$, or vice versa. Secondly, the resource constraint holds as an equality. Thirdly, the downward adjacent self-selection constraints also hold as equalities, which means $u(c^{h+1}, \ell^{h+1}) = u(c^h, z^h/n^{h+1})$ for all $h < H$. According to this *chain property*, every person except the least productive must be indifferent between the pair of consumption and income intended for him and the pair intended for his left-hand neighbor. The chain property, illustrated by points A and B in FIGURE 1, is pretty clear because the government wishes to redistribute from top to bottom as much as possible. Together with $z^{h+1} \geq z^h$, the chain property

implies that *all* self-selection constraints are satisfied, so that the upward as well as the non-adjacent self-selection constraints can be neglected.

Let us now derive the optimal income tax, assuming, as both Revesz and Saez do, that there are neither corner solutions nor bunching, and that the tax function $T(z)$ is differentiable. Since non-satiation implies $c^h = z^h - T(z^h)$ at any individual optimum, person h maximizes utility $u(z^h - T(z^h), z^h/n^h)$ by selecting the best income. Differentiating with respect to income yields the first-order condition $u_c(c^h, \ell^h)(1 - T'(z^h)) + u_\ell(c^h, \ell^h)/n^h = 0$, and solving for the marginal tax rate, using (1), gives the tax rate as the familiar wedge between the slope of the resource constraint (unity) and the marginal rate of substitution:

$$(4) \quad T'(z^h) = 1 - mrs^h(n^h).$$

As has been argued above, problem (3) is equivalent to maximizing the social objective subject to H equality constraints, namely the resource constraint and the $H-1$ downward adjacent self-selection constraints. In the Appendix, the gradients of these constraints are shown to be linearly independent. Hence, we can make use of a well-known theorem (e.g. Bertsekas 1999, proposition 3.1.1) which states that, irrespective of the problem's convexity properties, there exist non-negative Lagrange multipliers p, μ^h associated with the constraints such that the usual first-order conditions are *necessary* for an optimum. Forming the Lagrangean

$$(5) \quad \mathcal{L} = \sum_{h=0}^H u(c^h, \ell^h) f^h + p \sum_{h=0}^H (z^h - c^h - R) f^h + \sum_{h=0}^{H-1} \mu^{h+1} (u(c^{h+1}, \ell^{h+1}) - u(c^h, z^h/n^{h+1})),$$

differentiating with respect to c^h and ℓ^h and rearranging terms yields

$$(6) \quad u_c(c^h, \ell^h)(f^h + \mu^h) = \mu^{h+1} u_c(c^h, z^h/n^{h+1}) + p f^h$$

$$(7) \quad \frac{u_\ell(c^h, \ell^h)}{n^h} (f^h + \mu^h) = \mu^{h+1} \frac{u_\ell(c^h, z^h/n^{h+1})}{n^{h+1}} - p f^h.$$

To shorten the notation, let $\hat{u}_c = u_c(c^h, z^h/n^{h+1})$. Adding the two equations, factoring out the marginal utilities of consumption and using (1) gives

$$(8) \quad u_c(c^h, \ell^h)(f^h + \mu^h)(1 - mrs^h(n^h)) = \mu^{h+1} \hat{u}_c(1 - mrs^h(n^{h+1}))$$

which, after substituting $u_c(c^h, \ell^h)(f^h + \mu^h) = \mu^{h+1} \hat{u}_c + p f^h$ from (6) can be written as

$$(9) \quad \mu^{h+1} \hat{u}_c(1 - mrs^h(n^h)) + p f^h(1 - mrs^h(n^h)) = \mu^{h+1} \hat{u}_c(1 - mrs^h(n^{h+1})).$$

Using $T'(z^h) = 1 - mrs^h(n^h)$ and $\Delta mrs^h = mrs^h(n^h) - mrs^h(n^{h+1})$, we already have the main result which, by necessity of the first-order conditions, characterizes every second-best optimum:

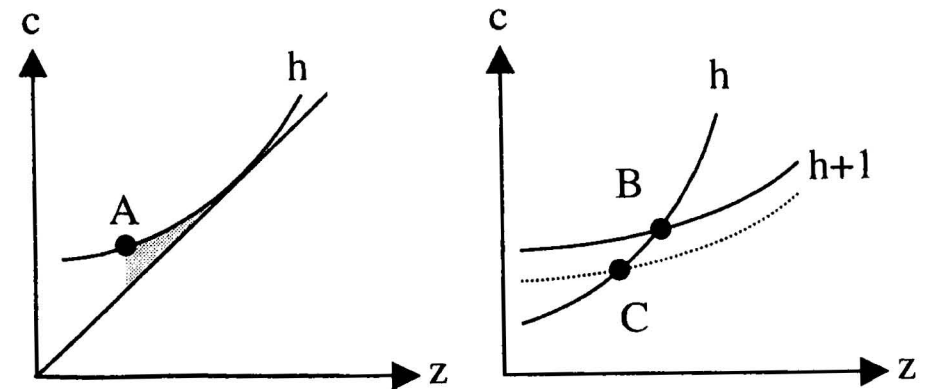
$$(10) \quad T'(z^h) = \frac{\mu^{h+1} \hat{u}_c}{p f^h} \Delta mrs^h.$$

The sign of the marginal tax rate, determined by Mirrlees in a complicated proof which also involved an important error, see Seade (1982), follows simply from the fact that the Lagrange multipliers are non-negative, whereas all other variables on the right-hand side are strictly positive. The multipliers (shadow prices) indicate changes in expected utility following a tax reduction (p) or an introduction of slack in one of the downward adjacent self-selection constraints (μ^h), respectively. However, the multiplier μ^{H+1} is not present in the Lagrangean, as person H has no right-hand neighbor who must be prevented from mimicking; hence (10) implies $T'(z^H) = 0$, the famous "no distortion at the top" result.

III. INTERPRETING THE FORMULA

No direct interpretation of the optimal income tax formula has been offered as yet. But there is one, and it is straightforward because formula (10) simply contrasts the cost and benefit of distortionary taxation. In the following argument we consider the consequences of slightly changing person h 's marginal tax rate, while keeping his utility constant.

FIGURE 2
Cost and Benefit of Distortionary Taxation



Positive tax rates have an efficiency cost: This is shown in the left-hand diagram in FIGURE 2. Point A corresponds to a positive marginal tax rate because the marginal rate

of substitution $mrs^h(n^h)$ falls short of the slope of the resource constraint. The shaded area is the familiar Harberger triangle, the deadweight loss from taxation. Starting from A and slightly changing z^h allows changing c^h by the same amount without violating the resource constraint. But person h only needs $mrs^h(n^h)$ as a compensation. Hence, $1 - mrs^h(n^h) = T'(z^h)$ is the marginal efficiency cost of distortionary taxation per individual, expressed in commodity terms. Multiplying by p , the shadow price of the resource constraint, transforms this into utility terms, and $T'(z^h)pf^h$ gives the aggregate marginal efficiency cost.

Positive tax rates have a redistributive benefit: Consider the right-hand diagram in FIGURE 2. Moving from B toward C, person h 's consumption can be reduced by $mrs^h(n^h)$ per income unit if his utility is to be kept constant. This reduction exceeds the quantity of consumption which could be taken from the potential mimicker $h + 1$, if the latter's utility were kept constant, by $mrs^h(n^h) - mrs^h(n^{h+1}) = \Delta mrs^h$. Thus, the mimicker's utility falls by $\hat{u}_c \Delta mrs^h$. The tax slackens the downward adjacent self-selection constraint, which is a good thing because more redistribution becomes possible. Since μ^{h+1} , the shadow price of the self-selection constraint, represents the social value of slackening this constraint by one unit, the marginal redistributive benefit from distortionary taxation equals $\mu^{h+1} \hat{u}_c \Delta mrs^h$.

An optimal income tax, of course, equates the marginal efficiency cost and the marginal redistributive benefit:

$$(11) \quad \begin{array}{l} T'(z^h)pf^h \\ \text{Marginal efficiency cost} \end{array} = \begin{array}{l} \mu^{h+1} \hat{u}_c \Delta mrs^h \\ \text{Marginal redistributive benefit} \end{array}$$

Dividing (11) by pf^h yields (10) which demonstrates that the optimal income tax formula has a direct economic meaning and can be constructed from simple economic reasoning. Indeed, the formula shows the famous equity-efficiency trade-off in its purest form. Positive marginal tax rates are a useful screening device that prevents the rich from mimicking the poor, facilitating more redistribution.

IV. RECOVERING MIRRLEES' ORIGINAL FORMULA

Now consider the continuum case where wage rates are distributed over some fixed interval $[n^{\min}, n^{\max}]$ according to a probability distribution function $F(n)$ with a continuous density $f(n) > 0$. An equidistant partition of this interval is a set $\{n^0, n^1, \dots, n^H\}$ such that $n^0 = n^{\min}, n^H = n^{\max}$ and, for all $h < H, n^{h+1} - n^h = \delta$, where $\delta = (n^{\max} - n^{\min})/H$. With probability masses $f^h = F(n^h) - F(n^{h-1})$ for $h > 0$ and $f^0 = F(n^0)$, each equidistant partition defines a finite economy. For instance, $H = 1$ defines an economy with two types of inhabitants, $H = 2$ an economy with three types, and so on. Letting $H = 1, 2, \dots$ gives a sequence of finite economies whose optimal tax schedules might look very different, but which are all characterized by formula (10). Applying the mean value theorem twice we write

$$(12) \quad \begin{aligned} f^h &= F(n^h) - F(n^{h-1}) = f\delta \quad \text{and} \\ \Delta mrs^h &= mrs^h(n^h) - mrs^h(n^{h+1}) = -\frac{dmrs}{dn}\delta, \end{aligned}$$

where f and $dmrs/dn$ must be evaluated in intervals $[n^h - \delta, n^h]$ and $[n^h, n^{h+1} + \delta]$, respectively. Substituting (12) into (10), δ cancels out, and when H grows without bound, the intervals collapse, \hat{u}_c approaches $u_c(c^h, \ell^h)$, and (10) becomes

$$(13) \quad T'(z) = \frac{-\mu u_c}{pf} \frac{dmrs}{dn}$$

This formula characterizes optimal tax rates in an economy with a dense skill distribution. It still states that an optimal income tax must equate the marginal deadweight loss $T'(z)pf$ with the marginal redistributive gain $-\mu u_c dmrs/dn$. To see the equivalence with Mirrlees' own expression, differentiate (1) with respect to n , keeping utility constant:

$$(14) \quad \frac{dmrs}{dn} = \frac{(u_{\ell\ell} - u_{\ell}/u_c u_{c\ell})\ell + u_{\ell}}{u_c n^2}$$

Mirrlees calls the nominator in (14) $-\Psi_y$, and our μ corresponds to his $-\Phi_n$. Substituting this and (14) into (13) yields $T'(z) = -\Phi_n \Psi_y / (pf n^2)$ which is a simple rearrangement of Mirrlees' formula (13) in the case of a linear resource constraint, where his w equals p . Later on, in his equation (27), Mirrlees writes $-\Phi_n$ in integrated form, but the emerging integrals, which are also used by Revesz and Saez, do not seem to yield additional economic insights.

V. RECOVERING THE ELASTICITY FORMULA

In his equations (23) and (24), Saez (2001) expresses the uncompensated elasticity (ζ^u) and the compensated elasticity (ζ^c) of income with respect to one minus the tax rate in terms of the utility function. He shows that

$$(15) \quad \frac{1 + \zeta^u}{\zeta^c} = \frac{(u_{\ell\ell} - u_{\ell}/u_c u_{c\ell})\ell + u_{\ell}}{u_{\ell}}$$

Comparing with (14) one recognizes that $dmrs/dn = (1 + \zeta^u)/\zeta^c u_{\ell}/(u_c n^2)$. Because $-u_{\ell}/(u_c n)$ equals $1 - T'(z)$ due to (4), we can write $-dmrs/dn = (1 + \zeta^u)/\zeta^c (1 - T'(z))/n$. Substituting this into (13) yields a characterization of the optimal tax rate in terms of elasticities:

$$(16) \quad \frac{T'(z)}{1 - T'(z)} = \frac{1 + \zeta^u}{\zeta^c n f} \frac{\mu u_c}{p}$$

The expression on the left-hand side represents the tax rate if the tax were levied as a consumption surcharge rather than as a payroll tax; for instance, a payroll tax of fifty per cent is equivalent to a consumption surcharge of one hundred per cent. Saez (2001), in his formula (15), denotes the first ratio on the right-hand side, multiplied by $1 - F(n)$, as $A(n)$, and writes the second ratio in integrated form, divided by $1 - F(n)$, as $B(n)$.

To summarize, our limit theorem shows that Mirrlees' (1971) formula is correct, Saez' (2001) formula is also correct, and Revesz' (1989) formula can be corrected by two minor modifications. It is important, however, to understand that neither formula yields an explicit solution of the optimal marginal income tax rate; they are all but characterizations. Equation (16) looks pretty much like an inverse elasticity rule but does not state that the optimal marginal tax rate varies inversely with the compensated elasticity in the sense of a causal relationship. Even when one makes the strong assumption of a constant compensated elasticity, changing this elasticity would generally change the uncompensated elasticity, the marginal utilities and the resulting shadow prices. Therefore, when doubling the compensated elasticity, it is by no means certain that the tax rate would fall to one half its previous value. This raises some doubts as to the usefulness of an elasticity approach: Firstly, if it comes to empirical applications, the elasticity approach is certainly not helpful. Saez (2001, p. 223) rightly remarks that his elasticity formula "cannot be directly applied using the empirical income distribution because the income distribution is affected by taxation. Therefore, it is useful to come back to the Mirrlees formulation". Secondly, concerning economic interpretations, the elasticity approach seems to be a rather roundabout method because equation (10) characterizes the optimal income tax directly in terms of marginal cost and benefit, which is much more straightforward than the economic reasoning behind (16). However, this may be a matter of personal opinion.

VI. TWO FINAL ISSUES: DIFFERENTIABILITY AND CIRCULARITY

The above proof of the optimal income tax formula by means of a limit theorem is rigorous, but in the derivation of (4) we assumed that the optimal tax schedule were differentiable. Mirrlees, Saez, and Revesz had to assume differentiability anyway, because they used control theory or calculus of variations, but Revesz (1989, p. 465) raised the question as to whether non-differentiable schedules could yield further improvements. Moreover, Revesz (2003) pointed out that the tax schedule $T(z)$ depends on income z which depends on the schedule itself. Owing to this circularity $T(z(T(z...)))$ he thinks that standard mathematical tools, which assume z to be exogenous and rule out such circles, cannot be used in the context of optimal income taxation. In this concluding section we would like to clarify the issues of differentiability and circularity, which are interconnected.

At the outset, it should be emphasized that the Lagrangean approach used above is not subject to Revesz' circularity criticism: The optimization (5) simply characterizes

second-best allocations, derived under the premise that the government has direct control over c^h and ℓ^h . After second-best allocations have been described in this way, the government looks, in a second step, for tax schedules to support them. A tax schedule is said to support an allocation if every person can reach the pair (c^h, z^h) intended for him and if no person can make himself better off by choosing some pair $(c, z) \neq (c^h, z^h)$. To state the difference between circular and non-circular approaches most clearly: Our approach would become circular if we substituted c^h by $z^h - T(z^h)$ in the Lagrangean (5). Differentiating with respect to z^h alone, then, would yield the same result, but this method would be subject to the circularity criticism.

Now, at each observable income z^h , a supporting tax schedule is uniquely defined by $T(z^h) = z^h - c^h$. At each non-observable income z , however, it is not uniquely defined because if some $T^*(z)$ does the job, every $T(z) > T^*(z)$ will do the job as well because it makes the pair $(z - T(z), z)$ less attractive. Hence, the set of tax schedules supporting a second-best allocation in the finite economy is uncountable. With a view to the limit theorem, it is most convenient to consider the point-wise minimum of all supporting tax schedules, since this has the nicest continuity and differentiability properties. Surely, this tax schedule need not be chosen, but it can be chosen without loss of generality. Put into symbols, the point-wise minimum of all tax schedules supporting the optimum is uniquely defined by

$$(17) \quad u(z - T(z), z) = u(c^{h+1}, \ell^{h+1}) \quad \text{for all } z \in [z^h, z^{h+1}] \text{ and all } h < H.$$

From the chain property we know that person $h+1$ must be indifferent between the pair intended for him and the pair intended for person h . The tax schedule (17) makes person $h+1$ also indifferent with respect to all pairs (c, z) between (c^h, z^h) and (c^{h+1}, z^{h+1}) . Agent monotonicity ensures that no third person prefers any of these intermediate pairs to his own. In FIGURE 1, the resulting budget line coincides with person $h+1$'s indifference curve between points A and B, and it coincides with person h 's indifference curve for incomes left to point A. Since h 's indifference curve is steeper, the budget line has a downward kink at point A, which means that the tax schedule has an upward kink. The figure also shows that a differentiable tax schedule supporting the optimum does not exist: Making an arbitrary straight line tangent to point A, either person h or person $h+1$ will find pairs of consumption and income which are strictly better than the pair intended for him.

The tax schedule (17) is continuous on its entire domain $[z^0, z^H]$ and differentiable almost everywhere. At the intermediate observable incomes $z^1 \dots z^{H-1}$, however, the schedule is only differentiable from the left and from the right. The symbol $T'(z^h)$ introduced naively in (10) turns out to be the left-derivative at such an observable income, whereas the right-derivative equals $T'(z^h) + \Delta mrs^h$. The marginal tax schedule looks like a saw-blade, with upward jumps Δmrs^h at each intermediate observable income. To repeat, the entire argument is perfectly non-circular because the government identifies second-best allocations in the first step, and then, by selecting an appropriate tax function

$T(z)$ in the second step, modifies the common budget set in such a way that everyone, by maximizing $u(z^h - T(z^h), z^h/n^h)$, voluntarily accepts the pair of consumption and income intended for him. Such a method does not solve the circularity problem, but avoids it from the outset.

Now, let us consider again the limit process introduced in Section IV. With δ denoting the difference between two adjacent wage rates, $\Delta mrs^h = mrs^h(n^h) - mrs^h(n^h + \delta)$ vanishes as the wage rate distribution becomes dense. Hence, the upward jumps in the marginal tax rate become smaller and smaller, and if some limit tax schedule $T(z)$ exists, it will be differentiable. Thus, nothing can be gained by relaxing the differentiability assumption; differentiability emerges as a result and need not be assumed. Since the derivatives (6) and (7) as well as the tax formulae are all evaluated at (c^h, z^h) , and neither of them at (c^{h+1}, z^{h+1}) , this result holds irrespective of possible gaps in the limit income distribution. But note that bunching has been ruled out here. If there is an interval $[n^h, n^k]$ of persons having the same income z at the optimum, the left-derivative equals $1 - mrs(n^h)$, the right-derivative $1 - mrs(n^k)$, and even the limit tax schedule will be non-differentiable at z . Bunching and corner solutions make the analysis a bit more difficult, but they do not affect the basic results, see Homburg (2001).

VII. CONCLUSION

This paper has shown that the theory of optimal non-linear income taxation can be made completely rigorous if one starts with a finite economy and obtains the results for the continuum case as limits. Such an approach does not suffer from the messiness and opaqueness of Mirrlees' original approach; moreover, it allows definite proofs and yields simple intuitive explanations of the economic forces at work. The approach also allows straightforward derivations of the elasticity expressions proposed by Revesz (1989) and Saez (2001) and highlights the non-circularity of the argument.

APPENDIX

Let k^0 denote the recourse constraint $\sum (z^h - c^h - R)f^h = 0$, and let $k^h (h > 0)$ denote the self-selection constraint $u(c^h, \ell^h) - u(c^{h-1}, z^{h-1}/n^h) = 0$. If the gradients of these constraints are linearly dependent, there exist numbers $\alpha^h (h = 0...H)$, not all zero, such that $\sum \alpha^h \text{grad } k^h = 0$. Differentiating with respect to c^0, ℓ^0, c^1 , and so on, yields the following sign pattern:

$$\begin{aligned} \text{grad } k^0 &= (- + - + - + \dots - +), \\ \text{grad } k^1 &= (- + + - 0 0 \dots 0 0), \\ \text{grad } k^2 &= (0 0 - + + - \dots 0 0), \\ &\dots \\ \text{grad } k^H &= (0 0 0 0 0 0 \dots + -). \end{aligned}$$

Assume $\alpha^0 > 0$. As the first column must vanish, $\alpha^1 < 0$. As the third column must also vanish, $\alpha^2 < 0$ and so forth until $\alpha^H < 0$. But then the sum of the numbers in the last column is strictly positive, which presents a contradiction. The premise $\alpha^0 < 0$ yields an analogous contradiction. Finally, $\alpha^0 = 0$ implies $\alpha^1 = 0$, which implies $\alpha^2 = 0$ and so forth, proving that the gradients are in fact linearly independent.

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