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Bertrand competition under network externalities

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Discussion Paper No. 884

BERTRAND COMPETITION UNDER NETWORK EXTERNALITIES

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Bertrand Competition under Network Externalities

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September 4, 2013

Abstract

Two sellers engage in price competition to attract buyers located on a network. The value of the good of either seller to any buyer depends on the number of neighbors on the network who consume the same good. For a generic specification of consumption externalities, we show that an equilibrium price equals the marginal cost if and only if the buyer network is complete or cyclic. When the externalities are approximately linear in the size of consumption, we identify the classes of networks in which one of the sellers monopolizes the market, or the two sellers segment the market.
Key words: graphs, networks, externalities, Bertrand, divide and conquer, discriminatory pricing, monopolization, segmentation.
Journal of Economic Literature Classification Numbers: C72, D82.

1 Introduction

Goods have network externalities when their value to each consumer depends on the consumption decisions of other consumers. The externalities may derive from physical connection to consumers adopting the same good as in the case of telecommunication devices, from provision of complementary goods as in the case of operating systems and softwares for computers, or from pure psychological factors as in the case of band-wagon and snob consumption. Despite their importance in reality, we only have limited understanding of network externalities particularly when those goods are supplied competitively. The objective of this paper is to study price competition in the presence of consumption externalities represented by a buyer

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network. Specifically, we represent a network by an undirected graph whose nodes correspond to buyers, and suppose that the degree of externalities experienced by any buyer is increasing in the number of buyers who are linked to them in the graph.

A more detailed description of our model is as follows: Two sellers each sell goods that are incompatible with each other. Consumers of either good experience larger positive externalities when more of his neighbors consume the same good. In stage 1, the two sellers post prices simultaneously. The prices can be perfectly discriminatory and can be negative. Upon publicly observing the price vectors posted by both sellers, the buyers in stage 2 simultaneously decide which good to buy or not to buy. The sellers have no cost of serving the market, and their payoffs simply equal the sum of prices offered to the buyers who choose to buy their goods.

In this framework, we find that the equilibrium outcome of price competition subtly depends on the network structure. Our first observation concerns the validity of marginal cost pricing. When no network externalities are present, it is clear that the unique subgame perfect equilibrium of this game has both sellers offer zero to all buyers. In the presence of externalities, however, we show that the marginal cost pricing is consistent with equilibrium only if either the externalities are linear (in the number of neighbors consuming the same good), or the network is either a complete graph or a cycle.[1] In any other network, if the externalities generic, there exists no equilibrium in which either seller captures the entire market by offering the same price to all buyers. This is so even in networks where all buyers have symmetric locations. Given this surprising result, we attempt to identify equilibrium prices under non-linear externalities.

Positive identification of equilibrium prices is possible when the externalities are close to linear and when the network satisfies certain properties as follows. First, we consider *bipartite* networks. A network is bipartite if the set of buyers can be divided into two subsets such that for every buyer in either subset, all his neighbors belong to the other subset. This is an important class of networks given that it corresponds to a *two-sided market* that has received much attention in the literature as discussed in the next section. We show that in a bipartite network, there exists an equilibrium in which one of the sellers captures the entire market (*i.e.,* buyers on both sides) by charging positive prices to all buyers on one side while subsidizing all buyers on the other side.

[1] A graph is complete if any pair of buyers are neighbors. The linear externalities in particular imply that the value of the good is zero to a buyer when none of his neighbors consumes it.

Next, we identify the class of networks for which market segmentation takes place in equilibrium. We say that a network is *separable* if the buyer set can be divided into two subsets such that every buyer in each subset has at least as many neighbors in the same subset as in the other subset, and some buyer in each subset has strictly more neighbors in the same subset than in the other subset. In a separable network, we show that market segmentation takes place in equilibrium with each seller making strictly positive profits.

The paper is organized as follows: After discussing the related literature in the next section, we formulate a model of price competition in Section 3. Section 4 considers the subgame played by the buyers that follows the public observation of prices posted by both sellers. The critical observations there are that this simultaneous-move game is one of strategic complementarity, and hence that there exist maximal and minimal Nash equilibria in each subgame. We present an algorithm to obtain these extreme equilibria and use them in our construction of a subgame perfect equilibrium of the entire game. We begin the analysis of a subgame perfect equilibrium in Section 5 and identify lower bounds on the sellers' payoff in such equilibrium. Section 6 examines the validity of marginal cost pricing in equilibrium. With the definition of approximate linearity, Section 7 discusses equilibrium in a bipartite network, which corresponds to a model of two-sided markets. Equilibrium market segmentation in separable networks is discussed in Section 8. We conclude in Section 9. All the proofs are collected in the Appendix. The Appendix also contains an analysis of the game when the buyers coordinate their actions by playing a strong Nash equilibrium in the stage 2 subgame.

2 Related Literature

Dybvig and Spatt (1983) are the first to theoretically study the provision of goods with network externalities.[2] The problems of a single supplier of a good with network externalities are subsequently studied by Cabral et al. (1999), Park (2004), Sekiguchi (2009), Ochs and Park (2010), Aoyagi (2013), among others. These papers focus on such issues as the construction of efficient or revenue maximizing adoption schemes under complete and incomplete information, intertemporal patterns of adoption decisions, as well as the validity of introductory pricing.

Segal (2003), Winter (2004) and Bernstein and Winter (2012) study a closely

[2]Rohlfs (1974) provides a very early discussion of network externalities.

related problem of contracting under externalities in which a single principal offers a contract to the set of agents whose participation decisions create externalities to other agents. They discuss the so-called *divide-and-conquer* strategy used by the principal: According to the strategy, the principal approaches agents one by one in some order. The contract offered to the first agent induces him to participate even if all other agents abstain. The contract offered to the second agent induces him to participate if all but the first agent abstain, and so on. In our analysis of an equilibrium, we use exactly the same argument: Given some price profile, we examine if it is profitable for either seller to approach the buyers one by one in some order with prices that induce them to choose the buyer provided that all their predecessors do the same. We note that the argument is essentially that of iterative elimination of strictly dominated strategies, and show that it can be used very effectively to examine if the given price profile is part of an equilibrium.

Competition between suppliers of goods with network externalities was first formulated by Katz and Shapiro (1985), and subsequently studied by Sundararajan (2003), Ambrus and Argenziano (2009), Bernaji and Dutta (2009), and Jullien (2011). These models are often couched in terms of *two-sided markets,* where the sellers are providers of *platforms* who offer a marketplace for agents on two sides such as sellers and buyers of some good. In such models, the utility of an agent on one side is an increasing function of the number of participants from the other side.[3] Ambrus and Argenziano (2009) analyze Bertrand competition between platforms in a two-sided market. Jullien (2011) applies the divide-and-conquer argument to his analysis of multi-sided markets, and derives a bound on the platforms' payoffs when they engage in Stackelberg price competition. Both Ambrus and Argenziano (2009) and Jullien (2011) formulate externalities differently from the present paper, and impose some non-trivial restrictions on the agents' strategies. Although these restrictions may appear natural under some price profiles, their full implications are not immediately clear. In contrast, our analysis of a subgame perfect equilibrium imposes no restriction on the buyers' strategies.

To the best of our knowledge, Banerji and Dutta (2009) are the only other paper that introduces graph structure into a model of price competition under network externalities. They identify conditions under which price competition leads to monopolization and market segmentation. They assume, however, that each seller sets the same price for all buyers and also place restrictions on the buyers' strategies. Be-

[3]See Armstrong (1998), and Laffont et al. (1998a,b).

cause of these differences in assumptions, their conclusions are difficult to compare with ours.

3 Model

Two sellers A and B compete for the set $I = \{1, \ldots, N\}$ of $N \geq 3$ buyers. Consumption of either seller's good generates externalities to the buyers according to a buyer network. Formally, a buyer network is represented by a simple undirected graph G whose nodes correspond to the buyers, and consumption externalities exist between buyers i and j if they are *adjacent* in the sense that there is a link between i and j. When buyer j is adjacent to buyer i, we also say that j is i's *neighbor.*

The buyer network G is *connected* in the sense that for any pair of buyers i and j, there exists a path from i to j. That is, there exist buyers $i_1, i_2, \ldots, i_m$, such that i_1 is adjacent to i, i_2 is adjacent to i_1, $\ldots$, and i_m is adjacent to j. For any buyer i in network G, denote by $N_i(G)$ (or simply N_i) the set of i's neighbors in G. The *degree* $d_i(G) = |N_i(G)|$ of buyer i in network G is the number of i's neighbors. Define also M to be the number of links in G. Since each link counts twice when aggregating the number of degrees in G, we have $M = \frac{1}{2} \sum_{i \in I} d_i$.

For $r = 2, \ldots, N-1$, the network G is *r-regular* if all buyers have the same degree r, and regular if it is r-regular for some r. G is *cyclic* if it is connected and 2-regular, and *complete* if it is $(N-1)$-regular, or equivalently, every pair of buyers are adjacent to each other. For any non-empty subset $J \subset I$ of buyers, denote by $G[J]$ the subnetwork induced from G: The set of nodes in $G[J]$ is J, and $G[J]$ has a link between $i \in J$ and $j \in J$ if and only if i and j are adjacent in the original network G.

The value of either seller's good to any buyer i is determined by the number of neighbors of i who consume the same good. We denote by v^n the value of either good to any consumer when n of his neighbors consume the same good. In particular, v^0 denotes the *stand-alone value,* or the value to any buyer of either good when none of his neighbors consumes the same good. The value does not depend on the identity of a buyer or the identity of the seller who supplies the good. The consumption externalities are non-negative in the sense that $0 \leq v^0 \leq v^1 \leq \cdots \leq v^{N-1}$.

Each seller produces his good at no fixed cost and a constant marginal cost. For simplicity, assume that the marginal costs also equals zero. Let p_i and q_i denote the prices offered to buyer i by seller A and seller B, respectively. The sellers can

perfectly price discriminate the buyers. They simultaneously quote price vectors $p = (p_i)_{i \in I} \in \boldsymbol{R}^N$ and $q = (q_i)_{i \in I} \in \boldsymbol{R}^N$. The buyers publicly observe (p, q), and then simultaneously decide whether to buy from either seller, or not buy. Buyer i's action x_i is hence an element of the set $\{A, B, \emptyset\}$, where $\emptyset$ represents no purchase. Each seller's strategy is hence an element of $\boldsymbol{R}^N$, whereas buyer i's strategy σ_i is a mapping from the set $\boldsymbol{R}^{2N}$ of price vectors (p, q) to the set $\{A, B, \emptyset\}$. Let $\sigma = (\sigma_i)_{i \in I}$ be the buyers' strategy profile, and for each choice profile $x = (x_i)_{i \in I}$ of buyers, let

$$I_A(x) = \{i \in I : x_i = A\}, \quad \text{and} \quad I_B(x) = \{i \in I : x_i = B\}$$

denote the set of buyers choosing seller A and the set of buyers choosing B, respectively. If we denote by $\pi_A(p, q, \sigma)$ and $\pi_B(p, q, \sigma)$ the payoffs of sellers A and B, respectively, under the strategy profile (p, q, σ), then they are given by

$$\pi_A(p, q, \sigma) = \sum_{i \in I_A(\sigma(p,q))} p_i,$$
$$\pi_B(p, q, \sigma) = \sum_{i \in I_B(\sigma(p,q))} q_i.$$

Recall that $G[J]$ $(J \subset I)$ is the subnetwork of G consisting of buyers in J, and that $d_i(G[J])$ is the degree of buyer i in $G[J]$. Given the price profile (p, q), buyer i's payoff under the action profile x is given by

$$u_i(x) = \begin{cases} v^{d_i(G[I_A(x)])} - p_i & \text{if } x_i = A, \\ v^{d_i(G[I_B(x)])} - q_i & \text{if } x_i = B, \\ 0 & \text{if } x_i = \emptyset, \end{cases} \tag{1}$$

and buyer i's payoff under the strategy profile (p, q, σ) is written as:

$$\pi_i(p, q, \sigma) = u_i(\sigma(p, q)).$$

A price vector (p^*, q^*) and a strategy profile $\sigma = (\sigma_i)_{i \in I}$ together constitute a *subgame perfect equilibrium* (SPE) if given any price vector $(p, q) \in \boldsymbol{R}^{2N}$, the action vector $(\sigma_i(p, q))_{i \in I}$ is a Nash equilibrium of the subgame following (p, q), and given σ, each component of the price vector (p^*, q^*) is optimal against the other:

$$\pi_i\left(p, q, \sigma(p, q)\right) \geq \pi_i\left(p, q, x_i, \sigma_{-i}(p, q)\right) \text{ for every } x_i,\ i \text{ and } (p, q),$$
$$\pi_A\left(p^*, q^*, \sigma(p^*, q^*)\right) \geq \pi_A\left(p, q^*, \sigma(p, q^*)\right) \text{ for every } p,$$
$$\pi_B\left(p^*, q^*, \sigma(p^*, q^*)\right) \geq \pi_B\left(p^*, q, \sigma(p^*, q)\right) \text{ for every } q.$$

4 Nash Equilibrium in the Buyers' Game

In this section, we fix the price vector (p, q), and consider an equilibrium of the buyers' subgame following (p, q). For the payoff function u_i defined in (1), the simultaneous-move game $(I, S = \{A, B, \emptyset\}^I, (u_i)_{i \in I})$ among the buyers is a supermodular game when the set of actions of each buyer is endowed with the ordering $A \succ \emptyset \succ B$. It follows that the game has pure Nash equilibria that are maximal and minimal with respect to the partial ordering on S induced by $\succ$.[4] We refer to the maximal equilibrium as the *A-maximal* equilibrium and denote it by x^A, and the minimal equilibrium as the *B-maximal* equilibrium and denote it by x^B. By definition, for any NE y and buyer i, $y_i = A$ implies $x_i^A = A$, and $y_i = B$ implies $x_i^B = B$.

We introduce some notation below in view of the fact that any NE must survive the iterative elimination of strictly dominated actions.

Define $T_0 = \emptyset$ and recursively define the subsets of buyers Y_k, Z_k, P_k, Q_k, R_k, and T_k as follows. For $k = 0, 1, 2, \ldots$, define $Y_{k+1} \subset I \setminus T_k$ to be the maximal set such that

$$\begin{aligned} Y_{k+1} = \Big\{ & i \in I \setminus T_k : \\ & u_i \left(x^*_{T_k}, x_{Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \geq 0 \Big\}. \end{aligned} \tag{2}$$

Given the action profile $x^*_{T_k}$ of buyers in set T_k, those buyers in Y_{k+1} can collectively choose A to enjoy non-negative payoffs from it. In other words, if $i \notin Y_{k+1}$, then $x_i = A$ is an iteratively dominated action for i. Note that maximality is well-defined since if Y and Y' both satisfy (2), then $Y \cup Y'$ also satisfies (2). If there is no such set, let $Y_{k+1} = \emptyset$. Likewise, define $Z_{k+1} \subset I \setminus T_k$ to be the maximal set of buyers who can collectively choose B to enjoy non-negative payoffs from it:

$$\begin{aligned} Z_{k+1} = \Big\{ & i \in I \setminus T_k : \\ & u_i \left(x^*_{T_k}, x_{Z_{k+1}} = (B, \ldots, B), x_{-T_k - Z_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \geq 0 \Big\}. \end{aligned} \tag{3}$$

If there is no such set, let $Z_{k+1} = \emptyset$. Again, if $i \notin Z_{k+1}$, then $x_i = B$ is iteratively strictly dominated for i. Define

$$R_{k+1} = (I \setminus T_k) \setminus (Y_{k+1} \cup Z_{k+1})$$

[4] See Topkis (1998).

to be the set of buyers i for whom $x_i = \emptyset$ is iteratively strictly dominant. Now define $P_{k+1} \subset I \setminus T_k$ by

$$\begin{aligned} P_{k+1} = \Big\{ i \in I \setminus T_k : u_i \left(x^*_{T_k}, x_i = B, x_{-T_k - i} = (\emptyset, \ldots, \emptyset) \right) \\ > u_i \left(x^*_{T_k}, x_{Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \Big\}, \end{aligned} \tag{4}$$

That is, if $i \in P_{n+1}$, buyer i is strictly better off choosing $x_i = B$ than choosing $x_i = A$ or $\emptyset$ even if seller A attracts all those buyers j for whom $x_j = A$ is not iteratively strictly dominated. In other words, if $i \in P_{k+1}$, then $x_i = B$ is an iteratively strictly dominant action for i. Likewise, define $Q_{k+1} \subset I \setminus T_k$ to be the set of buyers i for whom $x_i = A$ is iteratively strictly dominant:

$$\begin{aligned} Q_{k+1} = \Big\{ i \in I \setminus T_k : u_i \left(x^*_{T_k}, x_i = A, x_{-T_k - i} = (\emptyset, \ldots, \emptyset) \right) \\ > u_i \left(x^*_{T_k}, x_{Z_{k+1}} = (B, \ldots, B), x_{-T_k - Z_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \Big\}. \end{aligned} \tag{5}$$

1) If $P_{k+1} = Q_{k+1} = R_{k+1} = \emptyset$, then set $K = k$ and stop.

2) Otherwise, define

$$x^*_i = \begin{cases} B & \text{if } i \in P_{k+1}, \\ A & \text{if } i \in Q_{k+1}, \\ \emptyset & \text{if } i \in R_{k+1}. \end{cases}$$

and

$$T_{k+1} = T_k \cup (P_{k+1} \cup Q_{k+1} \cup R_{k+1}).$$

If $T_{k+1} = I$, then set $K = k + 1$ and stop. Otherwise, increase k by one and start over.

Since the above process starts over only when at least one buyer has an iteratively strictly dominant action, the maximal number of rounds K must satisfy $K \leq N$. For any NE x, we must have every buyer in T_K choosing his iteratively strictly dominant action so that

$$x_{T_K} = x^*_{T_K}.$$

Therefore the possible difference between any pair of NE arises only for buyers in $I \setminus T_K$. The following proposition states that the A-maximal and B-maximal NE can be constructed by letting the maximal number of buyers choose A or B among those buyers.

Proposition 1 *Let x^A and x^B be defined by*

$$x^A = (x^*_{T_K}, x_{Y_{K+1}} = (A, \ldots, A), x_{-T_K - Y_{K+1}} = (\emptyset, \ldots, \emptyset)), \quad \text{and}$$
$$x^B = (x^*_{T_K}, x_{Z_{K+1}} = (B, \ldots, B), x_{-T_K - Z_{K+1}} = (\emptyset, \ldots, \emptyset)).$$

Then x^A and x^B are the A-maximal and B-maximal NE, respectively.

Of course, when $T_K = I$ so that every buyer has a iteratively strictly dominant action, the NE is unique and given by $x^A = x^B$.

5 Subgame Perfect Equilibrium

We now turn to the original two-stage game including the sellers. The proposition below makes a simple observation that if a price vector (p^*, q^*) is sustained in some SPE, then it must be sustained in an SPE in which the buyers choose an extreme response to either seller's deviation: If seller A deviates from p^*, then all buyers coordinate on the B-maximal NE that least favors seller A, and vice versa. The proposition hence presents a bang-bang property of an SPE.

Proposition 2 *For any network G, (p^*, q^*) is an SPE price vector if and only if there exists buyers' strategy profile σ such that (p^*, q^*, σ) is an SPE and*

$$\sigma(p, q) = \begin{cases} \sigma^B(p, q) & \text{if } p \neq p^* \text{ and } q = q^*, \\ \sigma^A(p, q) & \text{if } p = p^* \text{ and } q \neq q^*. \end{cases}$$

Consider next seller A's best response p to B's price q when the buyers play the B-maximal strategy σ^B. Since $\sigma^B(p, q)$ is a B-maximal NE for any (p, q), seller A can attract buyer i if and only if $x_i = A$ is an iteratively strictly dominant action for buyer i: $i \in \cup_{k=1}^K Q_k$, where Q_k is as defined in (5). Hence,

$$\pi_A(p, q.\sigma^B) = \sum_{k=1}^{K} \sum_{i \in Q_k} p_i.$$

The following lemma shows that if seller A's price vector p is a best response to (q, σ^B), then no two buyers in Q_k defined under (p, q) are adjacent. In other words, in order to attract adjacent buyers i and j, seller A should approach them sequentially. Intuitively, this is because making choice A dominant for both buyers simultaneously

requires offering lower prices to both of them than making $x_i = A$ dominant for buyer i first, then making $x_j = A$ dominant for buyer j next conditional on the knowledge that i chooses $x_i = A$.

Lemma 3 *Let $(Q_k)_{k=1,\ldots,K}$ be as defined in (5) under the price vector (p, q). If p is a best response to (q, σ^B), then for every $k = 1, \ldots, K$,*

$$i, j \in Q_k \quad \Rightarrow \quad i \text{ and } j \text{ are not adjacent.}$$

We now derive a key result that establishes a lower bound for each seller's equilibrium payoff given the price vector of the other seller. As mentioned in the Introduction, the argument is one of *divide and conquer,* where seller A, say, approaches each buyer sequentially according to some ordered list, and presents them with a price which makes the choice A a dominant action given all his predecessors in the list choose A.

Formally, fix the price q^* of seller B, and suppose that the buyers choose A only when it is an iteratively strictly dominated action. Suppose further that seller A makes an ordered list of all buyers $i_1, \ldots, i_N$. Seller A first targets buyer i_1 by making it strictly dominant for buyer i_1 to choose $x_{i_1} = A$ by offering a sufficiently low price. In fact, seller A needs to offer p_{i_1} such that

$$v^0 - p_{i_1} > v^{d_{i_1}} - q^*_{i_1} \quad \text{and} \quad p_{i_1} < v^0,$$

or equivalently

$$p_{i_1} < \min\left\{v^0 - v^{d_{i_1}} + q^*_{i_1},\, v^0\right\}$$

to make $x_{i_1} = A$ strictly dominant. Let $H_1 = \{i_1\}$. Seller A next targets buyer i_2 by making i_2 strictly dominant. In this case, seller A must offer p_{i_2} such that

$$p_{i_2} < \min\left\{v^{s_{i_2}} - v^{d_{i_2} - s_{i_2}} + q^*_{i_2},\, v^{s_{i_2}}\right\},$$

where $s_{i_2} = 1$ if buyer i_2 is adjacent to i_1, and $= 0$ otherwise. Let $H_2 = \{i_1, i_2\}$. Proceeding iteratively, we see that against buyer i_k, seller A must offer p_{i_k} such that

$$p_{i_k} < \min\left\{v^{s_{i_k}} - v^{d_{i_k} - s_{i_k}} + q^*_{i_k},\, v^{s_{i_k}}\right\}, \tag{6}$$

where s_{i_k} is the number of neighbors of i_k in the set $H_{k-1} = \{i_1, \ldots, i_{k-1}\}$. s_{i_k} can be thought of the externalities buyer i_k can enjoy by choosing A when those buyers

in H_{k-1} have already chosen A. On the other hand, $d_i - s_{i_k}$ is the externalities i_k can enjoy from B when those buyers in $I \setminus H_{k-1}$ still choose B. Note that for any list $i_1, \ldots, i_N$ of buyers,

$$\sum_{k=1}^{N} s_{i_k} = M,$$

where M is the total number of links in G. Define S by

$$S = \Big\{ s = (s_i)_{i \in I} : s_{i_1} = 0 \text{ and } s_{i_k} = |N_{i_k} \cap \{i_1, \ldots, i_{k-1}\}| \text{ for } k \geq 2 \\ \text{for some relabeling } (i_1, \ldots, i_N) \text{ of buyers} \Big\}. \tag{7}$$

Note that if s corresponds to the list $i_1, \ldots, i_N$, then $d - s = (d_i - s_i)_{i \in I}$ corresponds to the reversed list $i_N, \ldots, i_1$. Hence, if $s \in S$, then $d - s \in S$ as well. We also observe that

$$H_k \subset \cup_{\ell=1}^{k} Q_\ell,$$

where Q_k is as defined in (5) and equals the set of buyers i for whom $x_i = A$ is iteratively strictly dominant in round k of the iteration process under the price profile (p, q^*). Hence, even if the buyers play the B-maximal equilibrium σ^B that least favors seller A, A can at least secure the payoff implied by the prices in (6). We hence have the following lemma that gives a lower bound for each seller's equilibrium payoff.

Lemma 4 *If (p^*, q^*, σ) is an SPE, then*

$$\begin{aligned} \pi_A(p^*, q^*, \sigma) &\geq \max_{s \in S} \sum_{i=1}^{N} \min \left\{ v^{s_i} - v^{d_i - s_i} + q_i^*,\ v^{s_i} \right\}, \\ \pi_B(p^*, q^*, \sigma) &\geq \max_{s \in S} \sum_{i=1}^{N} \min \left\{ v^{s_i} - v^{d_i - s_i} + p_i^*,\ v^{s_i} \right\}. \end{aligned} \tag{8}$$

Figures 1 and 2 illustrate the discussion for the line network of three buyers. In Figure 1, seller A approaches the buyers in the order $(i_1, i_2, i_3) = (1, 3, 2)$ when seller B offers $q^* = (q_1^*, q_2^*, q_3^*)$: When buyers 1 and 3 switch to A, their valuation of A's good is just v^0 (stand-alone value) since at that point they don't know if buyer 2 will switch as well. On the other hand, when buyer 2 switches to A, he knows that both his neighbors will choose A, and he expects that A's good is worth v^2. Hence,

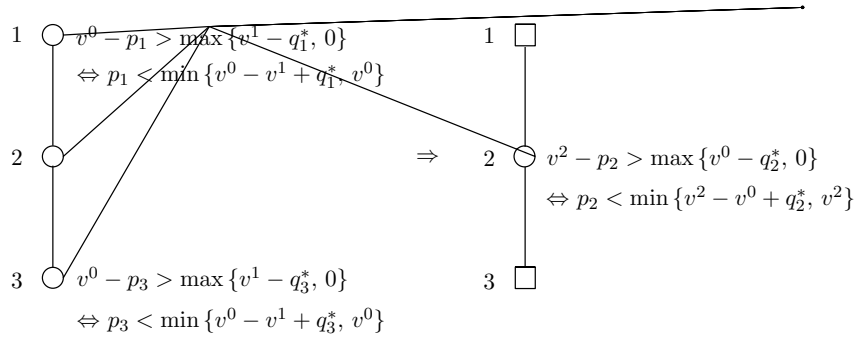

Figure 1: Divide-and-conquer by seller A with $(i_1, i_2, i_3) = (1, 3, 2)$.

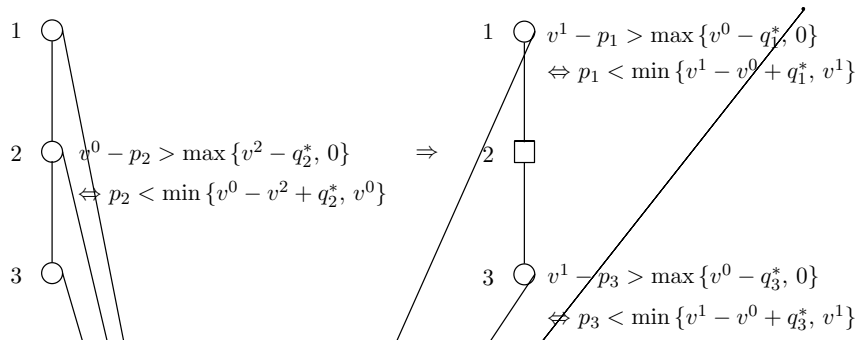

Figure 2: Divide-and-conquer by seller A with $(i_1, i_2, i_3) = (2, 1, 3)$.

even if the buyers play the B-maximal equilibrium σ^B, seller A's divide-and-conquer strategy with $(i_1, i_2, i_3) = (1, 3, 2)$ is profitable if

$$\begin{aligned} \min\{v^0 - v^1 + q_1^*, v^0\} + \min\{v^0 - v^1 + q_3^*, v^0\} \\ + \min\{v^2 - v^0 + q_2^*, v^2\} > 0. \end{aligned} \tag{9}$$

Likewise, his divide-and-conquer strategy with $(i_1, i_2, i_3) = (2, 1, 3)$ illustrated in Figure 2 is profitable if

$$\begin{aligned} \min\{v^0 - v^2 + q_2^*, v^0\} + \min\{v^1 - v^0 + q_1^*, v^1\} \\ + \min\{v^1 - v^0 + q_3^*, v^1\} > 0, \end{aligned} \tag{10}$$

and that with $(i_1, i_2, i_3) = (1, 2, 3)$ is profitable if

$$\begin{aligned} \min\{v^0 - v^1 + q_1^*, v^0\} + \min\{v^1 - v^1 + q_2^*, v^1\} \\ + \min\{v^1 - v^0 + q_3^*, v^1\} > 0. \end{aligned}$$

6 Marginal Cost Pricing

When there are no consumption externalities $0 < v^0 = \cdots = v^{N-1}$, it is clear that a subgame perfect equilibrium price (p^*, q^*) is unique and equal to the marginal cost: $(p^*, q^*) = (0, 0)$. In this section, we will examine if and how this result can be extended when there are externalities.

Let $D = D(G)$ be the highest degree in G:

$$D(G) = \max_{i \in I} d_i(G).$$

For the network G, hence, the relevant levels of externalities are $(v^0, \ldots, v^D)$. We say that the externalities $(v^0, \ldots, v^D)$ are *linear* if there exists $h > 0$ such that

$$v^k = kh \quad \text{for every } k = 0, 1, \ldots, D.$$

Note in particular that the stand-alone value v^0 is zero when the externalities are linear. In this sense, linearity implies pure network externalities and violates the formulation of weak externalities in Jullien (2011).[5]

Proposition 5 *Let G be an arbitrary buyer network. Under the linear externalities $(v^0, \ldots, v^D)$, $(p^*, q^*) = (0, 0)$ is an SPE price vector.*

We next consider some generic property of externalities. As will be seen, whether or not the marginal cost can be the equilibrium price depends crucially on the configuration of the buyer network in this case. Specifically, for S defined in (7), suppose that the externalities $(v^0, \ldots, v^D)$ satisfy the following condition:

$$s \in S \text{ and } d - s \text{ is not a permutation of } s \Rightarrow \sum_{i=1}^{N} v^{s_i} \neq \sum_{i=1}^{N} v^{d_i - s_i}. \tag{11}$$

(11) implies that the sum of externalities over buyers are different between the two goods when seller A attracts buyers by offering prices as described in (6). The set of $(v^0, \ldots, v^D)$ satisfying (11) is generic in the set

$$\left\{(v^0, \ldots, v^D) : 0 < v^0 \leq \cdots \leq v^D\right\}$$

of all externalities.

[5] Assumption 1 of Jullien (2011).

Lemma 4 implies that a seller's equilibrium payoff is closely linked to the value of

$$\max_{s \in S} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right).$$

It turns out that whether this quantity is positive or not under (11) depends crucially on the network configuration as seen in the following lemma.

Lemma 6 *Suppose that the externalities $v = (v^0, \ldots, v^D)$ satisfy (11). If the buyer network G is neither cyclic nor complete, then*

$$\max_{s \in S} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right) > 0. \tag{12}$$

The following lemma, which readily follows from Lemmas 4 and 6, provides some key observations on equilibrium pricing.

Lemma 7 *Suppose that (p^*, q^*, σ) is an SPE for the buyer network G which is neither complete nor cyclic, and that the externalities $v = (v^0, \ldots, v^D)$ satisfy (11). Then*

1) $\pi_A(p^*, q^*, \sigma) = 0 \Rightarrow \min_i q_i^* < 0$.

2) $\pi_A(p^*, q^*, \sigma) \leq \sum_i q_i^* \Rightarrow \max_i q_i^* > v^0$.

3) $I_B(\sigma(p^*, q^*)) = I \Rightarrow \max_i q_i^* > v^0$, $\min_i (v^{d_i} - q_i^*) \geq v^0$, *and* $v^D > 2v^0$.

While the first two statements of Lemma 7 are true regardless of whether the market is monopolized or segmented in equilibrium, the implications of the lemma are seen most clearly for a monopolization equilibrium. Suppose that G is neither cyclic nor complete, and that seller B captures the entire market in equilibrium: $I_B(\sigma(p^*, q^*)) = I$. Then seller B must subsidize at least one buyer, and must charge some buyer strictly above the stand-alone value:

$$\min_i q_i^* < 0 \leq v^0 < \max_i q_i^* < v^D - v^0.$$

Furthermore, for any such equilibrium to exist, the externalities cannot be too small: $v^D > 2v^0$. This is a non-trivial restriction for networks in which every buyer has a small degree as in line networks. We summarize this observation as a proposition below.

Proposition 8 *Let a buyer network G be given and the externalities $v = (v^0, \ldots, v^D)$ satisfy (11). Then there exists no SPE in which one of the sellers monopolizes the market by charging the same price to every buyer.*

The impossibility of uniform pricing is counter-intuitive in networks which are not cyclic or complete, but are symmetric with respect to every buyer. For example, buyer locations are exactly symmetric in the 4-regular network depicted in Figure 3.

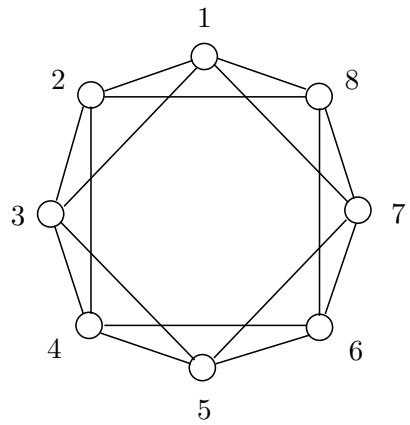

Figure 3: 4-regular network with eight buyers

We are now ready to state our main result on marginal cost pricing. Suppose that both sellers offer zero to all the buyers. In this case, both sellers' payoffs equal zero regardless of whether or not they capture a positive portion of the market. Hence, this price profile cannot be an equilibrium by Lemma 7(1) unless the network is complete or cyclic. The following proposition shows that when the network is complete or cyclic, there indeed exists an SPE of the type presented in Proposition 2 in which both sellers offer zero.

Proposition 9 *Let a buyer network G be given and the externalities $v = (v^0, \ldots, v^D)$ satisfy (11). $(p^*, q^*) = (0, 0)$ is an SPE price vector if and only if G is either cyclic or complete.*

For illustration of the impossibility of marginal cost pricing, return to the example of the three-buyer line network depicted in Figures 1 and 2. Suppose that

$q^* = 0$. In this case, we have

$$\begin{aligned} (9) \quad &\Leftrightarrow \quad 2v^1 - v^2 - v^0 < 0, \\ (10) \quad &\Leftrightarrow \quad 2v^1 - v^2 - v^0 > 0. \end{aligned}$$

Hence, if

$$2v^1 \neq v^2 + v^0, \tag{13}$$

seller A can profitably divide and conquer the buyers against $q^* = 0$. Note that (13) corresponds to (12) in Lemma 6: It fails under the linear externalities $v^0 = 0$, $v^1 = h$ and $v^2 = 2h$, but is true under generic specifications of v^0, v^1 and v^2.

7 Monopolization on a Bipartite Network

The result in the preceding section shows that if one of the sellers monopolizes the market, then he must employ discriminatory pricing. Since it appears difficult to derive a general conclusion on equilibrium pricing for an arbitrary network, we we will focus on certain classes of networks in what follows. In this section, we identify a class of networks in which monopolization takes place in equilibrium through discriminatory pricing.

Our analysis in the subsequent sections assumes that the externalities are approximately linear in the following sense: For $h > 0$, the externalities $(v^0, \ldots, v^D)$ are *ε-close to linear* if

$$|v^k - kh| < \varepsilon \text{ for } k = 0, 1, \ldots, D.$$

Since the condition holds for any $\varepsilon > 0$ when the externalities are exactly linear, our conclusions in what follows hold with no change in models of linear externalities. In conjunction with Proposition 5, then, this implies the multiplicity of equilibria in these markets.

The buyer network is *bipartite* if the buyer set I is partitioned into two disjoint subsets I_1 and I_2 such that every neighbor of $i \in I_1$ belongs to I_2 and every neighbor of $i \in I_2$ belongs to I_1. Line and star networks are simple examples of a bipartite network. For example, the line network in Figures 1 and 2 is bipartite with the partition $I_1 = \{1, 3\}$ and $I_2 = \{2\}$. A cycle network with an even number of buyers is also bipartite.

Bipartite networks are particularly important in their connection to two-sided markets. For example, we can think of I_1 as the set of sellers and I_2 as the set of

buyers of a certain good. In this case, the sellers A and B are interpreted as the platforms that offer marketplace to these sellers and buyers, and their prices are interpreted as *participation fees* into their platforms. Our conclusion on a bipartite network hence translates to that on equilibrium determination of participation fees in a two-sided market where two platforms compete.

Proposition 10 *Suppose that the buyer network G is bipartite. For any $h > 0$, there exists $\bar{\varepsilon} > 0$ such that if the externalities are ε-close to h-linear for $\varepsilon < \bar{\varepsilon}$, then there exists an SPE (p^*, q^*, σ) in which one seller captures all the buyers.*

The equilibrium constructed in the proof is described as follows: Let I_1 and I_2 be the partition of the buyer set, and suppose that seller B captures the market. Seller B offers $q_i = v^{d_i} - v^0$ to each buyer i in set I_1 and $q_i = v^0 - v^{d_i}$ to each buyer i in set I_2 provided that yields a non-negative payoff.[6] In other words, the monopolizing seller taxes every buyer on one side, and subsidizes every buyer on the other side. Seller A offers the same price to each buyer as seller B. When either seller deviates, the buyers play the extreme equilibrium which is least favorable to the deviating seller as in Proposition 2. It is shown that this price vector leaves no room for seller A to profitably attract any buyers. Figure 4 illustrates the equilibrium pricing of Proposition 10 in a star network with five buyers when the externalities satisfy approximate linearity and

$$v^4 - v^0 \geq 4(v^1 - v^0). \tag{14}$$

It can be seen that the hub buyer 1 is charged a positive price whereas all the peripheral buyers are subsidized. In other words, the subsidies to the peripheral buyers are a protection against the inducement by the other seller. Since (14) holds when the externalities are marginally increasing, we can understand this pricing behavior from the fact that it is relatively more difficult for the other seller to take away the hub buyer. When the inequality (14) is reversed, then the prices are (-1) times those listed in Figure 4. In this case of marginally decreasing externalities, hence, the hub buyer needs to be protected as it is relatively easier for the other seller to take him away. As seen in this example, which buyer(s) should be protected with subsidies in a bipartite network depends on the specification of externalities.

[6] Recall that d_i denotes the degree of buyer i. If this price profile yields a negative payoff, the equilibrium prices are simply $-q_i$ for each i.

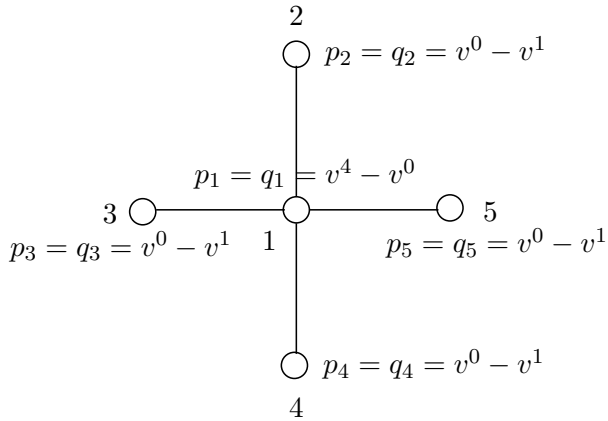

Figure 4: Monopolization through discriminatory pricing on a star network when $v^4 - v^0 \geq 4(v^1 - v^0)$.

8 Segmentation on a Separable Network

Under the same assumption of approximately linear externalities as in the previous section, we now examine the possibility of an equilibrium in which market segmentation takes place. For this, we consider a class of buyer networks that have roughly the opposite property as the bipartite networks introduced in the previous section: In this class of networks, the buyer set is again partitioned into two disjoint subsets, but each buyer has at least as many neighbors in the same subset than in the other subset. Formally, the buyer network is *separable* if there exists a two-way partition (I_1, I_2) of the set I of buyers such that for m, $n = 1$, 2, and $m \neq n$,

$$|N_i \cap I_n| \geq |N_i \cap I_m| \text{ for every } i \in I_n, \text{ and}$$
$$|N_i \cap I_n| > |N_i \cap I_m| \text{ for some } i \in I_n.$$

Intuitively, in a separable network with partition (I_1, I_2), we can classify buyers in I_1 or I_2 into *core* and *peripheral* buyers: The core buyers are those who have strictly more neighbors in the same set than in the other set, while the peripheral buyers have as many neighbors in the same set as in the other set. We can see that any line network with four or more buyers is separable: For example, a line network of four or more buyers is separable. The regular network in Figure 3 is also separable when we take $I_1 = \{1, 2, 3, 4\}$ and $I_2 = \{5, 6, 7, 8\}$. Buyer 2 and 3 are core buyers for I_1 and buyers 6 and 7 are core buyers for I_2.

Proposition 11 *Suppose that G is separable. For any $h > 0$, there exists $\bar{\varepsilon} > 0$*

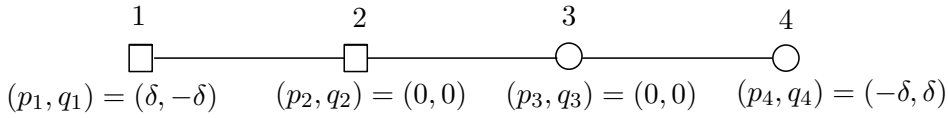

Figure 5: Segmentation on a line network ($\delta = |v^2 + v^1 - 2v^0| > 0$): A captures $\{1, 2\}$ and B captures $\{3, 4\}$.

such that if the externalities are ε-close to h-linear for $\varepsilon < \bar{\varepsilon}$, there exists an SPE in which buyers in I_1 choose seller A and buyers in I_2 choose seller B.

The proof of this proposition constructs an equilibrium in which each seller charges a small but positive price to one of the core buyers in his segment of the market. Specifically, recalling that s is a sequence of degrees of externalities as defined in (7), we specify the price to be charged to this core buyer by

$$\delta = \max_{s \in S} \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i} \right),$$

which is strictly positive for generic externalities (Lemma 6), but is small for approximately linear externalities. Each seller charges zero to all other buyers in their segment of the market. Figure 5 illustrates the equilibrium for a line network of four buyers.[7] As in Proposition 2, any deviation by either seller results in the extreme equilibrium that is least favorable to the deviating seller. Each core buyer who is charged the positive price will not switch to the other seller since he enjoys strictly higher externalities under the present seller. Furthermore, each seller enjoys strictly positive profits in equilibrium, and has no incentive to engage in divide-and-conquer taking advantage of the non-generic externalities as in the case of marginal cost pricing.

9 Conclusion

In this paper, we formulate a model of price competition between two sellers when each one of their goods exhibits local network externalities as represented by a graph-theoretic network of buyers. We show that whether a given price profile is consistent

[7] As seen in Figure 5, each seller charges $-\delta$ to the core buyer in the other segment who is charged δ by the other seller. This is to make the sum of the prices of each seller equal to zero.

with a subgame perfect equilibrium of the two-stage game depends crucially on the exact specifications of network structure and externalities. In the non-generic case of linear externalities, the marginal cost pricing of both sellers quoting zero to every buyer is consistent with an SPE for any network. In the generic specification of externalities, however, it is consistent with an SPE if and only if the network is either cyclic or complete. That is, in any other networks, some form of price discrimination is expected even if every buyer has exactly symmetric locations in those networks. Given these results, we proceed to the identification of an SPE when the externalities are approximately linear. In a bipartite network which corresponds to a two-sided market, we show that there exists an SPE in which one of the sellers monopolizes the market by charging a positive price to every buyer on one side, and a negative price to every buyer on the other side. The pricing strategy there gives us a hint as to which buyer needs to be protected from the inducement by the other seller. In a separable network in which each buyer has more neighbors on his side than on the other side, on the other hand, we show that there exists an equilibrium in which the two sellers segment the market.

In the present model, the goods of the two sellers are assumed symmetric and incompatible with each other. A natural extension would involve introducing asymmetry or a positive degree of compatibility between them. It would also be interesting to study endogenous determination of compatibility levels by the sellers. Although some of these issues are investigated in the literature,[8] it will be useful to examine them under the alternative specifications of externalities and equilibrium as in the present paper.

Appendix I: Proofs

Proof of Proposition 1. We show that x^A is an A-maximal NE. The symmetric argument shows that x^B is a B-maximal NE. We begin with the following lemma.

Lemma 12 *a)* $u_i(x^A) \geq 0$ *for every* i.

b) For any n, $\{i \in I \setminus T_n : x_i^A = A\} \subset Y_{n+1}$ *and* $\{i \in I \setminus T_n : x_i^A = B\} \subset Z_{n+1}$.

c) For any n, $J \subset I \setminus T_n$, *and* y_J *such that* $u_i(y_J, x_{-J}^A) \geq 0$ *for every* $i \in J$,

$$\{i \in J : y_i = A\} \subset Y_{n+1} \quad \textit{and} \quad \{i \in J : y_i = B\} \subset Z_{n+1}. \tag{15}$$

[8] See Jullien (2011).

Proof of Lemma 12. a) Suppose that $i \in P_{n+1}$ for some n. Then

$$\begin{aligned} u_i(x^A) &= u_i(x^A_{T_n}, x_i = B, x^A_{-T_n-i}) \\ &\geq u_i\left(x^A_{T_n}, x_i = B, x_{-T_n-i} = (\emptyset, \ldots, \emptyset)\right) \\ &> u_i\left(x^A_{T_n}, x_{Y_{n+1}} = (A, \ldots, A), X_{-T_n-Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq 0. \end{aligned}$$

The proof is similar if $i \in Q_{n+1}$. If $i \in R_{n+1}$ or $i \in I \setminus T_n$ for n such that $P_{n+1} = Q_{n+1} = R_{n+1} = \emptyset$, then the inequality follows from the definition of x^A.

b) Denote $K = \{i \in I \setminus T_n : x^A_i = A\}$. Then for any $i \in K$, we have

$$\begin{aligned} &u_i\left(x^A_{T_n}, x_{K \cup Y_{n+1}} = (A, \ldots, A), x_{-T_n-K-Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq u_i\left(x^A_{T_n}, x_K = (A, \ldots, A), x_{-T_n-K} = (\emptyset, \ldots, \emptyset)\right) \\ &= u_i(x^A) \geq 0, \end{aligned}$$

and for any $i \in Y_{n+1}$,

$$\begin{aligned} &u_i\left(x^A_{T_n}, x_{K \cup Y_{n+1}} = (A, \ldots, A), x_{-T_n-K-Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq u_i\left(x^A_{T_n}, x_{Y_{n+1}} = (A, \ldots, A), x_{-T_n-Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq 0. \end{aligned}$$

This contradicts the maximality of Y_{n+1}.

c) Denote $K = \{i \in J : y_i = A\}$. Suppose that $K \not\subset Y_{n+1}$. Then for $i \in K$,

$$\begin{aligned} &u_i\left(x^A_{T_n}, x_{K \cup Y_{n+1}} = (A, \ldots, A), x_{-T_n-Y_{n+1}-K} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq u_i\left(x^A_{T_n}, x_J, x^A_{-T_n-J}\right) \\ &= u_i(x^A_{-J}, x_J) \\ &\geq 0, \end{aligned}$$

where the first inequality follows from Lemma 12(b), and for any $i \in Y_{n+1}$,

$$\begin{aligned} &u_i\left(x^A_{T_n}, x_{K \cup Y_{n+1}} = (A, \ldots, A), x_{-T_n-Y_{n+1}-K} = (\emptyset, \ldots, \emptyset)\right) \\ &u_i\left(x^A_{T_n}, x_{Y_{n+1}} = (A, \ldots, A), x_{-T_n-Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq 0. \end{aligned}$$

This again contradicts the maximality of Y_{n+1}. ■

We now return to the proof of Proposition 1.

1) x^A is a NE.

Since $u_i(x^A) \geq 0$ by Lemma 12(a), $x_i' = \emptyset$ cannot be a profitable deviation for any i, and moreover a profitable deviation, if any, must yield a strictly positive payoff.

Take any $i \in P_{n+1}$ so that $x_i^A = B$, and consider a deviation $x_i' = A$. If $u_i(x_i', x_{-i}^A) \geq 0$, then $i \in Y_{n+1}$ by Lemma 12(c) and hence

$$\begin{aligned} u_i(x_i', x_{-i}^A) &\leq u_i\left(x_{T_n}^A, x_{Y_{n+1}} = (A, \ldots, A), x_{-Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &< u_i\left(x_{T_n}^A, x_i^A, x_{-T_n-i} = (\emptyset, \ldots, \emptyset)\right) \\ &\leq u_i(x^A). \end{aligned}$$

Hence the deviation is not profitable. Likewise, no profitable deviation exists for $i \in Q_{n+1}$. Suppose next that $i \in R_{n+1}$ so that $x_i^A = \emptyset$. $x_i' = A$ is not profitable since $i \in R_{n+1}$ implies that $i \notin Y_{n+1}$ and hence $u_i(x_i', x_{-i}^A) < 0$ by Lemma 12(c). Likewise, the deviation $x_i' = B$ is not profitable. Finally, suppose that $i \in I \setminus T_n$ and that $P_{n+1} = Q_{n+1} = R_{n+1} = \emptyset$. In this case, $x_i^A = A$ if $i \in Y_{n+1}$ and $x_i^A = \emptyset$ otherwise. If $x_i' = B$, then

$$\begin{aligned} u_i(x_i', x_{-i}^A) &= u_i\left(x_{T_n}^A, x_i', x_{-T_n-i}^A\right) \\ &= u_i\left(x_{T_n}^A, x_i', x_{-T_n-i} = (\emptyset, \ldots, \emptyset)\right) \\ &\leq u_i\left(x_{T_n}^A, x_{Y_{n+1}} = (A, \ldots, A), x_{-T_n-Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &= u_i(x^A), \end{aligned}$$

where the inequality follows since $i \notin P_{n+1}$. Hence, $x_i' = B$ is not a profitable deviation. If $i \in I \setminus T_n \setminus Y_{n+1}$ and $x_i' = A$, then $u_i(x_i', x_{-i}^A) < 0$ by Lemma 12(c).

2) x^A is A-maximal.

Take any NE y. Clearly, $u_i(y) \geq 0$ for every i. We first show that $y_i = x_i^A$ if $i \in T_1$. To see that $y_i = B$ for any $i \in P_1$, suppose $y_i = A$. Then by setting $n = 0$ and $J = I$ in Lemma 12(c), we see that $\{i : y_i = A\} \subset Y_1$ so that

$$\begin{aligned} u_i(y_i, y_{-i}) &\leq u_i\left(x_{Y_1} = (A, \ldots, A), x_{-Y_1} = (\emptyset, \ldots, \emptyset)\right) \\ &< u_i\left(x_i = B, x_{-i} = (\emptyset, \ldots, \emptyset)\right) \\ &\leq u_i(x_i = B, y_{-i}), \end{aligned}$$

where the second inequality follows from the definition of P_1. Hence $x_i = B$ is a profitable deviation. Likewise, $y_i = A$ holds for any $i \in Q_1$. If $i \in R_1$, then $y_i = \emptyset$ must hold since $i \notin Y_1 \cup Z_1$.

As an induction hypothesis, suppose that $y_i = x_i^A$ if $i \in T_n$. We show that $y_i = x_i^A$ if $i \in T_{n+1} \setminus T_n$. If $i \in P_{n+1}$, then $y_i = B$: If $y_i = A$, then $\{i \in I \setminus T_n : y_i = A\} \subset Y_{n+1}$ by Lemma 12(c) (set $J = I \setminus T_n$) so that

$$\begin{aligned} u_i(y_i, y_{-i}) &\leq u_i\left(x_{T_n}^A, x_{Y_{n+1}} = (A, \ldots, A), x_{-T_n - Y_{n+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &< u_i\left(x_{T_n}^A, x_i = B, x_{-T_n - i} = (\emptyset, \ldots, \emptyset)\right) \\ &\leq u_i(x_i = B, y_{-i}), \end{aligned}$$

where the second inequality follows from the definition of P_{n+1}. Hence $x_i = B$ is a profitable deviation. Likewise, $y_i = A$ for any $i \in Q_{n+1}$. If $i \in R_{n+1}$, then $y_i = \emptyset$ must hold since $i \notin Y_{n+1} \cup Z_{n+1}$. Finally, if $i \in I \setminus T_n$ and $P_{n+1} = Q_{n+1} = R_{n+1} = \emptyset$, then $y_i = A$ implies $i \in Y_{n+1}$ by Lemma 12(c), but $x_i^A = A$ for any such i by definition. We can therefore conclude that x^A is an A-maximal NE. ■

Proof of Proposition 2. If there exists such a strategy profile σ of buyers, then (p^*, q^*) is clearly an SPE price vector. Conversely, suppose that (p^*, q^*) is an SPE price vector. Then there exists $\hat{\sigma}$ such that $(p^*, q^*, \hat{\sigma})$ is an SPE. Define σ as follows:

$$\sigma(p, q) = \begin{cases} \hat{\sigma}(p, q) & \text{if } (p, q) = (p^*, q^*), \text{ or } p \neq p^* \text{ and } q \neq q^*, \\ \sigma^A(p, q) & \text{if } p = p^* \text{ and } q \neq q^*, \\ \sigma^B(p, q) & \text{if } p \neq p^* \text{ and } q = q^*. \end{cases}$$

Then (p^*, q^*, σ) is an SPE: The definition of σ^B and the equilibrium property of $\hat{\sigma}$ together imply

$$\pi_A(p, q^*, \sigma) \leq \pi_A(p, q^*, \hat{\sigma}) \leq \pi_A(p^*, q^*, \hat{\sigma}) = \pi_A(p^*, q^*, \sigma).$$

Likewise, the definition of σ^A and the equilibrium property of $\hat{\sigma}$ together imply $\pi_B(p^*, q, \sigma) \leq \pi_B(p^*, q^*, \sigma)$. ■

Proof of Lemma 3. For simplicity, let $k = K$ and suppose to the contrary that $1, 2 \in Q_K$ and 1 and 2 are adjacent. Then it must be the case that

$$v^{\alpha_1^K} - p_1 > \max\{v^{d_1 - \alpha_1^K} - q_1, 0\} \quad \text{and} \quad v^{\alpha_2^K} - p_2 > \max\{v^{d_2 - \alpha_2^K} - q_2, 0\},$$

where

$$\alpha_1^K = \left| N_1 \cap \cup_{\ell=1}^{K-1} Q_\ell \right|, \quad \text{and} \quad \alpha_2^K = \left| N_2 \cap \cup_{\ell=1}^{K-1} Q_\ell \right|$$

are the numbers of neighbors of 1 and 2, respectively, for whom x_i is iteratively strictly dominant in round $K-1$ or earlier. Hence,

$$p_1 < v^{\alpha_1^K} - \max\{v^{d_1-\alpha_1^K} - q_1, 0\} \quad \text{and} \quad p_2 < v^{\alpha_2^K} - \max\{v^{d_2-\alpha_2^K} - q_2, 0\}.$$

On the other hand, let p' be such that $p'_i = p_i$ for $i \neq 2$, and

$$p_2 < p'_2 < v^{\alpha_2^K+1} - \max\{v^{d_2-\alpha_2^K-1} - q_2, 0\}.$$

Denote by Q'_k the set of buyers for whom $x_1 = A$ is an iteratively dominant action in round k under (p', q) as defined in (5). We then have $Q'_k = Q_k$ for $k = 1, \ldots, K-1$ and $Q'_K = Q_K \cup \{1\}$ so that $\alpha_2^K + 1$ of 2's neighbors have chosen A in round K or earlier. Since

$$v^{\alpha_2^K+1} - p'_2 > \max\{v^{d_2-\alpha_2^K-1} - q_2, 0\},$$

$Q'_{K+1} = \{2\}$. Furthermore, since $p'_2 > p_2$, $\pi_A(p', q, \sigma_B) > \pi_A(p, q, \sigma_B)$. ■

Proof of Lemma 4. Fix any relabeling of buyers $i_1, \ldots, i_N$. Let $s = (s_i)_{i \in I}$ be defined by

$$s_{i_1} = 0 \quad \text{and} \quad s_{i_k} = |N_{i_k} \cap \{i_1, \ldots, i_{k-1}\}| \text{ for } k = 2, \ldots, N.$$

Let $\varepsilon > 0$ be given, and define the price vector $p = (p_i)_{i \in I}$ by

$$p_i = \min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\} - \varepsilon. \tag{16}$$

As explained in the text, by offering p, seller A makes $x_{i_1} = A$ a strictly dominant action for buyer i_1, and in any subsequent step, $x_{i_k} = A$ an iteratively strictly dominant action for buyer i_k under (p, q^*). Hence, seller A's payoff under (p, q^*, σ) satisfies

$$\pi_A(p, q^*, \sigma) \geq \sum_{i=1}^{N} \min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\} - N\varepsilon.$$

Since $\varepsilon > 0$ and $s \in S$ are arbitrary, if (8) does not hold, then we would have a contradiction

$$\pi_A(p, q^*, \sigma) > \pi_A(p^*, q^*, \sigma).$$

The symmetric argument proves the inequality for seller B's payoff. ■

Proof of Proposition 5. We first show that $(p^*, q^*) = (0, 0)$ is an SPE price. Let σ^A and σ^B be the A-maximal and B-maximal equilibria as defined earlier, and let σ be the buyers' strategy profile such that

$$\sigma(p,q) = \begin{cases} (B, \ldots, B) & \text{if } (p,q) = (0,0), \\ \sigma^B(p,q) & \text{if } p \neq 0 \text{ and } q = 0, \\ \sigma^A(p,q) & \text{if } p = 0 \text{ and } q \neq 0. \end{cases}$$

Now consider a deviation from $p^* = 0$ to $p \neq 0$ by seller A. Let Q_k $(k = 1, \ldots, K)$ be as defined in (5) under (p, q^*). It then follows that

$$I_A(\sigma^A(p, q^*)) = \cup_{k=1}^K Q_k$$

for some $K \leq N$. In other words, any buyer attracted by seller A with p must choose A as his iteratively strictly dominant action. Hence, seller A's payoff under (p, q^*, σ) can be written as:

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^{K} \sum_{i \in Q_k} p_i. \tag{17}$$

Now let

$$\alpha_i^k = \left| N_i \cap \cup_{\ell=1}^{k-1} Q_\ell \right|$$

denote the number of neighbors of buyer i who have already chosen seller A in rounds prior to k. If $i \in Q_k$, then $x_i = A$ must be a dominant action in round k for buyer i so that

$$v^{\alpha_i^k} - p_i > v^{d_i - \alpha_i^k} \quad \Leftrightarrow \quad p_i < v^{\alpha_i^k} - v^{d_i - \alpha_i^k}. \tag{18}$$

Note now that

$$\begin{aligned} &\sum_{k=1}^{K} \sum_{i \in Q_k} \alpha_i^k \\ &= \sum_{k=1}^{K} \left(\#\text{links between } Q_k \text{ and } \cup_{\ell=1}^{k-1} Q_\ell \right) \\ &\leq \#\text{links in the subnetwork } G\left[\cup_{k=1}^K Q_k\right] \\ &= \frac{1}{2} \sum_{k=1}^{K} \sum_{i \in Q_k} d_i \left(G\left[\cup_{k=1}^K Q_k\right] \right) \\ &\leq \frac{1}{2} \sum_{k=1}^{K} \sum_{i \in Q_k} d_i. \end{aligned} \tag{19}$$

Substituting (18), (19) and the linearity of the externalities into (17), we obtain

$$\begin{aligned}\pi_A(p,q^*,\sigma) &< \sum_{k=1}^{K}\sum_{i\in Q_k}\left(v^{\alpha_i^k}-v^{d_i-\alpha_i^k}\right)\\ &= h\sum_{k=1}^{K}\sum_{i\in Q_k}\left(2\alpha_i^k-d_i\right)\le 0.\end{aligned}$$

Therefore, the deviation p is not profitable. By the symmetric argument, no deviation q by seller B is profitable either. ■

Proof of Lemma 6. Note that (12) follows if we show that $d-s$ is not a permutation of s for some s: (11) implies that either $\sum_{i=1}^{N}\left(v^{s_i}-v^{d_i-s_i}\right)>0$ or <0. If the latter holds, then let $i'_k=i_{N-k+1}$ for $k=1,\ldots,N$ and define $t=(t_i)_{i\in I}$ by setting $t_{i'_k}$ equal to the number of neighbors of i'_k in $\{i'_1,\ldots,i'_{k-1}\}$:

$$t_{i'_1}=0 \quad\text{and}\quad t_{i'_k}=|N_{i'_k}\cap\{i'_1,\ldots,i'_{k-1}\}| \text{ for } k=2,\ldots,N. \tag{20}$$

Then we can verify that

$$\sum_{i=1}^{N}\left(v^{t_i}-v^{d_i-t_i}\right)=-\sum_{i=1}^{N}\left(v^{s_i}-v^{d_i-s_i}\right)>0.$$

We will consider the following two cases separately.

1) G is not regular.

Take a pair of buyers i and j such that i is adjacent to j, $d_i=D$ and $d_j<D$, where $D\ge 2$ is the highest degree in G. Take another buyer k that is adjacent to i but not to j. To see that there exists such a buyer, suppose to the contrary that every buyer $\neq j$ that is adjacent to i is also adjacent to j. Then j has at least D neighbors, a contradiction. Let $i_1=k$, $i_2=i$ and $i_3=j$, and define $i_4,\ldots,i_N\notin\{i,j,k\}$ arbitrarily. Then

$$\begin{aligned}&(s_{i_1},s_{i_2},s_{i_3})=(0,1,1)\,,\\ &(d_{i_1}-s_{i_1},d_{i_2}-s_{i_2},d_{i_3}-s_{i_3})=(d_k,D-1,d_j-1)\,.\end{aligned}$$

If s is not a permutation of $d-s$, then we are done. Suppose then that s is a permutation of $d-s$, and define $i'_1=k$, $i'_2=j$, $i'_3=i$, and $i'_\ell=i_\ell$ for $\ell\ge 4$, and let $t=(t_i)_{i\in I}$ be defined by (20) for these $i'_1,\ldots,i'_N$. Then

$$\begin{aligned}&\left(t_{i'_1},t_{i'_2},t_{i'_3}\right)=(0,0,2)\,,\\ &\left(d_{i'_1}-t_{i'_1},d_{i'_2}-t_{i'_2},d_{i'_3}-t_{i'_3}\right)=(d_k,d_j,D-2)\,.\end{aligned}$$

Since $i'_\ell = i_\ell$ for $\ell \geq 4$, we have

$$\begin{aligned} \left|\left\{\ell \geq 4 : \, d_{i_\ell} - s_{i_\ell} = 0\right\}\right| &= \left|\left\{\ell \geq 4 : \, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right|, \\ \left|\left\{\ell \geq 4 : \, s_{i_\ell} = 0\right\}\right| &= \left|\left\{\ell \geq 4 : \, t_{i'_\ell} = 0\right\}\right|. \end{aligned} \tag{21}$$

a) $d_j = 1$.

In this case,

$$\left|\left\{\ell \leq 3 : \, d_{i_\ell} - s_{i_\ell} = 0\right\}\right| = \left|\left\{\ell \leq 3 : \, s_{i_\ell} = 0\right\}\right| = 1.$$

Hence, since $d - s$ is a permutation of s, we must have

$$|\{\ell \geq 4 : \, d_{i_\ell} - s_{i_\ell} = 0\}| = |\{\ell \geq 4 : \, s_{i_\ell} = 0\}|.$$

It then follows from (21) that

$$\left|\left\{\ell \geq 4 : \, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| = \left|\left\{\ell \geq 4 : \, t_{i'_\ell} = 0\right\}\right|. \tag{22}$$

However,

$$\left|\left\{\ell \leq 3 : \, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| \leq 1 < 2 = \left|\left\{\ell \leq 3 : \, t_{i'_\ell} = 0\right\}\right|. \tag{23}$$

(22) and (23) together show that $d - t$ cannot be a permutation of t.

b) $d_j \geq 2$.

In this case, we have $D \geq 3$ since $D > d_j \geq 2$, and also

$$\left|\left\{\ell \leq 3 : \, d_{i_\ell} - s_{i_\ell} = 0\right\}\right| = 0 < 1 = \left|\left\{\ell \leq 3 : \, s_{i_\ell} = 0\right\}\right|.$$

Hence, since $d - s$ is a permutation of s,

$$|\{\ell \geq 4 : \, d_{i_\ell} - s_{i_\ell} = 0\}| = |\{\ell \geq 4 : \, s_{i_\ell} = 0\}| + 1.$$

It then follows from (21) that

$$\left|\left\{\ell \geq 4 : \, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| = \left|\left\{\ell \geq 4 : \, t_{i'_\ell} = 0\right\}\right| + 1. \tag{24}$$

However,

$$\left|\left\{\ell \leq 3 : \, d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right| = \left|\left\{\ell \leq 3 : \, t_{i'_\ell} = 0\right\}\right| - 2 \tag{25}$$

(24) and (25) together imply that $d - t$ is not a permutation of t.

2) G is r-regular with $2 < r < N-1$.

Since G is connected and not complete, we can take a pair of buyers i_1 and i_2 such that i_1 and i_2 are adjacent, and take another buyer i_3 who is adjacent to i_2 but not to i_1. To see that this is possible, suppose to the contrary that for any pair of adjacent buyers i and j, any buyer $k \neq i$ adjacent to j is also adjacent to i. We then show that G must be complete. Take any pair of buyers i and j. Since G is connected, there is a path $k_1 = i \to k_2 \to \cdots \to k_{m-1} \to k_m = j$. Since k_2 is adjacent to $i = k_1$ and k_3 is adjacent to k_2, k_3 is adjacent to i as well by the above. Now since k_4 is adjacent to k_3, it is also adjacent to i. Proceeding the same way, we conclude that $j = k_m$ is adjacent to $i = k_1$, implying that G is complete.

We now label buyers other than $\{i_1, i_2, i_3\}$ as $i_4, \ldots, i_N$ in an arbitrary manner. For our choice of i_1, i_2 and i_3, we have

$$\begin{aligned}
&(s_{i_1}, s_{i_2}, s_{i_3}) = (0, 1, 1)\,, \\
&(d_{i_1} - s_{i_1}, d_{i_2} - s_{i_2}, d_{i_3} - s_{i_3}) = (r, r-1, r-1)\,.
\end{aligned}$$

If $d - s$ is a not permutation of s, then we are done. Suppose then that $d - s$ is a permutation of s. We then must have

$$\Big|\{\ell : s_{i_\ell} = 0\}\Big| = \Big|\{\ell : d_{i_\ell} - s_{i_\ell} = 0\}\Big|. \tag{26}$$

Let $i'_1 = i_1$, $i'_2 = i_3$, $i'_3 = i_2$ and $i'_\ell = i_\ell$ for $\ell \geq 4$, and let $t = (t_i)_{i \in I}$ be defined by (20) for these $i'_1, \ldots, i'_N$. Note that

$$\begin{aligned}
&\left(t_{i'_1}, t_{i'_2}, t_{i'_3}\right) = (0, 0, 2)\,, \\
&\left(d_{i'_1} - t_{i'_1}, d_{i'_2} - t_{i'_2}, d_{i'_3} - t_{i'_3}\right) = (r, r, r-2)\,.
\end{aligned}$$

Since $r > 2$, if (26) holds, then the same argument as in the non-regular case shows that

$$\left|\left\{\ell : t_{i'_\ell} = 0\right\}\right| \neq \left|\left\{\ell : d_{i'_\ell} - t_{i'_\ell} = 0\right\}\right|,$$

implying that $d - t$ is not a permutation of t. ■

Proof of Lemma 7. We first show that if (p^*, q^*, σ) is an SPE, then

$$\pi_A(p^*, q^*, \sigma) > \sum_{i=1}^{N} \min\{q_i^*,\, v^0\} \quad \text{and} \quad \pi_B(p^*, q^*, \sigma) > \sum_{i=1}^{N} \min\{p_i^*,\, v^0\}. \tag{27}$$

By Lemma 4, for any $s \in S$, seller A's payoff under (p^*, q^*) satisfies

$$\pi_A(p^*, q^*, \sigma) \geq \sum_{i=1}^{N} \min\{v^{s_i} - v^{d_i - s_i} + q_i^*, v^{s_i}\}.$$

Rearranging, we get for any $s \in S$,

$$\begin{aligned}\pi_A(p^*, q^*, \sigma) &\geq \sum_{i=1}^{N} \left(v^{s_i} - v^{d_i - s_i}\right) + \sum_{i} \min\{q_i^*,\, v^{d_i - s_i}\} \\ &\geq \sum_{i=1}^{N} (v^{s_i} - v^{d_i - s_i}) + \sum_{i=1}^{N} \min\{q_i^*,\, v^0\}.\end{aligned}$$

When G is neither cyclic or complete, there exists by Lemma 6 an $s \in S$ such that the first term on the right-hand side is > 0. Hence, the first inequality in (27) must hold. The proof for the second inequality is similar.

a) If $\min_i q_i^* \geq 0$, then $\pi_A(p^*, q^*, \sigma) = 0 \leq \sum_i \min\{q_i^*, v^0\}$, contradicting (27).

b) If $\max_i q_i^* \leq v^0$, then $\pi_A(p^*, q^*, \sigma) \leq \sum_i q_i^* = \sum_i \min\{q_i^*, v^0\}$, contradicting (27).

c) The inequality $\max_i q_i^* > v^0$ follows from (b) above since $I_B(p^*, q^*, \sigma) = I$ implies $\pi_A(p^*, q^*, \sigma) = 0$ and $0 \leq \pi_B(p^*, q^*, \sigma) = \sum_i q_i^*$. If $v^{d_i} - q_i^* < v^0$ for some i, then any p such that $p_i = v^0 - \varepsilon$ and $p_{-i} = 0$ for $0 < \varepsilon < q_i^* - v^{d_i} + v^0$ would induce buyer i to switch to A and hence is a profitable deviation for seller A. To see that $v^D > 2v^0$, note first that $\min_i (v^{d_i} - q_i^*) \geq v^0$ in particular implies that $\max_i q_i^* \leq v^D - v^0$. Hence, if $v^D \leq 2v^0$, we have a contradiction to the first statement since $\max_i q_i^* \leq v^D - v^0 \leq v^0$. ■

Proof of Proposition 8. Suppose that G is neither cyclic or complete, and suppose that seller B attracts all the buyers in an SPE (p^*, q^*, σ) such that $q_1^* = \cdots = q_N^*$. Then since $\pi_A(p^*, q^*, \sigma) = 0$, Lemma 7(1) implies that $q_1^* = \cdots = q_N^* = \min_i q_i^* < 0$. Then, however, $\pi_B(p^*, q^*, \sigma) < 0$, a contradiction. ■

Proof of Proposition 9 It suffices to show that $(p^*, q^*) = (0, 0)$ coupled with the following strategy profile σ of the buyers is an SPE in each class of networks:

$$\sigma(p, q) = \begin{cases} (B, \ldots, B) & \text{if } (p, q) = (p^*, q^*), \\ \sigma^A(p, q) & \text{if } p = p^* \text{ and } q \neq q^*, \\ \sigma^B(p, q) & \text{if } p \neq p^* \text{ and } q\ q^*. \end{cases}$$

In other words, all buyers choose B under $(p^*, q^*) = (0, 0)$, and when one of the firms deviates to a non-zero price vector, the buyers coordinate on the NE which is least favorable to the deviating seller. In what follows, we show that seller A has no incentive to deviate. A symmetric argument shows that seller B has no incentive to deviate.

1) G is a cycle.

Suppose that seller A deviates to $p \neq p^*$. Let Q_k be as defined in (5) under (p, q^*). Since

$$I_A(\sigma(p, q^*)) = \cup_{k=1}^{K} Q_k$$

for some $K \leq N$, if $\sigma_i(p, q^*) = A$, then $i \in Q_k$ for some $k \leq K$. Recall that N_i is the set of neighbors of i in G, and that $d_i = |N_i| = 2$ since G is cyclic. Let

$$\alpha_i^k = \left| N_i \cap \cup_{\ell=1}^{k-1} Q_\ell \right| \in \{0, 1, 2\}$$

denote the number of i's neighbors who have chosen A in rounds prior to k. If $i \in Q_k$, then $x_i = A$ is a dominant action in round k for buyer i so that

$$v^{\alpha_i^k} - p_i > v^{2-\alpha_i^k} \quad \Leftrightarrow \quad p_i < v^{\alpha_i^k} - v^{2-\alpha_i^k}. \tag{28}$$

In particular, buyer i is attracted by seller A in round 1 if $p_i < v^0 - v^2$, and attracted by A in round $k > 1$ either if (i) $p_i < 0$ and exactly one of his two neighbors has already chosen A ($\alpha_i^k = 1$), or (ii) $p_i < v^2 - v^0$ and both his neighbors have already chosen A ($\alpha_i^k = 2$). Note also that only in round 1 does any buyer choose A when neither of his neighbors have already chosen A.

Seller A's payoff under (p, q^*, σ) hence satisfies

$$\begin{aligned}\pi_A(p, q^*, \sigma) &= \sum_{k=1}^{K} \sum_{i \in Q_k} p_i \\ &< |Q_1|(v^0 - v^2) + (v^2 - v^0) \sum_{k=2}^{K} \left| \{ i \in I \setminus \left(\cup_{\ell=1}^{k-1} Q_\ell \right) : \alpha_i^k = 2 \} \right|.\end{aligned}$$

Since no buyer chooses A in round $k \geq 2$ if neither of his neighbors has already chosen A, the number of components in $G[\cup_{\ell=1}^{k-1} Q_\ell]$ is less than or equal to that in $G[Q_1]$ for any k. It follows that

$$\sum_{k=2}^{K} |\{ i \in I \setminus \left(\cup_{\ell=1}^{k-1} Q_\ell \right) : \alpha_i^k = 2 \}| \leq |Q_1|.$$

We can therefore conclude that $\pi_A(p, q^*, \sigma) \leq 0$ and hence that p is not a profitably deviation.

2) G is complete.

Define Q_k $(k = 1, \ldots, K)$ as above. Since G is complete, for any buyer i, the number α_i^k of i's neighbors who have chosen A equals the number α_k of buyers who have chosen A in rounds $1, \ldots, k-1$:

$$\alpha_i^k = \left| N_i \cap \cup_{\ell=1}^{k-1} Q_\ell \right| = \sum_{\ell=1}^{k-1} |Q_\ell| \equiv \alpha_k.$$

Furthermore, by Proposition 3, we only need consider p such that each Q_k contains a single buyer. (If Q_k contains two or more buyers, then since G is complete, those buyers are adjacent.) Hence, without loss of generality, $Q_k = \{k\}$ for each $k = 1, \ldots, N$. For $k = 1, \ldots, K$, we also have

$$p_k < v^{\alpha_k} - v^{N-1-\alpha_k}.$$

Seller A's payoff under (p, q^*, σ) hence satisfies

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^{N} \sum_{k=1}^{N} p_k < \sum_{k=1}^{K} \left(v^{\alpha_k} - v^{N-1-\alpha_k} \right). \tag{29}$$

It is then straightforward to verify that the right-hand side equals zero. Hence, seller A has no profitable deviation.

■

Proof of Proposition 10. We will construct an SPE (p^*, q^*, σ) in which seller B captures all the buyers: $I_B(\sigma(p^*, q^*)) = I$. Let the buyer set be partitioned into I_1 and I_2 so that links exist only between I_1 and I_2. Suppose without loss of generality that

$$\sum_{i \in I_1} (v^{d_i} - v^0) - \sum_{i \in I_2} (v^{d_i} - v^0) \geq 0. \tag{30}$$

Let

$$p_i^* = q_i^* = \begin{cases} v^{d_i} - v^0 & \text{if } i \in I_1, \\ v^0 - v^{d_i} & \text{if } i \in I_2, \end{cases}$$

and

$$\sigma(p, q) = \begin{cases} \sigma^B(p, q) & \text{if } q = q^*, \\ \sigma^A(p, q) & \text{otherwise.} \end{cases}$$

By (30), seller B's payoff under (p^*, q^*, σ) is non-negative:

$$\pi_B(p^*, q^*, \sigma) = \sum_{i \in I} q_i^* \geq 0.$$

By the definition of the B-maximal NE, if seller A deviates to p, then the set of buyers he captures equals $I_A(\sigma(p, q^*)) = \cup_{k=1}^{K} Q_k$, where Q_k is the set of buyers i for whom $x_i = A$ is a strictly dominant strategy in round k under (p, q^*) as defined in (5).

Suppose first that $Q_1 \subset I_1$. we then have

$$\sum_{i \in Q_1} p_i < \sum_{i \in Q_1} \min\{v^0 - v^{d_i} + q_i^*, v^0\} = 0.$$

Therefore, no p such that $K = 1$ and $Q_1 \subset I_1$ under (p, q^*) is profitable. Since $q_i^* < 0$ for $i \in I_2$, it is clear that no p such that $K = 1$ under (p, q^*) is profitable either.

Suppose next that $K = 2$ and that $Q_1 \subset I_1$. Then $Q_2 \cap I_1 = \emptyset$ since in round 2, every buyer $i \in Q_1$ must be adjacent to some buyer in $Q_1 \subset I_1$. (Otherwise, i would have been in Q_1.) It follows that $q_i^* < 0$ for each $i \in Q_2$, and hence that

$$\begin{aligned}\sum_{i \in Q_2} p_i &< \sum_{i \in Q_2} \min\{v^{\alpha_i^2} - v^{d_i - \alpha_i^2} + q_i^*, v^{\alpha_i^2}\} \\ &= \sum_{i \in Q_2} \left(v^{\alpha_i^2} - v^{d_i - \alpha_i^2} - v^{d_i} + v^0\right) \leq 0,\end{aligned}$$

where $\alpha_i^2 = |N_i \cap Q_1|$ is the number of i's neighbors in Q_1. Therefore, no deviation p is profitable if $K = 2$, $Q_1 \subset I_1$ and $Q_2 \subset I_2$ under (p, q^*). It is then also clear that no deviation p is profitable if $K = 2$ and $Q_2 \subset I_2$.

We next show that no deviation p is profitable if $K \geq 2$ and $Q_k \cap I_1 \neq \emptyset$ for some $k \geq 2$. Together with the above observations, this would imply that no deviation p is profitable if $K = 2$. Furthermore, if $K \geq 3$, then it must be the case that either $Q_2 \cap I_1 \neq \emptyset$ or $Q_3 \cap I_1 \neq \emptyset$ since G is bipartite, and since every $i \in Q_3$ is adjacent to some buyer in Q_2. It would hence follow that no deviation p is profitable.

Let $j \in Q_k \cap I_1$ for some $k \geq 2$. Then

$$\begin{aligned}
\pi_A(p, q^*, \sigma) &= \sum_{\ell=1}^{K} \sum_{i \in Q_\ell} p_i \\
&< \sum_{\ell=1}^{K} \sum_{i \in Q_\ell} \min \left\{ v^{\alpha_i^\ell} - v^{d_i - \alpha_i^\ell} + q_i^*,\, v^{\alpha_i^\ell} \right\} \qquad (31) \\
&\leq \sum_{\ell=1}^{K} \sum_{i \in Q_\ell} \left(v^{\alpha_i^\ell} - v^{d_i - \alpha_i^\ell} + q_i^* \right) + v^{\alpha_j^k} - \left(v^{d_j} - v^0 + q_j \right),
\end{aligned}$$

where

$$\alpha_i^\ell = \left| N_i \cap \left(\cup_{\kappa=1}^{\ell-1} Q_\kappa \right) \right|$$

is the number of i's neighbors who have chosen A prior to round ℓ. We now use approximate linearity to evaluate the right-hand side of (31) term by term. First, since $q_j^* = v^{d_j} - v^0$,

$$v^{\alpha_j^k} - \left(v^{d_j} - v^0 + q_j^* \right) \leq -v^{d_j} + 2v^0 < -hd_j + 3\varepsilon. \qquad (32)$$

Observe next that

$$\begin{aligned}
&\sum_{\ell=1}^{K} \sum_{i \in Q_\ell} \left\{ v^{\alpha_i^\ell} - v^{d_i - \alpha_i^\ell} \right\} \\
&= \sum_{\ell=1}^{K} \sum_{i \in Q_\ell} \left[\left\{ v^{\alpha_i^\ell} - \alpha_i^\ell h \right\} - \left\{ v^{d_i - \alpha_i^\ell} - (d_i - \alpha_i^\ell) h \right\} \right] \qquad (33) \\
&- \sum_{\ell=1}^{K} \sum_{i \in Q_\ell} \left((d_i - \alpha_i^\ell) - \alpha_i^\ell \right) h \\
&\leq 2\varepsilon \left| \cup_{\ell=1}^{K} Q_\ell \right| - hm,
\end{aligned}$$

where

$$m = \#\text{links between } \cup_{\ell=1}^{K} Q_\ell \text{ and } I \setminus \left(\cup_{\ell=1}^{K} Q_\ell \right).$$

Observe finally that

$$
\begin{aligned}
&\sum_{\ell=1}^{K} \sum_{i \in Q_\ell} q_i^* \\
&= \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_1} (v^{d_i} - v^0) - \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_2} (v^{d_i} - v^0) \\
&= \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_1} (v^{d_i} - v^0 - d_i h) - \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_2} (v^{d_i} - v^0 - d_i h) \\
&+ h \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_1} d_i - h \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_2} d_i.
\end{aligned} \tag{34}
$$

Since the externalities are ε-close to linear,

$$
\sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_1} (v^{d_i} - v^0 - d_i h) - \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_2} (v^{d_i} - v^0 - d_i h) \leq 2\varepsilon \left| \cup_{\ell=1}^{K} Q_\ell \right|. \tag{35}
$$

We also have

$$
\begin{aligned}
\sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_1} d_i &= \#\text{links between } \cup_{\ell=1}^{K} Q_\ell \cap I_1 \text{ and } I_2 \\
\sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_2} d_i &= \#\text{links between } \cup_{\ell=1}^{K} Q_\ell \cap I_2 \text{ and } I_1 \\
&\geq \#\text{links between } \cup_{\ell=1}^{K} Q_\ell \cap I_2 \text{ and } \cup_{\ell=1}^{K} Q_\ell \cap I_1
\end{aligned}
$$

so that

$$
\begin{aligned}
&\sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_1} d_i - \sum_{\ell=1}^{K} \sum_{i \in Q_\ell \cap I_2} d_i \\
&\leq \#\text{links between } \cup_{\ell=1}^{K} Q_\ell \text{ and } I_2 \setminus \left(\cup_{\ell=1}^{K} Q_\ell \right) \\
&\leq \#\text{links between } \cup_{\ell=1}^{K} Q_\ell \text{ and } I \setminus \left(\cup_{\ell=1}^{K} Q_\ell \right) \\
&= m.
\end{aligned} \tag{36}
$$

Substituting (35) and (36) into (34), we obtain

$$
\sum_{\ell=1}^{K} \sum_{i \in Q_\ell} q_i^* \leq 2\varepsilon \left| \cup_{\ell=1}^{K} Q_\ell \right| + hm. \tag{37}
$$

Substituting (32), (33) and (37) into (31), we see that

$$\begin{aligned}\pi_A(p,q^*,\sigma) &< 2\varepsilon\left|\cup_{\ell=1}^K Q_\ell\right| - hm + 2\varepsilon\left|\cup_{\ell=1}^K Q_\ell\right| + hm - hd_j + 3\varepsilon \\ &= \varepsilon\left(4\left|\cup_{\ell=1}^K Q_\ell\right| + 3\right) - hd_j \\ &\leq \varepsilon\left(4N+3\right) - hd_j.\end{aligned}$$

Hence, if we set $\bar{\varepsilon} = h/\left(4N+3\right)$, then $\pi_A(p,q^*,\sigma) < 0$ when $\varepsilon < \bar{\varepsilon}$. ■

Proof of Proposition 11. Let

$$\delta = \max_{s\in S}\sum_{i=1}^N\left(v^{s_i} - v^{d_i-s_i}\right).$$

When the externalities are ε-close to h-linear,

$$\begin{aligned}\sum_{i=1}^N\left(v^{s_i} - v^{d_i-s_i}\right) &= \sum_{i=1}^N\left\{(v^{s_i} - s_ih) - \left(v^{d_i-s_i} - (d_i-s_i)h\right) - h\left((d_i-s_i)-s_i\right)\right\} \\ &< 2N\varepsilon,\end{aligned}$$

and hence

$$\delta < 2N\varepsilon. \tag{38}$$

Since G is separable, let (I_1, I_2) be the partition of the buyer set I, and let $i_A \in I_1$ and $i_B \in I_2$ be such that

$$|N_{i_A}\cap I_1| > |N_{i_A}\cap I_2| \text{ and } |N_{i_B}\cap I_2| > |N_{i_B}\cap I_1|.$$

We specify (p^*, q^*, σ) as follows:

$$(p_i^*, q_i^*) = \begin{cases}(\delta, -\delta) & \text{if } i = i_A, \\ (-\delta, \delta) & \text{if } i = i_B, \\ (0,0) & \text{otherwise,}\end{cases}$$

and

$$\sigma(p,q) = \begin{cases}(\underbrace{A,\ldots,A}_{I_1},\underbrace{B,\ldots,B}_{I_2}) & \text{if } (p,q) = (p^*,q^*), \\ \sigma^B(p,q) & \text{if } p \neq p^*, \\ \sigma^A(p,q) & \text{if } p = p^* \text{ and } q \neq q^*.\end{cases}$$

Note that $\pi_A(p^*,q^*,\sigma) = \pi_B(p^*,q^*,\sigma) = \delta$.

We first show that the buyers' action profile following (p^*, q^*) is a NE. If $i \in I_1 \setminus \{i_A\}$, then $x_i = A$ is a best response since

$$v^{|N_i \cap I_1|} - p_i = v^{|N_i \cap I_1|} \geq v^{|N_i \cap I_2|} = v^{|N_i \cap I_2|} - q_i.$$

If $i = i_A$, then $|N_i \cap I_1| > |N_i \cap I_2|$ so that

$$\begin{aligned}
&v^{|N_i \cap I_1|} - v^{|N_i \cap I_2|} \\
&= \left(v^{|N_i \cap I_1|} - h|N_i \cap I_1|\right) - \left(v^{|N_i \cap I_2|} - h|N_i \cap I_2|\right) + h\left\{|N_i \cap I_1| - |N_i \cap I_2|\right\} \\
&\geq h - 2\varepsilon.
\end{aligned}$$

Hence, if we take

$$\bar{\varepsilon} = \frac{h}{2(2N+1)}, \tag{39}$$

then for any $\varepsilon < \bar{\varepsilon}$, (38) implies that

$$v^{|N_i \cap I_1|} - p_i = v^{|N_i \cap I_1|} - \delta > v^{|N_i \cap I_2|} + \delta = v^{|N_i \cap I_2|} - q_i.$$

The symmetric argument shows that $x_i = B$ is a best response for each $i \in I_2$ following (p^*, q^*).

We will next show that seller A has no profitable deviation. Let p be any deviation by seller A, and denote by Q_k the set of buyers who will choose A as an iteratively dominant action in round k under (p, q^*) as defined in (5). Since the buyers play σ^B following (p, q^*), buyer i will choose A only if $x_i = A$ is iteratively dominant: $i \in \cup_{k=1}^{K} Q_k$. By Proposition 3, we may assume that no buyers in Q_k are adjacent.

If $i \in Q_k$, then

$$p_i < \min\left\{v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*,\, v^{\alpha_i^k}\right\} \leq v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^*,$$

where

$$\alpha_i^k = \left|N_i \cap \left(\cup_{\kappa=1}^{k-1} Q_\kappa\right)\right|$$

is the number of i's neighbors who have chosen A prior to round k. Suppose first that $\cup_{k=1}^{K} Q_k \subsetneq I$. Since the externalities are ε-close to h-linear, we have

$$\begin{aligned}
&v^{\alpha_i^k} - v^{d_i - \alpha_i^k} \\
&= \left(v^{\alpha_i^k} - \alpha_i^k h\right) - \left(v^{d_i - \alpha_i^k} - (d_i - \alpha_i^k)h\right) - \left((d_i - \alpha_i^k) - \alpha_i^k\right) h \\
&< 2\varepsilon - \left((d_i - \alpha_i^k) - \alpha_i^k\right) h.
\end{aligned}$$

Hence,

$$\begin{aligned}
\pi_A(p, q^*, \sigma) &= \sum_{k=1}^{K} \sum_{i \in Q_k} p_i \\
&\leq \sum_{k=1}^{K} \sum_{i \in Q_k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* \right) \\
&< \sum_{k=1}^{K} \sum_{i \in Q_k} \left\{ 2\varepsilon - \left((d_i - \alpha_i^k) - \alpha_i^k \right) h \right\} + \sum_{k=1}^{K} \sum_{i \in Q_k} q_i^* \\
&= 2\varepsilon \left| \cup_{k=1}^{K} Q_k \right| - h \sum_{k=1}^{K} \sum_{i \in Q_k} \left((d_i - \alpha_i^k) - \alpha_i^k \right) + \sum_{k=1}^{K} \sum_{i \in Q_k} q_i^*.
\end{aligned}$$

Since $\cup_{k=1}^{K} Q_k \subsetneq I$ by assumption and since G is connected,

$$\sum_{k=1}^{K} \sum_{i \in Q_k} \left((d_i - \alpha_i^k) - \alpha_i^k \right) = \#\text{links between } \cup_{k=1}^{K} Q_k \text{ and } I \setminus \cup_{k=1}^{K} Q_k \geq 1.$$

It hence follows from (38) that

$$\pi_A(p, q^*, \sigma) < 2\varepsilon \left| \cup_{k=1}^{K} Q_k \right| - h + 2N\varepsilon < 4N\varepsilon - h,$$

which is < 0 for $\varepsilon < \bar{\varepsilon}$ when $\bar{\varepsilon}$ is given in (39).

Suppose next that $\cup_{k=1}^{K} Q_k = I$. In this case, $\sum_{k=1}^{K} \sum_{i \in Q_k} q_i^* = 0$. Hence the definition of δ implies that

$$\pi_A(p, q^*, \sigma) = \sum_{k=1}^{K} \sum_{i \in Q_k} p_i \leq \sum_{k=1}^{K} \sum_{i \in Q_k} \left(v^{\alpha_i^k} - v^{d_i - \alpha_i^k} + q_i^* \right) \leq \delta = \pi_A(p^*, q^*, \sigma).$$

In either case, hence, the deviation p is not profitable. ■

Appendix II: Strong Equilibrium in the Buyers' Game

Our discussion in the text has placed no restriction on the buyers' strategies other than that implied by a Nash equilibrium. In this Appendix, we ask what happens when the buyers actively coordinate their actions. While there can be many different formulations of action coordination, one simple and extreme way is to suppose that any subset of the buyers may choose a joint deviation whenever that yields each one of them a strictly higher payoff than adhering to the proposed action profile.

In other words, we will require that in each subgame, the buyers' action profile constitutes a strong Nash equilibrium.[9] We find that the marginal cost pricing is not consistent with a strong Nash equilibrium even if the network is complete or cyclic.

Formally, the buyers action profile x^* is a *strong Nash equilibrium* (strong NE) if for any nonempty subset $J \subset I$ of buyers, and for any x_J,

$$u_i(x^*) \geq u_i(x_J, x^*_{-J}) \text{ for some } i \in J. \tag{40}$$

In other words, an action profile is a strong NE if, whenever a coalition of buyers contemplate a joint deviation, there is a member in the coalition who cannot strictly benefit from the deviation. If x^* is a strong NE, then it is clearly a NE. Note also that x^* is a strong NE if and only if (40) holds for any non-empty $J \subset I$ and any x_J such that $x_j \neq x^*_j$ for *every* $j \in J$.[10]

A strong NE x of the buyers' subgame is *A-maximal* if for any strong NE y, $y_i = A$ implies $x_i = A$, and *B-maximal* if $y_i = B$ implies $x_i = B$. We can find these maximal strong NE using the iteration procedure similar to that used to find the A-maximal and B-maximal NE.

For any action profiles x and y, identify $u_i(x_\emptyset, y)$ with $u_i(y)$. Let $T_0 = \emptyset$, and define the subsets of buyers T_k, P_k, Q_k, R_k, Y_k and Z_k $(n = 1, 2, \ldots)$ recursively as follows.

For $k = 0, 1, 2, \ldots$, define $Y_{k+1} \subset I \setminus T_k$ to be the maximal set such that

$$Y_{k+1} = \Big\{ i \in I \setminus T_k : u_i \left(x^*_{T_k}, x_{Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \geq 0 \Big\}.$$

Y_{k+1} is the set of buyers not in T_k who, given $x^*_{T_k}$, can collectively choose A and enjoy non-negative payoffs from it.[11] If there is no such set, let $Y_{k+1} = \emptyset$. Likewise, define $Z_{k+1} \subset I \setminus T_k$ to be the maximal set such that

$$Z_{k+1} = \Big\{ i \in I \setminus T_k : u_i \left(x^*_{T_k}, x_{Z_{k+1}} = (B, \ldots, B), x_{-T_k - Z_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \geq 0 \Big\}.$$

[9] By the property of the payoff functions of the buyers' game, we can verify that any strong Nash equilibrium is a coalition-proof Nash equilibrium in the sense of Bernheim *et al.* (1987).

[10] To see this, suppose that x^* is not a strong NE. Then there exist $J \neq \emptyset$ and x_J such that $u_j(x^*) < u_j(x_J, x^*_{-J})$ for every $j \in J$. Then $J' \equiv \{j \in J : x_j \neq x^*_j\} \neq \emptyset$. Moreover,

$$u_j(x_{J'}, x^*_{-J'}) = u_j(x_{J'}, x_{J \setminus J'}, x_{-J}) = u_j(x_J, x^*_{-J}) > u_j(x^*)$$

for every $j \in J'$ since $j \in J \setminus J'$ implies $x_j = x^*_j$. Hence, the coalition J' also has a profitable joint deviation such that $x_j \neq x^*_j$ for every $j \in J'$.

[11] With the possible difference in T_k and x_{T_k}, hence, the definition of Y_{k+1} is the same as in (2).

If there is no such set, then let $Z_{k+1} = \emptyset$. Let also R_{k+1} be defined by

$$R_{k+1} = (I \setminus T_k) \setminus (Y_{k+1} \cup Z_{k+1}).$$

As before, R_{k+1} is the set of buyers i for whom $x_i = \emptyset$ is iteratively strictly dominant given $x^*_{T_k}$. Now define $P_{k+1} \subset I \setminus T_k$ to be the maximal set such that

$$\begin{aligned} P_{k+1} = \Big\{ i \in I \setminus T_k : u_i \left(x^*_{T_k}, x_{P_{k+1}} = (B, \ldots, B), x_{-T_k - P_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \\ > u_i \left(x^*_{T_k}, x_{Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \Big\}. \end{aligned}$$

If there exists no such set, then let $P_{k+1} = \emptyset$. P_{k+1} is the set of buyers whose payoffs from collectively choosing $x_i = B$ are strictly higher than those from the maximal coordination on A or from $\emptyset$. Likewise, define $Q_{k+1} \subset I \setminus T_k$ to be the maximal set of buyers whose payoffs from collectively choosing $x_i = A$ are strictly higher than those from the maximal coordination on B or from $\emptyset$:

$$\begin{aligned} Q_{k+1} = \Big\{ i \in I \setminus T_k : u_i \left(x^*_{T_k}, x_{Q_{k+1}} = (A, \ldots, A), x_{-T_k - Q_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \\ > u_i \left(x^*_{T_k}, x_{Z_{k+1}} = (B, \ldots, B), x_{-T_k - Z_{k+1}} = (\emptyset, \ldots, \emptyset) \right) \Big\}. \end{aligned}$$

Again, if there exists no such set, then let $Q_{k+1} = \emptyset$.

1) If $P_{k+1} = Q_{k+1} = R_{k+1} = \emptyset$, then let $k = K$ and stop.
2) Otherwise, let

$$T_{k+1} = T_k \cup (P_{k+1} \cup Q_{k+1} \cup R_{k+1}),$$

and

$$x^*_i = \begin{cases} B & \text{if } i \in P_{k+1}, \\ A & \text{if } i \in Q_{k+1}, \\ \emptyset & \text{if } i \in R_{k+1}. \end{cases}$$

If $T_{k+1} = I$, then set $K = k + 1$ and stop. Otherwise, increase k by one and start over.

Given that the above process starts over only when there is a buyer who has a joint dominant action, the maximal number of iteration $K \leq N$.

Proposition 13 *Let x^A and x^B be defined by*

$$\begin{aligned} x^A &= (x^*_{T_K}, x_{Y_{K+1}} = (A, \ldots, A), x_{-T_K - Y_{K+1}} = (\emptyset, \ldots, \emptyset)), \quad \textit{and} \\ x^B &= (x^*_{T_K}, x_{Z_{K+1}} = (B, \ldots, B), x_{-T_K - Z_{K+1}} = (\emptyset, \ldots, \emptyset)). \end{aligned}$$

Then x^A and x^B are the A-maximal and B-maximal strong NE, respectively.

(p^*, q^*, σ) is a *strong SPE* if for every (p, q), $\sigma(p, q)$ is a strong NE of the buyers' subgame, and $\pi_A(p^*, q^*, \sigma) \geq \pi_A(p, q^*, \sigma)$ and $\pi_B(p^*, q^*, \sigma) \geq \pi_B(p^*, q, \sigma)$ for every p and q.

Proposition 14 *Let a buyer network G be given and the externalities $v = (v^0, \ldots, v^D)$ satisfy (11). There exists no buyers' strategy profile σ such that for $p^* = q^* = 0$ (p^*, q^*, σ) is a strong SPE.*

Proof of Proposition 13. We show that x^A is an A-maximal strong NE. The symmetric argument shows that x^B is a B-maximal strong NE. We begin by making some preliminary observations as follows:

Lemma 15 *a) $u_i(x^A) \geq 0$ for every i.*

b) For any $k = 1, \ldots, K$,

$$\{i \in I \setminus T_k : x_i^A = A\} \subset Y_{k+1}, \quad \textit{and} \quad \{i \in I \setminus T_k : x_i^A = B\} \subset Z_{k+1}. \tag{41}$$

c) For any k, $J \subset I \setminus T_k$ and x_J, if $u_i(x_J, x_{-J}^A) \geq 0$ for every $i \in J$, then

$$\{i \in J : x_i = A\} \subset Y_{k+1} \quad \textit{and} \quad \{i \in J : x_i = B\} \subset Z_{k+1}. \tag{42}$$

In particular, for any y_{-T_k} such that $u_i(y_{-T_k}, x_{T_k}^A) \geq 0$ for every $i \in I \setminus T_k$,

$$\{i \in I \setminus T_k : y_i = A\} \subset Y_{k+1} \quad \textit{and} \quad \{i \in I \setminus T_k : y_i = B\} \subset Z_{k+1}. \tag{43}$$

Proof of Lemma 15. a) Suppose $i \in P_{k+1}$. Then

$$\begin{aligned} u_i(x^A) &= u_i\left(x_{T_k}^A, x_{P_{k+1}} = (B, \ldots, B), x_{-T_k - P_{k+1}}^A\right) \\ &\geq u_i\left(x_{T_k}^A, x_{P_{k+1}} = (B, \ldots, B), x_{-T_k - P_{k+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &> u_i\left(x_{T_k}^A, x_{Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y_{k+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq 0, \end{aligned}$$

where the last inequality holds trivially if $i \in (I \setminus T_k) \setminus Y_{k+1}$ and by the definition of Y_{k+1} if $i \in Y_{k+1}$. $u_i(x^A) \geq 0$ holds also when $i \in Q_{k+1}$, R_{k+1} or $I \setminus T_K$.

b) Let $Y = \{i \in I \setminus T_k : x_i^A = A\}$. If $Y \not\subset Y_{k+1}$, then

$$\begin{aligned} &u_i\left(x_{T_k}^A, x_{Y \cup Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y - Y_{k+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq u_i\left(x_{T_k}^A, x_Y = (A, \ldots, A), x_{-T_k - Y} = (\emptyset, \ldots, \emptyset)\right) = u_i(x^A) \geq 0 \end{aligned}$$

for $i \in Y$, and

$$\begin{aligned}&u_i\left(x^A_{T_k}, x_{K\cup Y_{k+1}} = (A,\ldots,A), x_{-T_k-K-Y_{k+1}} = (\emptyset,\ldots,\emptyset)\right)\\ &\geq u_i\left(x^A_{T_k}, x_{Y_{k+1}} = (A,\ldots,A), x_{-T_k-Y_{k+1}} = (\emptyset,\ldots,\emptyset)\right) \geq 0\end{aligned}$$

for $i \in Y_{k+1}$. This contradicts the maximality of Y_{k+1}.

c) Let $Y = \{i \in J : x_i = A\}$. If $Y \not\subset Y_{k+1}$, then

$$u_i\left(x^A_{T_k}, x_{Y\cup Y_{k+1}} = (A,\ldots,A), x_{-Y-Y_{k+1}} = (\emptyset,\ldots,\emptyset)\right) \geq u_i\left(x_J, x^A_{-J}\right) \geq 0$$

for every $i \in Y$ because of (41), and

$$\begin{aligned}&u_i\left(x^A_{T_k}, x_{Y\cup Y_{k+1}} = (A,\ldots,A), x_{-Y-Y_{k+1}} = (\emptyset,\ldots,\emptyset)\right)\\ &\geq u_i\left(x^A_{T_k}, x_{Y_{k+1}} = (A,\ldots,A), x_{-T_k-Y_{k+1}} = (\emptyset,\ldots,\emptyset)\right)\\ &\geq 0\end{aligned}$$

for every $i \in Y_{k+1}$. This contradicts the maximality of Y_{k+1}. Hence (42) holds. (42) implies (43) if we set $J = I \setminus T_k$. ■

We now return to the proof of Proposition 13.

In what follows, denote by J the deviating coalition of buyers. We first show that x^A is a strong NE by verifying (40) for each choice of J specified below.

1) First take J such that $J \cap T_1 \neq \emptyset$.

   If $J \cap P_1 \neq \emptyset$, take $i \in J \cap P_1$. Then $x^A_i = B$ by the definition of x^A. If x_J is such that $x_i = A$ and $u_j(x_J, x^A_{-J}) \geq 0$ for every $j \in J$, then

$$\begin{aligned}u_i(x_J, x^A_{-J}) &\leq u_i\left(x_{Y_1} = (A,\ldots,A), x_{-Y_1} = (\emptyset,\ldots,\emptyset)\right)\\ &< u_i\left(x_{P_1} = (B,\ldots,B), x_{-P_1} = (\emptyset,\ldots,\emptyset)\right)\\ &\leq u_i(x^A),\end{aligned}$$

   where the first inequality follows from (41) and (42). Likewise, (40) holds for any J such that $J \cap Q_1 \neq \emptyset$. If J is such that $J \cap R_1 \neq \emptyset$, then take $i \in J \cap R_1$. By definition, $x^A_i = \emptyset$. If $x^A_i = A$, then since $i \notin Y_1$, we have $u_i(x_J, x^A_{-J}) < 0 = u_i(x^A)$. We also have $u_i(x_J, x^A_{-J}) < 0 = u_i(x^A)$ if $x_i = B$ since $i \notin Z_1$.

2) As an induction hypothesis, suppose that (40) holds for any J such that $J \cap T_k \neq \emptyset$.

   Suppose that we take J such that $J \cap T_k = \emptyset$ but $J \cap T_{k+1} \neq \emptyset$.

   If $J \cap P_{k+1} \neq \emptyset$, take $i \in J \cap P_{k+1}$. Then $x_i^A = B$ by the definition of x^A. If x_J is such that $x_i = A$ and $u_j(x_J, x_{-J}^A) \geq 0$ for every $j \in J$, then

$$\begin{aligned} u_i(x_J, x_{-J}^A) &= u_i\left(x_{T_k}^A, x_J, x_{-J-T_k}^A\right) \\ &\leq u_i\left(x_{T_k}^A, x_{Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y_{k+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &< u_i(x_{T_k}^A, x_{P_{k+1}} = (B, \ldots, B), x_{-T_k - P_{k+1}} = (\emptyset, \ldots, \emptyset)) \\ &\leq u_i(x^A), \end{aligned}$$

   where the second line follows from (41) and (42). By the similar argument, (40) holds for any J such that $J \cap Q_{k+1} \neq \emptyset$. If $J \cap R_{k+1} \neq \emptyset$, take $i \in J \cap R_{k+1}$. Then $x_i^A = \emptyset$ by definition. If $x_i = A$, then since $i \notin Y_{k+1}$, $u_i(x_J, x_{-J}^A) < 0 = u_i(x^A)$. We also have $u_i(x_J, x_{-J}^A) < u_i(x^A)$ if $x_i = B$ since $i \notin Z_{k+1}$.

3) Finally, suppose that $J \subset I \setminus T_K$. Since $P_{K+1} = \emptyset$, if $x_J = (B, \ldots, B)$, there exists $i \in J$ such that

$$\begin{aligned} u_i(x^A) &= u_i\left(x_{T_K}^A, x_{Y_{K+1}} = (A, \ldots, A), x_{-T_K - Y_{K+1}} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq u_i\left(x_{T_K}^A, x_J = (B, \ldots, B), x_{-T_K - J} = (\emptyset, \ldots, \emptyset)\right) \\ &= u_i(x_J = (B, \ldots, B), x_{-J}^A), \end{aligned}$$

   where the last equality holds because $x_i^A \neq B$ for any $i \in I \setminus T_K$ by definition. Clearly, no other joint deviation x_J by J yields a higher payoff for i than $u_i(x_J = (B, \ldots, B), x_{-J}^A)$. Hence, (40) holds for any J such that $J \subset I \setminus T_K$.

We next show that the strong NE x^A is A-maximal. Take any strong NE y. Clearly, $u_i(y) \geq 0$ for every $i \in I$. If $i \in P_1$, then $y_i = B$: If $y_i \neq B$, then

$$\begin{aligned} u_i\left(x_{P_1} = (B, \ldots, B), y_{-P_1}\right) &> u_i\left(x_{Y_1} = (A, \ldots, A), x_{-Y_1} = (\emptyset, \ldots, \emptyset)\right) \\ &\geq u_i(y), \end{aligned}$$

where the last inequality from (43) for $k = 0$. Hence, y violates (40). Likewise, we can conclude that

$$y_i = \begin{cases} A & \text{if } i \in Q_1, \\ \emptyset & \text{if } i \in R_1. \end{cases}$$

Hence, $y_{T_1} = x^A_{T_1}$. As an induction hypothesis, suppose that $y_{T_k} = x^A_{T_k}$. If $i \in P_{k+1}$ then $y_i = B$: If $y_i \neq B$, then

$$\begin{aligned}
&u_i\left(y_{T_k}, x_{P_{k+1}} = (B, \ldots, B), y_{-T_k - P_{k+1}}\right) \\
&> u_i\left(y_{T_k}, x_{Y_{k+1}} = (A, \ldots, A), x_{-T_k - Y_{k+1}} = (\emptyset, \ldots, \emptyset)\right) \\
&\geq u_i(y_{T_k}, y_{-T_k}),
\end{aligned}$$

where the last inequality follows from (43) since $y_{T_k} = x^A_{T_k}$ by the induction hypothesis. Hence, y violates (40). We also have

$$y_i = \begin{cases} A & \text{if } i \in Q_{k+1}, \\ \emptyset & \text{if } i \in R_{k+1}. \end{cases}$$

Hence, $y_{T_{k+1}} = x^A_{T_{k+1}}$. Suppose finally that $i \in I \setminus T_K$. Since $\{i \in I \setminus T_K :, y_i = A\} \subset Y_{K+1}$ by (**??**), $y_i = A$ implies $x^A_i = A$, showing that x^A is A-maximal. ■

Proof of Proposition 14. In view of Proposition 9, it suffices to check the existence of a strong NE when G is either cyclic or complete.

1) G is a cycle.

Suppose that for $\varepsilon > 0$ small, p is given by

$$p_i = \begin{cases} v^1 - v^2 - \varepsilon & \text{if } i = 1,\ N-1, \\ -\varepsilon & \text{if } i = 2, \ldots, N-2, \\ v^2 - v^0 - \varepsilon & \text{if } i = N. \end{cases}$$

We can then verify that under (p, q^*), $Q_1 = \{1, \ldots, N-1\}$: For $i = 1$ and $N-1$,

$$\begin{aligned}
u_i\left(x_{Q_1} = (A, \ldots, A), x_N = \emptyset\right) &= v^1 - p_i \\
&= v^2 + \varepsilon \\
&> u_i\left(x = (B, \ldots, B)\right),
\end{aligned}$$

and for $i = 2, \ldots, N-2$,

$$\begin{aligned}
u_i\left(x_{Q_1} = (A, \ldots, A), x_N = \emptyset\right) &= v^2 - p_i \\
&= v^2 + \varepsilon \\
&> u_i\left(x = (B, \ldots, B)\right),
\end{aligned}$$

but for $i = N$,

$$\begin{aligned} u_N\left(x = (A, \ldots, A)\right) &= v^2 - p_N \\ &= v^0 + \varepsilon \\ &< u_N\left(x = (B, \ldots, B)\right). \end{aligned}$$

We can also verify that $P_1 = R_1 = \emptyset$. Given $T_1 = I \setminus \{N\}$, $Q_2 = \{N\}$:

$$u_N(x_{T_1}, x_N = A) = v^2 - p_N = v^0 + \varepsilon > u_N(x_{T_1}, x_N = B).$$

Therefore, $Q_1 \cup Q_2 = I$ and seller A's payoff under (p, q^*) equals

$$\pi_A(p, q^*, \sigma) = 2(v^1 - v^2 - \varepsilon) + v^2 - v^0 - \varepsilon = 2v^1 - v^0 - v^2 - 3\varepsilon,$$

which is strictly positive if $2v^1 - v^0 - v^2 > 0$ and ε is sufficiently small. If $2v^1 - v^0 - v^2 < 0$, then we can verify that $\pi_A(p, q^*, \sigma) > 0$ if we take $\varepsilon > 0$ small and p such that

$$p_i = \begin{cases} v^2 - v^1 - \varepsilon & \text{if } i = 1,\ N-1, \\ -\varepsilon & \text{if } i = 2, \ldots, N-2, \\ v^0 - v^2 - \varepsilon & \text{if } i = N. \end{cases}$$

2) Suppose next that G is complete.

Consider p such that

$$p_i = \begin{cases} v^{N-2} - v^{N-1} - \varepsilon & \text{if } i = 1, \ldots, N-1, \\ v^{N-1} - v^0 - \varepsilon & \text{if } i = N. \end{cases}$$

Then $Q_1 = \{1, \ldots, N-1\}$ and $Q_2 = \{N\}$, and

$$\begin{aligned} \pi_A(p, q^*, \sigma) &= (N-1)(v^{N-2} - v^{N-1} - \varepsilon) + v^{N-1} - v^0 - \varepsilon \\ &= (N-1)v^{N-2} - (N-2)v^{N-1} - v^0 - N\varepsilon, \end{aligned}$$

which is strictly positive if $(N-1)v^{N-2} - (N-2)v^{N-1} - v^0 > 0$ and ε is sufficiently small. If $(N-1)v^{N-2} - (N-2)v^{N-1} - v^0 < 0$, then $\pi_A(p, q^*, \sigma) > 0$ if $\varepsilon > 0$ is small and p is given by

$$p_i = \begin{cases} v^{N-1} - v^{N-2} - \varepsilon & \text{if } i = 1, \ldots, N-1, \\ v^0 - v^{N-1} - \varepsilon & \text{if } i = N. \end{cases}$$

In either case, hence, seller A can profitably deviate from $p^* = 0$ against $q^* = 0$. ■

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