Juselius, Katarina

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Testing for Near I(2) Trends When the Signal to Noise Ratio Is Small

Katarina Juselius

Abstract
Researchers seldom find evidence of I(2) in exchange rates, prices, and other macroeconomics time series when they test the order of integration using univariate Dickey-Fuller tests. In contrast, when using the multivariate ML trace test we frequently find double unit roots in the data. Our paper demonstrates by simulations that this often happens when the signal-to-noise-ratio is small.

JEL C1 C18 C22 C32 C52
Keywords Univariate and multivariate unit root tests; double unit roots; near I(2)

Authors
Katarina Juselius, ✉ University of Copenhagen, Denmark, katarina.juselius@econ.ku.dk

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1 Introduction

The global financial and economic crisis that began in 2007 has led to an increased interest in the mechanisms that cause asset prices to undergo persistent swings away and towards long-run equilibrium values. One important implication of the recent theory of Imperfect Knowledge Economics (IKE) by Frydman and Goldberg (2007, 2011) is that fully rational behavior under imperfect knowledge will show such tendencies. While these wide swings are typical of most prices subject to speculation, such as nominal exchange rates and stock prices, they also characterize many variables in the real economy, for example unemployment rates, suggesting a close two-way interrelationship between the financial market and the real economy.

The purpose of this paper is to discuss different econometric characterizations of these persistent swings, in particular focussing on nominal and real exchange rates in periods of currency float. Figure 1, upper panel, illustrates how the German mark/US dollar nominal exchange rate has shown a tendency to move in long persistent swings around its long-run purchasing power parity (PPP) value as given by the prices differential between the two countries. The graphs illustrate that the longest swing took place from approximately 1976 to 1988, followed by shorter swings. Figure 1, lower panel, shows the real exchange rate (measured as $p - p^* - s$) together with the long bond rate differential. It appears that the long swings in the real exchange rates (inherited from the nominal exchange rate) move almost in parallel with the swings in the bond rate differential.

Given the assumption that purchasing power parity holds as a stationary condition one would a priori expect relative prices and nominal exchange rates to be I(1) and the real exchange rate to be I(0). However, econometric tests often find many real and nominal exchange rates to be I(1) or even more persistent. For example, Engel and Hamilton (1990) found that the random walk model is strongly rejected in favor of a segmented-trends model for nominal exchange rates.[1] Shocks to a segmented-trends process display a high degree of persistence because the segmented trends have a long-lasting impact on both the level and the first difference of the variable. This is in contrast to shocks to a random walk series which are persistent only in the level. Also an I(2) process has a tendency to generate longer-lasting swings similar to a segmented-trends process, because the shocks have a persistent impact both on the levels and the first differences (Johansen, 1997, 2006a, Paruolo and Rahbek, 1999).

[1] Other studies that reject the random walk in favor of a segmented-trends model include Engel (1994) and Cheung and Erlandsson (2005).

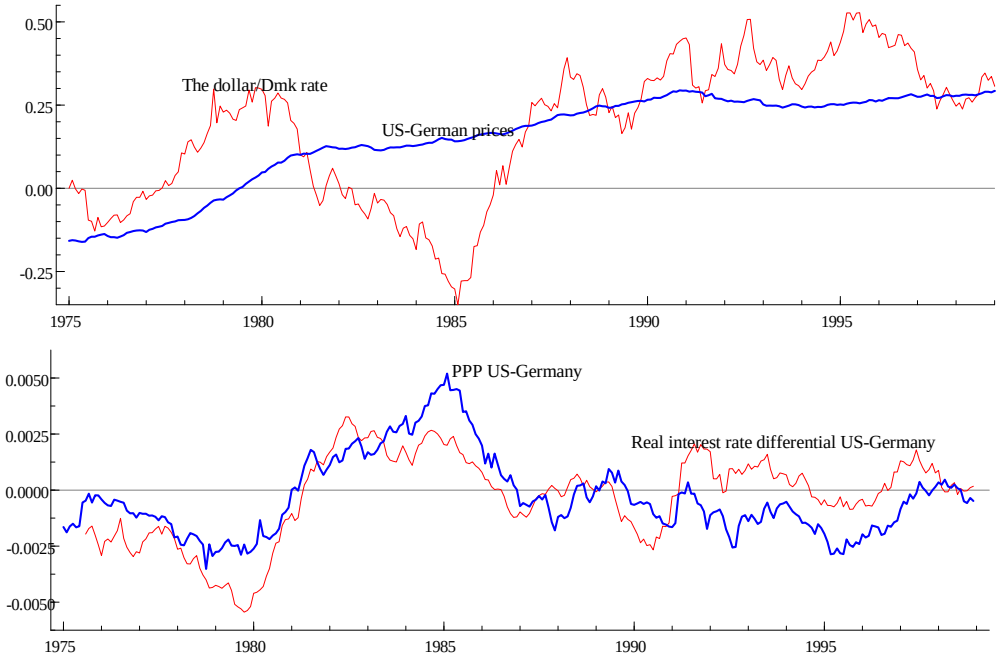

Figure 1: The graphs of the (mean and range adjusted) German-US price differential, pp, and the nominal exchange rate, s_{12} (upper panel), and the $ppp = pp - s_{12}$ and the real bond rate differential(lower panel).

In line with this, nominal and real exchange rates have frequently been well approximated by an I(2) process (Juselius, 1995, 2006, Johansen et al. 2011) but only when using multivariate unit root tests. Based on univariate Dickey-Fuller tests they are mostly found to be at most I(1). To explore why this is the case, we have simulated data designed to replicate typical features of relative prices and nominal exchange rates relying on results in Frydman and Goldberg (2007). Given an assumption that economic actors make forecasts under imperfect information they show that real exchange rates are likely to have a small signal-to-noise ratio. When testing the order of integration of the simulated series the results show that the univariate DF test tends to reject the second (near) unit root in almost all cases, whereas the multivariate test almost always finds it. The former result can be explained by the low power of the univariate DF tests to detect a second unit root when the shocks to the drift term of the differenced process are small compared to the shocks to the differenced process itself, i.e. when the signal-to-noise ratio is small.

We apply the above results to a data set consisting of the German Mark/US dollar exchange rate, German and US prices and interest rates (1975-1999). Based on the multivariate tests we could not reject the I(2) type characterization of the

nominal and real exchange rate at high significance levels against the null of an I(1) characterization. Based on the univariate Dickey-Fuller tests the double unit root was rejected.

2 Testing for a double unit root when the signal-to noise ratio is small

We first introduce the baseline autoregressive model of order k and discuss the augmented D-F model, the Engle-Hamilton segmented trends model and the Imperfect Knowledge model. We then simulate time series with double (near) unit roots and a small signal-to-noise ratio using parameter values that closely replicate the characteristic features of actual prices and exchange rates. To start with, just one single case is used to illustrate in detail why it is often difficult to discover a second unit root when the signal-to-noise ratio is small. We also illustrate the difference between a random walk, a near I(2) and an I(2) process and then proceed to test the series using univariate versus multivariate test procedures.

2.1 Univariate models

The pronounced persistence of many real exchange rates is often associated with a unit root in the data and mostly tested based on univariate Dickey-Fuller type models such as:

$$\Delta q_t = \mu - \rho_1 q_{t-1} + \rho_{11}\Delta q_{t-1} + ... + \rho_{1k}\Delta q_{t-k} + \varepsilon_{q_t} \tag{1}$$

where Δ is the first difference operator, q_t stands for the log of the real exchange rate in a period of currency float, and ε_{q_t} is an error term. The null of a unit root in q_t is formulated as the composite null hypothesis ($\rho_1 = 0$ and $\mu = 0$) where the second condition reflects the fact that one would not consider deterministic linear trends in real exchange rates to be economically plausible. When the null cannot be rejected we conclude that there is at least one unit root in the real exchange rates. To test whether there is a double unit root, (1) is respecified as:

$$\Delta^2 q_t = -\rho_2 \Delta q_{t-1} + \rho_{21}\Delta^2 q_{t-1} + ... + \rho_{2,k-1}\Delta^2 q_{t-k+1} + \varepsilon_{q_t} \tag{2}$$

where $\rho_2 = 0$ is the null. In this case, the real exchange rate would exhibit a very pronounced persistence as its change is a unit root process and the real exchange rates would be likely to drift off from it its long-run value for extended periods of time. If, instead, ρ_2 deviates from zero with a very small amount, then Δq_t would be mean-reverting, but the mean-reversion would be very slow. In this

case, moderately sized realizations of q_t could easily exhibit a similar persistence as an I(2) process and, for practical purposes, would be difficult to distinguish from an I(2) process.

Another possibility is to model the long swings in real exchange rate as a combination of unit roots and piecewise linear trends as in Engle and Hamilton (1990) implying the following change in the specification of (1):

$$\Delta q_t = \mu_t + \rho_{11}\Delta q_{t-1} + ... + \rho_{1k}\Delta q_{t-k} + \varepsilon_{q_t}, \quad \varepsilon_{q_t} \sim N(0, \sigma^2_{\varepsilon_q}) \tag{3}$$

where $\mu_t = \gamma_i \, i = 1, 2, 3, ...$ In this case, the real exchange rate would be described by piecewise linear trends with shifting slope parameters γ_i. If the shifts in the secular trends take place at ever smaller intervals, μ_t could instead be modelled as a random walk

$$\mu_t = \mu_{t-1} + \varepsilon_{\mu_t} \quad \varepsilon_{\mu_t} \sim N(0, \sigma^2_{\varepsilon_\mu}) \tag{4}$$

where ε_{μ_t} would describe the change in slope parameter from time t to $t+1$ and the piece-wise linear trend specification (3) would converge to a double unit root process.

The Imperfect Knowledge Economics (IKE) model described in Frydman and Goldberg (2007, 2011) resembles the secular trends model but differs with respect to the specification of μ_t:

$$\mu_t = \rho_t \mu_{t-1} + \varepsilon_{\mu_t} \quad \varepsilon_{\mu_t} \sim N(0, \sigma^2_{\varepsilon_\mu}) \tag{5}$$

where μ_t is a drift term measuring the change in the real exchange rate due to a change in individuals' forecasting strategies and in the underlying fundamentals. When q_t is in the neighborhood of the long-run benchmark value individuals change their forecasts conservatively and we would expect $\rho_t \simeq 1$, whereas when q_t is far away from the benchmark, we would expect $\rho_t < 1$. Since the far-from equilibrium period is likely to be much shorter than the "close-to-neighborhood" period, the average value, $\bar{\rho}$, of ρ_t for $t = 1, ..., T$ is likely to be close to unity. This is in particular so when T is sufficiently large to cover several long swings in the real exchange rate.[2]

Another important feature of the IKE model to be subsequently discussed is that the signal-to-noise ratio $\sigma^2_{\varepsilon_\mu}/\sigma^2_{\varepsilon_q}$ can be very small implying that the second near unit root associated with (5) may not be easily detectable.

2.2 Illustrating different degrees of persistence

Because the drift term (5) is unobservable in actual time-series, it is useful to simulate the persistency properties of processes described by (3) - (5) under different

[2] For varying sub-sample periods, the average value of ρ_t may of course vary to some extent.

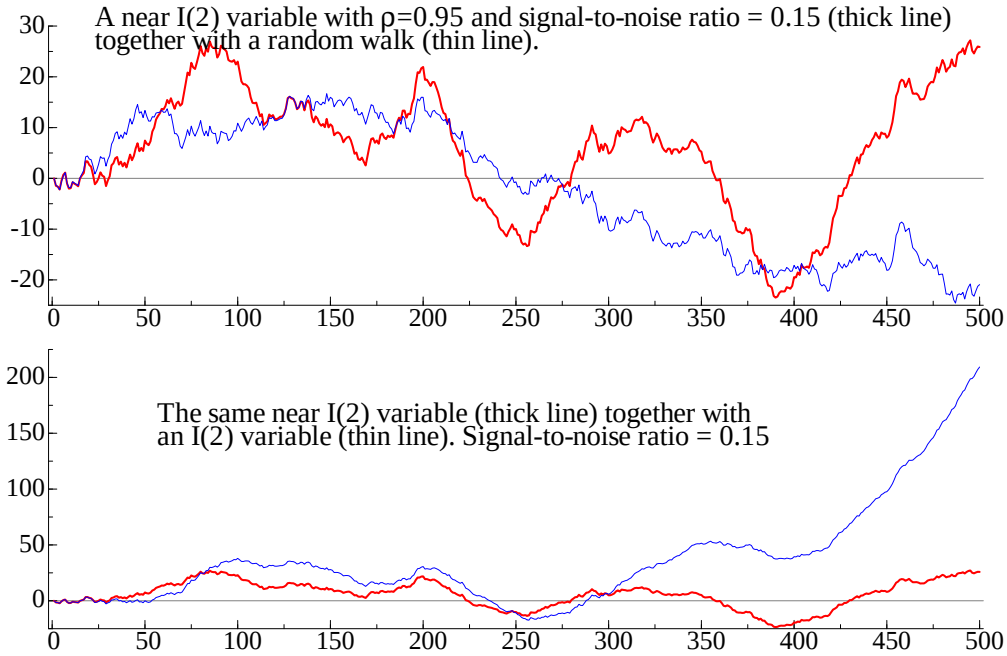

Figure 2: The graph of a near-$I(2)$ variable together with a random walk (upper panel) and the same near-$I(2)$ variable together with an $I(2)$ variable

assumptions of the drift term. For this purpose, we have generated time-series according to:

$$\Delta q_t = \mu_t + \varepsilon_{q_t} \quad \varepsilon_{q_t} \sim N(0,1) \tag{6}$$

and

$$\mu_t = \bar{\rho}\mu_{t-1} + \varepsilon_{\mu_t}, \quad \varepsilon_{\mu_t} \sim N(0, 0.15^2) \tag{7}$$

where $\bar{\rho}$ is {0.0, 0.95, 1.0} and lagged differences have been set to zero without loss of generality. The length of the simulated sample is set to 500 corresponding to roughly 40 years of monthly observations.

Figure 2, upper panel, illustrates swings that have been generated by a random walk ($\rho = 0$) and a near-$I(2)$ process ($\rho = 0.95$). The lower panel compares the same near-$I(2)$ process ($\rho = 0.95$) with an $I(2)$ process ($\rho = 1.00$). The range of variation of the near-$I(2)$ process is 50 compared to 220 for the $I(2)$ process, which explains the difference in appearance of the identical near I(2) variables in the two panels. To isolate the effect of the persistency parameter ρ, all three series have been generated from an identical realization of the random shocks ε_{q_t} and ε_{μ_t} and the signal-to-noise ratio is 0.15 for the near I(2) and the I(2) series.

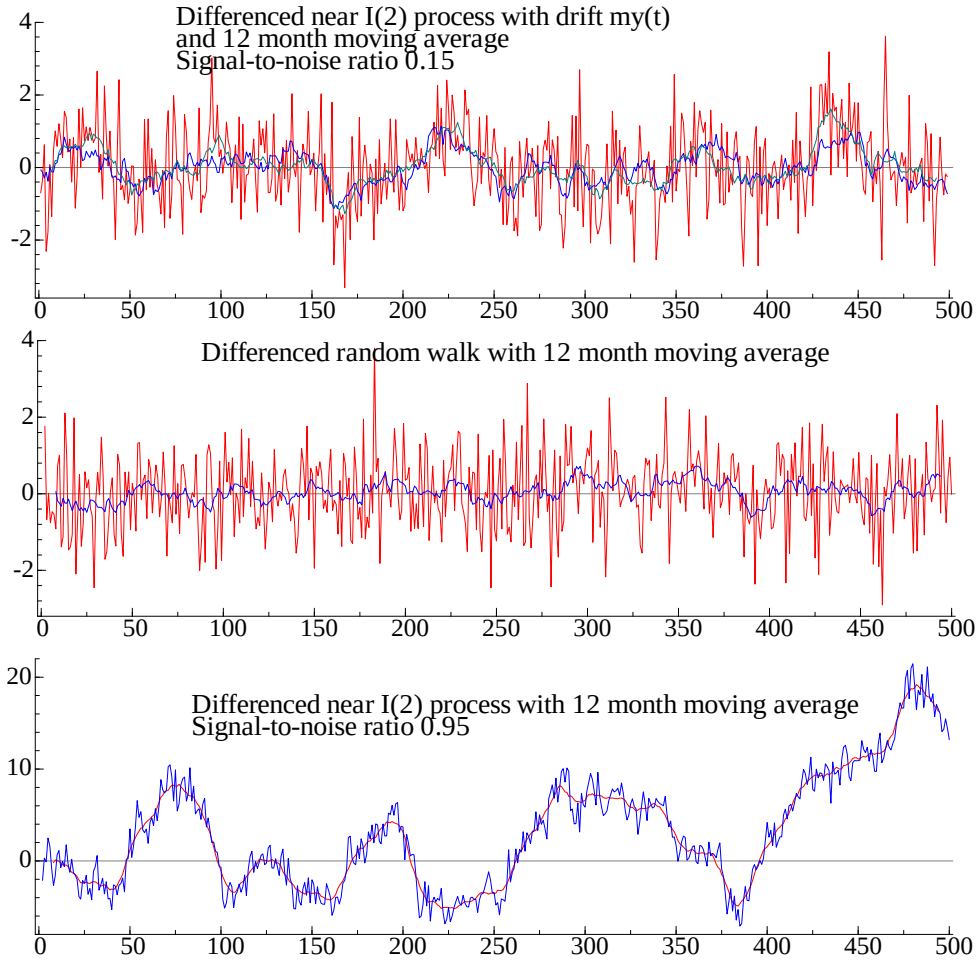

Figure 3: The graphs of Δq_t together with a 12 months MA when $\bar{\rho} = 0.95$ (upper panel), $\bar{\rho} = 0$ (middle panel) and $\bar{\rho} = 1$ (lower panel)

While both series in the upper panel exhibit persistent swings, they are much more pronounced for the near-$I(2)$ series compared to the random walk. The two series in the lower panel exhibit persistent swings, but the swings of the $I(2)$ series are less bounded, signifying the absence of significant mean reversion in the changes of an I(2) process. This becomes even more apparent at the end of the sample illustrating that we often need a long sample period to distinguish a near I(2) from an I(2) process.

A small, but persistent, drift term can be almost hidden for the eye when variance of the first differences is large and may not be easily detectable in a time graph. A (sufficiently long) MA of the original series will smoothen out the highly volatile short-term movements and, therefore, can provide a first rough indication

of a persistent drift term in the data. To illustrate this, Figure 3 upper panel, shows the graph of μ_t together with a 12 period MA of Δq_t for the simulated process (6) with $\bar{\rho} = 0.95$. While not identical, the moving average component seems to provide a fairly good description of the near I(2) drift term μ_t. But, as successive moving average values share *2k-2* identical observations, a moving average component is inherently a time dependent process. It is, therefore, likely to exhibit swings also when there are no swings in the data. For example, when q_t is a random walk process, i.e. when $\mu_t = 0$ and Δq_t is temporally independent, its *k* length moving average, $\overline{\Delta q_t} = f(\Delta q_{t-k}, ..., \Delta q_{t+k})$, is not independent. This is illustrated in Figure 3, middle panel, showing the differenced random walk of Figure 2 together with its 12 period moving average. As expected, the latter exhibits persistent fluctuations but compared to the moving average of the differenced near I(2) series they stay bounded within much more narrow bands around the mean.

To illustrate the difference between a small and a large signal-to-noise ratio, the lower panel shows a differenced near-$I(2)$ series that was generated with a large ratio of $\sigma_{\varepsilon_\mu}/\sigma_{\varepsilon_q}$. The persistence of the drift term of this series is identical to that of the near-I(2) series in the upper panel, i.e. both are generated with $\rho = 0.95$. Hence the series in the upper panel and the lowest panel differ only in terms of the signal to noise ratio, i.e. by the relative magnitude of the shocks ε_{μ_t} and ε_{q_t}. When the shocks to μ_t and Δq_t are of similar magnitude the difference is striking: no moving average is needed to see the persistent drift in the data.

Finally, Figure 4 illustrate how different realizations of $\varepsilon_{\mu t}$ and ε_{q_t} can produce series that look widely different. Both panels show a near I(2) ($\rho = 0.95$) and an I(2) ($\rho = 1.00$) series with a signal-to-noise ratio of 0.15, so the difference between the panels is only due to different stochastic realizations of the process. The two series in the upper panel exhibit long swings around the zero line whereas in the lower panel they show a much more pronounced tendency to move away from the zero line. As expected, the I(2) series tend to drift away from the zero line more persistently than the near I(2) series. Also, the divergence of the near I(2) and the I(2) series tend to be much stronger at the end of the sample. As will be demonstrated in Section 3 based on a simulation study of 5000 replications, a long sample is often needed to be able to statistically discriminate between the two. The "true" underlying process that have generated an observed economic variable is of course much more complex than all the models discussed in Section 2.1. The graphs of the simulated series suggest that a near I(2) process (with a small signal-to-noise ratio) can reproduce the long swings behavior we often see in actual real exchange rate data and that similar behavior also can be generated by an I(2) process for moderately sized samples. Whether these models are sufficiently close approximations to allow us to make inference on the broad features of the underlying data generating process will be studied by simulation in Section 3.

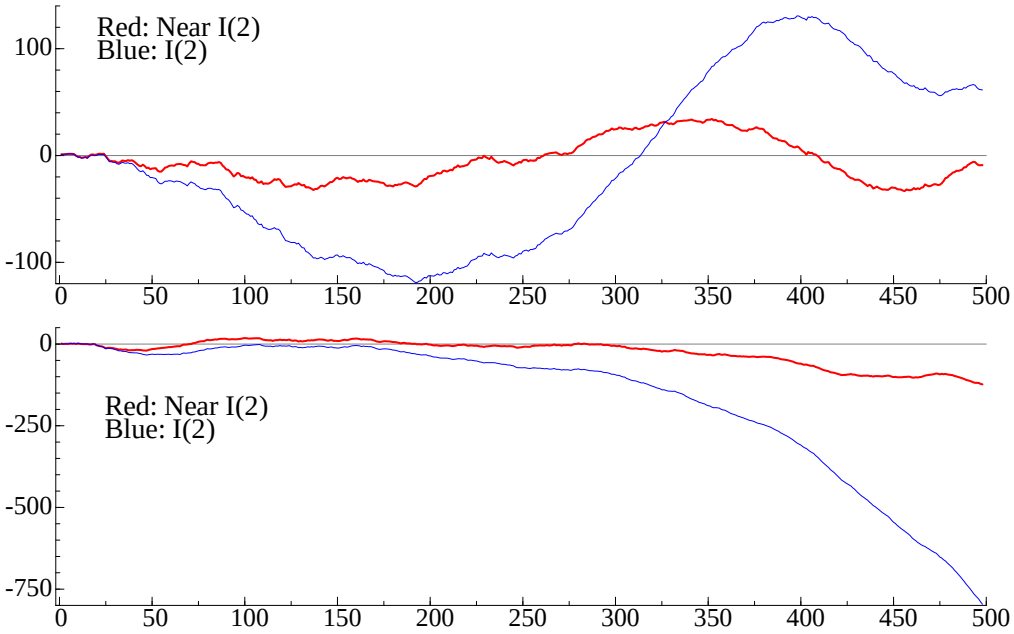

Figure 4: Two different realization of a near I(2) and an I(2) process

Another issue to discuss is how the asymptotic $I(2)$ inference is affected when data are near $I(2)$ rather than exactly $I(2)$. It is useful to distinguish between the case when a near unit root is treated as (1) stationary or (2) nonstationary. In the first case Johansen (2006b) showed by simulation that some inference (on steady-state values) became very fragile when a near unit root was treated as stationary. For example, up to 5000 observations were needed for the empirical distribution to converge to Students t when the near unit root was 0.998.

In the second case, Elliot (1998) showed both analytically and by simulations that the asymptotic distribution is no longer mixed Gaussian and that standard asymptotic inference can be misleading. However, Corollary 1 in Johansen (1997) can be used to show that inference on β and α in the $I(2)$ model is efficient and unbiased also in the near $I(2)$ case[3]. Since all results discussed in the subsequent sections have been obtained by cointegration analysis in the $I(2)$ model, the corollary result allows us to attach a fair degree of confidence to our empirical findings. Nonetheless, robustness is an important issue which needs to be further studied.

[3] This is because the second reduced rank condition (which is associated with the I(2) model property) does not affect the asymptotic efficiency of the ML estimator of β and α.

Table 1: Testing the order of integration with a Dickey-Fuller test

DF tests of	$q_t \sim I(2),\ \bar{\rho} = 1.0$		$q_t \sim nearI(2),\ \bar{\rho} = 0.95$		$q_t \sim I(1),\ \bar{\rho} = 0.0$	
	$\hat{\rho}_i$	$\tau - ratio$	$\hat{\rho}_i$	$\tau - ratio$	$\hat{\rho}_i$	$\tau - ratio$
$\sigma_v/\sigma_\varepsilon$	0.15		0.15		-	
$I(1): \rho_1 = 0$	0.007	5.57	-0.004	-0.93	-0.0008	-0.23
$I(2): \rho_2 = 0$	-0.31	-6.48	-0.72	-12.8	-0.96	-21.3
$\sigma_v/\sigma_\varepsilon$	1.0		1.0			
$I(1): \rho_1 = 0$	0.0006	1.96	0.0006	1.94		
$I(2): \rho_2 = 0$	-0.021	-1.90	-0.031	-2.74		

2.3 Univariate Dickey-Fuller Tests

The near I(2) and the I(2) series were all simulated for a fairly small signal-to-noise ratio (0.15) and the drift term was not easily detectable as it was well hidden in the very volatile first differences. Both processes contain two large characteristic roots, one associated with a high error variance, $\sigma^2_{\varepsilon_q}$, the other with a small error variance, $\sigma^2_{\varepsilon_\mu}$. In this case, the power of univariate unit root tests to detect the second (near) unit root is likely to be low because the estimated residual is a function of ε_{q_t} and ε_{μ_t}. With a small signal-to-noise ratio, ε_{q_t} will completely dominate ε_{μ_t} and the small but persistent drift that is associated with the second large root becomes hard to detect. To illustrate, we test the null of a unit root with the augmented Dickey-Fuller test based on (1) for the three series depicted in Figure 2.

Table 1 presents the results of testing the null of a unit root in the levels ($\gamma_1 = 0$) and first differences ($\gamma_2 = 0$) of the simulated series using Dickey-Fuller regressions.[4] The results in the upper part of the table are for the simulated $I(2)$, near $I(2)$ with $\rho = 0.95$, both with a signal-to-noise ratio of 0.15, and for the $I(1)$ series. The lower part of the table, compare the results for similarly simulated $I(2)$ and near $I(2)$ series but with a signal-to-noise ratio of 1.0. The results show that the null of a second unit root is strongly rejected for both the $I(2)$ and near-$I(2)$ cases when $\sigma_{\varepsilon_\mu}/\sigma_{\varepsilon_q}$ is small, whereas it cannot be rejected when $\sigma_{\varepsilon_\mu}/\sigma_{\varepsilon_q}$ is large.

The results indicate that the univariate Dickey-Fuller test has great difficulty detecting the large root associated with the persistent drift term, μ_t when the signal-to-noise ratio is small. The null of a double unit root was strongly rejected both when $q_t \sim I(2)$ and when it was a near I(2) series with $\bar{\rho} = 0.95$. When q_t was a random walk it was also rejected but now correctly so. In all cases the

[4] The regressions for levels and first differences were $\Delta q_t = \rho_1 q_{t-1} + \rho_{11}\Delta q_{t-1} + \rho_0 + \varepsilon_t$ and $\Delta^2 q_t = \rho_2 \Delta q_{t-1} + \varepsilon_t$, respectively.

(correct) null of (at least) one unit root could not be rejected. Section 3 reports a more comprehensive simulation study.

2.4 Multivariate I(2) trace tests

In a univariate model, it is straightforward to determine whether a variable has two large roots (with a modulus that is large but less than one) whereas in a multivariate model we can only determine the number of common stochastic trends in the system and whether they are of first order or second order. The classification of variables according to their order of integration is, therefore, more involved in the latter case which might suggest the use of univariate tests.[5] But, as demonstrated in Section 3, univariate Dickey-Fuller type of tests have low power to detect a second unit root, in particular when the signal to noise ratio is small. Since the multivariate model accounts for all information in the data a multivariate model is more powerful in this respect. Besides,the order of integration of individual series can also be determined within this model as shown in Section 4.3.

To give the intuition for the main issues we first use one set of simulations to analyze a three-dimensional VAR model in detail and then improve the generality of our findings based on a much larger simulation study. The VAR model has two lags and is conveniently expressed in acceleration rates, changes and levels:

$$\Delta^2 x_t = \Gamma \Delta x_{t-1} + \Pi x_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t \tag{8}$$

where Γ, Π are $p \times p$ matrices, μ_0, μ_1 are $p \times 1$ vectors and ε_t is $NID(0, \sigma_\varepsilon^2)$. The matrices Γ and Π are variation free but μ_0 and μ_1 are restricted to exclude the possibility of quadratic trends.

The hypothesis that $x_t \sim I(2)$ is formulated as two reduced rank hypothesis:

$$\Pi = \alpha \beta' \tag{9}$$

where α, β are $p \times r$, with r the cointegration rank, and

$$\alpha'_\perp \Gamma \beta_\perp = \xi \eta' \tag{10}$$

where $\beta_\perp, \alpha_\perp$ are $p \times p-r$ orthogonal complements to β, α, and ξ, η are $p-r \times p-r-s_2$. The number of common stochastic trends is $p-r = s_1 + s_2$ of which s_1 are integrated of order one and s_2 of order two. If $s_2 = 0$, (10) is a full rank matrix and $x_t \sim I(1)$. Thus testing whether $x_t \sim I(2)$ amounts to testing $s_2 > 0$.

Paruolo and Rahbek (1999) suggested the following parameterization of (8):

[5] For an extensive discussion and analysis of CVAR models, see Juselius (2006, 2013).

$$\Delta^2 x_t = \alpha(\beta' x_{t-1} + \delta' \Delta x_{t-1}) + \zeta \tau' \Delta x_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t$$

where $\tau = [\beta, \beta_{\perp 1}]$, δ is a $p \times s_2$ matrix of polynomially cointegrating parameters, such that $(\beta' x_{t-1} + \delta' \Delta x_{t-1}) \sim I(0)$, and ζ is a $p \times p - s_2$ matrix of medium run adjustment coefficients.

The three variables have been simulated to reflect typical time series properties of the real exchange rate, q_t, the interest rate differential between the two countries, b_t, and the long-term drift term, μ_t (measured for example by the change in nominal exchange rate, Δs_t). They are simulated in accordance with the IKE models (3) and (5):

$$\begin{aligned}
q_t &= q_{t-1} + \mu_t + \varepsilon_{q,t} = \sum_{i=1}^{t} \sum_{s=1}^{i} \overline{\rho}^{i-s} \varepsilon_{\mu,s} + \sum_{i=1}^{t} \varepsilon_{q,i} + \overline{\rho} \mu_0 \sum_{i=1}^{t} \overline{\rho}^{i} + q_0 \\
b_t &= b_{t-1} + \mu_t + \varepsilon_{b,t} = \sum_{i=1}^{t} \sum_{s=1}^{i} \overline{\rho}^{i-s} \varepsilon_{\mu,s} + \sum_{i=1}^{t} \varepsilon_{b,i} + \overline{\rho} \mu_0 \sum_{i=1}^{t} \overline{\rho}^{i} + b_0 \\
\mu_t &= \overline{\rho} \mu_{t-1} + \varepsilon_{\mu,t} = \sum_{i=1}^{t} \overline{\rho}^{t-i} \varepsilon_{\mu,i} + \overline{\rho}^{t} \mu_0
\end{aligned}$$

where $\sigma_{\varepsilon_q} = 1.0, \sigma_{\varepsilon_b} = 0.2, q_t$, $\sigma_{\varepsilon_\mu} = 0.15$, and $\overline{\rho} = 0.95$. Thus, the variables q_t and b_t share the same realization of μ_t . For an application to a problem with similar characteristics, see Johansen et al. (2010). As in the univariate case, we have generated 500 observations.

Provided the near unit root $\overline{\rho} = 0.95$ is approximated with a unit root in an empirical CVAR application, then the results would be consistent with $r = 1$, $s_1 = 1$, and $s_2 = 1$. In this case, q_t and b_t would share one common stochastic near I(2) trend and

$$q_t - b_t = \sum_{i=1}^{t} \varepsilon_{q,i} - \sum_{i=1}^{t} \varepsilon_{b,i} + q_0 - b_0 \tag{11}$$

would be $CI(2,1)$. Since the two I(1) trends in (11) cannot cancel by any linear combination $\delta' \Delta x_t$, (11) corresponds to the $\beta'_{\perp 1} x_t$ relation $(s_1 = 1)$ which can only become stationary by differencing. The polynomially cointegrated relation $(r = 1)$ corresponds to $\mu_t - \delta_1 \Delta q_t - \delta_2 \Delta b_t$ where $\delta_1 + \delta_2 = 1$.

Table 2 reports the multivariate rank test results where the first row shows the trace test for $s_2 = 3,2,1,0$, given $r = 0$ and the second row for $s_2 = 2,1,0$, given $r = 1$. The hypothesis $\{r = 1, s_2 = 1\}$ cannot be rejected based on a p-value of 0.23. It implies three unit roots in the characteristic polynomial. The lower part of Table 2 reports the modulus of the four largest roots in the characteristic

Table 2: Determination of the two rank indices in the bivariate model

Rank Test Statistics						
$p-r$	r	$s_2=3$	$s_2=2$	$s_2=1$	$s_2=0$	
3	0	1169.66 [0.00]	727.76 [0.00]	375.24 [0.00]	334.49 [0.00]	
2	1		366.12 [0.00]	**28.42** [0.23]	17.33 [0.40]	
3	2			7.92 [0.84]	3.69 [0.78]	
The characteristic roots:						
Unrestricted VAR			0.99	0.99	0.96	0.07
$r=1,s_1=2,s_2=0$			1.0	1.0	0.96	0.07
$r=1,s_1=1,s_2=1$			1.0	1.0	1.0	0.07

Table 3: The estimated values of $\beta,\beta_{\perp 1},\beta_{\perp 2}$ for the simulated I(2) process

	q_t	b_t	μ_t	t		Δq_t	Δb_t	$\Delta \mu_t$	1
The polynomially cointegrated relation $\beta' x_t + \delta' \Delta x_t$									
β'	−0.00 [−0.21]	−0.00 [−0.28]	1.00 [*NA*]	0.00 [1.69]	δ'	−0.49	−0.46	−0.00	0.00
The medium run cointegrated relation $\beta'_{\perp 1} x_t$									
$\beta'_{\perp 1}$	−0.93	1.00	0.00	0.02					
The non-cointegrating relation $\beta'_{\perp 2} x_t$									
$\beta'_{\perp 2}$	1.00	0.93	0.00						

polynomial. The unrestricted VAR contains two roots almost exactly equal to one and a third root, 0.96, which is very close to the simulated value of 0.95. If we approximate the latter with a unit root then the highest unrestricted root is 0.07 for the choice of $\{r=1,s_1=1,s_2=1\}$. But if we ignore the possibility of I(2) then the model would contain an unrestricted root of 0.96 for the choice of $\{r=1,s_1=2,s_2=0\}$. Such a large root is likely to jeopardize standard inference on stationarity at least for some hypotheses (Johansen, 2006b).

Table 3 report the estimates of $\beta,\beta_{\perp 1},\beta_{\perp 2}$.The test of the proportionality of q_t and b_t, formulated as the hypothesis:

$$H\tau = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta' x_t \\ \beta'_{\perp 1} x_t \end{bmatrix}$$

could not be rejected based on $\chi^2(2) = 1.60[0.45]$ consistent with the true data generating process.

To conclude the main difference between testing the order of integration based on a univariate versus a multivariate model is that the former is generally unable to detect a large root in μ_t when the signal to noise ratio is small, whereas the multivariate trace test is able to do so. Also, approximating a root of 0.95 with a unit root allows us to structure the data in terms of polynomial cointegration and common trends of different order, thereby exploiting the different persistency profiles of the data. Evidence of the second large root can also be found by checking the characteristic roots of the multivariate model whereas such evidence is usually not present in the roots of a univariate model, in particular when the signal-to-noise ratio is small.

3 A simulation study

We consider now the IKE model (6) - (7)

$$\begin{aligned} x_{1,t} &= x_{1,t-1} + \mu_t + \varepsilon_{1,t} && (12)\\ x_{2,t} &= x_{2,t-1} + \mu_t + 0.2\varepsilon_{2,t} && (13)\\ \mu_t &= 0.95\mu_{t-1} + 0.15\varepsilon_{u,t}. && (14) \end{aligned}$$

where $x_{1,t}$ represents the real exchange rate and $x_{2,t}$ the real bond rate differential. They share the same drift term but differs in terms of the short-term volatility with $x_{1,t}$ fluctuating much more than $x_{2,t}$ reflecting typical behavior of real exchange rates versus real interest rate differentials. The drift term μ_t is highly persistent and the signal-to-noise ratio between $\varepsilon_{1,t}$ and $\varepsilon_{u,t}$ is 1.0/0.15. The model (12)-(14) represents the baseline model. Since μ_t is generally not directly observable we have simulated three versions of the three-dimensional process $x_t' = [x_{1,t}, x_{2,t}, x_{3,t}]$ which differs with respect to the choice of $x_{3,t}$:

$$\begin{aligned} \textit{Case } 1 &: \; x_{3,t}^{(1)} = \mu_t \\ \textit{Case } 2 &: \; x_{3,t}^{(2)} = \mu_t + 0.2\varepsilon_{4,t} \\ \textit{Case } 3 &: \; x_{3,t}^{(3)} = \mu_t + \varepsilon_{5,t} \end{aligned}$$

where $\varepsilon_{i,t} \sim NID(0, \sigma_i^2)$, $i = 1, ..., 5$.

Table 4 reports the simulated results of testing the hypotheses (r, s_2) for $r = 0, 1, 2, 3$ and $s_2 = p - r, p - r - 1, ..., 0$ based on the multivariate rank test in a *CVAR* model. The latter is estimated for the three cases $S_1 = (x_{1,t}, x_{2,t}, x_{3,t}^{(1)})$, $S_2 = (x_{1,t}, x_{2,t}, x_{3,t}^{(2)})$, $S_3 = (x_{1,t}, x_{2,t}, x_{3,t}^{(3)})$ using two lags. S_1 represents the case when

the drift term, μ_t, is known, S_2 when it can be measured with a small error, S_3 when it is very imprecisely measured.

The results show generally that the multivariate I(2) test:

- rarely rejects cointegration, except when μ_t is imprecisely measured (S_3),
- seldom accepts two or three cointegration relations and no I(2) trends,
- rarely rejects I(2) when $T = 50$ or 100 provided μ_t is either known (S_1) or measured with a small error (S_2),
- rarely rejects the preferred hypothesis $H(1,1)$ unless T is large(500) and μ_t is imprecisely measured (S_3),
- frequently fails to reject the hypothesis $H(1,0)$ when T is large and μ_t is imprecisely measured (S_3), i.e. $\bar{\rho} = 0.95$ is found to be significantly different from a unit root, and
- seldom accepts that the process can be stationary.

In general, the larger the sample the smaller the standard errors and the higher the power of the test to discriminate between a root of 0.95 and 1.00. Thus, with a sufficiently large sample size even a tiny deviation from unity will be found significant even though the drift term μ_t is highly persistent. Also, the more imprecisely μ_t is measured, the more difficult it is to detect the large near I(2) root in the model.

Table 5 reports simulated univariate augmented Dickey-Fuller tests of the hypothesis $x_{1,t} \sim I(2), I(1), I(0)$ based on two respective three lags for $T = 50, 100, 250$ and 500. Each case has been replicated 5.000 times. The simulations of $x_{1,t}$ are identical to the ones being analyzed in Table 3 above. The upper part of the table reports the results for $\rho = 1.0$ and the lower part for $\rho = 0.95$. The results show that the ADF test

- rejects I(2) in essentially all cases whether $\rho = 1$ or 0.95,
- fails to reject I(1) in the absolute majority of cases

Thus, the results suggest that the univariate Dickey-Fuller test will essentially never detect a double unit root when the signal-to-noise ratio is small.

Table 4: Simulated frequences of testing the cases ($r = i$ and $s_2 = 3-i-j$), $i = 0,1,2,3$ and $j = 3-i,\ldots,0$

	$T=50$			$T=100$			$T=250$			$T=500$		
$H(r,s_2)$	S_1	S_2	S_3	S_1	S_2	S_3	S_1	S_2	S_3	S_1	S_2	S_3
An I(2) process $\rho = 1.0$												
$H(0,3)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$H(0,2)$	0.1	0.3	0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$H(0,1)$	9.0	16.4	24.0	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0
$H(0,0)$	2.6	3.4	4.2	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0
$H(1,2)$	7.9	8.1	5.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0
$H(1,1)$	**72.1**	**62.8**	**48.6**	**93.4**	**89.6**	**61.4**	**94.4**	**90.0**	**48.7**	**94.8**	**89.2**	**45.1**
$H(1,0)$	0.9	1.5	4.0	0.6	1.6	9.3	0.7	1.5	11.8	0.4	1.6	12.3
$H(2,1)$	5.0	5.3	9.0	4.1	5.5	17.6	3.2	5.2	23.1	3.0	5.4	24.0
$H(2,0)$	1.6	1.4	3.3	1.2	2.4	8.7	1.2	2.7	13.7	1.3	2.8	15.4
$H(3,0)$	0.6	0.8	1.4	0.7	0.9	2.5	0.5	0.7	2.6	0.5	1.1	3.2
A near I(2) process $\rho = 0.95$												
$H(0,3)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$H(0,2)$	0.1	0.3	0.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$H(0,1)$	9.5	17.2	26.2	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0
$H(0,0)$	5.1	6.2	6.5	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0
$H(1,2)$	7.7	7.6	4.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$H(1,1)$	**70.1**	**60.6**	**46.0**	**94.2**	**89.8**	**55.7**	**89.6**	**73.4**	**14.5**	**69.6**	**28.7**	**0.2**
$H(1,0)$	1.0	2.2	7.0	0.7	3.9	26.7	4.0	18.6	67.3	22.5	63.5	82.8
$H(2,1)$	5.2	4.7	6.9	4.0	5.4	14.6	3.8	6.0	14.7	3.7	5.1	13.6
$H(2,0)$	0.8	0.8	1.5	0.8	0.6	2.1	2.1	1.6	2.9	3.8	2.4	2.9
$H(3,0)$	0.5	0.4	0.6	0.2	0.3	0.5	0.5	0.4	0.6	0.5	0.3	0.5

Table 5: Simulated frequences for testing the order of integration based on the ADF

	$T=50$		$T=100$		$T=250$		$T=500$	
Lags	2	3	2	3	2	3	2	3
$x_{1,t}$ is an I(2) process								
$I(2)$	0.00	0.11	0.00	0.00	0.00	0.00	0.00	0.00
$I(1)$	0.94	0.83	0.95	0.95	0.91	0.91	0.76	0.76
$I(0)$	0.06	0.06	0.05	0.05	0.09	0.09	0.24	0.24
$x_{1,t}$ is an I(2) process ($\rho = 0.95$)								
$I(2)$	0.00	0.11	0.00	0.00	0.00	0.00	0.00	0.00
$I(1)$	0.94	0.83	0.95	0.95	0.95	0.95	0.95	0.95
$I(0)$	0.06	0.06	0.05	0.05	0.06	0.05	0.05	0.05

4 Empirical illustration: the dollar Dmk rate

This section illustrates the test procedures based on actual real exchange data for USA and Germany for a sample from May 1975 to December 1998 comprising the post Bretton Woods period of currency float. Section 4.1 first reports the univariate Dickey-Fuller test to determine the order of integration of the real and nominal exchange rate, US and German prices and long-term interest rates. Section 4.2 then reports the order of integration and cointegration based on the multivariate trace test in a five-dimensional VAR model of $x_t' = [p_{1,t}, p_{2,t}, s_{12,t}, b_{1,t}, b_{2,t}]$ where p stands for prices, s for the nominal dollar-Dmk rate, b for long-term bond rates and the subscript 1 stands for USA and 2 for Germany. Finally Section 4.3 reports tests of the order of integration of the individual series within the multivariate model.

4.1 Univariate Dickey-Fuller tests

Table 6 report the Dickey-Fuller univariate tests of I(1) based on (1), and of I(2) based on (2) for the five variables as well as for the following transformations: real exchange, $p_{1,t} - p_{2,t} - s_{12,t}$, the interest rate differential, $b_{1,t} - b_{2,t}$ and relative prices, $p_{1,t} - p_{2,t}$. The I(1) test failed to reject the null of one unit root in all cases except for the relative price for which $\hat{\rho}_1 = -0.005$ was found to be significant due to a very small standard error. The I(2) version of the Dickey-Fuller test failed to detect a second (near) unit root in almost all cases except for US long-term bond rate and US prices (borderline). For both the nominal and real exchange rate a double unit root was strongly rejected with almost identical test statistics signifying the fact that the two variables have moved almost in parallel. That a second large root was detected in the US bond rate is interesting and suggests that

Table 6: Testing the order of integration with a Dickey-Fuller test

D-F tests of	$p_{1,t}$		$p_{2,t}$		$s_{12,t}$		$p_{1,t}-p_{2,t}$	
	$\hat{\rho}_i$	τ-ratio	$\hat{\rho}_i$	τ-ratio	$\hat{\rho}_i$	τ-ratio	$\hat{\rho}_i$	τ-ratio
$I(1)$: $\rho_1 = 0$	-0.002	-1.76	-0.01	-1.63	-0.01	-1.47	-0.005	-4.18
$I(2)$: $\rho_2 = 0$	-0.21	-3.95	-0.67	-13.1	-0.90	-11.9	-0.62	-12.2
	$q_t = p_1 - p_2 - s_{12}$		$b_{1,t}$		$b_{2,t}$		$b_{1,t}-b_{2,t}$	
	$\hat{\rho}_i$	τ-ratio	$\hat{\rho}_i$	τ-ratio	$\hat{\rho}_i$	τ-ratio	$\hat{\rho}_i$	τ-ratio
$I(1)$: $\rho_1 = 0$	-0.02	-1.85	-0.01	-0.43	-0.01	-1.75	-0.03	-2.75
$I(2)$: $\rho_2 = 0$	-0.91	-11.9	0.11	2.70	-0.66	-11.2	-1.03	-13.9

it has exhibited similar long swings as the real exchange rates while not exhibiting the same short-term volatility. That the test borderline failed to reject a double unit root in the US CPI is also interesting as consumer prices are not likely to be subject to speculation in any significant manner and, hence, should not be affected by the long swings drift term μ_t. Therefore, a double root in US CPI prices suggests that CPI shocks, while not necessary large, have been persistent over this period and that the signal-to-noise ratio is rather high. This suggests that the power of Dickey-Fuller test to detect a double unit root is reasonable provided the signal-to-noise ratio is not too small.

4.2 Multivariate VAR based unit root tests

We estimate the VAR model (8) for $x_t' = [p_{1,t}, p_{2,t}, s_{12,t}, b_{1,t}, b_{2,t}]$ augmented with a few dummy variables, primarily to account for the German reunification as explained in Johansen et al. (2010) and in Juselius (2012).

Table 7 reports the $I(2)$ trace tests as well as the characteristic roots of the model. Since all versions for $r = 0$ were strongly rejected, they are not reported in the table. The estimated characteristic roots in Table 7 suggest a total number of five large roots in the unrestricted VAR, four of which are almost exactly on the unit circle, while the fifth, while large (0.88), is not equally close to one. Juselius (2012) shows that the case $(r = 2, s_1 = 1, s_2 = 2)$ is theoretically consistent with an IKE-based model. The first acceptable choice is $\{r = 1, s_1 = 3, s_2 = 1\}$. Both the preferred and the first acceptable case are consistent with five unit roots in the characteristic polynomial. Jensen (2013) derives a Likelihood ratio test for the choice between two non-nested models with equal number of unit roots, $LR\{\mathscr{H}(r-1, s_1+2) \mid \mathscr{H}(r, s_1)\}$. The test of $\mathscr{H}(1,3)$ against $\mathscr{H}(2,1)$ gives a test statistic of $13.6 > Q(.95) = 13.3$. Since $\mathscr{H}(2,1)$ is consistent with our theoretical prior we continue with the case $\{r = 2, s_1 = 1, s_2 = 2\}$. The given this

Table 7: Determination of the two rank indices

Rank Test Statistics								
$p-r$	r	$s_2=5$	$s_2=4$	$s_2=3$	$s_2=2$	$s_2=1$	$s_2=0$	
4	1		489.2 [0.00]	291.6 [0.00]	149.5 [0.00]	**84.1** [0.328]	**77.19** [0.137]	
3	2			140.4 [0.00]	**56.96** [0.93]	**42.85** [0.93]	**37.22** [0.77]	
2	3				**36.7** [0.94]	**17.4** [1.00]	**15.3** [0.96]	
Six largest characteristic roots:								
Unrestricted VAR			**0.99**	**0.99**	**0.98**	**0.98**	**0.82**	0.50
$r=2, p-r=3$			1.0	1.0	1.0	**0.96**	**0.96**	0.50
$r=2, s_1=2, s_2=1$			1.0	1.0	1.0	1.0	**0.96**	0.51
$r=2, s_1=1, s_2=2$			1.0	1.0	1.0	1.0	1.00	0.51

choice is 0.51. If, instead, we had chosen $s_2=0$ (and treated the variables in the model as $I(1)$), our model would contain two large roots with a modulus of 0.96 and any inference on stationarity would have been jeopardized.

To summarize: the case $\{r=2, s_1=1, s_2=2\}$ is supported by the data, is able to account for all five large roots in the unrestricted VAR, and is consistent with the economic prior.

4.3 Testing the order of integration of individual variables

As discussed above, the multivariate trace tests are informative about the order of integration of the vector process but are generally uninformative about the individual variables. It is, however, straightforward to formulate and test hypotheses of the order of integration of these variables given the choice of r, s_1 and s_2. We first define $\tau = \{\beta, \beta_{\perp,1}\}$. The hypothesis that a variable is $I(1)$ in the $I(2)$ CVAR model can be formulated as a known vector b_1 in $\tau = (b_1, b_{1\perp}\varphi)$ where $b_{1\perp}\varphi$ defines the other vectors to be restricted to lie in the orthogonal space of b_1. For example $b_1 = [0,0,1,0,0,0,0]$ is a test of the hypothesis that the nominal exchange rate, $s_{12,t}$, is a unit vector in τ. If not rejected, it can be considered $I(1)$, otherwise $I(2)$. See Johansen et al. (2006) for further details.

As prices generally are subject to linear deterministic (as well as stochastic) trends we need to test the hypothesis that the price as well as the trend-adjusted price is $I(1)$. As discussed in Johansen et al. (2010) this can be formulated as a test on $\beta = (H_1\varphi_1, H_2\varphi_2)$. For example, $H_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ would be a test of trend-adjusted US price being I(1). As the VAR model was specified to

Table 8: Testing hypotheses of I(1) versus I(2)

		$p_{1,t}$	$p_{2,t}$	$s_{12,t}$	$b_{1,t}$	$b_{2,t}$	t_{91}	t	$\chi^2(v)$	$p-val$
Is the price differential $I(1)$?										
$\mathscr{H}_1$	β_1'	1.0	-1.0	-	-	-	-	-	64.09 (4)	0.00
Is the nominal exchange rate $I(1)$?										
$\mathscr{H}_2$	β_1'	-	-	1.0	-	-	-	-	23.6 (4)	0.00
Is the US trend-adjusted price $I(1)$?										
$\mathscr{H}_3$	β_1'	1.0	-	-	-	-	*	*	39.1 (3)	0.00
Is the German trend-adjusted price $I(1)$?										
$\mathscr{H}_4$	β_1'	-	1.0	-	-	-	*	*	48.02 (3)	0.00
Is the bond rate differential $I(1)$?										
$\mathscr{H}_8$	β_1'	-	-	-	1.0	-1.0	-	-	11.2 (4)	0.02
Is the US bond rate $I(1)$?										
$\mathscr{H}_9$	β_1'	-	-	-	1.0	-	-	-	12.4 (4)	0.01
Is the German bond rate $I(1)$?										
$\mathscr{H}_{10}$	β_1'	-	-	-	-	1.0	-	-	5.5(4)	0.24
Is the real exchange rate $I(1)$?										
$\mathscr{H}_{11}$	β_1'	1.0	-1.0	-1.0	-	-	-	-	10.4 (4)	0.03

allow for a change in the slope of the linear trend at the time of the reunification we formulate the test to allow also for this possibility. Table 8 reports the Likelihood Ratio test results. Except for the German bond rate, all hypotheses were strongly rejected, indicating that the differenced processes are persistent enough to reject the $I(1)$ hypothesis in favor of an I(2)-type characterization. [6]

5 Conclusions

Macro-financial data (nominal exchange rates, stock prices, etc.) are typically characterized by very volatile short-run changes around smooth persistent trends. In this case, the drift term in the changes of the process is a very persistent process with a small variance compared to the large variance of the short-run changes, i.e. the process is (near) I(2) with a small signal-to-noise ratio. A priori we expect near-I(2) trends to be prevalent in asset prices strongly affected by financial speculation, such as exchange rates, stock prices, and commodity prices. However, univariate Dickey-Fuller tests seldom find evidence of a double unit root in such

[6] The inability to reject the I(1) hypothesis for the German bond rate with a p-value of 0.20 indicates that the German bond rate has moved in a slightly less persistent manner than the other variables.

data whereas multivariate trace tests frequently do. Our paper demonstrates by simulations that this is often the case when the signal-to-noise-ratio is small.

6 References

Bacchiocchi, E. and L. Fanelli (2005), "Testing the Purchasing Power Parity through I(2) Cointegration Techniques," *Journal of Applied Econometrics* 20, 749—770.

Cheung, Yin-Wong and Ulf G. Erlandsson (2005), "Exchange Rates and Markov Switching Dynamics," J*ournal of Business and Economic Statistics*, 23, 3, 314-320.

Cheung, Y., K.S. Lai, and M. Bergman (2004), "Dissecting the PPP Puzzle: The Unconventional Roles of Nominal Exchange Rate and Price Adjustments," *Journal of International Economics* 64, 135-150.

Elliot, G. (1998), The Robustness of Cointegration Methods when Regressors Almost Have Unit Roots, *Econometrica*, 66, pp. 149-58.

Engel, C. and James D. Hamilton (1990), "Long Swings in the Exchange Rate: Are They in the Data and Do Markets Know It?," *American Economic Review*, September, 689-713.

Engel, Charles (1994), "Can the Markov Switching Model Forecast 28 Exchange Rates," *Journal of International Economics*, 36, 151-165.

Frankel, Jeffrey. A. (1979), "On the Mark: A Theory of Floating Exchange Rate Based On Real Interest Differentials," *American Economic Review* 69, September, 610-22.

Frydman, R. and M. Goldberg (2007), *Imperfect Knowledge Economics: Exchange rates and Risk*, Princeton, NJ: Princeton University Press.

Frydman, R. and M. Goldberg (2011), *Beyond Mechanical Markets: Risk and the Role of Asset Price Swings*, Princeton University Press.

Jensen, A.N. (2013), On the nesting structure of the I(2) models. Unpublished PhD thesis chapter.

Johansen, S. (1992), "An I(2) Cointegration Analysis of the Purchasing Power Parity between Australia and the United States," in Hargreaves, C. ed., *Macroeconomic Modelling of the Long Run*, 229—248. London: Edward Elgar.

Johansen, Søren (1997), "Likelihood Analysis of the I(2) Model," *Scandinavian Journal of Statistics* 24, 433-462.

Johansen, Søren (2006a), "Statistical Analysis of Hypotheses on the Cointegrating Relations in the I(2) Model," *Journal of Econometrics* 132, 81—115.

Johansen, S. (2006b), "Confronting the Economic Model with the Data," in Colander, D. (ed.) *Post Walrasian macroeconomics: Beyond the dynamic*

stochastic general equilibrium model. Cambridge: Cambridge University Press, 2006., pp. 287-300.

Johansen, Søren., Katarina Juselius, Roman Frydman, and Michael D. Goldberg (2010), "Testing Hypotheses in an I(2) Model With Piecewise Linear Trends. An Analysis of the Persistent Long Swings in the Dmk/$ Rate," *Journal of Econometrics*, 158, 117-29.

Juselius, Katarina (1995), "Do the Purchasing Power Parity and the Uncovered Interest Rate Parity Hold in the Long Run? - An Application of Likelihood Inference in a Multivariate Time Series Model," *Journal of Econometrics* 69, 211-240.

Juselius, Katarina (2007), The Cointegrated VAR Model: Methodology and Applications, Oxford: Oxford University Press.

Juselius, K. (2012) Testing the Rational Expectations Hypothesis Against the Imperfect Knowledge Hypothesis in a Model for Foreign Exchange: A Scenario Analysis. Working paper, Department of Economics, University of Copenhagen.

Juselius, K. (2013). "Imperfect Knowledge, Asset Price Swings and Structural Slumps: A Cointegrated VAR Analysis of Their Interdependence", (eds.) E. Phelps and R. Frydman, *Rethinking Expectations: The Way Forward for Macroeconomics*, Princeton University Press, Princeton.

Kongsted, H.C. (2003), "An I(2) Cointegration Analysis of Small Country Import Price Determination," *Econometrics Journal* 12, 163—178.

Kongsted, H.C. (2005), "Testing the Nominal-to-Real Transformation," *Journal of Econometrics* 124, 205—225.

Kongsted, H.C. and H. B. Nielsen (2004), "Analyzing I(2) Systems by Transformed Vector Autoregressions," *Oxford Bulletin of Economics and Statistics* 66, 379—397.

Paruolo, P. and A. Rahbek (2007), Weak Exogeneity in I(2) VAR systems, *Journal of Econometrics*, 93:281-308

Taylor, Alan and Mark P. Taylor (2004), "The Purchasing Power Parity Debate," *Journal of Economic Perspectives*, 18, 4, 135—158.