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Ehlert, Andree; Schlather, Martin

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A Constructive Proof for the Extremal Coefficient of a Dissipative Max-Stable Process on Z being a Set Covariance

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Andree Ehlert, Martin Schlather

May 2010

Platz der Göttinger Sieben 3 · 37073 Goettingen · Germany
Phone: +49-(0)551-3914066 · Fax: +49-(0)551-3914059

Email: crc-peg@uni-goettingen.de Web: http://www.uni-goettingen.de/crc-peg

A Constructive Proof for the Extremal Coefficient of a Dissipative Max-Stable Process on $\mathbb{Z}$ being a Set Covariance

Andree Ehlert* & Martin Schlather†

Georg-August-Universität Göttingen
Goldschmidtstr. 7, D–37077 Göttingen

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Abstract

For any extremal coefficient function ϕ of a dissipative max-stable process on $\mathbb{Z}$ there exists a measurable set $S \subset \mathbb{R}$ such that

$$2 - \phi(h) = |S \cap (S + h)|, \qquad h \in \mathbb{Z}.$$

We present a constructive proof by giving a monotonic sequence of sets S_n that converge to the set S.

Keywords: Set covariance function; extremal coefficient function; extremal dependence; extreme value theory

*ehlert@math.uni-goettingen.de
†schlather@math.uni-goettingen.de

1 Introduction

It is well-known from [1] and [4] that a dissipative max-stable process Y on $\mathbb{Z}$ with standard Fréchet margins has the representation $Y_t = \max_{i\in\mathbb{N}} U_i f(t-z_i)$, $t \in \mathbb{Z}$. Here, $f : \mathbb{R} \to \mathbb{R}_+$ with $\int f(s)\,ds = 1$, and $\{(U_i, z_i)\}_{i=1}^{\infty}$ is a Poisson point process on $[0,\infty)\times\mathbb{R}$ with intensity measure $u^{-2}\mathbf{1}(u > 0)du \times dz$. In particular, the so-called spectral function f completely characterizes the dependence structure of Y. A corresponding summary measure with properties similar to the Gaussian autocovariance function is given by the extremal coefficient function

$$\phi_f(h) = \int \max\{f(s), f(s+h)\}\,ds, \quad h \in \mathbb{Z}, \tag{1}$$

that has been proposed by [3]. We will consider a sequence $(f_n)_{n\in\mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, of non-negative step functions such that $f_n \uparrow \gamma$ for a suitable function $\gamma : \mathbb{R} \to \mathbb{R}_+$ with $\phi_\gamma = \phi_f$, and hence $\phi_{f_n} \to \phi_f$ as $n \to \infty$. Our main result will be the construction of a bounded monotonic sequence of sets, i.e. $(S_n)_{n\in\mathbb{N}_0} \uparrow S$, $|S| < \infty$, associated to (f_n) such that

$$2\int f_n(s)\,ds - \phi_{f_n}(h) = |S_n \cap (S_n - h)|, \quad n \in \mathbb{N}_0, h \in \mathbb{Z}. \tag{2}$$

Hence, our results imply that for any extremal coefficient function (1) on $\mathbb{Z}$ an equivalent representation as a set covariance function $|S \cap (S-h)|$, $h \in \mathbb{Z}$, given by a certain set $S \subset \mathbb{R}$, $|S| < \infty$, exists. The reverse is straightforward [2]. Consequently, the ensembles for set covariance and extremal coefficient functions for dissipative processes can be shown to coincide on a grid. An application of our result can also be found in [2].

To be specific, let $(f_n)_{n\in\mathbb{N}_0} \uparrow \gamma$ be a monotonically increasing sequence of step functions with nonnegative coefficients a_{nki}, $n \in \mathbb{N}_0$, $k \in K_n = \{-n,\ldots,n\}$, $i = (i_1,\ldots,i_n) \in \{0,1\}^n$, and all other coefficients zero. Here, $i_s \in \{0,1\}$, $s = 1,\ldots,n$, and $i = i_0 = \emptyset$ if $n = 0$. Throughout, we will put $[i]_2 = \sum_{j=1}^n i_j 2^{n-j}$ and $i|_\xi = (i_1,\ldots,i_\xi)$, $\xi = 1,\ldots,n$, where $i|_0 = \emptyset$. Note that the use of a binary number for the index i will be advantageous later on. In particular, for all $x \in \mathbb{R}$ we put $\lfloor x\rfloor = \max\{n \in \mathbb{Z} : n \le x\}$, and we will assume from now on that

$$f_n(x) = \sum_{s=0}^{n} a_{s,k,i|_s}, \text{ where } k = \lfloor x\rfloor \text{ and } i \in \{0,1\}^n \text{ with } [i]_2 = \lfloor 2^n(x-k)\rfloor. \tag{3}$$

Only in Corollary 3 this assumption will be abandoned. According to (1) let $\phi_{f_n}(h)$, $h \in \mathbb{Z}$, $n \in \mathbb{N}_0$, denote the extremal coefficient function of the stationary dissipative process Y_{f_n} generated by the spectral function f_n where by (3) we find that

$$\begin{aligned}\phi_{f_n}(h) &= 2^{-n+1}\sum_{k\in K_n}\sum_{i\in\{0,1\}^n}\sum_{s=0}^{n} a_{s,k,i|_s} - \\ &\quad 2^{-n}\sum_{k\in K_n}\sum_{i\in\{0,1\}^n}\min\left\{\sum_{s=0}^{n} a_{s,k,i|_s}, \sum_{s=0}^{n} a_{s,k+h,i|_s}\right\}, \quad h \in \mathbb{Z}.\end{aligned} \tag{4}$$

Example 1. For the continuous spectral function $f = \gamma$ given in Fig. 1 we sketch the first three elements of a monotonic sequence of step functions $(f_n)_{n\in\mathbb{N}_0} \uparrow f$ given by (3) with

$$\begin{array}{llllll} a_{0,0,\emptyset} & = \frac{1}{10} & a_{1,0,0} & = \frac{1}{15} & a_{2,-2,(0,0)} & = \frac{1}{5} \\ a_{1,-1,0} & = \frac{1}{30} & a_{1,1,0} & = \frac{1}{15} & a_{2,0,(1,0)} & = \frac{1}{30} \\ a_{1,-1,1} & = \frac{2}{15} & a_{1,1,1} & = \frac{1}{30} & a_{2,2,(1,0)} & = \frac{1}{15}. \end{array}$$

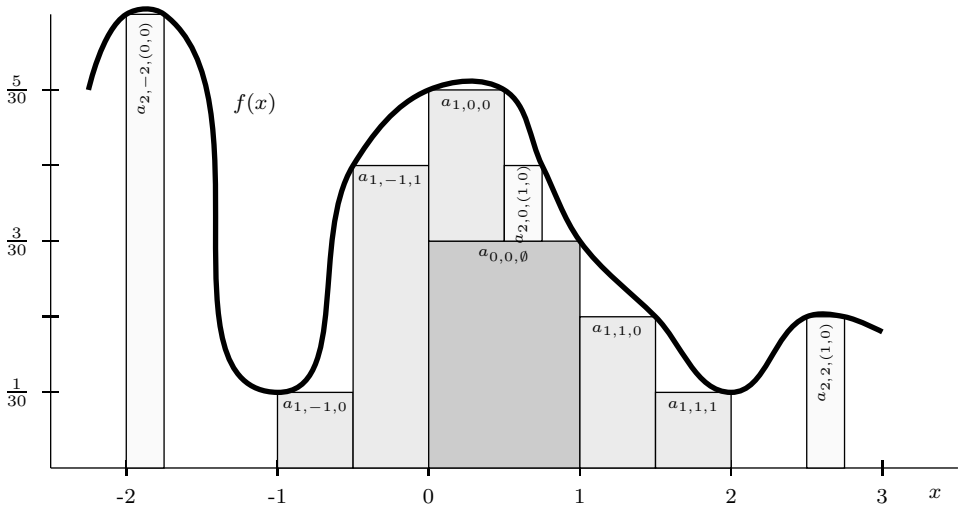

Figure 1: Approximation of a continuous spectral function f by the step function f_2 defined in (3). The corresponding coefficients are given in Example 1. Note that the bars a_{nki} do not necessarily touch the graph of f.

To give an idea of our approach to construct a suitable sequence (S_n) consider the sets A_{ski}, $s = 0, 1, 2$, $k \in K_s$, $i \in \{0,1\}^s$, given in Fig. 2. We put

$$S_n = \bigcup_{s=0}^{n} \bigcup_{k \in K_s} \bigcup_{i \in \{0,1\}^s} (A_{ski} + k), \quad n \in \mathbb{N}_0, \tag{5}$$

such that (S_n) is monotonic, and we will show in Theorem 1, cf. Section 5, that (2) holds. For the above small set of coefficients the latter may be readily verified using Figs. 1 and 2. We point out that in (2) the requirement of monotonicity for (S_n) appears to be a fundamental restriction. More precisely, the determination of an arbitrary sequence (S_n) such that (2) holds is straightforward. Throughout the rest of our analysis we will mainly be concerned with the construction of suitable sets A_{ski} as well as the discussion of their properties. However, note that in Fig. 2 we also include intervals B_{sbi}, $s = 0, 1, 2$, $i \in \{0,1\}^s$, for some tedious index b that will be discussed below. At this point the intervals B_{sbi} may best be thought of as placeholders. In particular, they indicate allowable locations for the sets A_{ski}. Note further that the intervals B_{sbi} will have to be constructed jointly with A_{ski}. Roughly speaking, the construction of $B_{s+1,a,j}$ for $j \in \{(i,0),(i,1)\}$ and some suitable index a will be shown in (16) below to depend on certain intersections of A_{ski} on the index k ranging over particular orderable subsets of K_s. The latter will specifically be reflected by the index a. To conclude the example, note that in Theorem 1 we will essentially make use of the fact that

$$2^{-n} f_n(k + [i]_2 2^{-n}) = \left| \bigcup_{s=0}^{n} A_{s,k,i|s} \cap \bigcup_{b} B_{nbi} \right| \tag{6}$$

for all $k \in K_n$ and all $i \in \{0,1\}^n$. It will be helpful later on to check at this point that (6) holds for the above example using Figs. 1 and 2.

To fix some notations let us denote a proper inclusion by "$\subset$". We shall use "$\subseteq$" for an inclusion that does not preclude equality. Further, we will understand $[x, y) = \emptyset$ if $y < x$,

and $A^0 = \emptyset$ for any set A. For $n \in \mathbb{N}_0$ let $\mathbb{B}_n = \{0,1\}^{n^2}$, $b = (b_1, \ldots, b_n) \in \mathbb{B}_n$, where $b_s \in \{0,1\}^{2s-1}$, $s = 1, \ldots, n$, $b = b_0 = \emptyset$ if $n = 0$, and N_{b_s} is the set of indices corresponding to zeros in b_s, e.g. $N_{b_s} = \{2,5\}$ for $b_s = (1,0,1,1,0)$. Let $E_{n,0} = \mathbb{B}_n \times \{0,1\}^n$ and $E_n = E_{n,0} \setminus (\{0\} \times \{0,1\}^n)$. We will put $b|_\xi = (b_0, \ldots, b_\xi)$, and accordingly $(b,i)|_\xi = (b|_\xi, i|_\xi)$ for $\xi = 0, \ldots, n$. Our approach will be organized as follows. In Section 2 we will define suitable intervals B_{nbi} and discuss their relevant properties. To this end, we shall study an order on the joint index $(b,i) \in E_{n,0}$ that will later refer to the allocation of the intervals B_{nbi} on the line. We will formally introduce the order in (7). The nature of the order will then be largely revealed by part 2 of Lemma 1. It will be shown in Lemma 2 that the order is total on a suitable subset of $E_{n,0}$. In particular, the actual definition of B_{nbi} in (16) will be restricted to this subset in a natural way. As we will be able to draw some important conclusions on the intervals B_{nbi} even for arbitrary sets A_{nki} we will defer the actual joint definition of B_{nbi} and A_{nki} to Section 3. There, in Corollaries 1 and 2 we will show that the assertions of two auxiliary assumptions made for step n, cf. (A1) and (A2) in Section 2, hold true by induction in step $n+1$. In Section 3 we will further discuss two important properties of A_{nki} in Lemmata 5 and 6. Thereafter we will study a decomposition of A_{nki} in Section 4 that will eventually be useful in the proof of Theorem 1 in Section 5 where we will show that (2) holds. Finally, for any given spectral function f we will find that there is a suitable function γ as a limit of step functions where $\phi_f = \phi_\gamma$, cf. Corollary 3.

2 A Sequence of Auxiliary Sets

To begin with, we will equip the sets $E_{n,0}$, $n \in \mathbb{N}$, with the following partial order "$\prec_p$". For $(b,i) \in E_{n,0}$ let

$$\begin{aligned}\{(a,j) \in E_{n,0} : (a,j) \prec_p (b,i)\} = \Big\{&(a,j) \in E_{n,0} : \exists \xi \leq n \text{ such that } a|_\xi = b|_\xi \\ &\text{and } a|_\kappa \neq b|_\kappa, a|_\kappa, b|_\kappa \neq 0 \text{ for all } \xi < \kappa \leq n. \text{ Further, } [j|_\xi]_2 < [i|_\xi]_2, \\ &\text{or } j|_\xi = i|_\xi \text{ and } N_{a_{\xi+1}} \subset N_{b_{\xi+1}}\Big\} \cup \Big\{(a,j) \in E_{n,0} : \exists \delta \leq n \text{ such} \\ &\text{that } b|_\delta = 0, a|_\delta \neq 0 \text{ and } b|_\lambda \neq 0 \text{ for all } \delta < \lambda \leq n\Big\}. \end{aligned} \tag{7}$$

For later reference note that by (7), in particular,

$$(b,i) \prec_p (0,j), \quad \text{for all } j \in \{0,1\}^n \text{ and } (b,i) \in E_n. \tag{8}$$

Further, we have that

$$(b,i) \in E_n \text{ if } (b,i) \prec_p (a,j) \text{ for any } (a,j) \in E_n. \tag{9}$$

As indicated above, we will show in Lemma 2 below that for all $n \in \mathbb{N}_0$ the functions $f_0, \ldots, f_n$ generate a suitable subset $\tilde{E}_{n,0} = \tilde{E}_{n,0,f} \subseteq E_{n,0}$ for which the above order is total. An essential step to the construction of $\tilde{E}_{n,0}$ is provided by the following lemma whose proof is obvious.

Lemma 1. *For $k = 1, \ldots, K$, let $q_k \in [0,1]$ and put $q_\infty = 1$. Define $\max_{k \in \emptyset} q_k = 0$ and $\min_{k \in \emptyset} q_k = 1$ and let $x_b = \max_{k \in N_b} q_k$ and $y_b = \min_{k \notin N_b} q_k$ for all $b \in \{0,1\}^K$. Put $M_u = \{l : q_l \leq u\}$, $u \in \mathbb{R}$, and $U_k = \{b \in \{0,1\}^K : N_b = M_u, u \in \{q_l : q_l < q_k\} \cup \{0\}\}$. For all $k \in \{1, \ldots, K\} \cup \{\infty\}$ we have that*

1. *A partition of* $[0, q_k)$ *is given by* $\{[x_b, y_b),\, b \in U_k\}$

2. $\{N_b,\, b \in U_k\}$ *is strictly totally ordered under inclusion*

3. $y_a \le x_b$ *for* $N_a \subset N_b$, $a, b \in U_k$, *or for* $a = b \notin U_k$

4. $U_k = \{a \in U_\infty : N_a \subseteq N_b\}$, $b \in \arg\max_{a \in U_k} |N_a|$.

Example 2. For $K = 6$ and $q_1 = 0.2$, $q_2 = 0.3$, $q_3 = 0$, $q_4 = 0.4$, $q_5 = 0.1$ and $q_6 = 0.2$ we consider the partition of $[0, q_2)$ given by part 1 of Lemma 1. We have $\{q_l : q_l < q_2\} \cup \{0\} = \{0, 0.1, 0.2\}$ and $M_0 = \{3\}$, $M_{0.1} = \{3, 5\}$ and $M_{0.2} = \{1, 3, 5, 6\}$ such that $U_2 = \{(1,1,0,1,1,1), (1,1,0,1,0,1), (0,1,0,1,0,0)\}$. In particular, part 1 of Lemma 1 yields $[0, 0.3) = [0, 0.1) \cup [0.1, 0.2) \cup [0.2, 0.3)$, and parts 2 and 3 are obvious. Concerning part 4 of the lemma we have

$$U_\infty = \{(1,1,0,1,1,1), (1,1,0,1,0,1), (0,1,0,1,0,0), (0,0,0,1,0,0), (0,0,0,0,0,0)\}.$$

To verify the assertion note finally that $\arg\max_{a \in U_2} |N_a| = \{(0,1,0,1,0,0)\}$.

For later reference by parts 1 to 3 of Lemma 1 for all $b \in U_\infty$ we have

$$[x_b, y_b) = \sum_{\substack{a \in U_\infty : \\ N_a \subset N_b}} |[x_a, y_a)| + [0, y_b - x_b). \tag{10}$$

For all $n \in \mathbb{N}_0$ we will next define successively the sets $\tilde{E}_{n,0}$ and B_{nbi}. For that purpose we shall frequently apply the notation introduced in Lemma 1. Note carefully, however, that we will necessarily extend the subscripts by the indices $n \in \mathbb{N}_0$ and $(b, i) \in \tilde{E}_{n,0}$. Consequently, for all $k \in K_n \cup \{\infty\}$ and $(b, i) \in E_{n,0}$ let now

$$U_{nkbi} = \left\{a \in \{0,1\}^{2n-1} : N_a = M_{nubi}, u \in \{q_{nlbi} : q_{nlbi} < q_{nkbi}\} \cup \{0\}\right\} \tag{11}$$

where, as above, $M_{nubi} = \{k : q_{nkbi} \le u\}$, $u \in \mathbb{R}$. Further, we put

$$q_{nkbi} = |A_{nki} \cap B_{nbi}| \tag{12}$$

and

$$q_{n,\infty,b,i} = |B_{nbi}| \tag{13}$$

for arbitrary sets A_{nki} that will be chosen to depend on $f_0, \ldots, f_{n-1}$ in Eq. (35) below, i.e. q_{nkbi}, $k \in K_n \cup \{\infty\}$, are arbitrary numbers up to $q_{nkbi} \le |B_{nbi}|$. Note that by (11) we have

$$0 \notin U_{nkbi} \quad \text{if } k \in K_n, \text{ (i.e. } k \ne \infty). \tag{14}$$

Let now

$$\begin{aligned} \tilde{E}_{n,0} \;=\; & \left\{((b,u),(i,j)),(b,i) \in \tilde{E}_{n-1,0}, u \in U_{n-1,\infty,b,i}, j \in \{0,1\}\right\} \\ & \cup (\{0\} \times \{0,1\}^n) \end{aligned} \tag{15}$$

and put $\tilde{E}_n = \tilde{E}_{n,0} \setminus (\{0\} \times \{0,1\}^n)$. We will discuss below that the union in (15) can in fact be disjoint. In particular, for any $n \in \mathbb{N}_0$ we will have that $0 \notin U_{\infty,0,i}$, $i \in \{0,1\}^n$, cf. the proof of Corollary 1. Note that in the following we shall occasionally truncate the above indexation where no confusion may arise.

Lemma 2. *For all $n \in \mathbb{N}$ the order "$\prec_p$" given in (7) is total on $\tilde{E}_{n,0}$.*

Proof. Let the order "$\prec_p$" be total on $\tilde{E}_{n-1,0}$ and let $(a,j),(b,i) \in \tilde{E}_{n-1,0}$. By (15) it is sufficient to show that either $((a,\alpha),(j,\iota)) = ((b,\beta),(i,\epsilon))$ or $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ or $((a,\alpha),(j,\iota)) \succ_p ((b,\beta),(i,\epsilon))$ for all $\alpha \in U_{\infty,a,j} \cup \{0\}$, $\beta \in U_{\infty,b,i} \cup \{0\}$ and all $\iota,\epsilon \in \{0,1\}$. By symmetry we may assume that $(a,j) \preceq (b,i)$. Let first $(a,j) = (b,i)$. Then, $U_{\infty,a,j} = U_{\infty,b,i}$, and the following threefold distinction is a partition of all $((a,\alpha),(j,\iota))$ and $((b,\beta),(i,\epsilon))$ with $(a,j) = (b,i)$. In either case we will show that an ordering by "$\prec_p$" exists where we will omit the trivial relation of equality.

1. Let $\alpha = \beta \in U_{\infty,a,j}$ and $\iota,\epsilon \in \{0,1\}$ such that $[(j,\iota)]_2 < [(i,\epsilon)]_2$. Then, $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (7) for $\xi = n$. (δ does not exist.)

2. Let $(a,\alpha),(b,\beta) \neq 0$, $\alpha,\beta \in U_{\infty,a,j}$, $\alpha \neq \beta$, and $\iota,\epsilon \in \{0,1\}$. Then, $(a,\alpha)|_\xi = (b,\beta)|_\xi$ and $(a,\alpha)|_\kappa \neq (b,\beta)|_\kappa$ for all $\xi < \kappa \leq n$, only if $\xi = n-1$. Further, $(j,\iota)|_{n-1} = j = i = (i,\epsilon)|_{n-1}$, and $N_\alpha \subset N_\beta$ (or $N_\alpha \supset N_\beta$) by part 2 of Lemma 1 and the fact that $\alpha,\beta \in U_{\infty,a,j}$. ($\delta$ does not exist.)

3. Let $(b,\beta) = 0$, $\alpha \in U_{\infty,a,i}$, $\alpha \neq 0$, and $\iota,\epsilon \in \{0,1\}$. Then, $\delta = n$, and the fact that $(a,\alpha) \neq 0$ yields $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (7). (ξ does not exist.)

Next, let $(a,j) \prec_p (b,i)$. Then, by (7) there is $\xi \leq n-1$ such that $a|_\xi = b|_\xi$ and $a|_\kappa \neq b|_\kappa$, $a|_\kappa, b|_\kappa \neq 0$ for all $\xi < \kappa \leq n-1$, or there is $\delta \leq n-1$ such that $b|_\delta = 0$, $a|_\delta \neq 0$ and $b_\lambda \neq 0$ for all $\delta < \lambda \leq n-1$. According to (7) we may distinguish three cases that yield the ordering $(a,j) \prec_p (b,i)$. We will consider them separately and show that in either case also $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ for all $\alpha,\beta \in \{0,1\}^{2n-1}$ and all $\iota,\epsilon \in \{0,1\}$.

1. Let $a|_\xi = b|_\xi$, $a|_\kappa \neq b|_\kappa$, $a|_\kappa, b|_\kappa \neq 0$ for all $\xi < \kappa \leq n-1$, and $[j|_\xi]_2 < [i|_\xi]_2$. Now, $(a,\alpha)|_{\xi+1} = (b,\beta)|_{\xi+1}$ only if $\xi = n-1$. (δ does not exist.) Then, the fact that $[(j,\iota)|_{\xi+1}]_2 < [(i,\epsilon)|_{\xi+1}]_2$ yields $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (7). If $(a,\alpha)|_{\xi+1} \neq (b,\beta)|_{\xi+1}$ also $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (7) using that $(a,\alpha)|_\xi = (b,\beta)|_\xi$ and $[(j,\iota)|_\xi]_2 < [(i,\epsilon)|_\xi]_2$.

2. Let $a|_\xi = b|_\xi$, $a|_\kappa \neq b|_\kappa$, $a|_\kappa, b|_\kappa \neq 0$ for all $\xi < \kappa \leq n$, and $j|_\xi = i|_\xi$, $N_{a_{\xi+1}} \subset N_{b_{\xi+1}}$. ($\delta$ does not exist.) Then, $\xi < n$, and for all $\alpha,\beta \in \{0,1\}^{2n-1}$ and all $\iota,\epsilon \in \{0,1\}$ we trivially also have $(a,\alpha)|_\xi = (b,\beta)|_\xi$, $[(j,\iota)|_\xi]_2 = [(i,\epsilon)|_\xi]_2$ and $N_{(a,\alpha)_{\xi+1}} \subset N_{(b,\beta)_{\xi+1}}$. The above yields further that $(a,\alpha)|_{\xi+1} \neq (b,\beta)|_{\xi+1}$ and $(a,\alpha)|_{\xi+1}, (b,\beta)|_{\xi+1} \neq 0$. Comparing with (7) we readily find that $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$.

3. Finally, let $b_\delta = 0$, $a|_\delta \neq 0$ and $b|_\lambda \neq 0$ for all $\delta < \lambda \leq n$. (ξ does not exist.) Now, also $(b,\beta)|_\delta = 0$ and $(a,\alpha)|_\delta \neq 0$. If $(b,\beta)|_{\delta+1} \neq 0$ then $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ is immediate by (7). If $(b,\beta)|_{\delta+1} = 0$ then $\delta = n-1$, and $(a,\alpha)|_{\delta+1} \neq 0$ yields by (7) that $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$.

□

We will denote the respective total order by "$\prec$". For $(b,i) \in \tilde{E}_{n,0}$ let

$$B_{nbi} = \begin{cases} \left[x_b, \dfrac{x_b+y_b}{2}\right) + \displaystyle\sum_{(a,j)\prec(b,i)|_{n-1}} |B_{n-1,a,j}|\,, & b \neq 0,\ i_n = 0, \\ \left[\dfrac{x_b+y_b}{2}, y_b\right) + \displaystyle\sum_{(a,j)\prec(b,i)|_{n-1}} |B_{n-1,a,j}|\,, & b \neq 0,\ i_n = 1, \\ 2^{-n}\left[0, \max\limits_{k\in\mathbb{Z}} f_n\left(k+[i]_2 2^{-n}\right) - \max\limits_{k\in\mathbb{Z}} f_{n-1}\left(k+[i]_2 2^{-n}\right)\right) \\ \quad + \displaystyle\sum_{(b,j)\prec(0,i)} |B_{nbj}|, & b = 0, \end{cases} \tag{16}$$

where retaining the notation of Lemma 1 we put

$$x_b = x_{n,b,i|_{n-1}} = \max_{k\in N_{b_n}-n} q_{n-1,k,(b,i)|_{n-1}} \tag{17}$$

and

$$y_b = y_{n,b,i|_{n-1}} = \min_{k\notin N_{b_n}-n} q_{n-1,k,(b,i)|_{n-1}}. \tag{18}$$

Note that part 3 of Lemma 1, and (15) yield in particular that by applying (16) to any $(b,i) \notin \tilde{E}_{n,0}$ we get $B_{nbi} = \emptyset$. In Fig. 2 we give a successive construction of B_{nbi} up to $n = 2$. There, we use the coefficients discussed in Example 1 and we anticipate (35) in order to fix A_{nki}. Next, note that (16) for all $n \in \mathbb{N}_0$ yields

$$|B_{naj}| = |B_{nai}| \quad \text{for all } (a,j),(a,i) \in \tilde{E}_n \text{ with } j|_{n-1} = i|_{n-1} \tag{19}$$

where we point out that (19) does not hold for $a = 0$, cf. the intervals $B_{1,0,0}$ and $B_{1,0,1}$ in Fig. 2. As indicated above we shall now assume that for a fixed $n \in \mathbb{N}_0$, we have

$$B_{m,0,i} = \bigcupdot_{\substack{(a,j)\in\tilde{E}_{m+1}:\\ (a,j)|_m=(0,i)}} B_{m+1,a,j} \quad \text{for all } m < n \text{ and } i \in \{0,1\}^m. \tag{A1}$$

Lemma 3. *Assume (A1). Then, for all $m \leq n$ the following holds.*

1. *For $(b,i) \in \tilde{E}_{m,0}$ we have*

$$B_{mbi} = [0, |B_{mbi}|) + \sum_{(a,j)\prec(b,i)} |B_{maj}|. \tag{20}$$

2. *For $(b,i) \in \tilde{E}_m$ we have*

$$B_{mbi} = \bigcupdot_{\substack{(a,j)\in\tilde{E}_{m+1}:\\ (a,j)|_m=(b,i)}} B_{m+1,a,j}. \tag{21}$$

Proof. For the proof of (20) we may restrict to the case $b \neq 0$ as (20) is immediate from (16) for $b = 0$. In the following, let $(a,j) \in \tilde{E}_{m-1,0}$ for any $m \leq n$. To begin with, in (22) to (24) we will discuss simple but important preliminaries that follow easily from the above setup. Let first $\gamma \in U_{\infty,a,j} = U_{m-1,\infty,a,j}$ be arbitrary. Using (17) and (18) we then have by (10) that

$$\left[x_{(a,\gamma)}, \frac{x_{(a,\gamma)}+y_{(a,\gamma)}}{2}\right) = \sum_{\substack{\beta\in U_{\infty,a,j}:\\ N_\beta\subset N_\gamma}} \left|[x_{(a,\beta)}, y_{(a,\beta)})\right| + \left[0, \frac{y_{(a,\gamma)}-x_{(a,\gamma)}}{2}\right) \tag{22}$$

and, accordingly,

$$\left[\frac{x_{(a,\gamma)}+y_{(a,\gamma)}}{2}, y_{(a,\gamma)}\right) = \sum_{\substack{\beta\in U_{\infty,a,j}:\\ N_\beta\subset N_\gamma}} \left|[x_{(a,\beta)}, y_{(a,\beta)})\right| + \frac{y_{(a,\gamma)}-x_{(a,\gamma)}}{2} + \left[0, \frac{y_{(a,\gamma)}-x_{(a,\gamma)}}{2}\right). \tag{23}$$

Further, we get by (15) that

$$\begin{aligned}\sum_{\substack{\beta\in U_{\infty,a,j},\\ N_\beta\subset N_\gamma}} \sum_{l\in\{0,1\}} |B_{m,(a,\beta),(j,l)}| &= \sum_{\substack{(c,l)\in\tilde{E}_{m,0}: N_{c_m}\subset N_\gamma,\\ (c,l)|_{m-1}=(a,j)}} |B_{mcl}| \\ &= \sum_{\substack{(c,l)\in\tilde{E}_{m,0}:(c,l)\prec((a,\gamma),(j,0)),\\ (c,l)|_{m-1}=(a,j)}} |B_{mcl}| \end{aligned} \tag{24}$$

where the latter equality holds by (7). Next, let $\gamma\in U_{\infty,a,j}$ such that $(a,\gamma)\neq 0$. Note that depending on the above choice of $(a,j)\in\tilde{E}_{m-1,0}$ this constitutes an additional restriction on γ only if $a=0$. Now, using (16) in (22) and (23), we find that for any $i\in\{(j,0),(j,1)\}$

$$\begin{aligned} B_{m,(a,\gamma),i} &= \sum_{\substack{\beta\in U_{\infty,a,j}:\\ N_\beta\subset N_\gamma}} \sum_{l\in\{0,1\}} |B_{m,(a,\beta),(j,l)}| + \mathbf{1}(i_m=1)|B_{m,(a,\gamma),(j,0)}| \\ &\quad + [0, |B_{m,(a,\gamma),i}|) + \sum_{\substack{(c,l)\in\tilde{E}_{m-1,0}:\\ (c,l)\prec(a,j)}} |B_{m-1,c,l}| \\ &= \sum_{\substack{(c,l)\in\tilde{E}_m:(c,l)\prec((a,\gamma),i),\\ (c,l)|_{m-1}=(a,j)}} |B_{mcl}| + [0, |B_{m,(a,\gamma),i}|) + \sum_{\substack{(c,l)\in\tilde{E}_{m-1,0}:\\ (c,l)\prec(a,j)}} |B_{m-1,c,l}| \end{aligned} \tag{25}$$

where the latter equality holds by (24). Note that by (9) and the fact that $(a,\gamma)\neq 0$ we may restrict to $\tilde{E}_m$ instead of $\tilde{E}_{m,0}$ for the first sum in (25). We next consider the special case $(c,l)\in\tilde{E}_{m-1}$. Then,

$$\begin{aligned}[0, |B_{m-1,c,l}|) &= \bigcup_{\beta\in U_{\infty,c,l}}^{\cdot} \left\{\left[x_{(c,\beta)}, \frac{x_{(c,\beta)}+y_{(c,\beta)}}{2}\right) \cup \left[\frac{x_{(c,\beta)}+y_{(c,\beta)}}{2}, y_{(c,\beta)}\right)\right\} \\ &= \bigcup_{\beta\in U_{\infty,c,l}}^{\cdot} \bigcup_{i\in\{0,1\}}^{\cdot} B_{m,(c,\beta),(l,i)} - \sum_{\substack{(a,i)\in\tilde{E}_{m-1}:\\ (a,i)\prec(c,l)}} |B_{m-1,a,i}| \\ &= \bigcup_{\substack{(a,i)\in\tilde{E}_m:\\ (a,i)|_{m-1}=(c,l)}}^{\cdot} B_{mai} - \sum_{\substack{(a,i)\in\tilde{E}_{m-1}:\\ (a,i)\prec(c,l)}} |B_{m-1,a,i}| \end{aligned} \tag{26}$$

where the first equality is immediate by part 1 of Lemma 1 and (13), and the second equality holds by (16) as we have $c\neq 0$ by assumption. The third equality is a consequence of (15). Note that in the second and third equality we also make use of (9) in order to justify the sets $\tilde{E}_{m-1}$ instead of $\tilde{E}_{m-1,0}$. Now, by (26) and (A1), we have for all $(c,l)\in\tilde{E}_{m-1,0}$ that

$$|B_{m-1,c,l}| = \sum_{\substack{(a,i)\in\tilde{E}_m:\\ (a,i)|_{m-1}=(c,l)}} |B_{mai}|, \tag{27}$$

and (20) for $b \neq 0$ holds by (25) and (27). Note that we use (27) in order to resolve the last sum in (25), and that, in particular, (A1) is a necessary assumption even though $(a, \gamma) \neq 0$. In order to proof (21) consider (26) for $m+1$ instead of m, i.e.

$$[0, |B_{mci}|) = \bigcup_{\substack{(a,j)\in\tilde{E}_{m+1}:\\ (a,j)|_m=(c,i)}}^{\cdot} B_{m+1,a,j} - \sum_{\substack{(a,j)\in\tilde{E}_m:\\ (a,j)\prec(c,i)}} |B_{maj}|\,, \quad (c,i) \in \tilde{E}_m, \tag{28}$$

and the assertion follows by (20). □

In particular, by (20) we have for $m \leq n$ that

$$B_{mbi} \cap B_{maj} = \emptyset, \quad (b,i),(a,j) \in \tilde{E}_{m,0},\ (b,i) \neq (a,j). \tag{29}$$

Using (A1), and (21) repeatedly yields for all $(b,i) \in \tilde{E}_{m,0}$, $m < n$, that

$$B_{mbi} = \bigcup_{\substack{(a,j)\in\tilde{E}_n:a|_{m+1}\neq 0,\\ (a,j)|_m=(b,i)}}^{\cdot} B_{naj}. \tag{30}$$

Note that the restriction $a_{m+1} \neq 0$ affects the case $b = 0$ only, cf. (A1) where the index (a,j) does not run over $a = 0$. Further, using (19) repeatedly we have by (30) for all $(b,i) \in \tilde{E}_{m,0}$, $m < n$, and any $j \in \{0,1\}^n$ with $j|_m = i$ that

$$2^{m-n}|B_{mbi}| = \sum_{\substack{a\in\mathbb{B}_n:(a,j)\in\tilde{E}_n,\\ a|_m=b,a|_{m+1}\neq 0}} |B_{naj}|. \tag{31}$$

Note that in (31) the summation is over $a \in \mathbb{B}_n$ only, and j is fixed. We shall assume next that for a fixed $n \in \mathbb{N}_0$ and all $m < n$, $k \in K_m$ and $j \in \{0,1\}^m$ there is a unique $(g,(j,1)) = (g_{mkj},(j,1)) \in \tilde{E}_{m+1}$ such that

$$\sum_{\substack{(a,i)\in\tilde{E}_{m+1}:i|_m=j,\\ (a,i)\preceq(g,(j,1))}} |B_{m+1,a,i}| = 2^{-m}\sum_{p=0}^{m} a_{p,k,j|_p}. \tag{A2}$$

Lemma 4. *Let $n \in \mathbb{N}_0$, $k \in K_n$, $i \in \{0,1\}^n$ and $j = i|_{n-1}$. Assume (A1) and let $(g,(j,1)) = (g_{n-1,k,j},(j,1)) \in \tilde{E}_n$ according to (A2). We have*

$$\sum_{a:(g,(j,1))\prec(a,i)} |B_{nai}| \geq 2^{-n} a_{nki}.$$

Proof. By (31) for all $i \in \{0,1\}^n$, $n \in \mathbb{N}_0$, we get that

$$\begin{aligned}\sum_{m=0}^{n} 2^m |B_{m,0,i|_m}| &= 2^n \sum_{m=0}^{n-1} \sum_{\substack{a:(a,i)\in\tilde{E}_n,a|_m=0,\\ a|_{m+1}\neq 0}} |B_{nai}| + 2^n|B_{n,0,i}| \\ &= 2^n \sum_{a:(a,i)\in\tilde{E}_{n,0}} |B_{nai}|. \end{aligned} \tag{32}$$

Let $k^* = k^*_{i,n} \in \arg\max_k f_n(k + [i]_2 2^{-n})$. Then, by (3) we have

$$\begin{aligned} f_n(k^* + [i]_2 2^{-n}) &= \sum_{s=0}^{n} a_{s,k^*,i|_s} = \sum_{p=0}^{n} \left(\max_k \sum_{s=0}^{p} a_{s,k,i|_s} - \max_k \sum_{s=0}^{p-1} a_{s,k,i|_s} \right) \\ &= \sum_{p=0}^{n} 2^p |B_{p,0,i|_p}| \end{aligned} \tag{33}$$

where the third equality holds by (16). Combining (32) and (33) yields for all $k \in K_n$, $i \in \{0,1\}^n$ and $j = i|_{n-1}$ that

$$\begin{aligned} \sum_{a:(g,(j,1))\prec(a,i)} |B_{nai}| &= 2^{-n} \sum_{s=0}^{n} a_{s,k^*,i|_s} - \sum_{a:(a,i)\preceq(g,(j,1))} |B_{nai}| \\ &= 2^{-n} \sum_{s=0}^{n} a_{s,k^*,i|_s} - 2^{-n} \sum_{s=0}^{n-1} a_{s,k,i|_s} \geq 2^{-n} a_{nki} \end{aligned} \tag{34}$$

where we use (A2) and (19) for the second equality. □

3 The Sequence S_n: Building Blocks and Properties

The sets S_n, $n \in \mathbb{N}_0$, are given in (5) where at this point for all $n \in \mathbb{N}_0$, $k \in K_n$, $i \in \{0,1\}^n$ and for $(g,(i|_{n-1},1)) = (g_{n-1,k,i|_{n-1}},(i|_{n-1},1)) \in \tilde{E}_n$ according to (A2), we will define essentially

$$A_{nki} = \bigcup_{\substack{b:(b,i)\\ \in \tilde{E}_{n,0}}}^{\bullet} \left\{ B_{nbi} \cap \left([0, 2^{-n} a_{nki}) + \sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| + \sum_{\substack{a:(a,i)\\ \preceq(g,(i|_{n-1},1))}} |B_{nai}| \right) \right\}. \tag{35}$$

Here, the union is disjoint by (20). For the final definition of A_{nki} we refer to Corollary 2 below. In Fig. 2 we depict the sets A_{nki} and B_{nbi} up to $n = 2$ for the coefficients discussed in Example 1. Note that for all $(b,i) \in \tilde{E}_{n,0}$ and all $(c,j) \in \tilde{E}_{m,0}$, $m \leq n$, we have

$$\begin{aligned} &\sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| + \sum_{a:(a,i)|_m \preceq (c,j)} |B_{nai}| \\ &\quad = \begin{cases} \displaystyle\sum_{(a,j)\prec(b,i)} |B_{naj}| - \sum_{\substack{a:(c,j)\prec(a,i)|_m,\\ (a,i)\prec(b,i)}} |B_{nai}|, & (c,j) \prec (b,i)|_m \\ \displaystyle\sum_{(a,j)\prec(b,i)} |B_{naj}| + \sum_{\substack{a:(b,i)\preceq(a,i),\\ (a,i)|_m \preceq (c,j)}} |B_{nai}|, & \text{else,} \end{cases} \end{aligned} \tag{36}$$

such that, by (20) and (35), we get

$$A_{nki} \cap B_{nbi} = \begin{cases} \displaystyle\sum_{(a,j)\prec(b,i)} |B_{naj}| + \Bigg[0, \min\Bigg\{ 2^{-n} a_{nki} \\ \qquad \displaystyle - \sum_{\substack{a:(g,(i|_{n-1},1))\prec\\ (a,i)\prec(b,i)}} |B_{nai}|,\ |B_{nbi}| \Bigg\}\Bigg), & (g,(i|_{n-1},1)) \prec (b,i), \\ \emptyset, & \text{else.} \end{cases} \tag{37}$$

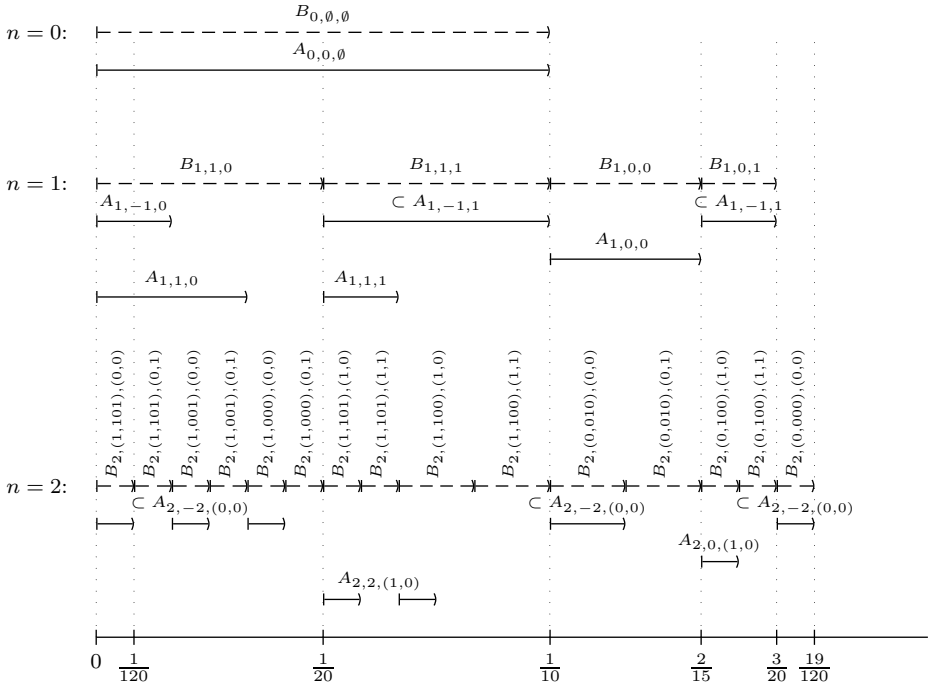

Figure 2: Jointly successive construction of B_{nbi} and A_{nki} (top to bottom) using (16) and (35) up to $n = 2$. The corresponding coefficients are given in Example 1. In the figure by "$\subset X$" we denote a subset of a set X.

Next, we find that by Lemma 4 and (37) for all $i \in \{0,1\}^n$, $k \in K_n$, there is a unique $(\tilde{g}, i) = (\tilde{g}_{nki}, i) \succ (g, (i|_{n-1}, 1)) = (g_{n-1,k,i|_{n-1}}, (i|_{n-1}, 1))$, $(\tilde{g}, i) \preceq (0, i)$, such that

$$0 \leq 2^{-n} a_{nki} - \sum_{(g,(i|_{n-1},1)) \prec (a,i) \prec (\tilde{g},i)} |B_{nai}| \leq |B_{n,\tilde{g},i}|. \tag{38}$$

Hence, by (37) and (38)

$$A_{nki} \cap B_{nbi} = \begin{cases} B_{nbi}, & (g, (i|_{n-1}, 1)) \prec (b, i) \prec (\tilde{g}, i), \\ \sum\limits_{(a,j) \prec (\tilde{g},i)} |B_{naj}| + [0, |A_{nki} \cap B_{nbi}|], & (b, i) = (\tilde{g}, i), \\ \emptyset, & \text{else.} \end{cases} \tag{39}$$

Note that (39) and (35) yield

$$A_{nki} \subseteq \bigdot\bigcup_{b:(g,(i|_{n-1},1)) \prec (b,i) \preceq (\tilde{g},i)} B_{nbi}, \tag{40}$$

such that, in particular,

$$A_{nki} = \bigdot\bigcup_{b:(g,(i|_{n-1},1)) \prec (b,i) \preceq (\tilde{g},i)} A_{nki} \cap B_{nbi}. \tag{41}$$

Further, by (37) and (41) we find for later reference that

$$|A_{nki}| = 2^{-n} a_{nki}. \tag{42}$$

Corollary 1. *Assume (A1) and (A2). For all $i \in \{0,1\}^n$ we have*

$$B_{n0i} = \bigcup_{\substack{(a,j)\in\tilde{E}_{n+1}:\\ (a,j)|_n=(0,i)}}^{\cdot} B_{n+1,a,j}. \tag{43}$$

Proof. Let $k^* = k^*_{i,n}$ as in the proof of Lemma 4, and let

$$(g,(i|_{n-1},1)) = (g_{n-1,k^*,i|_{n-1}},(i|_{n-1},1)) \in \tilde{E}_n$$

as in (A2). We now get by (34) for $k = k^*$ that for any $i \in \{0,1\}^n$

$$2^{-n}a_{n,k^*,i} = \sum_{a:(g,(i|_{n-1},1))\prec(a,i)} |B_{nai}| = \sum_{\substack{a:(g,(i|_{n-1},1))\prec\\ (a,i)\preceq(0,i)}} |B_{nai}| \tag{44}$$

where the second equality holds by (8). Using further that from (A2) and (8) we have $(g,(i|_{n-1},1)) \prec (0,i)$ we get by (37) that

$$\begin{aligned} &A_{n,k^*,i} \cap B_{n,0,i} \\ &= \sum_{(a,j)\prec(0,i)} |B_{naj}| + \left[0, \min\left\{2^{-n}a_{n,k^*,i} - \sum_{\substack{a:(g,(i|_{n-1},1))\\ \prec(a,i)\prec(0,i)}} |B_{nai}|,\, |B_{n,0,i}|\right\}\right) \\ &= \sum_{(a,j)\prec(0,i)} |B_{naj}| + [0,|B_{n,0,i}|) = B_{n,0,i}. \end{aligned} \tag{45}$$

Here, the second equality follows by (44) and the last equality corresponds to (20). By (45), using (12) and (13) we find that $q_{n,k^*,0,i} = q_{n,\infty,0,i}$ such that by (11), in particular,

$$U_{k^*,0,i} = U_{\infty,0,i}. \tag{46}$$

Further, (46) and (14) yield

$$0 \notin U_{\infty,0,i}. \tag{47}$$

Now,

$$\begin{aligned} [0,|B_{n,0,i}|) &= [0,q_{n,k^*,0,i}) = \bigcup_{\beta\in U_{k^*,0,i}}^{\cdot} [x_{(0,\beta)},y_{(0,\beta)}) = \bigcup_{\beta\in U_{k^*,0,i}}^{\cdot}\ \bigcup_{j\in\{0,1\}}^{\cdot} B_{n+1,(0,\beta),(i,j)} \\ &\quad - \sum_{(a,j)\prec(0,i)} B_{naj} = \bigcup_{\substack{(a,j)\in\tilde{E}_{n+1}:\\ (a,j)|_n=(0,i)}}^{\cdot} B_{n+1,a,j} - \sum_{(a,j)\prec(0,i)} B_{naj}. \end{aligned} \tag{48}$$

Here, the first equality follows from (45) and (13), the second equality holds by part 1 of Lemma 1, and the third equality is a consequence of (16) where we use that by (14) $0 \notin U_{k^*,0,i}$. The last equality holds by (46) and (15) where we again may restrict the union to $\tilde{E}_{n+1}$ instead of $\tilde{E}_{n+1,0}$ by (14). Finally, (20) and (48) yield the assertion. □

Corollary 2. *Assume (A1) and (A2). For all $k \in K_n$ and $j \in \{0,1\}^n$ there is a unique $(g,(j,1)) = (g_{nkj},(j,1)) \in \tilde{E}_{n+1}$ such that*

$$\sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j,\\ (a,i)\preceq(g,(j,1))}} |B_{n+1,a,i}| = 2^{-n}\sum_{p=0}^{n} a_{p,k,j|_p}. \tag{49}$$

Proof. Let $k \in K_n$ and $(\tilde{g}, j) = (\tilde{g}_{nkj}, j) \in \tilde{E}_{n,0}$ as in (38). Further, let $z \in \arg\max_{\beta \in U_{k\tilde{g}j}} |N_\beta|$. Then, $z \neq 0$ by (14), and z is unique by part 2 of Lemma 1. By part 4 of Lemma 1 we have

$$U_{k\tilde{g}j} = \{\beta \in U_{\infty,\tilde{g},j} : N_\beta \subseteq N_z\}\,. \tag{50}$$

Further,

$$\begin{aligned}
[0, q_{n,k,\tilde{g},j}) &= \bigcup_{\beta \in U_{k,\tilde{g},j}}^{\cdot} [x_{(\tilde{g},\beta)}, y_{(\tilde{g},\beta)}) = \bigcup_{\substack{\beta \in U_{\infty,\tilde{g},j}: \\ N_\beta \subseteq N_z}}^{\cdot} [x_{(\tilde{g},\beta)}, y_{(\tilde{g},\beta)}) \\
&= \bigcup_{\substack{\beta \in U_{\infty,\tilde{g},j}: \\ N_\beta \subseteq N_z}}^{\cdot} \bigcup_{l \in \{0,1\}}^{\cdot} B_{n+1,(\tilde{g},\beta),(j,l)} - \sum_{(a,i) \prec (\tilde{g},j)} |B_{nai}|
\end{aligned} \tag{51}$$

where the first equality holds by part 1 of Lemma 1 and the second equality is a consequence of (50). The third equality then follows by (16) and (14). Next, by (39) and (51) we get that

$$\begin{aligned}
A_{nkj} \cap B_{n\tilde{g}j} &= \bigcup_{\substack{\beta \in U_{\infty,\tilde{g},j}: \\ N_\beta \subseteq N_z}}^{\cdot} \bigcup_{l \in \{0,1\}}^{\cdot} B_{n+1,(\tilde{g},\beta),(j,l)} \\
&= \bigcup_{\substack{(a,i) \in \tilde{E}_{n+1}:(a,i)|_n = (\tilde{g},j), \\ N_{a_{n+1}} \subseteq N_z}}^{\cdot} B_{n+1,a,i} = \bigcup_{\substack{(a,i) \in \tilde{E}_{n+1}:(a,i) \preceq ((\tilde{g},z),(j,1)), \\ (a,i)|_n = (\tilde{g},j)}}^{\cdot} B_{n+1,a,i}
\end{aligned} \tag{52}$$

where the second equality holds by (15). Here, by (50) and (14) we have that $\beta \neq 0$ such that (8) justifies the restriction to $\tilde{E}_{n+1}$ instead of $\tilde{E}_{n+1,0}$. The third equality is a consequence of (7) and Lemma 2. Further, by (41) and (39)

$$\begin{aligned}
A_{nkj} &= \bigcup_{\substack{b:(g_{n-1,k,j|_{n-1}},(j|_{n-1},1)) \\ \prec (b,j) \prec (\tilde{g},j)}}^{\cdot} B_{nbj} \quad \cup \quad (A_{nkj} \cap B_{n\tilde{g}j}) \\
&= \bigcup_{\substack{(a,i) \in \tilde{E}_{n+1}:i|_n = j, \\ (g_{n-1,k,j|_{n-1}},(j|_{n-1},1)) \prec (a,i)|_n \prec (\tilde{g},j)}}^{\cdot} B_{n+1,a,i} \cup \bigcup_{\substack{(a,i) \in \tilde{E}_{n+1}:(a,i) \preceq ((\tilde{g},z),(j,1)), \\ (a,i)|_n = (\tilde{g},j)}}^{\cdot} B_{n+1,a,i} \\
&= \bigcup_{\substack{(a,i) \in \tilde{E}_{n+1}:i|_n = j, (a,i) \preceq ((\tilde{g},z),(j,1)), \\ (g_{n-1,k,j|_{n-1}},(j|_{n-1},1)) \prec (a,i)|_n}}^{\cdot} B_{n+1,a,i}
\end{aligned} \tag{53}$$

where the second equality follows from (21) and (52). Thus,

$$\sum_{\substack{(a,i) \in \tilde{E}_{n+1}:i|_n = j, (a,i) \preceq ((\tilde{g},z),(j,1)), \\ (g_{n-1,k,j|_{n-1}},(j|_{n-1},1)) \prec (a,i)|_n}} |B_{n+1,a,i}| = |A_{nkj}| = 2^{-n} a_{nkj} \tag{54}$$

where the second equality holds by (42). Further, (A2) yields

$$\begin{aligned}
2^{-n+1} \sum_{p=0}^{n-1} a_{p,k,j|_p} &= \sum_{\substack{(b,l) \in \tilde{E}_n:l|_{n-1} = j|_{n-1}, \\ (b,l) \preceq (g_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{nbl}| \\
&= \sum_{\substack{(a,i) \in \tilde{E}_{n+1}:i|_{n-1} = j|_{n-1}, \\ (a,i)|_n \preceq (g_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{n+1,a,i}|
\end{aligned} \tag{55}$$

where the second equality is a consequence of (21). Note that by (19)

$$\frac{1}{2} \sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_{n-1}=j|_{n-1},\\ (a,i)|_n\preceq(g_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{n+1,a,i}| =$$

$$\sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j,\\ (a,i)|_n\preceq(g_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{n+1,a,i}| = 2^{-n}\sum_{p=0}^{n-1} a_{p,k,j|_p}. \tag{56}$$

Here, the second equality holds by (55). The assertion follows by (56) and (54) where we put

$$g_{nkj} = (\tilde{g}_{nkj}, z). \tag{57}$$

□

Now, for (43) and (49) to hold for all $n \in \mathbb{N}_0$, by induction on Corollaries 1 and 2 it is sufficient to note that (A1) and (A2) hold trivially in the case $n = 0$. In particular, for all $n \in \mathbb{N}_0$, we now have

$$B_{n0i} = \bigcupdot_{\substack{(a,j)\in\tilde{E}_{n+1}:\\ (a,j)|_n=(0,i)}} B_{n+1,a,j}, \tag{58}$$

and for all $n \in \mathbb{N}_0$, $k \in K_n$ and $j \in \{0,1\}^n$ there is a unique $(g,(j,1)) = (g_{nkj},(j,1)) \in \tilde{E}_{n+1}$ such that

$$\sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j,\\ (a,i)\preceq(g,(j,1))}} |B_{n+1,a,i}| = 2^{-n}\sum_{p=0}^{n} a_{p,k,j|_p}. \tag{59}$$

Combining (58) and (21) yields that for all $n \in \mathbb{N}_0$

$$B_{nbi} = \bigcupdot_{\substack{(a,j)\in\tilde{E}_{n+1}:\\ (a,j)|_n=(b,i)}} B_{n+1,a,j}, \quad (b,i) \in \tilde{E}_{n,0}, \tag{60}$$

and by a repeated applicaction of (60) we get that for all $m, n \in \mathbb{N}_0$, $m < n$,

$$B_{mbi} = \bigcupdot_{\substack{(a,j)\in\tilde{E}_n:a|_{m+1}\neq 0,\\ (a,j)|_m=(b,i)}} B_{naj}, \quad (b,i) \in \tilde{E}_{m,0}. \tag{61}$$

Lemma 5. *For all $m, n \in \mathbb{N}_0$, $k \in K_m$ and $j \in \{0,1\}^m$, $i \in \{0,1\}^n$ with $n \neq m$ or $j \neq i$ we have*

$$A_{nki} \cap A_{mkj} = \emptyset.$$

Proof. For $m = n$ the assertion is immediate by (40) and (29). To prove the case $m < n$ let $(b,i) \in \tilde{E}_{n,0}$ such that $(b,i)|_{m+1} \preceq (g_{m,k,i|_m},(i|_m,1))$. Then, by (38) we have

$$(g_{m,k,i|_m},(i|_m,1)) \prec (\tilde{g}_{m+1,k,i|_{m+1}}, i|_{m+1}) = (g_{m+1,k,i|_{m+1}},(i|_{m+1},1))|_{m+1}$$

where the equality holds by (57). Now, $(b,i)|_{m+1} \prec (g_{m+1,k,i|_{m+1}},(i|_{m+1},1))|_{m+1}$, and by the second part in the proof of Lemma 2 the latter implies that $(b,i)|_{m+2} \prec (g_{m+1,k,i|_{m+1}},(i|_{m+1},1))$ if $m+2 \leq n$. Proceeding iteratively, we have that

$$(b,i) \prec (g_{n-1,k,i|_{n-1}},(i|_{n-1},1)) \quad \text{if } (b,i)|_{m+1} \preceq (g_{m,k,i|_m},(i|_m,1)). \tag{62}$$

Further, note that

$$A_{mkj} = \bigcup_{\substack{(b,i)\in\tilde{E}_{m+1}:i|_m=j,(b,i)\preceq(g_{mkj},(j,1)),\\(g_{m-1,k,j|_{m-1}},(j|_{m-1},1))\prec(b,i)|_m}} B_{m+1,b,i} = \bigcup_{\substack{(b,i)\in\tilde{E}_n:i|_m=j,(b,i)|_{m+1}\preceq(g_{mkj},(j,1)),\\(g_{m-1,k,j|_{m-1}},(j|_{m-1},1))\prec(b,i)|_m}} B_{nbi} \tag{63}$$

where the first equality follows from (53) and the second equality is a consequence of (61). Now, comparing (40) and (63) for $j = i|_m$ yields the assertion by (62). To finalize the proof let $j \neq i|_m$. By (35) we find that

$$A_{mkj} \subseteq \bigcup_{b:(b,j)\in\tilde{E}_{m,0}} B_{mbj} = \bigcup_{(c,l)\in\tilde{E}_{n,0}:l|_m=j} B_{ncl} \tag{64}$$

where the equality holds by (61). Using (29) the assertion follows by (35) and (64). The case $m > n$ follows by symmetry. □

Lemma 6. *For all $n \in \mathbb{N}_0$ and $k \in K_n$ we have*

$$\bigcup_{s=0}^{n} \bigcup_{i\in\{0,1\}^s} A_{ski} \subseteq [0,1). \tag{65}$$

Proof. For $m < n \in \mathbb{N}$ we have by (61) that

$$\bigcup_{\substack{(b,i)\in\tilde{E}_n:b|_{m+1}\neq 0,\\(b,i)|_m=(0,j)}} B_{nbi} = B_{m0j}, \quad j \in \{0,1\}^m, \tag{66}$$

such that (29) with (66) yields

$$B_{m0j} \cap B_{p0l} = \emptyset \quad \text{for } m \neq p \text{ or } j \neq l. \tag{67}$$

Consequently, we get that

$$\bigcup_{m=0}^{n-1} \bigcup_{j\in\{0,1\}^m} B_{m0j} = \bigcup_{m=0}^{n-1} \bigcup_{j\in\{0,1\}^m} \bigcup_{\substack{(b,i)\in\tilde{E}_n:b|_{m+1}\neq 0,\\(b,i)|_m=(0,j)}} B_{nbi} = \bigcup_{(b,i)\in\tilde{E}_n} B_{nbi}, \tag{68}$$

and by (20) and (7) for all $i \in \{0,1\}^n$ it holds that

$$\bigcup_{(b,j)\in\tilde{E}_n} B_{nbj} \cup \bigcup_{[j]_2<[i]_2} B_{n0j} = \bigcup_{(b,j)\prec(0,i)} B_{nbj}. \tag{69}$$

Now, from (68) and (69) we find that for all $i \in \{0,1\}^n$

$$\sum_{(b,j)\prec(0,i)} |B_{nbj}| = \sum_{m=0}^{n-1} \sum_{j\in\{0,1\}^m} |B_{m0j}| + \sum_{[j]_2<[i]_2} |B_{n0j}| \tag{70}$$

such that by (16) and (70)

$$B_{n0i} = [0, |B_{n0i}|) + \sum_{m=0}^{n-1} \sum_{j\in\{0,1\}^m} |B_{m0j}| + \sum_{[j]_2<[i]_2} |B_{n0j}|. \tag{71}$$

Next, using (66) we have for all $i \in \{0,1\}^n$

$$\bigdot\bigcup_{\substack{b:(b,i)\in\tilde{E}_n, b|_m=0,\\ b|_{m+1}\neq 0}} B_{nbi} \subseteq B_{m,0,i|_m}, \tag{72}$$

and (67) and (72) yield

$$\bigdot\bigcup_{m=0}^{n} B_{m,0,i|_m} \supseteq \bigdot\bigcup_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi}, \quad \text{for all } i \in \{0,1\}^n. \tag{73}$$

We get from (35) that $A_{nki} \subseteq \bigdot\bigcup_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi}$ for all $k \in K_n$, $i \in \{0,1\}^n$, such that using Lemma 5

$$\begin{aligned}
\bigdot\bigcup_{s=0}^{n} \bigdot\bigcup_{i\in\{0,1\}^s} A_{ski} &\subseteq \bigdot\bigcup_{s=0}^{n} \bigdot\bigcup_{i\in\{0,1\}^s} \bigdot\bigcup_{b:(b,i)\in\tilde{E}_{s,0}} B_{sbi} \subseteq \bigdot\bigcup_{s=0}^{n} \bigdot\bigcup_{m=0}^{s} \bigdot\bigcup_{i\in\{0,1\}^s} B_{m,0,i|_m} \\
&= \bigdot\bigcup_{s=0}^{n} \bigdot\bigcup_{m=0}^{s} \bigdot\bigcup_{j\in\{0,1\}^m} B_{m,0,j} = \bigdot\bigcup_{s=0}^{n} \bigdot\bigcup_{i\in\{0,1\}^s} B_{s,0,i} \\
&= \left[0, \sum_{s=0}^{n} \sum_{i\in\{0,1\}^s} |B_{s,0,i}|\right), \quad k \in K_n,
\end{aligned} \tag{74}$$

where the second inclusion follows by (73), the second equality holds by (67), and the last equality is a consequence of (71). Next, note that

$$\begin{aligned}
\sum_{s=0}^{n} \sum_{i\in\{0,1\}^s} |B_{s,0,i}| &= \sum_{s=0}^{n} \sum_{i\in\{0,1\}^s} 2^{-s} \left(\max_{k\in\mathbb{Z}} f_s\left(k + [i]_2 2^{-s}\right) - \max_{k\in\mathbb{Z}} f_{s-1}\left(k + [i]_2 2^{-s}\right)\right) \\
&\leq \sum_{s=0}^{n} \sum_{i\in\{0,1\}^s} \sum_{k\in\mathbb{Z}} 2^{-s} \left(f_s\left(k + [i]_2 2^{-s}\right) - f_{s-1}\left(k + [i]_2 2^{-s}\right)\right) \\
&= \sum_{s=0}^{n} \sum_{i\in\{0,1\}^n} \sum_{k\in\mathbb{Z}} 2^{-n} \left(f_s\left(k + [i]_2 2^{-n}\right) - f_{s-1}\left(k + [i]_2 2^{-n}\right)\right) \\
&= \sum_{i\in\{0,1\}^n} \sum_{k\in\mathbb{Z}} 2^{-n} f_n\left(k + [i]_2 2^{-n}\right) = \int f_n(x)\, dx \leq 1.
\end{aligned} \tag{75}$$

Here, the first equality follows directly from (16), and the first inequality is a consequence of the fact that for $a_k \leq b_k \in \mathbb{R}$ we have $0 \leq \max_k b_k - \max_k a_k \leq \sum_k (b_k - a_k)$. As to the second equality we use that for any $j \in \{0,1\}^s$ we have $f_s\left(k + [j]_2 2^{-s}\right) = f_s\left(k + [i]_2 2^{-n}\right)$, $i \in \{0,1\}^n$, $i|_s = j$, and $|\{i \in \{0,1\}^n : i|_s = j\}| = 2^{n-s}$. The last inequality reflects the assumption of unit Fréchet margins of the max-stable process generated by f. Finally, (74) and (75) yield the assertion. □

4 A Useful Decomposition for the Sets A_{nki}

Recall from (16) and (35) that the sets B_{mbi} and A_{mki} are defined in a joint successive way. The following notion of $D_{n,mki}$ will generalize the sets A_{mki}. In contrast to (35), however, for

$m < n$ they will require the corresponding sets B_{nbi} to be already defined. More precisely, we put

$$D_{n,mki} = \bigcupdot_{b:(b,i)\in\tilde{E}_{n,0}} \Big\{ B_{nbi} \cap \Big([0, 2^{-n} a_{m,k,i|_m}) + \sum_{\substack{a:(a,i)\in\tilde{E}_{n,0}:\\ (a,i)|_m \preceq (g,(i|_{m-1},1))}} |B_{nai}| + \sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| \Big) \Big\} \tag{76}$$

for all $n \in \mathbb{N}_0$, $m \leq n$, $k \in K_m$ and $i \in \{0,1\}^n$ where $(g, (i|_{m-1}, 1)) = (g_{m-1,k,i|_{m-1}}, (i|_{m-1}, 1))$ as in (59). In particular, we readily find by (76) that

$$D_{n,mki} \subseteq \bigcupdot_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi}, \tag{77}$$

and (76) and (35) yield that $D_{n,nki} = A_{nki}$. Further, for $i, j \in \{0,1\}^n$, $i \neq j$, we get by (29) and (76) that

$$D_{n,mki} \cap D_{n,p,k+h,j} = \emptyset, \quad \text{for all } m, p \leq n \in \mathbb{N}_0,\ h \in \mathbb{N}_0. \tag{78}$$

Next, using (77) and (61) we have for all $m < n$, $(b, j) \in \tilde{E}_{m,0}$ and all $i \in \{0,1\}^n$ with $i|_m = j$ that

$$D_{n,mki} \cap B_{mbj} = D_{n,mki} \cap \bigcupdot_{\substack{a:(a,i)\in\tilde{E}_n, a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}} B_{nai} = \bigcupdot_{\substack{a:(a,i)\in\tilde{E}_n, a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}} (B_{nai} \cap D_{n,mki})$$

such that, for later reference,

$$\bigcupdot_{\substack{i\in\{0,1\}^n:\\ i|_m=j}} (D_{n,mki} \cap B_{mbj}) = \bigcupdot_{\substack{i\in\{0,1\}^n:\\ i|_m=j}} \bigcupdot_{\substack{a:(a,i)\in\tilde{E}_{n,0}, (a,i)|_m=(b,j),\\ (a,i)|_{m+1}\neq(0,i|_{m+1})}} (D_{n,mki} \cap B_{nai}) \tag{79}$$

where the union on i is disjoint by (76) and (29).

Lemma 7. *For all $m < n \in \mathbb{N}_0$, $k \in K_m$, $i \in \{0,1\}^n$ and $j = i|_m$ we have*

$$D_{n,mki} \subseteq \bigcupdot_{\substack{a:(a,j)\in\tilde{E}_{m,0},\\ (g,(j|_{m-1},1))\prec(a,j)\preceq(\tilde{g},j)}} B_{maj}$$

where $(g, (j|_{m-1}, 1)) = (g_{m-1,k,j|_{m-1}}, (j_{m-1}, 1))$ and $(\tilde{g}, j) = (\tilde{g}_{mkj}, j)$.

Proof. Note first that (76) and (20) give

$$D_{n,mki} \cap B_{nbi} = \Big(\sum_{(a,j)\prec(b,i)} |B_{naj}| + [0, |B_{nbi}|) \Big) \cap \Big([0, 2^{-n} a_{m,k,i|_m}) + \sum_{\substack{a:(a,i)\in\tilde{E}_{n,0},\\ (a,i)|_m\preceq(g,(i|_{m-1},1))}} |B_{nai}| + \sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| \Big). \tag{80}$$

Next, applying (36) to (80) we find similar as in (37) that

$$D_{n,mki} \cap B_{nbi} = \begin{cases} \sum\limits_{(a,j)\prec(b,i)} |B_{naj}| + \Big[0, \min\Big\{2^{-n} a_{m,k,i|_m} \\ \quad - \sum\limits_{\substack{a:(g,(i|_{m-1},1))\prec(a,i)|_m,\\ (a,i)\prec(b,i)}} |B_{nai}|,\, |B_{nbi}|\Big\}\Big), & (g,(i|_{m-1},1)) \prec (b,i)|_m, \\ \emptyset, & \text{else.} \end{cases} \tag{81}$$

Further, using (19) repeatedly we get

$$\sum_{(g,(j|_{m-1},1))\prec(a,j)\preceq(\tilde{g},j)} |B_{maj}| = 2^{n-m} \sum_{c:(g,(j|_{m-1},1))\prec(c,i)|_m\preceq(\tilde{g},j)} |B_{nci}| \tag{82}$$

such that by (82) and (38) for all $j \in \{0,1\}^m$ and all $i \in \{0,1\}^n$ with $i|_m = j$

$$2^{-n} a_{mkj} - \sum_{\substack{c:(c,i)\in\tilde{E}_{n,0},\\ (g,(j|_{m-1},1))\prec(c,i)|_m\preceq(\tilde{g},j)}} |B_{nci}| \le 0. \tag{83}$$

Now, by (81) and (83) for all $(b,i) \in \tilde{E}_{n,0}$ with $(b,i)|_m \preceq (g,(i|_{m-1},1))$ or $(\tilde{g},i|_m) \prec (b,i)|_m$ we have

$$D_{n,mki} \cap B_{nbi} = \emptyset. \tag{84}$$

Finally, (77) and (84) yield that for all $i \in \{0,1\}^n$ with $i|_m = j$

$$D_{n,mki} \subseteq \bigdot\bigcup_{\substack{b:(b,i)\in\tilde{E}_{n,0},\\ (g,(j|_{m-1},1))\prec(b,i)|_m\preceq(\tilde{g},j)}} B_{nbi} = \bigdot\bigcup_{\substack{a:(a,j)\in\tilde{E}_{m,0},\\ (g,(j|_{m-1},1))\prec(a,j)\preceq(\tilde{g},j)}} B_{maj} \tag{85}$$

where the equality holds by (61). □

By (81) and (85) we may now state for later reference that

$$|D_{n,mki}| = 2^{-n} a_{m,k,i|_m}. \tag{86}$$

Lemma 8. *For all $m < n \in \mathbb{N}_0$, $k \in K_m$ and $j \in \{0,1\}^m$ we have*

$$A_{mkj} = \bigdot\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}} D_{n,mki}.$$

Proof. Note that by (40) and Lemma 7 it is sufficient to show for all $(b,j) \in \tilde{E}_{m,0}$ with $(g,(j|_{m-1},1)) \prec (b,j) \preceq (\tilde{g},j)$ that

$$B_{mbj} \cap A_{mkj} = B_{mbj} \cap \bigdot\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}} D_{n,mki}. \tag{87}$$

To this end, we shall consider a twofold case differentiation. First, let

$$2^{-m} a_{mkj} - \sum_{a:(g,(j|_{m-1},1))\prec(a,j)\prec(b,j)} |B_{maj}| \le |B_{mbj}|.$$

Then, by (37)

$$\begin{aligned} B_{mbj} \cap A_{mkj} &= \sum_{(a,i) \prec (b,j)} |B_{mai}| + \left[0, 2^{-m} a_{mkj} - \sum_{a:(g,(j|_{m-1},1)) \prec (a,j) \prec (b,j)} |B_{maj}|\right) \\ &= \bigcup_{\substack{i \in \{0,1\}^n: \\ i|_m = j}} \bigcup_{\substack{a:(a,i)|_m = (b,j), \\ (a,i)|_{m+1} \preceq (g,(j,1))}} B_{nai} \end{aligned} \tag{88}$$

where the second equality follows by (63) and (40). Now, (88) and a repeated application of (19) yield that for all $i \in \{0,1\}^n$ with $i|_m = j$

$$\begin{aligned} \sum_{\substack{a:(a,i)|_m = (b,j), \\ (a,i)|_{m+1} \preceq (g,(j,1))}} |B_{nai}| &= 2^{m-n} \left(2^{-m} a_{mkj} - \sum_{\substack{a:(g,(j|_{m-1},1)) \prec \\ (a,j) \prec (b,j)}} |B_{maj}|\right) \\ &= 2^{-n} a_{mkj} - \sum_{\substack{c:(g,(j|_{m-1},1)) \prec \\ (c,i)|_m \prec (b,j)}} |B_{nci}| \end{aligned} \tag{89}$$

and by (89), in particular,

$$\sum_{\substack{c:(c,i) \preceq (a,i), \\ (c,i)|_m = (b,j)}} |B_{nci}| \leq 2^{-n} a_{mkj} - \sum_{c:(g,(j|_{m-1},1)) \prec (c,i)|_m \prec (b,j)} |B_{nci}| \tag{90}$$

for all $a \in \mathbb{B}_n$ with $(a,i)|_m = (b,j)$ and $(a,i)|_{m+1} \preceq (g,(j,1))$. Further, by (89) we have for $(g,(j,1)) \prec (a,i)|_{m+1}$ that

$$2^{-n} a_{mkj} - \sum_{c:(g,(j|_{m-1},1)) \prec (c,i)|_m \prec (b,j)} |B_{nci}| - \sum_{\substack{c:(c,i) \prec (a,i), \\ (c,i)|_m = (b,j)}} |B_{nci}| \leq 0. \tag{91}$$

Now, by (20) we get

$$\begin{aligned} \bigcup_{\substack{a:(a,i)|_m = (b,j), \\ (a,i)|_{m+1} \preceq (g,(j,1))}} B_{nai} &= \bigcup_{\substack{a:(a,i)|_m = (b,j), \\ (a,i)|_{m+1} \preceq (g,(j,1))}} \left([0, |B_{nai}|) + \sum_{(c,l) \prec (a,i)} |B_{ncl}|\right) \\ &= \bigcup_{a:(a,i)|_m = (b,j)} \Bigg(\sum_{(c,l) \prec (a,i)} |B_{ncl}| + \Bigg[0, \min \Bigg\{ 2^{-n} a_{mkj} \\ &\quad - \sum_{c:(g,(j|_{m-1},1)) \prec (c,i)|_m \prec (b,j)} |B_{nci}| - \sum_{\substack{c:(c,i) \prec (a,i), \\ (c,i)|_m = (b,j)}} |B_{nci}|, |B_{nai}| \Bigg\}\Bigg)\Bigg) \\ &= \bigcup_{a:(a,i)|_m = (b,j)} (D_{n,mki} \cap B_{nai}) \end{aligned} \tag{92}$$

where we use (89) to (91) for the second equality and (81) for the last equality. Next, by (88) and (92) we find that

$$\begin{aligned} B_{mbj} \cap A_{mkj} &= \bigcup_{\substack{i \in \{0,1\}^n: \\ i|_m = j}} \bigcup_{a:(a,i)|_m = (b,j)} (D_{n,mki} \cap B_{nai}) \\ &= B_{mbj} \cap \bigcup_{\substack{i \in \{0,1\}^n: \\ i|_m = j}} D_{n,mki} \end{aligned} \tag{93}$$

where the last equality holds by (79). To conclude the proof consider now the case $2^{-m}a_{mkj} - \sum_{a:(g,(j|_{m-1},1))\prec(a,j)\prec(b,j)} |B_{maj}| > |B_{mbj}|$. Then, we have by (37) that

$$\begin{aligned} B_{mbj} \cap A_{mkj} &= \sum_{(a,i)\prec(b,j)} |B_{mai}| + [0, |B_{mbj}|) \\ &= B_{mbj} = \bigcupdot_{\substack{i\in\{0,1\}^n:\\ i|_m=j}} \bigcupdot_{\substack{a:(a,i)\in\tilde{E}_{n,a|_{m+1}\neq 0},\\ (a,i)|_m=(b,j)}} B_{nai} \end{aligned} \tag{94}$$

where the second equality is a consequence of (20) and the last equality holds by (61). In particular, using (94) and applying (19) repeatedly we find that

$$\sum_{\substack{a:(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}} |B_{nai}| = 2^{m-n}|B_{mbj}| < 2^{-n}a_{mkj} - \sum_{\substack{c:(g,(j|_{m-1},1))\prec\\ (c,i)|_m\prec(b,j)}} |B_{nci}| \tag{95}$$

where the inequality merely reflects the above assumption for the second case. Now, for any $a \in \mathbb{B}_n$ such that $(a,i)|_m = (b,j)$ we have by (95) that

$$2^{-n}a_{mkj} - \sum_{\substack{c:(g,(j|_{m-1}))\prec\\ (c,i)|_m\prec(b,j)}} |B_{nci}| - \sum_{\substack{c:(c,i)\prec(a,i),\\ (c,i)|_m=(b,j)}} |B_{nci}| > |B_{nai}|. \tag{96}$$

Hence, we get that

$$\begin{aligned} \bigcupdot_{\substack{a:(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}} B_{nai} &= \bigcupdot_{\substack{(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j))}} \left(\sum_{(c,l)\prec(a,i)} |B_{ncl}| + [0, |B_{nai}|) \right) \\ &= \bigcupdot_{\substack{(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}} \sum_{(c,l)\prec(a,i)} |B_{ncl}| + \Bigg[0, \min\Bigg\{2^{-n}a_{mkj} \\ &\quad - \sum_{\substack{c:(g,(j|_{m-1},1))\prec\\ (c,i)|_m\prec(b,j)}} |B_{nci}| - \sum_{\substack{c:(c,i)\prec(a,i),\\ (c,i)|_m=(b,j)}} |B_{nci}|, |B_{nai}|\Bigg\}\Bigg) \\ &= \bigcupdot_{\substack{a:(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}} (D_{n,mki} \cap B_{nai}) \end{aligned} \tag{97}$$

where the first equality holds by (20) and the second equality follows from (96). The last equality corresponds to (81). Now, similar to the above, the result follows by combining (94) and (97) first, and using (79) to conclude. □

Note that by (77) and (29) we get that

$$\bigcupdot_{i\in\{0,1\}^n} D_{n,mki} \cap \bigcupdot_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi} = D_{n,mki}, \quad i \in \{0,1\}^n, \tag{98}$$

such that (98) and Lemma 8 yield an alternative representation of $D_{n,mki}$, namely

$$D_{n,mki} = A_{m,k,i|_m} \cap \bigcupdot_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi} \quad \text{for all } i \in \{0,1\}^n.$$

By Lemma 8 and Lemma 5 we have

$$\bigcupdot_{j\in\{0,1\}^m} A_{mkj} = \bigcupdot_{i\in\{0,1\}^n} D_{n,mki}, \quad m \le n \in \mathbb{N}_0, k \in K_m. \tag{99}$$

Further, Lemma 8 yields that $D_{n,mki} \subseteq A_{mkj}$ for all $j \in \{0,1\}^m$, $k \in K_m$ and all $i \in \{0,1\}^n$ with $i|_m = j$. Then, the fact that for all $n \in \mathbb{N}_0$, $k \in K_n$ and $i \in \{0,1\}^n$ we have

$$D_{n,mki} \cap D_{n,pki} = \emptyset, \quad m < p \le n, \tag{100}$$

holds by Lemma 5.

5 Main Result

In the following theorem we shall make use of the sets S_n given in (5) where the unions are now seen to be disjoint by Lemmata 5 and 6.

Theorem 1. *The sequence of sets $(S_n)_{n\in\mathbb{N}_0} \uparrow S$ is monotonic, and*

$$2^{-n+1} \sum_{k\in K_n} \sum_{i\in\{0,1\}^n} \sum_{s=0}^{n} a_{s,k,i|_s} - \phi_{f_n}(h) = |S_n \cap (S_n - h)|, \quad n \in \mathbb{N}_0,\, h \in \mathbb{Z}.$$

Proof. By Lemma 6 we have

$$\begin{aligned}
|S_n \cap (S_n - h)| &= \sum_{k\in\mathbb{Z}} \left| \bigcupdot_{s=0}^{n} \bigcupdot_{i\in\{0,1\}^s} A_{ski} \quad \cap \quad \bigcupdot_{s=0}^{n} \bigcupdot_{i\in\{0,1\}^s} A_{s,k+h,i} \right| \\
&= \sum_{k\in\mathbb{Z}} \left| \bigcupdot_{i\in\{0,1\}^n} \bigcupdot_{s=0}^{n} D_{n,ski} \quad \cap \quad \bigcupdot_{i\in\{0,1\}^n} \bigcupdot_{s=0}^{n} D_{n,s,k+h,i} \right| \\
&= \sum_{k\in\mathbb{Z}} \left| \bigcupdot_{i\in\{0,1\}^n} \left(\bigcupdot_{s=0}^{n} D_{n,ski} \quad \cap \quad \bigcupdot_{s=0}^{n} D_{n,s,k+h,i} \right) \right| \\
&= \sum_{k\in\mathbb{Z}} \sum_{i\in\{0,1\}^n} \left| \bigcupdot_{s=0}^{n} D_{n,ski} \quad \cap \quad \bigcupdot_{s=0}^{n} D_{n,s,k+h,i} \right|
\end{aligned} \tag{101}$$

where the second equality holds by (99), and (78) gives the third equality. Next, note that (76) and (100) yield

$$\begin{aligned}
\bigcupdot_{s=0}^{n} D_{n,ski} &= \bigcupdot_{b:(b,i)\in\tilde{E}_{n,0}} \Bigg\{ B_{nbi} \cap \Bigg(\sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| \\
&\qquad + \bigcupdot_{s=0}^{n} \Bigg([0, 2^{-n} a_{s,k,i|_s}) + \sum_{\substack{a:(a,i)\in\tilde{E}_{n,0}:\\ (a,i)|_s \preceq (g,(i|_{s-1},1))}} |B_{nai}| \Bigg) \Bigg) \Bigg\} \\
&= \bigcupdot_{b:(b,i)\in\tilde{E}_{n,0}} \Bigg\{ B_{nbi} \cap \Bigg(\sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| + \Bigg[0, 2^{-n} \sum_{s=0}^{n} a_{s,k,i|_s} \Bigg) \Bigg) \Bigg\}
\end{aligned} \tag{102}$$

where the second equality holds by (59) and (19). Using (102) we get that

$$\bigdot\bigcup_{s=0}^{n} D_{n,ski} \quad \cap \quad \bigdot\bigcup_{s=0}^{n} D_{n,s,k+h,i} = \bigdot\bigcup_{s=0}^{n} D_{n,ski}$$

if and only if

$$\sum_{s=0}^{n} a_{s,k,i|_s} \le \sum_{s=0}^{n} a_{s,k+h,i|_s},$$

for any $n \in \mathbb{N}_0$, $h \in \mathbb{Z}$, $k \in K_n$ and $i \in \{0,1\}^n$. By (86) we further have

$$\left| \bigdot\bigcup_{s=0}^{n} D_{n,ski} \right| = 2^{-n} \sum_{s=0}^{n} a_{s,k,i|_s}. \tag{103}$$

Now, comparing (101) and (4), and using (103) yields the assertion. □

Corollary 3. *For any extremal coefficient function ϕ of a dissipative max-stable process on $\mathbb{Z}$ there exists a measurable set $S \subset \mathbb{R}$ such that*

$$2 - \phi(h) = |S \cap (S+h)|, \qquad h \in \mathbb{Z}.$$

Proof. By the results of [1], see the introduction, there is a measurable function f such that

$$2 - \phi(h) = \int \min\{f(s), f(s+h)\}\, ds, \qquad h \in \mathbb{Z}. \tag{104}$$

Starting from this function f we will construct a function γ which satisfies (104) instead of f, and may be approximated by (3). Let (ξ_j) be a sequence of simple functions with values $0 = k_0^{(j)} < k_1^{(j)} < \ldots < k_{m_j}^{(j)} < k_{m_j+1}^{(j)} = \infty$, and $\xi_j \uparrow f$. We may assume without loss of generality that

$$\xi_j(x) = \sum_{i=0}^{m_j} k_i^{(j)} \mathbf{1}\left(k_i^{(j)} \le f(x) < k_{i+1}^{(j)}\right). \tag{105}$$

Let the mapping $\iota : \mathbb{R}^{\mathbb{Z}\times[0,1)} \to \mathbb{R}^{\mathbb{R}}$, $\tilde{f} \mapsto f$, be defined by $f(x) = \tilde{f}(\lfloor x \rfloor, x - \lfloor x \rfloor)$. Consider the set $K_j = \{k_0^{(j)}, \ldots, k_{m_j+1}^{(j)}\}^{\mathbb{Z}}$ and impose an ordering on $\mathbb{R}^{\mathbb{Z}}$, hence on $\bigcup_j K_j$, such that the ordering of the margins is preserved, i.e., for all $k \in \mathbb{Z}$, $w_l, w_k^{(1)}, w_k^{(2)} \in \mathbb{R}$, $l \ne k$, and $w_k^{(1)} < w_k^{(2)}$ we have

$$(\ldots, w_{k-2}, w_{k-1}, w_k^{(1)}, w_{k+1}, w_{k+2}, \ldots) < (\ldots, w_{k-2}, w_{k-1}, w_k^{(2)}, w_{k+1}, w_{k+2}, \ldots).$$

Let, for $w = (\ldots, w_{-1}, w_0, w_1, \ldots) \in K_j$,

$$d_w^{(j)} = \left| \bigcap_{i\in\mathbb{Z}} \left(\iota^{-1}(\xi_j)(i,\cdot)\right)^{-1} (\{w_i\}) \right|$$

and put

$$\iota^{-1}(\gamma_j)(i,x) = \sum_{w\in K_j} w_i \mathbf{1}\left(0 < x - \sum_{\tilde{w}\in K_j, \tilde{w}<w} d_{\tilde{w}}^{(j)} \le d_w^{(j)}\right).$$

Then, we have for $h \in \mathbb{Z}$ that

$$\begin{aligned}&\int \min\{\gamma_j(s), \gamma_j(s+h)\}\, ds \\ &= \sum_{w \in K_j} d_w^{(j)} \sum_{z \in \mathbb{Z}} \min\{w_z, w_{z+h}\} = \int \min\{\xi_j(s), \xi_j(s+h)\}\, ds.\end{aligned}$$

Furthermore, the definition (105) of the ξ_j and the introduced ordering imply that

$$\sum_{\tilde{w} \in K_j : \tilde{w} < w'} d_{\tilde{w}}^{(j)} \leq \sum_{\tilde{w} \in K_{j+1} : \tilde{w} < w} d_{\tilde{w}}^{(j)} \leq \sum_{\tilde{w} \in K_{j+1} : \tilde{w} \leq w} d_{\tilde{w}}^{(j)} \leq \sum_{\tilde{w} \in K_j : \tilde{w} < w''} d_{\tilde{w}}^{(j)}$$

for all $w \in K_{j+1}$ and $w', w'' \in K_j$ with $w' \leq w < w''$. Hence, γ_j is monotonically increasing to some measurable function γ. Since for each j the function γ_j can be approximated by a sequence of step functions f_n satisfying (3) the limit γ can also be approximated by such a sequence. This proofs the assertion of the corollary. □

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