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Bargaining and Power

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Summary

Given a simple game, a power configuration specifies the power of each player in each winning coalition. We introduce a new power configuration which takes into account bargaining among players in coalitions. We show that under very weak conditions on a bargaining solution there is a power configuration which is stable with respect to renegotiations. We further show that given this power configuration there is a coalition which is both internally and Nash stable. We consider two different bargaining solutions on apex games and show under which conditions there are core stable coalitions. Finally, we investigate how infeasible coalition might affect the outcome and apply our model to the German parliament.

Keywords: Coalition Formation, Power, Bargaining

JEL Classification: C71, D71

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Bargaining and Power

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Abstract. Given a simple game, a power configuration specifies the power of each player in each winning coalition. We introduce a new power configuration which takes into account bargaining among players in coalitions. We show that under very weak conditions on a bargaining solution there is a power configuration which is stable with respect to renegotiations. We further show that given this power configuration there is a coalition which is both internally and Nash stable. We consider two different bargaining solutions on apex games and show under which conditions there are core stable coalitions. Finally, we investigate how infeasible coalition might affect the outcome and apply our model to the German parliament.

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1 Introduction

Consider a committee or a parliament which has to make decisions. Usually, players or parties with similar interests form coalitions which are able to enforce their will. Based on possible coalitions a player might join, one can make statements about his power in the committee or in a coalition he is member of.

The measurement of a member's power in a committee has been the subject of many articles. Famous examples are the Shapley-Shubik power index (SSPI, Shapley and Shubik, 1954) and various versions of the Banzhaf-Coleman power index (BCPI, Banzhaf, 1965). However, these indices per se do not consider players' power in coalitions apart from the whole player set. The same holds true for the values presented in de Clippel (2008) and Dutta et al. (2010) which apply to games with externalities.

Shenoy (1979) introduced the power of players in each coalition based on the Shapley-Shubik index. In particular, the author considered a coalition formation game where players' preferences over coalitions depend on their power in coalitions. This concept has been further developed and generalized by Dimitrov and Haake (2006, 2008a,b). However, these ideas of power within a coalition did not take into account anything outside of this coalition.

The Owen value (OV, Owen, 1977) and the Casajus value (CV, Casajus, 2009) are adaptations of the Shapley-Shubik index which take into account the partition of the player set into coalitions. The first one has been used by Hart and Kurz (1983, 1984) to introduce a similar coalition formation game as Shenoy (1979), but under consideration of the behavior of players outside of a fixed coalition. Although the power of a player therefore depends on other coalitions as well, the power a player has one coalition is completely independent of his power in any other coalition.

We interpret power as a payoff of players, for instance in a parliament where a government of several parties has to agree on the allocation of cabinet seats among parties. In this case, a player can use the power in one coalition

to claim a certain power in another coalition. In other words: Power in one coalition can be used to bargain about power in other coalitions. We illustrate this idea in the following example.

Example 1.1. The German Bundestag currently consists of five parliamentary parties, CDU/CSU (1), FDP (2), SPD (3), Linke (4), and B90/Grüne (5). A coalition has the absolute majority if and only if it contains at least one of the following coalitions: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, or $\{2, 3, 4\}$. In the model of Shenoy (1979) there is no stable outcome of this game; in the model of Hart and Kurz (1984) each of these four coalitions is stable.

The current government consists of CDU/CSU and FDP. Under the assumption that parties in the opposition do not collaborate, the above mentioned power indices deliver the following values for the governmental parties.

	SSPI	BCPI	OV	CV
CDU	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$
FDP	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$

The idea that these two parties are equally powerful, as SSPI and BCPI suggest, is not very convincing, given that there are two other parties outside of the government which each have the absolute majority together with CDU/CSU. OV distinguishes between the two parties, but if we assume that the remaining parties work together, i.e. if the partition of the player set changes, then we have $OV_{CDU} = OV_{FDP} = \frac{1}{2}$ although CDU has much better chances to find a different party for a government coalition than FDP. The cabinet consists of 16 ministers of which 11 are members of CDU/CSU and 5 belong to FDP. Hence, in this example the outside option value is closest to the actual distribution of power between parties.

The models mentioned thus far always made the assumption that the power of players in coalitions is specified ex ante and leads to a coalition formation game. Nevertheless, in reality power is a result from bargaining:

players identify their options in various coalitions use them to renegotiate. Hence, a separation of the coalition formation process from the power distribution does not seem convincing.

Our model brings together these two concepts; the power of a player within a coalition depends on two things:

1. His marginal contribution: A player who is needed in a winning coalition is more powerful than a player who could leave the coalition without effect.
2. His outside option: A player who is very powerful in another coalition is more powerful than a player who has no other options.

We assume that in each winning coalition a bargaining problem (Nash, 1950) occurs and that the allocation of power in coalitions follows a fixed bargaining rule which takes into account marginal contributions and outside options of all players. After any negotiation, the outside options of players may have changed and lead to a new negotiation. We do not focus on this dynamic process, but on the question whether we can find an allocation which is stable with respect to renegotiation. In this case an application of the bargaining rule would not change the result. We will show in Section 2 that under very weak conditions on a bargaining rule such an allocation exists. We also consider a special bargaining rule and show that under some restrictions this stable allocation is even unique.

We can interpret such an allocation as the result of exploratory talks between all groups of parties. As this allocation will not be renegotiated, the preferences of players over coalitions based on this allocation are very robust. In Section 3 we give conditions for the existence of a coalition which is both internally stable (i.e. no group of players would leave it to stay alone) and Nash stable (no player would leave the coalition to join another one). We also apply our model to the class of apex games. Karos (2012) considered the coalition formation game after application of the Shapley-Shubik index or the

normalized Banzhaf-Coleman index. It has been shown that each coalition contains a group of players which can improve by leaving and joining the players outside of this coalition. In the model we develop in this article the existence of a coalition which will not be left by any players is guaranteed for various bargaining rules.

In Section 4 we further extend our model. Especially in parliaments not all coalitions which could reach the absolute majority are likely to occur. There are parties which will never collaborate due to their political interests. Milchtaich and Winter (2002) introduced a model on which the distance between players in a property space is used to develop a coalition formation theory. In our case we consider only two cases. Either players in coalition are similar enough to collaborate or they are not. In the latter case we speak of an infeasible coalition. As the stable allocation in our model depends on outside options, we have to ensure that the allocation in an infeasible coalition does not affect the allocation in any other coalition. We show that if each player can chose to stay alone, a stable allocation still exists. In particular, if there is at least one possible winning coalition, then we can find a coalition which is stable as before. We apply our model to the German parliament and compare the results of different bargaining rules with the actual government and the distribution of cabinet seats among them. In Section 5 we give some concluding remarks and possible further developments of our model.

2 The Model

For $i \in N$ and $S \subseteq N$ the *marginal contribution* of player i to S in v is defined by

$$d_i^m(S) = v(S) - v(S \setminus \{i\}).$$

Note, that $d_i^m(S)$ depends on v ; we skip v in the notation for convenience, though.

Definition 2.1. Let v be a simple game. A *power configuration* $x = (x_i)_{i \in N}$

for v is a vector of maps $x_i : \mathcal{P}_i \rightarrow \mathbb{R}$ such that $x_i(S) = 0$ for all $i \in S$ if $v(S) = 0$ and $\sum_{i \in S} x_i(S) \leq v(S)$ for all $S \subseteq N$. A power configuration x is called *efficient* if $\sum_{i \in S} x_i(S) = v(S)$ for all $S \subseteq N$, and *individually rational* if $x_i(S) \geq v(\{i\})$ for all $S \in \mathcal{P}_i$. The collection of all power configurations for v is denoted by $\Delta(v)$ and the collection of all individually rational power configuration is denoted by $\Delta_{ir}(v)$.

We can think of a power configuration as a set of agreements which clarify in each coalition S how power is distributed in S . Let x be a power configuration. The condition $x_i(S) = 0$ for losing coalitions and all $i \in S$ reflects the idea that a player should not have any power if he is member of a losing coalition. Let S be a winning coalition and let $i \in S$. Player i is contained in many other coalitions (for instance $(N \setminus S) \cup \{i\}$), in particular, each coalition $T \subseteq (N \setminus S) \cup \{i\}$ with $i \in T$ ensures him power $x_i(T)$. We define player i 's *outside option in S* as

$$d_i^o(S, x) = \max_{T \subseteq N \setminus S} x_i(T \cup \{i\}).$$

When the members of S are negotiating on how power within S should be shared, the two values d_i^m and d_i^o are crucial for the bargaining position of i . The next definition specifies what we mean by bargaining.

Definition 2.2. A *disagreement point* for S is a vector $d = d(S) \in \mathbb{R}^S$. A *bargaining solution* is a map F such that $F(S, v(S), d(S)) \in \mathbb{R}^S$,

$$\sum_{i \in S} F_i(S, v(S), d(S)) \leq v(S)$$

for each proper monotonic simple game v , each coalition $S \subseteq N$, and each disagreement point d for S ; and $F(\{i\}, v(\{i\}), d(\{i\})) = v(\{i\})$.

The triple $(S, v(S), d)$ is called a *bargaining problem*. It describes exactly the situation discussed above: The players in S negotiate about how

to distribute $v(S)$ where the disagreement point d represents their bargaining positions. We have mentioned before that the bargaining position of player i depends on two things, namely the marginal contributions $d_i^m(S)$ and the outside option $d_i^o(S, x)$, given a power configuration x . In our model we assume that disagreement points are convex combinations of d^m and d^o . Henceforth, let

$$d_i(S, x) = \alpha d_i^m(S) + (1 - \alpha) d_i^o(S, x)$$

be the disagreement point in the bargaining problem within coalition S where $\alpha \in [0, 1]$. It is clear that the outside option $d_i^o(S, x)$ of a player $i \in S$ can be positive only if $(N \setminus S) \cup \{i\}$ is winning. Because of properness of v this can be the case only if $S \setminus \{i\}$ is losing. Hence, a player $i \in S$ can only have a positive outside option if he is pivotal in S . The parameter α specifies how this outside option shall be weighted. Many of the further results do not depend on the choice of α . For convenience, we do not mention α in these cases, having in mind that $\alpha \in [0, 1]$ is fixed but arbitrary.

We are now facing the following problem: Given any power configuration x , we have a set of bargaining problems with disagreement points depending on x , hence, players renegotiate their power. After applying a bargaining solution F , we end up with a new power configuration which leads to renegotiation, again. We are looking for a power configuration which is stable with respect to renegotiation. The next definition formalizes this idea.

Definition 2.3. Let F be a bargaining solution and v be a proper monotonic simple game. A power configuration $x \in \Delta(v)$ is called *stable with respect to F* if for all winning coalitions $S \subseteq N$ and all $i \in S$ the following holds.

$$\begin{aligned} x_i(S) &= F_i(S, v(S), d(S, x)) \\ d_i(S, x) &= \alpha d_i^m(S) + (1 - \alpha) d_i^o(S, x) \\ d_i^o(S, x) &= \max_{T \subseteq N \setminus S} x_i(T \cup \{i\}). \end{aligned} \tag{1}$$

Note that for all power configurations $x \in \Delta(v)$, all winning $S \subseteq N$ and all $i \in S$ we have that $d_i^o(S, x) \geq v(\{i\}) \geq 0$, hence, $d_i(S, x) \geq 0$. For general bargaining solutions F we cannot assume that a stable payoff configuration exists for all proper monotonic simple games. The aim of the remainder of this section is to find sufficient conditions on F such that a stable power configuration exists.

Remark 2.4. In classical bargaining theory we have that $\sum_{i \in S} d_i(S) \leq v(S)$ for each bargaining problem. We do not restrict our attention to this case. If the disagreement point is such that it cannot be reached by any allocation of $v(S)$, one usually talks about a *bankruptcy problem* (see for instance Aumann and Maschler, 1985; Curiel et al., 1987).

We can also interpret our bargaining problems as *bargaining problems with claims* (Chun and Thomson, 1992). There in a coalition S each player i has a disagreement point he could reach if he does not join the S (which in our case would be $v(\{i\})$) and a claim point (in our case $d_i(S, x)$).

We will not distinguish between bargaining problems, bargaining problems with claims, or bankruptcy problems; henceforth we will talk only about bargaining problems and disagreement points. The following properties a bargaining solution might satisfy account for this and are therefore slightly different from definitions which can be found in literature on bargaining problems.

Definition 2.5. A bargaining solution F is called

1. *individually rational* if we have $F_i(S, v(S), d) \geq v(\{i\})$ for all bargaining problems $(S, v(S), d)$ and all $i \in S$.
2. *efficient* if we have $\sum_{i \in S} F_i(S, v(S), d) = v(S)$ for all bargaining problems $(S, v(S), d)$ and all $i \in S$.
3. *symmetric* if we have $F_i(S, v(S), d) = F_j(S, v(S), d)$ for all bargaining problems $(S, v(S), d)$ with $d_i = d_j$.

4. *continuous* if $F(S, v(S), \cdot)$ is continuous for all coalitions $S \subseteq N$ and all proper monotonic simple games v .
5. *fair* if for all bargaining problems $(S, v(S), d)$ there is a player $i \in S$ with $F_i(S, v(S), d) \geq d_i(S)$ only if $F_j(S, v(S), d) \geq d_j(S)$ for all players $j \in S$.

Individual rationality does not guarantee that all players are satisfied by their power in the sense that $F_i(S, v(S), d) \geq d_i$. It rather says that no player should have less power than if he stays alone. Since we do not assume that each player i can receive at least d_i , fairness ensures that all players are on the same side of d : A player i cannot get more than d_i in the same coalition another player j receives less than d_j . Efficiency is standard, it can be understood as a normalization such that the distributed power in each winning coalition sums up to 1. Continuity ensures that a small change in disagreement points cannot cause an arbitrarily large change in the bargaining outcome.

Example 2.6. Let v be a proper monotonic simple game.

1. The *egalitarian bargaining solution* is defined as

$$E_i(S, v(S), d(S)) = d_i(S) + \frac{1}{|S|} \left(v(S) - \sum_{j \in S} d_j(S) \right)$$

for all $i \in N$. Clearly, E is efficient, fair, and continuous. However, E is not individual rational, as $\sum_{j \in S} d_j(S)$ might be very large.

2. The *constrained egalitarian bargaining solution* (see for instance Curiel et al., 1987) is defined as

$$\tilde{E}_i(S, v(S), d(S)) = \max \{d_i - \lambda, 0\}$$

for all $i \in N$, where λ is such that $\sum_{i \in S} E_i(S, v(S), d(S)) = v(S)$. \tilde{E} is individual rational, efficient, and continuous. However, \tilde{E} is not fair.

3. The *proportional bargaining solution*¹ is defined as

$$P_i(S, v(S), d(S)) = \begin{cases} \frac{d_i(S)}{\sum_{i \in S} d_i(S)} v(S), & \text{if } \sum_{i \in S} d_i(S) \neq 0, \\ \frac{1}{|S|} v(S), & \text{if } \sum_{i \in S} d_i(S) = 0 \end{cases}$$

for all $i \in N$. We see on the first sight that P is individual rational, efficient, and fair. But P is not continuous at $d = 0$.

The following theorem focuses on continuous bargaining solutions and gives sufficient conditions for the existence of a stable power configuration. The proportional bargaining solution, which is not continuous, is considered in the next section.

Theorem 2.7. *Let F be a continuous bargaining solution, let $\alpha \in [0, 1]$ be fixed but arbitrary, and v be a proper monotonic simple game.*

1. *If F is individually rational then there is a stable power configuration $x \in \Delta_{ir}(v)$.*
2. *If F is fair and efficient then there is a stable power configuration $x \in \Delta(v)$.*

Proof. Let F be a bargaining solution and v be a proper monotonic simple game. We define the map $\hat{F} : \Delta(v) \rightarrow \Delta(v)$ as

$$\hat{F}_{i,S}(x) = F_i(S, v(S), d(S, x)).$$

A power configuration $x \in \Delta(v)$ is stable with respect to F if and only if $\hat{F}(x) = x$. Hence, we have to show that \hat{F} has a fixed point. Before we show that \hat{F} is in both cases a map from a compact convex set on itself, we show that if F is continuous then \hat{F} is continuous as well. For this purpose we

¹This is the proportional solution from bankruptcy games (Curiel et al., 1987); particularly, it is different from the proportional bargaining solution introduced by Kalai (1977)

need the following construction. Let $x \in \Delta(v)$, $S \subseteq N$ and $i \in S$. Then let $\mathcal{T}_i^S(x) \subseteq \mathcal{P}(N \setminus S)$ be such that

$$x_i(T \cup \{i\}) \geq x_i(T' \cup \{i\})$$

for all $T \in \mathcal{T}_i^S(x)$ and all $T' \subseteq N \setminus S$. That is, given the power configuration x , $\mathcal{T}_i^S(x)$ is the collection of optimal coalitions for player i outside of S . In particular, we have $d_i^o(S, x) = x_i(T_i)$ for all $T_i \in \mathcal{T}_i^S(x)$. Note that

$$\hat{F}_{i,S}(x) = F_i(S, v(S), \alpha d_i^m(S) + (1 - \alpha)(x_i(T_i))_{i \in S})$$

for all $T_i \in \mathcal{T}_i^S(x)$. Let now

$$[x] = \{y \in \Delta(v); \mathcal{T}_i^S(x) \cap \mathcal{T}_i^S(y) \neq \emptyset \text{ for all } S \subseteq N, i \in S\}.$$

Then \hat{F} is continuous on $[x]$ for all $x \in \Delta(v)$ as F is continuous. By definition of $\mathcal{T}_i^S(x)$ it is straightforward that $[x]$ is closed for all $x \in \Delta(v)$. As further N and $\mathcal{P}(N)$ are finite, there can only be a finite number of sets of this type, i.e. there are x_1, \dots, x_n such that

$$\Delta(v) = \bigcup_{k=1}^n [x_k].$$

As \hat{F} is continuous on $[x_k]$ and $[x_k]$ is closed for all $k = 1, \dots, n$, \hat{F} is continuous on $\Delta(v)$.

We show now that in both cases of the Theorem there is a compact convex subset of $\Delta(v)$ such that \hat{F} maps this set on itself.

1. Let now F be individual rational. Then we have $F(\Delta_{ir}(v)) \subseteq \Delta_{ir}(v)$.
2. Let F be fair and efficient. Since $F_i(\{i\}, v(\{i\}), d) = v(\{i\}) \geq 0$, we

have $d_i(S) \geq 0$ for all $S \subseteq N$. Let now

$$Q = \{x \in \Delta(v); -(|N| - 1) \leq x_i(S) \leq 1 \text{ for all } S \subseteq N, i \in S\}.$$

We show that $\hat{F}(Q) \subseteq Q$. Let therefore $x \in Q$ and $S \subseteq N$. We consider two cases:

(a) Let $F_i(S, v(S), d(S, x)) \geq d_i(S, x) \geq 0$ for all $i \in S$. Since

$$\sum_{i \in S} F_i(S, v(S), d(S, x)) \leq v(S) \leq 1,$$

we have that $0 \leq F_i(S, v(S), d(S, x)) \leq 1$.

(b) Let $F_i(S, v(S), d(S, x)) \leq d_i(S, x)$. Then

$$\begin{aligned} F_i(S, v(S), d(S, x)) &= v(S) - \sum_{j \in S \setminus \{i\}} F_j(S, v(S), d(S, x)) \\ &\geq - \sum_{j \in S \setminus \{i\}} d_j(S, x) \\ &\geq - \sum_{j \in S \setminus \{i\}} \max_{x \in Q, T \subseteq N \setminus S} x_j(T \cup \{i\}) \\ &\geq -(|N| - 1). \end{aligned}$$

Hence,

$$-(|N| - 1) \leq F_i(S, v(S), d(S, x)) \leq d_i(S, x) \leq \max_{i, T} x_i(T) \leq 1.$$

So, we have that $\hat{F}(x) \in Q$.

As $\Delta_{ir}(v)$ and Q are both compact and convex, we can apply Brouwer's fixed point theorem. Hence, there is a fixed point x of \hat{F} . Particularly, in the first case we have $x \in \Delta_{ir}(v)$. ■

Theorem 2.7 ensures the existence of stable power configurations for con-

tinuous bargaining solutions under very weak conditions. Together with Example 2.6 it implies the following corollary immediately.

Corollary 2.8. *Let v be a proper monotonic simple game and let $F = E$ or $F = \tilde{E}$. Then there is a stable payoff configuration $x \in \Delta(v)$ with respect to F .*

Although the existence of a stable power configuration for all proper monotonic simple games is a strong result, Theorem 2.7 does not guarantee uniqueness of the stable power configuration. The next example shows that in general the stable power configuration is not unique.

Example 2.9. Let $\alpha = 0$ so that $d(S, x) = d^o(S, x)$, and let v be the proper monotonic simple game on $N = \{1, 2, 3, 4, 5, 6\}$ with minimal winning coalitions $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{2, 4, 6\}$, and $\{3, 5, 6\}$. A stable power configuration with respect to P , E , and \tilde{E} is for instance given by $x_i(S) = \frac{v(S)}{|S|}$ for all $S \subseteq N$ and all $i \in S$. However, this is not the only stable power configuration. Let y be defined as follows:

$$y_i(S) = \begin{cases} 0, & \text{if } v(S) = 0 \text{ or } i \notin S \\ \frac{1}{|S|}, & \text{if } v(S) = 1 \text{ and } |S| \geq 5, \\ 1, & \text{if } (S = \{1, 2, 3, 6\} \text{ or } S = \{1, 4, 5, 6\}) \text{ and } i = 1, \\ 1, & \text{if } (S = \{1, 2, 4, 6\} \text{ or } S = \{1, 3, 5, 6\}) \text{ and } i = 6, \\ 1, & \text{if } v(S) = 1, |S| = 3, i = 1, \text{ and } 1 \in S, \\ 0, & \text{if } v(S) = 0, |S| = 3, i \neq 1, \text{ and } 1 \in S, \\ 1, & \text{if } v(S) = 1, |S| = 3, i = 6, \text{ and } 6 \in S, \\ 0, & \text{if } v(S) = 0, |S| = 3, i \neq 6, \text{ and } 6 \in S. \end{cases}$$

Then we have $\hat{P}(x) = \hat{E}(x) = \hat{\tilde{E}}(x) = x$, that is x is stable with respect to P , E , and \tilde{E} , too.

Although we cannot guarantee uniqueness of a stable power configuration, we can state some properties it must have.

Lemma 2.10. *Let v be a proper monotonic simple game, let F be a bargaining solution, and let $x \in \Delta(v)$ be stable with respect to F .*

1. *If F is efficient then x is efficient.*
2. *If F is individually rational then x is individually rational.*
3. *If F is symmetric and i, j are symmetric with respect to v then x' , defined as*

$$x'_i(S) = \begin{cases} x_j(S), & \text{if } i, j \in S, \\ x_j((S \setminus \{i\}) \cup \{j\}), & \text{if } i \in S \text{ and } j \notin S, \end{cases}$$

$$x'_j(S) = \begin{cases} x_i(S), & \text{if } i, j \in S, \\ x_i((S \setminus \{j\}) \cup \{i\}), & \text{if } j \in S \text{ and } i \notin S, \end{cases}$$

and $x'_k(S) = x_k(S)$ for all $k \neq i, j$ and all $S \subseteq N$, is stable with respect to F as well.

Proof. The first two parts of the lemma are obvious, we prove only the last part. Let $x \in \Delta(v)$ be stable with respect to F and let $i, j \in N$ be symmetric with respect to v . Let $p^{i,j} : N \rightarrow N$ be the permutation defined by

$$p^{i,j}(k) = \begin{cases} i, & \text{if } k = j, \\ j, & \text{if } k = i, \\ k, & \text{if } k \neq i, j. \end{cases}$$

Then $T \in \mathcal{T}_k^S(x)$ if and only if $p^{i,j}(T) \in \mathcal{T}_{p^{i,j}(k)}^{p^{i,j}(S)}(x')$. Hence,

$$\begin{aligned} F_{p^{i,j}(k)}(p^{i,j}(S), v(p^{i,j}(S)), d(p^{i,j}(S), x')) &= F_k(S, v(S), x(T)) \\ &= x_k(S) \\ &= x'_{p^{i,j}(k)}(S). \end{aligned}$$

■

The first two parts of the Lemma need no further explanation. For the last part one has to keep in mind that a stable power configuration need not to be unique. In particular, not every stable power configuration is symmetric, i.e. give the same to symmetric players. Lemma 2.10 guarantees that the set of all stable power configuration is symmetric, though. An easy consequence is the following corollary.

Corollary 2.11. *Let v be a proper monotonic simple game and let F be a symmetric bargaining solution such that there is a unique $x \in \Delta(v)$ which is stable with respect to F . Then $x_i(S) = x_j(S)$ for all $S \subseteq N$ with $i, j \in S$ and $x_i(S \cup \{i\}) = x_j(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.*

We have already mentioned that the proportional bargaining solution P is not continuous at $d = 0$, so that we cannot apply Theorem 2.7. Nevertheless, the following lemma already states some properties of stable power configurations with respect to P , if they exist. We will later use it to prove existence and even uniqueness under some additional conditions.

Lemma 2.12. *Let v be a proper monotonic simple game let $x \in \Delta(v)$ be stable with respect to P .*

1. *If $S \subseteq N$ is such that no $i \in S$ is pivotal in S with respect to v then $x_i(S) = \frac{v(S)}{|S|}$ for all $i \in S$.*
2. *If $\alpha > 0$ and if $S \subseteq N$ is such that there is at least one player in S who is pivotal in S with respect to v then $x_j(S) = 0$ for all $j \in S$ which are*

not pivotal in S . If, additionally, there is only one player $i \in S$ who is pivotal in S with respect to v then $x_i(S) = 1$ and $x_j(S) = 0$ for all $j \in S \setminus \{i\}$.

Proof.

1. Since no i is pivotal, we have $d_i^m(S) = 0$ for all $i \in S$. Further, by properness of v , we have $v((N \setminus S) \cup \{i\}) = 0$. Hence, $d_i^o(S, x) = 0$ for all $x \in \Delta(v)$. Thus, $d_i(S, x) = 0$ for all $x \in \Delta(v)$ and all $i \in S$ and therefore $P_i(S, v(S), d(S, x)) = \frac{v(S)}{|S|}$.
2. By the same arguments as in the first part we have that $d_j(S, x) = 0$ for all $x \in \Delta(v)$ and all $j \in S$ which are not pivotal. As $d_i^m(S) > 0$ for each pivotal player $i \in S$ we have that $d_i(S, x) > 0$ and thus, $P_j(S, v(S), d(S, x)) = \frac{0}{\sum_{i \in S} d_i(S)} = 0$. If i is the only pivotal player, we have $P_i(S, v(S), d(S, x)) = 1$ by efficiency of P .

■

Note that the last two results in Lemma 2.12 depend on the parameter α . For $\alpha = 1$ the stable power would be unique and very easy to find: For any winning coalition S let $S' \subseteq S$ be the set of players who are pivotal in S . Then

$$x_i(S) = \begin{cases} \frac{1}{|S'|}, & \text{if } i \in S', \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

In the next theorem we show that for strictly positive α a stable power configuration with respect to P always exists. Moreover, we give a lower bound for α such that this stable power configuration is unique.

Theorem 2.13. *Let v be a proper monotonic simple game.*

1. *Let $\alpha > 0$. Then there is $x \in \Delta(v)$ which is stable with respect to P .*
2. *Let $\alpha \geq \frac{|N|}{|N|+2}$. Then there is a unique $x \in \Delta(v)$ which is stable with respect to P .*

Proof.

1. Let $\alpha > 0$. It has been shown in Lemma 2.12 that $\hat{P}_{i,S}$ is constant and therefore continuous for all coalitions $S \subseteq N$ which do not contain at least two pivotal players. We show that $\hat{P}_{i,S}$ is also continuous for all coalitions $S \subseteq N$ which contain at least two pivotal players. We see that

$$\hat{P}_{i,S}(x) = \frac{d_i(S)v(S)}{\sum_{j \in S} d_j(S)} = \frac{d_i(S)v(S)}{\alpha \sum_{j \in S} d_j^m(S) + (1 - \alpha) \sum_{j \in S} d_j^o(S)}.$$

As $\sum_{j \in S} d_j^m(S) \geq 2$ and $\alpha > 0$, we have that \hat{P} is continuous for all $x \in \Delta(v)$. Since $\hat{P}(\Delta_{ir}(v)) \subseteq \Delta_{ir}(v)$, there must be a fixed point of \hat{P} in $\Delta_{ir}(v)$ by Brouwer's fixed point theorem.

2. Let $\alpha \geq \frac{|N|-2}{|N|}$. If v is a proper monotonic simple game such that there is $i \in N$ with $v(\{i\}) = 1$ then the only stable power configuration with respect to P is

$$x_k(S) = \begin{cases} 1, & \text{if } k = i \text{ and } i \in S, \\ 0, & \text{otherwise} \end{cases}$$

by Lemma 2.12. So let v be such that $v(\{i\}) = 0$ for all $i \in N$. We show that \hat{P} is a contraction on $\Delta_{ir}(v)$. For this purpose, note that for the partial derivatives of \hat{P} we have

$$\frac{\partial \hat{P}_{i,S}}{\partial x_j(T)} = 0$$

for all $S \subseteq N$ which do not contain at least two pivotal players and for all $S \subseteq N$ and all $i \in S$ which are not pivotal in S , for all $T \subseteq N$ and all $j \in T$. We also have $\frac{\partial \hat{P}_{i,N}}{\partial x_j(T)} = 0$ for all $i \in N$, all $T \subseteq N$ and all $j \in T$ as N does not contain any player with a positive outside option. Let therefore $S \subsetneq N$ contain at least two pivotal players and let $i \in S$

be pivotal. Let further $T_j^S(x) \in \mathcal{T}_j^S(x)$ for all $j \in S$. Then we have

$$\frac{\partial \hat{P}_{i,S}}{\partial x_j(T)} = \begin{cases} \frac{(1-\alpha)(\alpha \sum_{k \neq i} d_k^m(S) + (1-\alpha) \sum_{k \neq i} d_k^o(S))}{(\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S))^2}, & \text{if } i = j, T = T_i^S(x), \\ -\frac{(1-\alpha)(\alpha d_i^m(S) + (1-\alpha) d_i^o(S))}{(\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S))^2}, & \text{if } i \neq j, T = T_j^S(x), \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \sum_{j \in N, T \subseteq N} \left| \frac{\partial \hat{P}_{i,S}}{\partial x_j(T)} \right| &\leq \frac{1-\alpha}{\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S)} \\ &\quad + \frac{(1-\alpha)(|S|-1)(\alpha d_i^m(S) + (1-\alpha) d_i^o(S))}{(\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S))^2} \\ &= \frac{1-\alpha}{\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S)} \\ &\quad + \frac{(1-\alpha)|S|(\alpha d_i^m(S) + (1-\alpha) d_i^o(S))}{(\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S))^2} \\ &\quad - \frac{(1-\alpha)(\alpha d_i^m(S) + (1-\alpha) d_i^o(S))}{(\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S))^2} \\ &\leq \frac{(1-\alpha)(|S|+1)}{\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S)} \\ &\quad - \frac{1-\alpha}{(\alpha \sum_{k \in S} d_k^m(S) + (1-\alpha) \sum_{k \in S} d_k^o(S))^2} \\ &\leq \frac{(1-\alpha)(|S|+1)}{2\alpha} - \frac{1-\alpha}{|S|^2} \end{aligned}$$

For the Jacobian matrix $D_{\hat{P}}$ we therefore find

$$\begin{aligned} \|D_{\hat{P}}\|_{\infty} &= \max_{i \in N, S \subseteq N} \sum_{j \in N, T \subseteq N} \left| \frac{\partial \hat{P}_{i,S}}{\partial x_j(T)} \right| \\ &\leq \frac{(1-\alpha)|N|}{2\alpha} - \frac{1-\alpha}{(|N|-1)^2}. \end{aligned}$$

Since this bound is decreasing in α and since $\alpha \geq \frac{|N|}{|N|+2}$ we find

$$\|D_{\hat{P}}\|_{\infty} \leq 1 - \frac{2}{(|N|+2)(|N|-1)^2}.$$

As the matrix norm $\|\cdot\|_{\infty}$ is compatible with the vector norm $\|\cdot\|_{\infty}$ we have that

$$\begin{aligned} \left\| \hat{P}(x) - \hat{P}(y) \right\|_{\infty} &\leq \|D_{\hat{P}}\|_{\infty} \|x - y\|_{\infty} \\ &\leq \left(1 - \frac{2}{(|N|+2)(|N|-1)^2} \right) \|x - y\|_{\infty} \end{aligned}$$

for all $x, y \in \Delta_{ir}(v)$. Hence, \hat{P} is a contraction on $\Delta_{ir}(v)$ and has therefore a unique fixed point by Banach's fixed point theorem. ■

3 Coalition Formation

A *hedonic coalition formation game* (Drèze and Greenberg, 1980) is a set N together with a profile of preferences $(\succeq_i)_{i \in N}$. For $i \in N$ and $S, T \in \mathcal{P}_i$ let \succeq_i be defined by

$$S \succeq_i T \quad \text{if and only if} \quad x_i(S) \geq x_i(T).$$

The outcome of a hedonic coalition formation game is a partition of the player set. In our case we are interested in coalitions rather than partitions. Therefore we slightly adapt the classical definitions of stability, for the original versions see for instance Bogomolnaia and Jackson (2002).

Definition 3.1. Let v be a proper monotonic simple game, $x \in \Delta(v)$, and $S \subseteq N$ be winning. S is called *Nash stable* (with respect to x) if for each

$i \in S$ and each $T \subseteq N \setminus S$ it holds that $x_i(T \cup \{i\}) \leq x_i(S)$.² S is called *individually stable* (with respect to x) if for each $i \in S$ and each $T \subseteq N \setminus S$ it holds that either $x_i(T \cup \{i\}) \leq x_i(S)$ or there is $j \in T$ with $x_j(T \cup \{i\}) < 0$.

Roughly speaking, a winning coalition S is Nash stable if no player $i \in S$ has an incentive to leave S and join any coalition $T \subseteq N \setminus S$. The coalition S is individually stable if it is Nash stable or if for each player i who would like to change his coalition from S to T , there is at least one player $j \in T$ who would not agree as he would be negatively affected by player i 's move. Clearly, Nash stability implies individual stability; if $x \in \Delta_{ir}(v)$ the definitions are even equivalent.

Let $x \in \Delta_{ir}(v)$. It is easy to see that a winning coalition S which does not contain any pivotal player must be Nash stable as no player can improve by moving to a losing coalition. However, these coalitions do not always seem *credible* in the following sense: Although no player can improve by leaving the coalition, there might still be a group of players T inside of S which could improve by excluding the remaining players. This motivates the following definition.

Definition 3.2. A coalition $S \subseteq N$ is called *internally stable* (with respect to x) if for each $T \subseteq S$ there is $i \in T$ such that $x_i(S) \geq x_i(T)$.

The question is now: Can we find a coalition which is both Nash stable and internally stable? The answer is yes, in the following set up.

Theorem 3.3. *Let v be a proper monotonic simple game, let F be an individually rational and fair bargaining solution, let $\alpha = 0$, and let $x \in \Delta(v)$ be stable with respect to F . Then there is a coalition $S \subseteq N$ which is both Nash stable and internally stable.*

Proof. First we show that there is an internally stable winning coalition S . For this purpose note that $x_i(N) \geq d_i^o(N) = 0$ for all $i \in N$ by individual

²Note that we do not forbid the existence of a player $j \in N \setminus S$ who could improve by joining S . See also Remark 3.7.

rationality. Let now $S_0 = N$ and $S_k \subsetneq S_{k-1}$ such that $x_i(S_k) > x_i(S_{k-1})$ for all $i \in S_k$. If k is such that there is no S_{k+1} then $S = S_k$ is internally stable. As N is finite, such S must exist.

We show that there is a Nash stable and internally stable coalition. For this reason let S be internally stable. Since F is fair and $\alpha = 0$, we have either $x_i(S) \geq d_i^o(S, x)$ for all $i \in S$ or $x_i(S) < d_i^o(S)$ for all $i \in S$. In the first case this means

$$x_i(S) \geq \max_{T \subseteq N \setminus S} x_i(T \cup \{i\}),$$

hence, S is Nash stable. So, let $i \in S$ and let $x_i(S) < d_i^o(S)$. Let $T_1 \in \mathcal{T}_i^S(x)$. Since $d_i^o(S) > x_i(S) \geq 0$, T_1 must be a winning coalition. Because of individual rationality there is no losing $T' \subsetneq T_1 \cup \{i\}$ with $x_j(T') > x_j(T_1 \cup \{i\})$ for all $j \in T'$. As i is pivotal in T_1 , i is contained in each winning $T' \subseteq T_1 \cup \{i\}$. Since $T_1 \in \mathcal{T}_i^S(x)$, we have that $x_i(T_1 \cup \{i\}) \geq x_i(T')$. Thus, T_1 is internally stable. Now, either $x_i(T_1) \geq d_i^o(T_1, x)$ or $x_i(T_1) < d_i^o(T_1, x)$. In the first case T_1 is Nash stable as fairness implies $x_j(T_1) \geq d_j^o(T_1, x)$ for all $j \in T_1$. In the latter case we define

$$T_k \in \mathcal{T}_i^{T_{k-1}}(x)$$

for all $k \geq 2$. Then all T_k are internally stable and T_k is Nash stable if and only if $x_i(T_k) \geq x_i(T_{k+1})$. As N and therefore $\mathcal{P}(N)$ are finite, there is k such that $T_{k+1} = T_l$ for some $l \leq k$. Let k^* be the first such k^* . Then

$$x_i(T_{k^*}) \geq x_i(T_l) = x_i(T_{k^*+1}) = d_i^{T_{k^*}}(S)$$

and we see that T_{k^*} is Nash stable. ■

The remainder of this section is devoted to the class of apex games. An

apex game a_{iJ} on a player set $N = \{i\} \cup J$, where $|J| \geq 3$, is defined by

$$a_{iJ}(S) = \begin{cases} 1, & \text{if } (i \in S \text{ and } S \cap J \neq \emptyset) \text{ or } J \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

We will show that for each apex game there are unique stable power configuration with respect to the bargaining solutions E , \tilde{E} , and P . We also investigate the induced hedonic coalition formation game. We already know that we can find internally and Nash stable coalitions in case of $\alpha = 0$. Now, we consider arbitrary $\alpha \in [0, 1]$ and show under which conditions we can find coalitions which satisfy core stability.

Definition 3.4. Let v be a proper monotonic simple game, let $x \in \Delta(v)$, and let $S \subseteq N$ be winning. A *deviation* of S is a coalition T such that $x_i(T) > x_i(S)$ for all $i \in S \cap T$ and $x_i(T) > 0$ for each $i \in T \setminus S$. S is called *core stable* (with respect to x) if there is no deviation of S .

So, T is a deviation of S if each player in T prefers that T forms over the formation of S : Those players contained in both coalitions have more power in T than in S ; and those which are only contained in T are powerless if S forms but have positive power in T . If T is a deviation of S , we also say that T blocks S .

We already know that there is a stable power configuration with respect to the proportional solution if $\alpha > 0$. In case of apex games such a power configuration exists also for $\alpha = 0$. Moreover this power configuration is even unique for arbitrary α , as the following theorem shows.

Theorem 3.5. *Let a_{iJ} be an apex game. The unique $x \in \Delta(a_{iJ})$ which is*

stable with respect to P , is given by

$$x_i(S) = \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ \frac{|J|}{(1+\alpha)^{|J|+1-\alpha}}, & \text{if } |S \cap J| = 1, \\ 1, & \text{if } 2 \leq |S \cap J| \leq |J| - 1 \end{cases}$$

$$x_j(S) = \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ \frac{1}{|J|}, & \text{if } S = J, \\ \frac{\alpha^{|J|+1-\alpha}}{(1+\alpha)^{|J|+1-\alpha}}, & S \cap J = \{j\}, \\ 0, & \text{if } 2 \leq |S \cap J| \leq |J| - 1, \end{cases}$$

for all winning coalitions $S \subseteq N$ and all minor players $j \in S$.

Proof. It can easily be shown that x is stable with respect to P . We show that x is the unique stable power configuration. Let therefore $y \in \Delta(v)$ be stable with respect as well. By Lemma 2.12 we have $y_i(S) = 1$ for all winning $S \subseteq N$ with $2 \leq |S \cap J| \leq |J| - 1$. Consequently, $d_i(\{i, j\}, y) = 1$ for all $j \in J$. Hence,

$$y_j(\{i, j\}) = \frac{\alpha + (1 - \alpha)y_j(J)}{1 + \alpha + (1 - \alpha)y_j(J)} = 1 - \frac{1}{1 + \alpha + (1 - \alpha)y_j(J)}.$$

We also have that

$$y_j(J) = \frac{\alpha + (1 - \alpha)y_j(\{i, j\})}{\alpha|J| + (1 - \alpha)\sum_{k \in J} y_k(\{i, k\})}.$$

Let $Y = \sum_{k \in J} y_k(\{i, k\})$. Then

$$y_j(J) = \frac{1 - \frac{1-\alpha}{1+\alpha+(1-\alpha)y_j(J)}}{|J| - (1 - \alpha)Y}$$

for all $j \in J$. Hence, y_j does not depend on j , so we must have $y_j(J) = y_k(J)$ for all $j, k \in J$. By efficiency of P , $y_j(J) = \frac{1}{|J|}$. For the remaining coalitions

S it can now easily be shown that $x_k(S) = y_k(S)$ for all $k \in S$. ■

We see immediately that the only candidates for core stable coalitions are J and $\{i, j\}$ for all $j \in J$. Hence, the following corollary can easily be derived.

Corollary 3.6. *Let $a_{i,J}$ be an apex game on N and $x \in \Delta(a_{i,J})$ be stable with respect to P . Then there is a core stable coalition with respect to x . In particular,*

$$\begin{aligned} J \text{ is core stable} & \quad \text{if and only if} \quad |J| \leq \sqrt{\frac{1}{\alpha}} + 1, \\ \{i, j\} \text{ is core stable for all } j \in J & \quad \text{if and only if} \quad |J| \geq \sqrt{\frac{1}{\alpha}} + 1, \end{aligned}$$

and there are no other core stable coalitions.

The existence of a core stable coalition for each apex game is a very nice feature of the power configuration x . In particular, for a power configuration which is derived from the Shapley-Shubik index or the Banzhaf Coleman index such coalitions do not exist (Karos, 2012).

Before we turn to the egalitarian solution we give the following remark on the relation between different stability notions.

Remark 3.7. From Corollary 3.6 it becomes clear that core stability does not imply individual stability: The coalition $\{i, j\}$ can never be individually stable as $x_i(\{i\} \cup J \setminus \{j\}) > x_i(\{i, j\})$ for all $j \in J$. Coalition J is Nash stable if and only if $|J| < \sqrt{\frac{1}{\alpha}} + 1$. However, the original definition of Nash stability (Bogomolnaia and Jackson, 2002) applies on a partition of N and states that each coalition in this partition must be Nash stable in our sense. If we apply this definition on the partition $\{\{i\}, J\}$, we see that player i would prefer to join J . Hence, there is a discrepancy between the original notion and our Definition 3.1. This is not the case when we talk about individual stability: If S is a winning coalition and $x_i(S \cup \{i\}) > 0$, coalition S would never allow player i to join S as at least one player $j \in S$ would lose power.

From Theorem 2.7 we know that there is a stable power configuration with respect to E for each proper monotonic simple game. We will prove that it is unique if an apex game is under consideration. For this purpose we need the following upper bound for the power of a player i in any coalition.

Lemma 3.8. *Let v be a proper monotonic simple game and $x \in \Delta(v)$ be stable with respect to E . Then $x_i(S) \leq 1$ for all $S \subseteq N$ and all $i \in S$.*

Proof. Assume that there is $S \subseteq N$ and $i \in S$ such that $x_i(S) > 1$ and let $\varepsilon = x_i(S) - 1$. Since

$$\begin{aligned} 1 + \varepsilon &= x_i(S) \\ &\leq \alpha + (1 - \alpha) d_i^o(S) + \frac{1}{|S|} \left(1 - \alpha - (1 - \alpha) \sum_{j \in S} d_j^o(S) \right) \\ &\leq \frac{|S| - 1}{|S|} (1 - \alpha) d_i^o(S) + \frac{1}{|S|} + \frac{|S| - 1}{|S|} \alpha \end{aligned}$$

we find

$$d_i^o(S_1) \geq \frac{1 + \frac{|S|}{|S|-1} \varepsilon - \alpha}{1 - \alpha} \geq 1 + \frac{|S|}{|S|-1} \varepsilon.$$

Let $T_1 \in \mathcal{T}_i^S(x)$, i.e. $x_i(T_1) \geq 1 + \frac{|S|}{|S|-1} \varepsilon > x_i(S)$. Then we find for the same reasons as before $d_i^o(T_1) > x_i(T_1)$. Let now $T_{k+1} \in \mathcal{T}_i^{T_k}(x)$ for all $k \geq 1$. With the same arguments we have $x_i(T_{k+1}) > x_i(T_k)$ for all $k \geq 1$. But this is impossible since there is only a finite number of coalitions. \blacksquare

With this result at hand we can now calculate a stable power configuration and show that it is unique.

Theorem 3.9. *Let $a_{i,J}$ be an apex game on $N = \{i\} \cup J$.*

1. *If $|J| = 3$, the unique $x^* \in \Delta(a_{i,J})$ which is stable with respect to E is*

given by

$$\begin{aligned}
 x_i^*(S) &= \begin{cases} \frac{1}{4}, & \text{if } S = N, \\ \frac{1}{2} + \frac{\alpha^2 - 1}{2\alpha^2 - 4\alpha - 4}, & \text{if } |S \cap J| = 1, \\ \frac{1}{3} + \frac{1 + \alpha}{2 + 2\alpha - \alpha^2}, & \text{if } |S \cap J| = 2, \end{cases} \\
 x_j^*(S) &= \begin{cases} \frac{1}{4}, & \text{if } S = N, \\ \frac{1}{3}, & \text{if } S = J, \\ \frac{1}{2} - \frac{\alpha^2 - 1}{2\alpha^2 - 4\alpha - 4}, & \text{if } S \cap J = \{j\}, \\ \frac{2}{3} - \frac{1 + \alpha}{2 + 2\alpha - \alpha^2}, & \text{if } |S \cap J| = 2 \end{cases} \quad (3)
 \end{aligned}$$

for all winning coalitions $S \subseteq N$ and all minor players $j \in S$.

2. If $|J| \geq 4$, the unique $x^* \in \Delta(a_{iJ})$ which is stable with respect to E is given by

$$\begin{aligned}
 x_i^*(S) &= \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ 1 - \frac{\alpha}{2} - \frac{1 - \alpha}{2|J|}, & \text{if } |S \cap J| = 1, \\ 1, & \text{if } 2 \leq |S \cap J| \leq |J| - 2, \\ 1 - \frac{|J| - 1}{|J|} \left(\frac{\alpha(1 - \alpha)}{2} + \frac{(1 - \alpha)^2}{2|J|} \right), & \text{if } |S \cap J| = |J| - 1 \end{cases} \\
 x_j^*(S) &= \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ \frac{1}{|J|}, & \text{if } S = J, \\ \frac{\alpha}{2} + \frac{1 - \alpha}{2|J|}, & \text{if } |S \cap J| = 1, \\ 0, & \text{if } 2 \leq |S \cap J| \leq |J| - 2, \\ \frac{|J| - 1}{|J|} \left(\frac{\alpha(1 - \alpha)}{2} - \frac{(1 - \alpha)^2}{2|J|} \right), & \text{if } |S \cap J| = |J| - 1 \end{cases}
 \end{aligned}$$

for all winning coalitions $S \subseteq N$ and all minor players $j \in S$.

Proof.

1. It is easy to verify that x^* is stable with respect to E , we show that x^* is

unique. Let therefore x be stable with respect to E . Since $x_i(\{i, j\}) \leq 1$ for all $j \in J$, we have

$$\begin{aligned} x_i(\{i\} \cup J \setminus \{j\}) &= \frac{1}{3} + \frac{2}{3}\alpha + \frac{2}{3}(1 - \alpha)x_i(\{i, j\}) \\ &\geq x_i(\{i, j\}). \end{aligned}$$

We show that there is no $j \in J$ such that $x_i(\{i, j\}) > x_i(\{i, j, k\})$ for all $k \in J \setminus \{j\}$. Assume that there is such $j \in J$. In this case we have

$$x_i(\{i, j\}) > x_i(\{i, j, k\}) \geq x_i(\{i, l\})$$

for all $k, l \in J \setminus \{j\}$, $k \neq l$. Hence, we have $\{i, j\} \in \mathcal{T}_i^{\{i, k\}}$ for all $k \in J \setminus \{j\}$; thus $d_i^o(\{i, k\}) = x_i(\{i, j\})$. Therefore, x must solve the following equation system.

$$\begin{aligned} x_i(\{i, j\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_i(\{i, k, l\}) - \frac{1-\alpha}{2}x_j(J) \\ x_j(\{i, j\}) &= \frac{1}{2} - \frac{1-\alpha}{2}x_i(\{i, k, l\}) + \frac{1-\alpha}{2}x_j(J) \\ x_i(\{i, k\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_i(\{i, j\}) - \frac{1-\alpha}{2}x_k(J) \\ x_k(\{i, k\}) &= \frac{1}{2} - \frac{1-\alpha}{2}x_i(\{i, j\}) + \frac{1-\alpha}{2}x_k(J) \\ x_i(\{i, l\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_i(\{i, j\}) - \frac{1-\alpha}{2}x_l(J) \\ x_l(\{i, l\}) &= \frac{1}{2} - \frac{1-\alpha}{2}x_i(\{i, j\}) + \frac{1-\alpha}{2}x_l(J) \\ x_i(\{i, j, k\}) &= \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3}x_i(\{i, l\}) \\ x_i(\{i, j, l\}) &= \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3}x_i(\{i, k\}) \\ x_i(\{i, k, l\}) &= \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3}x_i(\{i, j\}) \\ x_j(J) &= \frac{1}{3} + \frac{2(1-\alpha)}{3}x_j(\{i, j\}) - \frac{1-\alpha}{3}x_k(\{i, k\}) - \frac{1-\alpha}{3}x_l(\{i, l\}) \\ x_k(J) &= \frac{1}{3} - \frac{1-\alpha}{3}x_j(\{i, j\}) + \frac{2(1-\alpha)}{3}x_k(\{i, k\}) - \frac{1-\alpha}{3}x_l(\{i, l\}) \\ x_l(J) &= \frac{1}{3} - \frac{1-\alpha}{3}x_j(\{i, j\}) - \frac{1-\alpha}{3}x_k(\{i, k\}) + \frac{2(1-\alpha)}{3}x_l(\{i, l\}) \end{aligned} \tag{4}$$

The unique solution of this system delivers

$$\begin{aligned} x_i(\{i, j\}) &= \frac{2\alpha^4 - 8\alpha^3 - 9\alpha^2 + 22\alpha + 11}{2\alpha^4 - 14\alpha^3 + 34\alpha + 14} \\ x_i(\{i, j, k\}) &= \frac{7\alpha^4 + 2\alpha^3 - 6\alpha^2 - 40\alpha - 17}{3\alpha^4 - 21\alpha^3 + 51\alpha + 21}. \end{aligned}$$

It can now be shown that $x_i(\{i, j, k\}) > x_i(\{i, j\})$ for all $\alpha \in [0, 1]$ in contradiction to our assumption. Hence, there is no $j \in J$ such that $x_i(\{i, j\}) > x_i(\{i, j, k\})$ for all $k \in J \setminus \{j\}$.

Assume now that there is $j \in J$ such that $x_i(\{i, j, l\}) \geq x_i(\{i, j\}) > x_i(\{i, j, k\})$ for $j, l \in J \setminus \{j\}$. In this case we have $d_i^o(\{i, k\}) = x_i(\{i, j, l\})$ and $d_i^o(\{i, l\}) = x_i(\{i, j\})$. Hence, the only rows that change in the equation system compared to (4) are

$$\begin{aligned} x_k(\{i, k\}) &= \frac{1}{2} - \frac{1-\alpha}{2}x_i(\{i, j, l\}) + \frac{1-\alpha}{2}x_k(J) \\ x_i(\{i, l\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_i(\{i, j\}) - \frac{1-\alpha}{2}x_l(J). \end{aligned}$$

In this case there is again for each $\alpha \in [0, 1]$ a unique solution, in particular we have

$$\begin{aligned} x_i(\{i, j\}) &= \frac{4\alpha^4 - 13\alpha^3 - 6\alpha^2 + 23\alpha + 10}{4\alpha^4 - 19\alpha^3 + 3\alpha^2 + 35\alpha + 13} \\ x_i(\{i, j, k\}) &= \frac{8\alpha^4 + 19\alpha^3 - 21\alpha^2 - 83\alpha - 31}{12\alpha^4 - 57\alpha^3 + 9\alpha^2 + 105\alpha + 39}. \end{aligned}$$

It can now be shown that $x_i(\{i, j, k\}) > x_i(\{i, j\})$ for all $\alpha \in [0, 1]$. Again a contradiction to our assumption, that is there is no $j \in J$ such that $x_i(\{i, j, l\}) \geq x_i(\{i, j\}) > x_i(\{i, j, k\})$ for $j, l \in J \setminus \{j\}$.

After we ruled out the previous two possibilities, it must now be the case that $d_i^o(\{i, j\}) = x_i(\{i\} \cup J \setminus \{j\})$ for all $j \in J$. Hence, a stable

power configuration must solve

$$\begin{aligned}
x_i(\{i, j\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_i(\{i, k, l\}) - \frac{1-\alpha}{2}x_j(J) \\
x_j(\{i, j\}) &= \frac{1}{2} - \frac{1-\alpha}{2}x_i(\{i, k, l\}) + \frac{1-\alpha}{2}x_j(J) \\
x_i(\{i, k\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_i(\{i, j, l\}) - \frac{1-\alpha}{2}x_k(J) \\
x_k(\{i, k\}) &= \frac{1}{2} - \frac{1-\alpha}{2}x_i(\{i, j, l\}) + \frac{1-\alpha}{2}x_k(J) \\
x_i(\{i, l\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_i(\{i, j, k\}) - \frac{1-\alpha}{2}x_l(J) \\
x_l(\{i, l\}) &= \frac{1}{2} - \frac{1-\alpha}{2}x_i(\{i, j, k\}) + \frac{1-\alpha}{2}x_l(J) \\
x_i(\{i, j, k\}) &= \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3}x_i(\{i, l\}) \\
x_i(\{i, j, l\}) &= \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3}x_i(\{i, k\}) \\
x_i(\{i, k, l\}) &= \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3}x_i(\{i, j\}) \\
x_j(J) &= \frac{1}{3} + \frac{2(1-\alpha)}{3}x_j(\{i, j\}) - \frac{1-\alpha}{3}x_k(\{i, k\}) - \frac{1-\alpha}{3}x_l(\{i, l\}) \\
x_k(J) &= \frac{1}{3} - \frac{1-\alpha}{3}x_j(\{i, j\}) + \frac{2(1-\alpha)}{3}x_k(\{i, k\}) - \frac{1-\alpha}{3}x_l(\{i, l\}) \\
x_l(J) &= \frac{1}{3} - \frac{1-\alpha}{3}x_j(\{i, j\}) - \frac{1-\alpha}{3}x_k(\{i, k\}) + \frac{2(1-\alpha)}{3}x_l(\{i, l\}).
\end{aligned}$$

We find that in this case the unique solution is given in (3).

2. It is straightforward to verify that x^* is stable with respect to E , we show that x^* is unique. Let therefore x be stable with respect to E and define

$$\mathcal{S} = \{S \subseteq N; i \in S, |S \cap J| = 2\}.$$

Let $S_1 \in \mathcal{S}$ be such that $x_i(S_1) \geq x_i(S)$ for all $S \in \mathcal{S}$ and let $S_2 \in \mathcal{S}$ such that $S_1 \cap S_2 = \{i\}$ and $x_1(S_2) \geq x_1(S)$ for all $S \in \mathcal{S}$ with $S \cap S_1 = \{i\}$. Then $d_j(S_k) = 0$ for $k = 1, 2$ and all $j \in S_k \cap J$. Hence,

$$\begin{aligned}
x_i(S_1) &\geq \alpha + (1 - \alpha)x_i(S_2) + \frac{1}{3}(1 - \alpha - (1 - \alpha)x_i(S_2)) \\
&= \frac{1 + 2\alpha}{3} + \frac{2 - 2\alpha}{3}x_i(S_2).
\end{aligned}$$

We also see for the same reasons that $x_i(S_2) \geq \frac{1+2\alpha}{3} + \frac{2-2\alpha}{3}x_i(S_1)$, so

that

$$\begin{aligned} x_i(S_1) &\geq \frac{1+2\alpha}{3} + \frac{2-2\alpha}{3} \left(\frac{1+2\alpha}{3} + \frac{2-2\alpha}{3} x_i(S_1) \right) \\ &= \frac{5+8\alpha-4\alpha^2}{9} + \frac{4-8\alpha+4\alpha^2}{9} x_i(S_1). \end{aligned}$$

Hence, $x_i(S_1) \geq 1$ and for the same reasons $x_i(S_2) \geq 1$. By Lemma 3.8 we have $x_i(S_1) = 1$.

If S is such that $2 \leq |S \cap J| \leq |J| - 2$ then $d_i(S) = 1$, $d_j(S) = 0$ for all $j \in S \cap J$, and hence, $x_i(S) = 1$ and $x_j(S) = 0$. Similar to the proof of Theorem 3.5 we show that $x_j(J) = x_k(J)$ for all $j, k \in J$ and conclude $x_j(J) = \frac{1}{|J|}$. Hence, we have

$$x_i(\{i, j\}) = 1 + \frac{1}{2} \left(1 - 1 - \alpha - (1 - \alpha) \frac{1}{|J|} \right) = 1 - \frac{\alpha}{2} - \frac{1 - \alpha}{2|J|}$$

for all $j \in J$. Hence, $x_j(\{i, j\}) = \frac{\alpha}{2} + \frac{1 - \alpha}{2|J|}$. Finally,

$$\begin{aligned} x_i(\{i\} \cup J \setminus \{j\}) &= \frac{1}{|J|} + \frac{|J| - 1}{|J|} \alpha + \frac{|J| - 1}{|J|} (1 - \alpha) \left(1 - \frac{\alpha}{2} - \frac{1 - \alpha}{2|J|} \right) \\ &= 1 - (1 - \alpha) \frac{|J| - 1}{|J|} \left(\frac{\alpha}{2} + \frac{1 - \alpha}{2|J|} \right) \end{aligned}$$

and therefore $x_k(\{i\} \cup J \setminus \{j\}) = (1 - \alpha) \frac{|J| - 1}{|J|} \left(\frac{\alpha}{2} + \frac{1 - \alpha}{2|J|} \right)$ for all $k \in J \setminus \{j\}$. ■

Note that in case of \tilde{E} we have that $d_k(S) \leq 1$ for all winning coalitions S and all $k \in S$. In particular, for each winning coalition S except J we find that $d_k(S) \geq -\frac{1}{|S|} (1 - \sum_{l \in S} d_l(S))$ for all $k \in S$. Hence, we have that for $x \in \Delta(v)$ which is stable with respect to \tilde{E} it holds true that $x_k(S) = E_i(S, v(S), d(S))$. With similar arguments as in the proof of The-

orem 3.9 it can be shown that $x_j(J) = \frac{1}{|J|}$ for all $j \in J$. Hence, stable payoff configurations with respect to E and with respect to \tilde{E} coincide on all apex games. We close this section with the following corollary on the existence of core stable coalitions.

Corollary 3.10. *Let $a_{i,J}$ be an apex game on $N = \{i\} \cup J$ and let $x \in \Delta(a_{i,J})$ be stable with respect to E .*

1. *Let $|J| = 3$. Then J is core stable if and only if $\alpha \leq \frac{\sqrt{3}-1}{2}$. In this case J is the only core stable coalition. Further $\{i, j\}$ is core stable for each $j \in J$ if and only if $\alpha = 1$. In this case there are no other core stable coalitions. If $\alpha \in \left(\frac{\sqrt{3}-1}{2}, 1\right)$ there are no core stable coalitions.*
2. *Let $|J| \geq 4$. Then J is core stable if and only if $|J| \leq \frac{1+\alpha}{\alpha}$. In this case J is the only core stable coalition. Further $\{i, j\}$ is core stable for each $j \in J$ if and only if $\alpha = 1$. In this case there are no other core stable coalitions. If $\alpha \in \left(\frac{1}{|J|-1}, 1\right)$ then there are no core stable coalitions.*

Proof. Let x be the unique stable power configuration with respect to $a_{i,J}$.

1. Let $|J| = 3$. Then $x_j(\{i, j\}) \leq \frac{1}{3} = x_j(J)$ if and only if $\alpha \leq \frac{\sqrt{3}-1}{2}$. As $x_j(\{i, j, k\}) \leq x_j(\{i, j\})$, we have that J is core stable if and only if $\alpha \leq \frac{\sqrt{3}-1}{2}$. In this case each coalition of type $\{i, j, k\}$ is blocked by J and each coalition of type $\{i, j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$. If $1 > \alpha > \frac{\sqrt{3}-1}{2}$ then J is blocked by $\{i, j\}$, $\{i, j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$, and $\{i\} \cup J \setminus \{j\}$ is blocked by J . If $\alpha = 1$ then J is blocked by $\{i, j\}$ and $\{i\} \cup J \setminus \{j\}$ is blocked by J . However, $\{i, j\}$ is not blocked by $\{i\} \cup J \setminus \{j\}$, since $x_k(\{i\} \cup J \setminus \{j\}) = 0$ for all $k \in J \setminus \{j\}$.
2. Let $|J| = 4$. Then $x_j(\{i, j\}) \leq \frac{1}{|J|} = x_j(J)$ if and only if $\alpha \leq \frac{1}{|J|-1}$. We have that $x_j(\{i\} \cup J \setminus \{k\}) \leq x_j(\{i, j\})$ for all $k \in J \setminus \{j\}$, hence, J is core stable if and only if $\alpha \leq \frac{1}{|J|-1}$ or equivalently $|J| \leq \frac{1+\alpha}{\alpha}$. In this case each winning coalition which contains i and at least two minor

players is blocked by J and $\{i, j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$. If $\alpha = 1$ then J is blocked by $\{i, j\}$ and $\{i\} \cup J \setminus \{j\}$ is blocked by J . However, $\{i, j\}$ is not blocked by $\{i\} \cup J \setminus \{j\}$, since $x_k(\{i\} \cup J \setminus \{j\}) = 0$ for all $k \in J \setminus \{j\}$. If $\frac{1}{|J|-1} < \alpha < 1$ then J is blocked by $\{i, j\}$, $\{i, j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$, and each coalition which contains i and at least two minor players is blocked by J .

■

4 Infeasible Coalitions

In many applications of simple games the formation of certain coalitions is impossible. This might be because of legal issues (such as antitrust legislation) or simply because some political parties have so different interests that they cannot work together. So far, we ignored such restrictions. However, as the disagreement points of players depend on their outside options, we should guarantee that a player cannot use his hypothetical power in a coalition which will never form.

We say that $\mathcal{R} \subseteq \mathcal{P}$ is a *coalition restriction* if $\{i\} \in \mathcal{R}$ for all $i \in N$. This condition simply says that each player can stay alone, in particular, each player has the outside option to stay alone.

Definition 4.1. Let F be a bargaining solution, \mathcal{R} be a coalition restriction, and v be a simple game. A power configuration $x \in \Delta(v)$ is called *stable with respect to F under \mathcal{R}* if for all winning coalitions $S \in \mathcal{R}$ and all $i \in S$ the following holds.

$$\begin{aligned} x_i(S) &= F_i(S, v(S), d(S, x)) \\ d_i(S, x) &= \alpha d_i^m(S) + (1 - \alpha) d_i^o(S, x) \\ d_i^o(S, x) &= \max_{T \in \mathcal{R}, T \subseteq N \setminus S} x_i(T \cup \{i\}). \end{aligned} \tag{5}$$

It is easy to show that the proofs in Section 2 hold true for each coalition

restriction \mathcal{R} . Theorem 3.3 remains true as well: Let v be a proper monotonic simple game. If \mathcal{R} contains at least one winning coalition and if x is stable with respect to a fair and individually rational bargaining solution for $\alpha = 0$ then there is a winning coalition $S \in \mathcal{R}$ which is both Nash and internally stable.

In this section we do not focus on the adaptation of the respective proofs but we will return to our initial Example 1.1. The political interests of the five parties in the German Bundestag make it impossible that FDP and Linke, or CDU/CSU and Linke will ever cooperate. Therefore, let

$$\mathcal{R} = \{S \subseteq N; \text{ if } 4 \in S \text{ then } 1, 2 \notin S\}.$$

It can be shown that a stable power configuration with respect to E under \mathcal{R} must satisfy the equation system

$$\begin{aligned} x_1(\{1, 2\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_1(\{1, 3, 5\}) \\ x_1(\{1, 3\}) &= \frac{1}{2} + \frac{1-\alpha}{2}x_1(\{1, 2, 5\}) \\ x_1(\{1, 3, 5\}) &= \frac{1+\alpha}{3} + \frac{2}{3}(1-\alpha)x_1(\{1, 2\}) \\ x_1(\{1, 2, 5\}) &= \frac{1+\alpha}{3} + \frac{2}{3}(1-\alpha)x_1(\{1, 3\}). \end{aligned}$$

The unique solution of this system is

$$\begin{aligned} x_1(\{1, 2\}) = x_1(\{1, 3\}) &= \frac{2-\frac{1}{2}\alpha^2}{2+2\alpha-\alpha^2} \\ x_1(\{1, 3, 5\}) = x_1(\{1, 2, 5\}) &= \frac{2}{2+2\alpha-\alpha^2}. \end{aligned}$$

We can further calculate

$$x_5(\{1, 3, 5\}) = x_5(\{1, 2, 5\}) = \frac{1-2\alpha}{3} - \frac{1}{3}x_i(\{1, 3\}) = -\frac{1}{6} \frac{4\alpha + 9\alpha^2 - 4\alpha^2}{2+2\alpha-\alpha^2} < 0.$$

Hence, $\{1, 3, 5\}$ and $\{1, 2, 5\}$ are neither internally stable nor individually

rational. Finally we have

$$x_2(\{1, 2\}) = x_3(\{1, 3\}) = 1 - \frac{2 - \frac{1}{2}\alpha^2}{2 + 2\alpha - \alpha^2}$$

and see that the coalitions $\{1, 2\}$ and $\{1, 3\}$ are the only core stable coalitions. This is in line with reality as the current government consists of players 1 and 2.

The power configuration x is not individually rational since $x_5(\{1, 3, 5\}) < 0$. Hence, in this case E and \tilde{E} do not coincide. Let us now focus on \tilde{E} . Let x be stable with respect to \tilde{E} under \mathcal{R} . Then

$$x_1(\{1, j\}) = \frac{1}{2} + \frac{1 - \alpha}{2}d_1(\{1, j\})$$

for $j = 2, 3$. First, assume $d_1^o(\{1, 2\}, x) = x_1(\{1, 3\})$ and $d_1^o(\{1, 3\}, x) = x_1(\{1, 2\})$. In this case x must solve

$$\begin{aligned} x_1(\{1, 2\}) &= \frac{1}{2} + \frac{1 - \alpha}{2}x_1(\{1, 3\}) \\ x_1(\{1, 3\}) &= \frac{1}{2} + \frac{1 - \alpha}{2}x_1(\{1, 2\}). \end{aligned}$$

We find that in this case $x_1(S) = \frac{1}{1+\alpha}$ for all winning coalitions $S \in \mathcal{R}$. Further $x_j(\{1, j\}) = x_j(\{1, 5, j\}) = \frac{\alpha}{1+\alpha}$ for $j = 2, 3$ and $x_5(S) = 0$ for all winning $S \in \mathcal{R}$. To verify that x is actually stable, we have to show that the initial choice of outside options is consistent with x , i.e. that each player uses his best outside option. But this is clear since the only player with a positive outside option is player 1 and $x_1(S) = \frac{1}{1+\alpha}$ for all winning $S \in \mathcal{R}$.

We have mentioned before that we do not have evidence how α should be chosen. However, given the fact that the German cabinet consists of 16 ministers of which 11 are member of CDU/CSU, we can at least get an idea of α . Under the assumption that the allocation of cabinets seats displays the

power of the two parties, we find that α must solve

$$x_i(\{1, 2\}) = \frac{1}{1 + \alpha} = \frac{11}{16}$$

for the constrained egalitarian solution. This delivers $\alpha \approx 0.455$. For the egalitarian solution we can use the same argument to find $\alpha \approx 0.487$.

5 Conclusion

Coalition formation in simple games contains two parts: The forming of coalitions and the distribution of power within coalitions. We built a model in which these two parts are interdependent, the distribution of power depends on the formation of coalitions and the formation of coalitions depends on the distribution of power. We interpreted the distribution problem as a bargaining (or bankruptcy) problem and showed that under very weak conditions on the bargaining solutions we can find a power configuration which is stable with respect to renegotiations.

We pointed out that essentially two issues are crucial for the power of a player within a coalition. First, his marginal contribution, as a pivotal player will always be more powerful than a player who is not necessary for the surviving of a coalition. Second, and this is the new approach, his outside option. We can also interpret the outside option as opportunity costs: A player who has a chance to be in a very powerful position in a different winning coalition must somehow be convinced not to leave. In the paper we showed several results for specific convex combinations between these two values. However, we do not have any empirical evidence yet, how they should be weighted.

Besides the very natural motivation of this stable power configuration, we showed that it has further useful properties: First of all, it allows to take into account that there might be coalitions which will never form for any

external reasons. Second, under some additional conditions it guarantees the existence of a coalition which is both internally and individually stable for each proper monotonic simple game.

We can think of several challenges which can now be targeted: Empirical evidence for the applicability of the model is the first. In particular, it will be interesting to investigate the implicit values of α . Second, the model might be extended to general transferable utility games. In this case we would interpret the outside option of a player in a coalition as opportunity costs. Particularly, these costs will depend on the partition rather than on a coalition. A transformation of the game in a hedonic version will therefore lead to a *hedonic game with externalities* (Bloch and Dutta, 2011).

References

- Aumann, R., Maschler, M., 1985. Game Theoretic Analysis of a Bankruptcy Problem from the Talmud. *Journal of Economic Theory* 36, 195–213.
- Banzhaf, J., 1965. Weighted Voting Doesn't Work: A Mathematical Analysis. *Rutgers Law Review* 19, 317–343.
- Bloch, F., Dutta, B., 2011. Formation of Networks and Coalition, in: Benhabib, J., Bisin, A., Jackson, M. (Eds.), *Handbook of Social Economics*. Elsevier, pp. 729–779.
- Bogomolnaia, A., Jackson, M., 2002. The Stability of Hedonic Coalition Structures. *Games and Economic Behavior* 38, 201–230.
- Casajus, A., 2009. Outside Options, Component Efficiency, and Stability. *Games and Economic Behavior* 65, 49–61.
- Chun, Y., Thomson, W., 1992. Bargaining Problems with Claims. *Mathematical Social Science* 24, 19–33.

- de Clippel, G. Serrano, S., 2008. Marginal Contributions and Externalities in the Value. *Econometrica* 76, 1413–1436.
- Curiel, I., Maschler, M., Tijs, S., 1987. Bankruptcy Games. *Zeitschrift für Operations Research* 31, 143–159.
- Dimitrov, D., Haake, C., 2006. Government versus Opposition: Who Should be Who in the 16th German Bundestag. *Journal of Economics* 89, 115–128.
- Dimitrov, D., Haake, C., 2008a. A Note on the Paradox of Smaller Coalitions. *Social Choice and Welfare* 30, 571–579.
- Dimitrov, D., Haake, C., 2008b. Stable Governments and the Semistrict Core. *Games and Economic Behavior* 62, 460–475.
- Drèze, J., Greenberg, J., 1980. Hedonic Coalitions: Optimality and Stability. *Econometrica* 48, 987–1003.
- Dutta, B., Ehlers, L., Kar, A., 2010. Externalities, potential, value, and consistency. *Journal of Economic Theory* 145, 2380–2411.
- Hart, S., Kurz, M., 1983. Endogenous Formation of Coalitions. *Econometrica* 51, 1047–1064.
- Hart, S., Kurz, M., 1984. Stable Coalition Structures, in: Holler, M. (Ed.), *Coalitions and Collective Actions*. Physica Verlag, pp. 235–258.
- Kalai, E., 1977. Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons. *Econometrica* 45, 1623–1630.
- Karos, D., 2012. Coalition Formation in General Apex Games. FEEM Working Paper No. 38.2012.
- Milchtaich, I., Winter, E., 2002. Stability and segregation in group formation. *Games and Economic Behavior* 38, 318–346.

- Nash, J., 1950. The Bargaining Problem. *Econometrica* 18, 155–162.
- Owen, G., 1977. Values of Games with A Priori Unions, in: Hein, R., Moeschlin, O. (Eds.), *Essays in Mathematical Economics and Game Theory*. Springer, pp. 76–88.
- Shapley, L., Shubik, M., 1954. A Method for Evaluating the Distribution of Power in a Committee System. *The American Political Science Review* 48, 787–792.
- Shenoy, P., 1979. On Coalition Formation: A Game-Theoretical Approach. *International Journal of Game Theory* 8, 133–164.

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