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## The integer cohomology of toric Weyl arrangements

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# LEM

## Working Paper Series

**The integer cohomology of toric Weyl arrangements**

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# The integer cohomology of toric Weyl arrangements

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## Abstract

A toric arrangement is a finite set of hypersurfaces in a complex torus, every hypersurface being the kernel of a character. In the present paper we prove that if  $\mathcal{T}_{\widetilde{W}}$  is the toric arrangement defined by the *cocharacters* lattice of a Weyl group  $\widetilde{W}$ , then the integer cohomology of its complement is torsion free.

## Keywords:

Arrangement of hyperplanes, toric arrangements, CW complexes, Salvetti complex, Weyl groups, integer cohomology

## MSC (2010):

52C35, 32S22, 20F36, 17B10

## Introduction

Let  $T = (\mathbb{C}^*)^n$  be a complex torus and  $X \subset \text{Hom}(T, \mathbb{C}^*)$  be a finite set of characters of  $T$ . The kernel of every  $\chi \in X$  is a hypersurface of  $T$ :

$$H_\chi := \{t \in T \mid \chi(t) = 1\}.$$

Then  $X$  defines on  $T$  the *toric arrangement*:

$$\mathcal{T}_X := \{H_\chi, \chi \in X\}.$$

Let  $\mathcal{R}_X$  be the *complement* of the arrangement:

$$\mathcal{R}_X := T \setminus \bigcup_{\chi \in X} H_\chi$$

The geometry and topology of  $\mathcal{R}_X$  have been studied by many authors, see for instance [8], [9], [4], [7], [12] and [13]. In particular Looijenga (see [10]) and De Concini and Procesi (see [3]) computed the De Rham cohomology of  $\mathcal{R}_X$  and, recently, Moci and Settepanella (see [14]) described a regular CW-complex homotopy equivalent to  $\mathcal{R}_X$ . This complex is similar to the one introduced by Salvetti (see [15]) for the complement of hyperplane arrangements.

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If  $\mathcal{T}_{\widetilde{W}}$  is the toric arrangement associated to an affine Weyl group  $\widetilde{W}$ , the complex  $T(\widetilde{W})$  homotopic to the complement

$$\mathcal{R}_W := T \setminus \bigcup_{H \in \mathcal{T}_{\widetilde{W}}} H$$

admits a very nice description which generalizes a construction introduced in [16] and [6]. In their paper Moci and Settepanella conjectured that the integer cohomology of  $T(\widetilde{W})$  (equivalently  $\mathcal{R}_W$ ) is torsion free. Hence it coincides with the De Rham cohomology described in [3] and it is known since the Betti numbers can be easily computed using results in [11].

In the present paper we prove this conjecture generalizing to toric arrangements a well known result for hyperplane ones. Indeed Arnol'd proved that the integer cohomology of braid arrangement is torsion free in 1969 (see [1]).

In order to prove it we use a filtration introduced in [5] and generalized to braid arrangements in [17] (see subsection 1.2).

In Section 2 we prove that the above filtration involves complexes with torsion free cohomology. While in Section 3 we rewrite it for toric arrangements and we prove the main result of the paper:

**Theorem 1** *The integer (co)-homology of the complement  $\mathcal{R}_W$  is torsion free.*

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## 1 Notations and recalls

In this section we recall basic construction about affine and toric arrangements coming from Coxeter systems.

### 1.1 Salvetti's complex for Coxeter arrangements

Let  $(W, S)$  be the Coxeter system associated to the finite reflection group  $W$  and

$$\mathcal{A}_W = \{H_{ws_iw^{-1}} \mid w \in W \text{ and } s_i \in S\}$$

the arrangement in  $\mathbb{C}^n$  obtained by complexifying the reflection hyperplanes of  $W$ , where, in a standard way, the hyperplane  $H_{ws_iw^{-1}}$  is simply the hyperplane fixed by the reflection  $ws_iw^{-1}$ .

It is well known (see, for instance, [6] [16]) that the  $k$ -cells of Salvetti's complex  $C(W)$  for arrangements  $\mathcal{A}_W$  are of the form  $E(w, \Gamma)$  with  $\Gamma \subset S$  of cardinality  $k$  and  $w \in W$ .

While the integer boundary map can be expressed as follows:

$$\partial_k(E(w, \Gamma)) = \sum_{s_j \in \Gamma} \sum_{\beta \in W_\Gamma^{\Gamma \setminus \{s_j\}}} (-1)^{l(\beta) + \mu(\Gamma, s_j)} E(w\beta, \Gamma \setminus \{s_j\}) \quad (1)$$

where  $W_\Gamma$  is the group generated by  $\Gamma$ ,

$$W_\Gamma^{\Gamma \setminus \{\sigma\}} = \{w \in W_\Gamma : l(ws) > l(w) \forall s \in \Gamma \setminus \{\sigma\}\}$$

and  $\mu(\Gamma, s_j) = \#\{s_i \in \Gamma | i \leq j\}$ . Here  $l(w)$  stands for the length of  $w$ .

**Remark 1.1** *Instead of the co-boundary operator we prefer to describe its dual, i.e. we define the boundary of a  $k$ -cell  $E(w, \Gamma)$  as a linear combination of the  $(k-1)$ -cells which have  $E(w, \Gamma)$  in their co-boundary, with the same coefficient of the co-boundary operator. We make this choice since the boundary operator has a nicer description than co-boundary operator in terms of the elements of  $W$ .*

This description holds also for Coxeter systems  $(\widetilde{W}, \widetilde{S})$  associated to Weyl groups  $\widetilde{W}$ .

## 1.2 A filtration for the complex $(C(W), \partial)$

It's known (see [2]) that for all  $\Gamma \subset S$  the group  $W$  splits as

$$W = W^\Gamma W_\Gamma$$

with

$$W^\Gamma = \{w^\Gamma \in W \mid l(w^\Gamma s_i) > l(w^\Gamma) \text{ for all } s_i \in W_\Gamma\}. \quad (2)$$

If  $w = w^\Gamma w_\Gamma$ ,  $w^\Gamma \in W^\Gamma$  and  $w_\Gamma \in W_\Gamma$ , then  $l(w\beta) = l(w^\Gamma) + l(w_\Gamma\beta) \forall \beta \in W_\Gamma$  and the boundary map verifies

$$\partial(E(w, \Gamma)) = w^\Gamma \cdot \partial(E(w_\Gamma, \Gamma)).$$

In [17] (see also [5]) author defines a map of complexes

$$i_m := i : \bigoplus_{j=1}^{m_1} C(W_{S_{m-1}}) \longrightarrow C(W)$$

as follows

$$\begin{aligned} i(j \cdot E(w_{S_{m-1}}, \Gamma)) &= i(W^{S_{m-1}}(j) \cdot E(w_{S_{m-1}}, \Gamma)) = \\ i(w^{S_{m-1}} \cdot E(w_{S_{m-1}}, \Gamma)) &= w^{S_{m-1}} \cdot i(E(w_{S_{m-1}}, \Gamma)) = w^{S_{m-1}} \cdot E(w_{S_{m-1}}, \Gamma) = E(w, \Gamma). \end{aligned}$$

Where  $m_1$  is the cardinality of  $W^{S_{m-1}}$ ,  $w^{S_{m-1}} = W^{S_{m-1}}(j)$  its  $j$ -th element in a fixed order and  $S_h = \{s_1, \dots, s_h\} \subset S = \{s_1, \dots, s_m\}$ .

The cokernel of the map  $i$  is the complex  $F_m^1(W)$  having as basis all  $E(w, \Gamma_1)$  for  $w \in W$  and  $\Gamma_1 \subset S$  s.t.  $s_m \in \Gamma_1$ .

She iterates this construction getting maps

$$\begin{aligned} i_m[k] &:= i : \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{S_{m-k-1}})[k] \longrightarrow F_m^k(W), \\ i(w^{S_{m-k-1}}.(E(w_{S_{m-k-1}}, \Gamma))) &= w^{S_{m-k-1}}.i(E(w_{S_{m-k-1}}, \Gamma)) \\ &= E(w, \Gamma \cup \{s_m, \dots, s_{m-k+1}\}) \end{aligned}$$

Each  $i_m[k]$  gives rise to the exact sequence of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{S_{m-k-1}})[k] \xrightarrow{i} F_m^k(W) \xrightarrow{j} F_m^{k+1}(W) \longrightarrow 0. \quad (3)$$

It is possible to filter the complex  $F_m^0(W) = C(W)$  in a similar way through maps:

$$\begin{aligned} i^m[k] &:= i : \bigoplus_{j=1}^{m^1 \cdots m^{k+1}} C(W_{S^{k+1}})[k] \longrightarrow F_m^k(W), \\ i(w^{S^{k+1}}.(E(w_{S^{k+1}}, \Gamma))) &= w^{S^{k+1}}.i(E(w_{S^{k+1}}, \Gamma)) = \\ &= E(w, \{s_1, \dots, s_k\} \cup \Gamma) \end{aligned} \quad (4)$$

for  $0 \leq k \leq m$ ,  $S^k = \{s_{k+1}, \dots, s_m\}$  and  $m^i$  the cardinality of  $W_{S^{i-1}}$ .

### 1.3 Salvetti's complex for toric Weyl arrangements

Let  $\Phi$  be a root system,  $\langle \Phi^\vee \rangle$  be the lattice spanned by the coroots, and  $\Lambda$  be its dual lattice (which is called the *cocharacters* lattice). Then we define a torus  $T = T_\Lambda$  having  $\Lambda$  as group of characters.

If  $\widetilde{W}$  is the affine Weyl group associated to  $\Phi$ , we can regard  $\Lambda$  as a subgroup of  $\widetilde{W}$ , acting by translations. It is well known that  $\widetilde{W}/\Lambda \simeq W$ , where  $W$  is the finite reflection group associated to  $\widetilde{W}$ . As a consequence, the toric Weyl arrangement can be described as:

$$\mathcal{T}_{\widetilde{W}} = \{H_{[w]s_i[w^{-1}]} \mid w \in W \text{ and } s_i \in \widetilde{S}\}$$

where two hypersurfaces  $H_{[w]s_i[w^{-1}]}$  and  $H_{[\overline{w}]s_i[\overline{w}^{-1}]}$  are equal if and only if there is a translation  $t \in \Lambda$  such that  $tws_i(tw)^{-1} = \overline{w}s_i\overline{w}^{-1}$ , i.e.  $\overline{w} = tw$ .

In [14] authors prove that the complement

$$\mathcal{R}_W := T \setminus \bigcup_{H \in \mathcal{T}_{\widetilde{W}}} H$$

has the same homotopy type of a CW-complex  $T(\widetilde{W})$  which admits a description similar to  $C(W)$ .

Indeed the  $k$ -cells of  $T(\widetilde{W})$  correspond to elements  $E([w], \Gamma)$  where  $[w] \in \widetilde{W}/\Lambda \simeq W$  is an equivalence class with one and only one representative  $w \in W$  and  $\Gamma = \{s_{i_1}, \dots, s_{i_k}\}$  is a subset of cardinality  $k$  in  $\widetilde{S}$ .

The integer boundary operator is

$$\partial_k(E([w], \Gamma)) = \sum_{\sigma \in \Gamma} \sum_{\beta \in W_{\Gamma}^{\Gamma \setminus \{\sigma\}}} (-1)^{l(\beta) + \mu(\Gamma, \sigma)} E([w\beta], \Gamma \setminus \{\sigma\}). \quad (5)$$

Let  $\Gamma \subset \tilde{S}$  be a proper subset and  $W_{\Gamma}$  be the finite reflection group generated by  $\Gamma$ . The group

$$(\tilde{W}/\Lambda)_{\Gamma} = \{[w] \in \tilde{W}/\Lambda \mid w \in W_{\Gamma}\} \simeq W_{\Gamma}$$

is a well defined subgroup of  $\tilde{W}/\Lambda$ . As in the finite case, we get

$$\tilde{W}/\Lambda = (\tilde{W}/\Lambda)^{\Gamma} (\tilde{W}/\Lambda)_{\Gamma}$$

and the toric boundary map verifies

$$\partial(E([w], \Gamma)) = [w^{\Gamma}].\partial(E([w_{\Gamma}], \Gamma))$$

where  $[w^{\Gamma}] \in (\tilde{W}/\Lambda)^{\Gamma}$ ,  $[w_{\Gamma}] \in (\tilde{W}/\Lambda)_{\Gamma}$  and  $[w] = [w^{\Gamma}][w_{\Gamma}] = [w^{\Gamma}w_{\Gamma}]$ .

Let us remark that  $(\tilde{W}/\Lambda)_{\Gamma}$  is isomorphic to a subgroup of  $W$  which is not, in general, a parabolic one. In these cases the set  $(\tilde{W}/\Lambda)^{\Gamma}$  doesn't admit a description similar to the one in (2).

Our main interest in the sequel of this paper is to construct a filtration for  $T(\tilde{W})$  similar to the one in subsection 1.2. Also if it is not necessary to know an explicit description of  $(\tilde{W}/\Lambda)^{\Gamma}$  in order to filter the complex  $T(\tilde{W})$ , nevertheless we believe that it would be useful to know a little bit more about it to have a better understanding of our construction. In particular, if  $\tilde{S} = \{s_0, \dots, s_m\}$ , we are interested in the cases in which  $\Gamma = \{s_k, \dots, s_m\}$  or  $\Gamma = \{s_0, \dots, s_h\}$ .

It is a simple remark that, if  $s_0 \notin \Gamma$ , then  $(\tilde{W}/\Lambda)_{\Gamma} \simeq W_{\Gamma}$  is a parabolic subgroup of  $W$ . While the case  $s_m \notin \Gamma$  is a little bit more complicated. Since to remove  $s_0$  or  $s_m$  is perfectly symmetric for  $\tilde{W} = \tilde{A}_m, \tilde{C}_m, \tilde{D}_m, \tilde{E}_6, \tilde{E}_7$ , then in these cases we always get that  $(\tilde{W}/\Lambda)_{\Gamma} \simeq W_{\Gamma}$  is a parabolic subgroup of  $W$ . Hence in the above situations  $(\tilde{W}/\Lambda)^{\Gamma} \simeq W^{\Gamma}$  admits a description as in (2).

Otherwise  $W_{\tilde{S} \setminus \{s_m\}}$  is still a finite reflection group but it is not of type  $W$ . For example if  $\tilde{W} = \tilde{B}_m$  then  $W_{\tilde{S} \setminus \{s_m\}} = D_m$  which is not  $B_m$ . In these cases if  $\Gamma \subset \tilde{S}$  is a given subset with  $s_m \notin \Gamma$  and  $s_0 \in \Gamma$ , then  $(\tilde{W}/\Lambda)_{\Gamma} \simeq W_{\Gamma}$  is a parabolic subgroup of  $W_{\tilde{S} \setminus \{s_m\}}$  and, by [11], we have exactly

$$\frac{|W|}{|W_{\tilde{S} \setminus \{s_m\}}|}$$

copies of  $W_{\tilde{S} \setminus \{s_m\}}$  in  $W$ .

Let  $W'$  be the subgroup of  $W$  such that  $W' \simeq W_{\tilde{S} \setminus \{s_m\}} \simeq (\tilde{W}/\Lambda)_{\tilde{S} \setminus \{s_m\}}$  then  $W^{\tilde{S} \setminus \{s_m\}}$  will denote the subset of  $W$  such that  $W = W^{\tilde{S} \setminus \{s_m\}}W'$  and we get

$$(\tilde{W}/\Lambda)^{\Gamma} \simeq W^{\tilde{S} \setminus \{s_m\}}W_{\tilde{S} \setminus \{s_m\}}^{\Gamma}$$

where  $W_{\tilde{S} \setminus \{s_m\}}^{\Gamma}$  is the subset of  $W_{\tilde{S} \setminus \{s_m\}}$  described in (2).

## 2 The cohomology of complexes $F_n^k(W)$

It is well known that the integer homology, and hence cohomology, of complexes  $C(W)$  is torsion free, while the (co)-homology  $H^*(F_n^k, \mathbb{Z})$  is not known. In this section we will prove that it is torsion free.

As above we will consider the boundary map instead of the (co)-boundary one.

The exact sequences (3) give rise to the corresponding long exact sequences in homology

$$\begin{aligned} \cdots \longrightarrow H_{*+1}(F_m^k(W), \mathbb{Z}) \xrightarrow{\Delta_*} \bigoplus_{j=1}^{m_1 \cdots m_k} H_{*-k}(C(W_{S_{m-k}}), \mathbb{Z}) \xrightarrow{i_*} \\ \xrightarrow{i_*} H_*(F_m^{k-1}(W), \mathbb{Z}) \xrightarrow{j_*} H_*(F_m^k(W), \mathbb{Z}) \longrightarrow \cdots \end{aligned} \quad (6)$$

where the map  $\Delta_*$  is induced by the map on complexes:

$$\begin{aligned} \Delta : F_m^k(W) &\longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_k} C(W_{S_{m-k}}) \\ \Delta(E(w, \Gamma \cup S^{m-k})) &= \sum_{\beta \in W_{\Gamma \cup S^{m-k}}^{\Gamma \cup S^{m-k+1}}} (-1)^{l(\beta)} E(w\beta, \Gamma). \end{aligned} \quad (7)$$

To simplify notation from now on we will use

$$l = m - k - 1$$

and  $\bigoplus C(W_{S_{m-k}})$  instead of  $\bigoplus_{j=1}^{m_1 \cdots m_k} C(W_{S_{m-k}})$  since the number of copy  $\prod_{i=1}^k m_i$  is completely determined by  $S_{m-k}$ .

We have the following theorem.

**Theorem 2** *The integer (co)-homology of complexes  $F_m^k(W)$  is torsion free for all  $k \leq m$ .*

We need the following key Lemma.

**Lemma 2.1** *Let  $v \in F_m^k(W)$  be a boundary then one of the following occurs:*

- i)  $v \in i(\bigoplus C(W_{S_l})[k])$
- ii)  $v \in F_m^k(W) \setminus i(\bigoplus C(W_{S_l})[k])$
- iii)  $v$  is a sum of two boundaries  $v' \in i(\bigoplus C(W_{S_l})[k])$  and  $v'' \in F_m^k(W) \setminus i(\bigoplus C(W_{S_l})[k])$ .

**Proof.** By construction any chain  $v \in F_m^k(W)$  is a sum of two chains

$$v = v' + v''$$

the first one in  $i(\bigoplus C(W_{S_l})[k])$  and the second one in  $F_m^k(W) \setminus i(\bigoplus C(W_{S_l})[k])$ . Let  $v$  be a boundary. If  $v'$  ( $v''$ ) is zero then ii) (i) follows.



Let  $v'$  and  $v''$  both not zero. Ordering in a suitable way rows and columns of the boundary matrix, we get a block matrix as follows:

$$\begin{bmatrix} \bigoplus i(\partial C(W_{S_i})[k]) & B_1 \\ 0 & B_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \partial(F_m^k(W) \setminus i(\bigoplus C(W_{S_i})[k])).$$

Then we can diagonalize the matrix by row and column operations in such a way that the rows of the first (second) block are combined only with rows in the same block.

As consequence any element  $v$  which is in the boundary is written as a sum of two boundaries, one obtained by combinations of row in the first block, i.e. a combination of elements in  $i(\bigoplus C(W_{S_i})[k])$ , and the second one by elements in  $F_m^k(W) \setminus i(\bigoplus C(W_{S_i})[k])$ .  $\square$

**Remark 2.2** *If  $v'$  and  $v''$  are boundaries in  $F_m^k(W)$  as in the above Lemma, then  $v' \in i(\bigoplus \partial C(W_{S_i})[k])$  while, obviously,  $v''$  is a linear combination of elements in  $F_m^k(W) \setminus i(\bigoplus C(W_{S_i})[k])$ , but it is not in its boundary.*

**Proof of Theorem 2** The integer cohomology of the complex  $F_m^0(W) = C(W)$  is torsion free. By induction let us assume that  $H^*(F_m^{k-1}(W), \mathbb{Z})$ , and hence  $H_*(F_m^{k-1}(W), \mathbb{Z})$ , are torsion free.

As the sequence (6) is exact and  $H_*(C(W_{S_{m-k}}), \mathbb{Z})$  and  $H_*(F_m^{k-1}(W), \mathbb{Z})$  are torsion free, then  $H_*(F_m^k(W), \mathbb{Z})$  (and hence  $H^*(F_m^k(W), \mathbb{Z})$ ) is torsion free if and only if the image of  $i_*$  doesn't give rise to  $p$ -torsion for  $p \in \mathbb{Z}$ , i.e.

$$p[v] \in i_*\left(\bigoplus_{j=1}^{m_1 \cdots m_k} H_*(C(W_{S_{m-k}}), \mathbb{Z})\right) \iff [v] \in i_*\left(\bigoplus_{j=1}^{m_1 \cdots m_k} H_*(C(W_{S_{m-k}}), \mathbb{Z})\right).$$

Let  $[v]$  be a generator in the free module  $H_*(F_m^{k-1}(W), \mathbb{Z})$ . By construction

$$[v] = z' + z'' + \partial_*(F_m^{k-1}(W))$$

for  $z' \in i(\bigoplus C(W_{S_i})[k])$  and  $z'' \in F_m^{k-1}(W) \setminus i(\bigoplus C(W_{S_i})[k])$ .  
Let us assume

$$p[v] = pz' + pz'' + \partial_*(F_m^{k-1}(W)) \in i_*\left(\bigoplus H_*(C(W_{S_{m-k}}), \mathbb{Z})\right).$$

Then  $p[v]$  has at list one representative in the image  $i(\bigoplus C(W_{S_i})[k])$  and hence there is an element

$$\omega = \omega' + \omega'' \in \partial_*(F_m^{k-1}(W))$$

such that  $pz' + pz'' + \omega \in i(\bigoplus C(W_{S_i})[k])$ , i.e.  $\omega' \in i(\bigoplus C(W_{S_i})[k])$  and  $\omega'' = -pz''$ .

By Lemma 2.1 we get that  $-\omega'' = pz'' \in \partial_*(F_m^{k-1}(W))$  and hence  $z'' \in \partial_*(F_m^{k-1}(W))$  since  $H_*(F_m^{k-1}(W))$  has no torsion by inductive hypothesis. Then

$$[v] = z' + z'' + \partial_*(F_m^{k-1}(W)) = z' + \partial_*(F_m^{k-1}(W))$$

i.e.  $[v] \in i_*\left(\bigoplus H_*(C(W_{S_{m-k}}), \mathbb{Z})\right)$   $\square$

**Remark 2.3** Obviously Theorem 2 holds also for complexes  $F_m^k(W)$  obtained filtering with the inclusions in (4)

An important consequence of the above theorem is that maps  $\Delta_*$  are map between finitely generated free modules such that

$$p[v] \in \Delta_*(H_*(F_m^k(W), \mathbb{Z})) \iff [v] \in \Delta_*(H_*(F_m^k(W), \mathbb{Z})).$$

A map between two free modules which satisfies the above condition will be called a *one-free* map and it can be diagonalized as:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where  $I$  is the identity matrix. It is a simple remark that composition of one-free maps is still a one-free map. This will be useful in the next section.

### 3 The integer cohomology of $\mathcal{R}_W$

In this section we prove that the (co)-homology of  $T(\widetilde{W})$  (i.e.  $\mathcal{R}_W$ ) is torsion free. In order to do it we construct a filtration of  $T(\widetilde{W})$  similar to the one of  $C(W)$ .

#### 3.1 A filtration for the complex $(T(\widetilde{W}), \partial)$

Let  $\widetilde{S} = \{s_0, \dots, s_m\}$  be the system of generators of  $\widetilde{W}$  and  $W$  the finite group associated. We will keep the notation  $S^k = \{s_{k+1}, \dots, s_m\} \subset \widetilde{S}$  while we introduce the new one  $\widetilde{S}_h = \{s_0, \dots, s_h\} \subset \widetilde{S}$ .

Let us consider the natural inclusion

$$i_m := i : \bigoplus_{j=1}^{m_1} C(W_{\widetilde{S}_{m-1}}) \longrightarrow T(\widetilde{W}),$$

defined as:

$$\begin{aligned} i(j.E(w_{\widetilde{S}_{m-1}}, \Gamma)) &= i(W^{\widetilde{S}_{m-1}}(j).E(w_{\widetilde{S}_{m-1}}, \Gamma)) = \\ i(w_{\widetilde{S}_{m-1}}.E(w_{\widetilde{S}_{m-1}}, \Gamma)) &= [w_{\widetilde{S}_{m-1}}].E([w_{\widetilde{S}_{m-1}}], \Gamma) = E([w], \Gamma) \end{aligned}$$

where  $m_1$  is the cardinality of the set  $W^{\widetilde{S}_{m-1}} = W^{\widetilde{S} \setminus \{s_m\}}$  defined in subsection 1.3 and  $w_{\widetilde{S}_{m-1}} = W^{\widetilde{S}_{m-1}}(j)$  its  $j$ -th element in a fixed order.

Let us remark that  $m_1$  could be also equal to 1 depending on the type of  $\widetilde{W}$  as seen in subsection 1.3.

The cokernel of the map  $i$  is the toric complex  $F_m^1(\widetilde{W})$  having as basis all  $E([w], \Gamma_1)$  for  $w \in W$  and  $\Gamma_1 \subset \widetilde{S}$  with  $|\Gamma_1| \leq m$  s.t.  $s_m \in \Gamma_1$ .

We can iterate this construction getting maps

$$\begin{aligned} i_m[k] := i : \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{\widetilde{S}_i})[k] &\longrightarrow F_m^k(\widetilde{W}), \\ i(w_{\widetilde{S}_i}.E(w_{\widetilde{S}_i}, \Gamma)) &= [w_{\widetilde{S}_i}].E([w_{\widetilde{S}_i}], \Gamma) = E([w], \Gamma \cup S^{m-k}) \end{aligned}$$

with  $l = m - k - 1$ .

Each  $i_m[k]$  gives rise to the exact sequence of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{\tilde{S}_l})[k] \xrightarrow{i} F_m^k(\tilde{W}) \xrightarrow{j} F_m^{k+1}(\tilde{W}) \longrightarrow 0. \quad (8)$$

In a similar way we can filter using the inclusion:

$$\begin{aligned} i^m &:= i : C(W_{S^0}) \longrightarrow T(\tilde{W}), \\ i(E(w, \Gamma)) &= E([w], \Gamma). \end{aligned}$$

Here  $C(W_{S^0})$  is the classical Salvetti's complex for the finite reflection group  $W_{S^0} = W$ . The cokernel of the map  $i$  is the toric complex  $F_m^1(\tilde{W})$  having as basis all  $E([w], \Gamma_1)$  for  $w \in W$  and  $\Gamma_1 \subset \tilde{S}$  with  $|\Gamma_1| \leq m$  s.t.  $s_0 \in \Gamma_1$ .

We can iterate this construction getting maps

$$\begin{aligned} i^m[k] &:= i : \bigoplus_{j=1}^{m^1 \cdots m^k} C(W_{S^k})[k] \longrightarrow F_m^k(\tilde{W}), \\ i(w^{S^k} \cdot (E(w_{S^k}, \Gamma))) &= [w^{S^k}] \cdot i(E([w_{S^k}], \Gamma)) = E([w], \Gamma \cup \tilde{S}_{k-1}). \end{aligned}$$

Each  $i^m[k]$  gives rise to the exact sequence of complexes

$$0 \longrightarrow \bigoplus_{j=1}^{m^1 \cdots m^k} C(W_{S^k})[k] \xrightarrow{i} F_m^k(\tilde{W}) \xrightarrow{j} F_m^{k+1}(\tilde{W}) \longrightarrow 0. \quad (9)$$

### 3.2 Computation of integer cohomology

The exact sequences (8) give rise to the corresponding long exact sequences in homology

$$\begin{aligned} \cdots \longrightarrow H_{*+1}(F_m^{k+1}(\tilde{W}), \mathbb{Z}) \xrightarrow{\tilde{\Delta}_*} \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} H_{*-k}(C(W_{\tilde{S}_l}), \mathbb{Z}) \xrightarrow{i_*} \\ \xrightarrow{i_*} H_*(F_m^k(\tilde{W}), \mathbb{Z}) \xrightarrow{j_*} H_*(F_m^{k+1}(\tilde{W}), \mathbb{Z}) \xrightarrow{\tilde{\Delta}_*} \cdots \end{aligned}$$

The map  $\tilde{\Delta}_*$  is the one induced by maps on complexes:

$$\begin{aligned} \tilde{\Delta} : F_m^{k+1}(\tilde{W}) &\longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} C(W_{\tilde{S}_l}) \\ \tilde{\Delta}(E([w], \Gamma \cup S^l)) &= \sum_{\beta \in W_{\Gamma \cup S^l}^{\Gamma \cup S^l+1}} (-1)^{l(\beta)} E([w\beta], \Gamma). \end{aligned} \quad (10)$$

If  $H_*(F_m^{k+1}(\tilde{W}), \mathbb{Z})$  are torsion free, then  $H_*(F_m^k(\tilde{W}), \mathbb{Z})$  are torsion free if and only if the maps  $\tilde{\Delta}_*$  are one-free maps, i.e. if a generator  $[u] \in \bigoplus_{j=1}^{m_1 \cdots m_{k+1}} H_{*-k}(C(W_{\tilde{S}_l}), \mathbb{Z})$  is such that  $p[u] \in \text{Im } \tilde{\Delta}_*$  for an integer  $p \in \mathbb{Z}$ , then  $[u] \in \text{Im } \tilde{\Delta}_*$ . We will prove it through an inductively argument.

When  $k = m - 1$  we get the last long exact sequence in homology

$$0 \longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_m} H_1(C(W_{\tilde{S}_0}), \mathbb{Z}) \xrightarrow{i_*} H_m(F_m^{m-1}(\tilde{W}), \mathbb{Z}) \xrightarrow{j_*} H_m(F_m^m(\tilde{W}), \mathbb{Z}) \xrightarrow{\tilde{\Delta}_*} \\ \xrightarrow{\tilde{\Delta}_*} \bigoplus_{j=1}^{m_1 \cdots m_m} H_0(C(W_{\tilde{S}_0}), \mathbb{Z}) \xrightarrow{i_*} H_{m-1}(F_m^{m-1}(\tilde{W}), \mathbb{Z}) \longrightarrow 0.$$

As in the affine case, we drop the indices  $m_i$  from the sum  $\bigoplus$  when no misunderstanding is possible.

The integer homology for affine arrangements is torsion free and

$$H_m(F_m^m(\tilde{W}), \mathbb{Z}) = F_m^m(\tilde{W}) \simeq F_m^m(W) = H_m(F_m^m(W), \mathbb{Z})$$

are the free modules generated by  $E([w], S^0) = E([w], S) \simeq E(w, S)$ . Moreover, by definition, the map  $\tilde{\Delta}$  acts on  $F_m^m(\tilde{W})$  as  $\Delta$  on  $F_m^m(W)$ .

Hence, if  $C(W_\emptyset)$  denotes the complex generated by the 0-cell  $E(1, \emptyset)$ , we get the following commutative diagram in homology:

$$\begin{array}{ccc} H_m(F_m^m(W), \mathbb{Z}) & \xrightarrow{\Delta_*} & \bigoplus_{j=1}^{\sharp W} H_0(C(W_\emptyset), \mathbb{Z}) \\ \wr \downarrow & & \downarrow i_* \\ H_m(F_m^m(\tilde{W}), \mathbb{Z}) & \xrightarrow{\tilde{\Delta}_*} & \bigoplus_{j=1}^{m_1 \cdots m_m} H_0(C(W_{\tilde{S}_0}), \mathbb{Z}) \end{array} \quad (11)$$

induced by the corresponding maps on complexes. Then, if  $k = m - 1$ ,  $\tilde{\Delta}_*$  is one-free as composition of two one-free maps  $\Delta_*$  and  $i_*$  and  $H_{m-1}(F_m^{m-1}(\tilde{W}), \mathbb{Z})$  is torsion free. This provide the base of induction.

We remark that  $H_m(F_m^{m-1}(\tilde{W}), \mathbb{Z})$  is torsion free since the map

$$0 \longrightarrow \bigoplus_{j=1}^{m_1 \cdots m_m} H_1(C(W_{\tilde{S}_0}), \mathbb{Z})$$

is obviously one-free.

We are interested in a slightly more general situation. For any two given subset  $\tilde{S}_h, S^k$  such that  $\sharp(\tilde{S}_h \cup S^k) \leq m$ , we consider the complexes  $F_m^{\tilde{S}_h \cup S^k}(\tilde{W})$  generated by cells  $E([w], \Gamma)$  such that  $\Gamma \supset \tilde{S}_h \cup S^k$ . Hence we define the inclusion maps:

$$i_m^h[l] := i : \bigoplus_{j=1}^{\tilde{m}_k} F_{k+1}^{h+1}(W_{\tilde{S}_k})[l] \longrightarrow F_m^{\tilde{S}_h \cup S^{k+1}}(\tilde{W})$$

as

$$i(j.E(w_{\tilde{S}_k}, \tilde{S}_h \cup \Gamma)) = i(W^{\tilde{S}_k}(j).E(w_{\tilde{S}_k}, \tilde{S}_h \cup \Gamma)) = \\ i(w^{\tilde{S}_k}.E(w_{\tilde{S}_k}, \tilde{S}_h \cup \Gamma)) = [w^{\tilde{S}_k}].E([w_{\tilde{S}_k}], \tilde{S}_h \cup \Gamma \cup S^{k+1}) = E([w], \tilde{S}_h \cup \Gamma \cup S^{k+1})$$

where  $W^{\tilde{S}_k}$  is the subset of  $W$  isomorphic to  $(\tilde{W}/\Lambda)^{\tilde{S}_k}$ ,  $\tilde{m}_k$  its cardinality and  $w^{\tilde{S}_k} = W^{\tilde{S}_k}(j)$  its  $j$ -th element in a fixed order.

They provide short exact sequences as in (3) and (8):

$$0 \longrightarrow \bigoplus_{j=1}^{\tilde{m}_k} F_{k+1}^{h+1}(W_{\tilde{S}_k})[l] \longrightarrow F_m^{\tilde{S}_h \cup S^{k+1}}(\tilde{W}) \longrightarrow F_m^{\tilde{S}_h \cup S^k}(\tilde{W}) \longrightarrow 0 \quad (12)$$

If  $\sharp(\tilde{S}_h \cup S^k) = m - 1$  then  $k = h + 1$  and, for  $l_h = m - h - 1$ , we get the last short exact sequence:

$$0 \longrightarrow \bigoplus_{j=1}^{\tilde{m}_{h+1}} F_{h+2}^{h+1}(W_{\tilde{S}_{h+1}})[l_h - 1] \xrightarrow{i} F_m^{\tilde{S}_h \cup S^{h+2}}(\tilde{W}) \xrightarrow{j} F_m^{\tilde{S}_h \cup S^{h+1}}(\tilde{W}) \longrightarrow 0.$$

It is a simple remark that

$$H_m(F_m^{\tilde{S}_h \cup S^{h+1}}(\tilde{W}), \mathbb{Z}) = F_m^{\tilde{S}_h \cup S^{h+1}}(\tilde{W}) \simeq \bigoplus_{j=1}^{\tilde{m}_h} F_{h+1}^{h+1}(W_{\tilde{S}_h})[l_h] = \bigoplus_{j=1}^{\tilde{m}_h} H^{h+1}(F_{h+1}^{h+1}(W_{\tilde{S}_h}), \mathbb{Z})$$

are the free modules generated by  $E([w], \tilde{S} \setminus \{s_{h+1}\}) = E([w], \tilde{S}_h \cup S^{h+1}) \simeq E(w, \tilde{S}_h) = w^{\tilde{S}_h} \cdot E(w_{\tilde{S}_h}, \tilde{S}_h)$ .

Moreover the map

$$\tilde{\Delta} : F_m^{\tilde{S}_h \cup S^{h+1}}(\tilde{W}) \longrightarrow \bigoplus_{j=1}^{\tilde{m}_{h+1}} F_{h+2}^{h+1}(W_{\tilde{S}_{h+1}})$$

splits as follows:

$$\begin{array}{ccc} \bigoplus_{j=1}^{\tilde{m}_h} F_{h+1}^{h+1}(W_{\tilde{S}_h})[l_h] & \xrightarrow{\Delta} & \bigoplus_{j=1}^{\sharp W} C(W_\emptyset) \\ \wr \downarrow & & \downarrow i \\ F_m^{\tilde{S}_h \cup S^{h+1}}(\tilde{W}) & \xrightarrow{\tilde{\Delta}} & \bigoplus_{j=1}^{\tilde{m}_{h+1}} F_{h+2}^{h+1}(W_{\tilde{S}_{h+1}}) \end{array}$$

and we get the commutative diagram in homology:

$$\begin{array}{ccc} \bigoplus_{j=1}^{\tilde{m}_h} H_{h+1}(F_{h+1}^{h+1}(W_{\tilde{S}_h}), \mathbb{Z}) & \xrightarrow{\Delta_*} & \bigoplus_{j=1}^{\sharp W} H_0(C(W_\emptyset), \mathbb{Z}) \\ \wr \downarrow & & \downarrow i_* \\ H_m(F_m^{\tilde{S}_h \cup S^{h+1}}(\tilde{W}), \mathbb{Z}) & \xrightarrow{\tilde{\Delta}_*} & \bigoplus_{j=1}^{\tilde{m}_{h+1}} H_{h+1}(F_{h+2}^{h+1}(W_{\tilde{S}_{h+1}}), \mathbb{Z}). \end{array}$$

Hence if  $\sharp(\tilde{S}_h \cup S^k) = m - 1$  the map  $\tilde{\Delta}_*$  is one-free since it is composition of one-free maps  $\Delta_*$  and  $i_*$ . So far we proved the base of a more general induction.

Going backwards on homology exact sequences induced by (12) we get maps

$$\tilde{\Delta}_* : H_{*+1}(F_m^{\tilde{S}_h \cup S^k}(\tilde{W}), \mathbb{Z}) \longrightarrow \bigoplus H_{*-l}(F_{k+1}^{h+1}(W_{\tilde{S}_k}), \mathbb{Z}). \quad (13)$$

Let us assume, by induction, that they are one-free maps for all  $\tilde{S}_h, S^k$  such that  $n < \sharp(\tilde{S}_h \cup S^k) \leq m - 1$  (i.e.  $H_*(F_m^{\tilde{S}_h \cup S^k}(\tilde{W}), \mathbb{Z})$  are free modules for  $n \leq \sharp(\tilde{S}_h \cup S^k) \leq m - 1$ ).

Let  $\sharp(\tilde{S}_h \cup S^k)$  be equal to  $n$ .

We can also filter  $F_m^{\tilde{S}_h \cup S^k}(\tilde{W})$  as follows:

$$i^m[h+1] := i : \bigoplus_{j=1}^{m^1 \dots m^{h+1}} F_{l_h}^{m-k}(W_{S^{h+1}})[h+1] \longrightarrow F_m^{\tilde{S}_h \cup S^k}(\tilde{W})$$

$$i(w^{S^{h+1}} E(w_{S^{h+1}}, \Gamma \cup S^k)) = E([w], \tilde{S}_h \cup \Gamma \cup S^k).$$

We get the exact sequences

$$0 \longrightarrow \bigoplus_{j=1}^{m^1 \dots m^{h+1}} F_{l_h}^{m-k}(W_{S^{h+1}})[h+1] \longrightarrow F_m^{\tilde{S}_h \cup S^k}(\tilde{W}) \longrightarrow F_m^{\tilde{S}_{h+1} \cup S^k}(\tilde{W}) \longrightarrow 0.$$

This is equivalent to say that for any cell  $E([w], \tilde{S}_h \cup \Gamma \cup S^k) \in F_m^{\tilde{S}_h \cup S^k}(\tilde{W})$  we have only two possibilities:

$$i)s_{h+1} \in \Gamma \text{ and hence } E([w], \tilde{S}_h \cup \Gamma \cup S^k) = E([w], \tilde{S}_{h+1} \cup \Gamma' \cup S^k) \in F_m^{\tilde{S}_{h+1} \cup S^k}(\tilde{W})$$

or

$$ii)s_{h+1} \notin \Gamma \text{ and hence } E([w], \tilde{S}_h \cup \Gamma \cup S^k) = i(w^{S^{h+1}} E(w_{S^{h+1}}, \Gamma \cup S^k)) \in$$

$$\in i\left(\bigoplus_{j=1}^{m^1 \dots m^{h+1}} F_{l_h}^{m-k}(W_{S^{h+1}})[h+1]\right).$$

As a consequence if  $\tilde{\Delta} : F_m^{\tilde{S}_h \cup S^k}(\tilde{W}) \longrightarrow \bigoplus_{j=1}^{\tilde{m}_k} F_{k+1}^{h+1}(W_{\tilde{S}_k})[l]$  is the map which induces the map  $\tilde{\Delta}_*$  in (13),  $\tilde{\Delta}$  splits as follows:

$$\begin{bmatrix} \tilde{\Delta}|_{F_m^{\tilde{S}_h \cup S^k}(\tilde{W})} & 0 \\ 0 & \Delta|_{\bigoplus_{j=1}^{\tilde{m}_k} F_{k+1}^{h+1}(W_{\tilde{S}_k})} \end{bmatrix}.$$

Here  $\tilde{\Delta}|_{F_m^{\tilde{S}_h \cup S^k}(\tilde{W})}$  is the map  $\tilde{\Delta}$  defined on  $F_m^{\tilde{S}_h \cup S^k}(\tilde{W})$ , i.e. on a complex such that  $\sharp(\tilde{S}_{h+1} \cup S^k) = n+1$  if  $\sharp(\tilde{S}_h \cup S^k) = n$ .

From now on we will denote this map  $\tilde{\Delta}_{n+1}$  in order to distinguish it from  $\tilde{\Delta}_n$ .

By previous consideration it follows that the diagram on complexes

$$\begin{array}{ccccccc} 0 \longrightarrow \bigoplus F_{l_h}^{m-k}(W_{S^{h+1}})[h+1] & \xrightarrow{\tilde{i}} & F_m^{\tilde{S}_h \cup S^k}(\tilde{W}) & \xrightarrow{\tilde{j}} & F_m^{\tilde{S}_{h+1} \cup S^k}(\tilde{W}) & \longrightarrow & 0 \\ & & \Delta \downarrow & & \tilde{\Delta}_n \downarrow & & \\ 0 \longrightarrow \bigoplus \bigoplus C(W_{\tilde{S}_k \setminus \tilde{S}_{h+1}})[l][h+1] & \xrightarrow{i} & \bigoplus F_{k+1}^{h+1}(W_{\tilde{S}_k})[l] & \xrightarrow{j} & \bigoplus F_{k+1}^{h+2}(W_{\tilde{S}_k})[l] & \longrightarrow & 0 \\ & & & & (14) & & \end{array}$$

is commutative.

Here  $i : \bigoplus \bigoplus C(W_{\tilde{S}_k \setminus \tilde{S}_{h+1}})[l][h+1] \longrightarrow \bigoplus_{j=1}^{\tilde{m}_k} F_{k+1}^{h+1}(W_{\tilde{S}_k})[l]$  is the map of type (4) such that  $i(w^{\tilde{S}_k \setminus \tilde{S}_{h+1}}.E(w_{\tilde{S}_k \setminus \tilde{S}_{h+1}}, \Gamma)) = w^{\tilde{S}_k} E(w_{\tilde{S}_k}, \tilde{S}_h \cup \Gamma)$ .

Let us remark that the sum

$$\bigoplus \bigoplus C(W_{\tilde{S}_k \setminus \tilde{S}_{h+1}})[l][h+1] = \bigoplus_{j=1}^{\#W/\#W_{\tilde{S}_k \setminus \tilde{S}_{h+1}}} C(W_{\tilde{S}_k \setminus \tilde{S}_{h+1}})[l][h+1]$$

splits in different ways depending if we are considering the horizontal exact sequence or the vertical map  $\Delta$ .

The diagram (14) gives rise to the following commutative diagram in homology:

$$\begin{array}{ccccccc} \longrightarrow & \bigoplus H_{*-h-1}(F_{l_h}^{m-k}(W_{S^{h+1}}), \mathbb{Z}) & \xrightarrow{\tilde{i}_*} & H_*(F_m^{\tilde{S}_h \cup S^k}(\tilde{W}), \mathbb{Z}) & \xrightarrow{\tilde{j}_*} & H_*(F_m^{\tilde{S}_{h+1} \cup S^k}(\tilde{W}), \mathbb{Z}) & \longrightarrow \\ & \Delta_* \downarrow & & \tilde{\Delta}_{n*} \downarrow & & \tilde{\Delta}_{n+1*} \downarrow & \\ \longrightarrow & \bigoplus H_{*-l-h-1}(C(W_{\tilde{S}_k \setminus \tilde{S}_{h+1}}), \mathbb{Z}) & \xrightarrow{i_*} & \bigoplus H_{*-l}(F_{k+1}^{h+1}(W_{\tilde{S}_k}), \mathbb{Z}) & \xrightarrow{j_*} & \bigoplus H_{*-l}(F_{k+1}^{h+2}(W_{\tilde{S}_k}), \mathbb{Z}) & \longrightarrow \end{array}$$

The maps  $i_*$ ,  $j_*$  and  $\Delta_*$  are one-free (see section 2).

Moreover  $H_{*-h-1}(F_{l_h}^{m-k}(W_{S^{h+1}}), \mathbb{Z})$  and  $H_*(F_m^{\tilde{S}_h \cup S^k}(\tilde{W}), \mathbb{Z})$  are free modules respectively by theorem 2 and by inductive hypothesis. Then the maps  $\tilde{i}_*$  and  $\tilde{j}_*$  in the diagram are one-free. Moreover  $\tilde{\Delta}_{n+1*}$  are one-free by induction and hence we get that maps  $\tilde{\Delta}_{n*}$  are one-free too.

So far we proved the main result of the paper:

**Theorem 3** *The integer (co)-homology of the complement  $\mathcal{R}_W$  is torsion free.*

As an immediate consequence of the above theorem,  $H^*(\mathcal{R}_W, \mathbb{Z})$  coincides with the De Rham cohomology described in [3] and the Betti numbers can be easily computed using results in [11].

In general we have the following

**Conjecture 3.1** *Let  $\mathcal{T}_X$  be a thick toric arrangement in the sense of [14]. Then the integer cohomology of the complement is torsion free (and hence it coincides with the De Rham cohomology computed in [3]).*

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