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Working Paper

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LEM Working Paper Series, No. 2012/15

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Laboratory of Economics and Management (LEM), Sant'Anna School of Advanced Studies

Suggested Citation: Manfredini, Sandro; Parveen, Saima; Settepanella, Simona (2012) : Braid groups in complex spaces, LEM Working Paper Series, No. 2012/15, Scuola Superiore Sant'Anna, Laboratory of Economics and Management (LEM), Pisa

This Version is available at:

<https://hdl.handle.net/10419/89542>

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WORKING PAPER SERIES

Braid groups in complex spaces

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2012/15

September 2012

Braid groups in complex spaces

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September 13, 2012

Abstract

We describe the fundamental groups of ordered and unordered k -point sets in \mathbb{C}^n generating an affine subspace of fixed dimension.

Keywords:

complex space, configuration spaces,
braid groups.

MSC (2010): 20F36, 52C35, 57M05, 51A20.

1 Introduction

Let M be a manifold and Σ_k be the symmetric group on k elements. The *ordered* and *unordered configuration spaces* of k distinct points in M , $\mathcal{F}_k(M) = \{(x_1, \dots, x_k) \in M^k \mid x_i \neq x_j, i \neq j\}$ and $\mathcal{C}_k(M) = \mathcal{F}_k(M)/\Sigma_k$, have been widely studied. It is well known that for a simply connected manifold M of dimension ≥ 3 , the *pure braid group* $\pi_1(\mathcal{F}_k(M))$ is trivial and the *braid group* $\pi_1(\mathcal{C}_k(M))$ is isomorphic to Σ_k , while in low dimensions there are non trivial pure braids. For example, (see [F]) the pure braid group of the plane \mathcal{PB}_n has the following presentation

$$\mathcal{PB}_n = \pi_1(\mathcal{F}_n(\mathbb{C})) \cong \langle \alpha_{ij}, 1 \leq i < j \leq n \mid (YB3)_n, (YB4)_n \rangle,$$

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where $(YB3)_n$ and $(YB4)_n$ are the Yang-Baxter relations:

$$\begin{aligned} (YB3)_n: \quad & \alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{ik}\alpha_{jk}\alpha_{ij} = \alpha_{jk}\alpha_{ij}\alpha_{ik}, \quad 1 \leq i < j < k \leq n, \\ (YB4)_n: \quad & [\alpha_{kl}, \alpha_{ij}] = [\alpha_{il}, \alpha_{jk}] = [\alpha_{jl}, \alpha_{jk}^{-1}\alpha_{ik}\alpha_{jk}] = [\alpha_{jl}, \alpha_{kl}\alpha_{ik}\alpha_{kl}^{-1}] = 1, \\ & 1 \leq i < j < k < l \leq n, \end{aligned}$$

while the braid group of the plane \mathcal{B}_n has the well known presentation (see [A])

$$\mathcal{B}_n = \pi_1(\mathcal{C}_n(\mathbb{C})) \cong \langle \sigma_i, 1 \leq i \leq n-1 \mid (A)_n \rangle,$$

where $(A)_n$ are the classical Artin relations:

$$\begin{aligned} (A)_n: \quad & \sigma_i\sigma_j = \sigma_j\sigma_i, \quad 1 \leq i < j \leq n-1, \quad j-i \geq 2, \\ & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i < n-1. \end{aligned}$$

Other interesting examples are the pure braid group and the braid group of the sphere $S^2 \approx \mathbb{C}P^1$ with presentations (see [B2] and [F])

$$\begin{aligned} \pi_1(\mathcal{F}_n(\mathbb{C}P^1)) & \cong \langle \alpha_{ij}, 1 \leq i < j \leq n-1 \mid (YB3)_{n-1}, (YB4)_{n-1}, D_{n-1}^2 = 1 \rangle \\ \pi_1(\mathcal{C}_n(\mathbb{C}P^1)) & \cong \langle \sigma_i, 1 \leq i \leq n-1 \mid (A)_n, \sigma_1\sigma_2 \dots \sigma_{n-1}^2 \dots \sigma_2\sigma_1 = 1 \rangle, \end{aligned}$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{24}\alpha_{34}) \dots (\alpha_{1k}\alpha_{2k} \dots \alpha_{k-1 k})$.

The inclusion morphisms $\mathcal{P}\mathcal{B}_n \rightarrow \mathcal{B}_n$ are given by (see [B2])

$$\alpha_{ij} \mapsto \sigma_{j-1}\sigma_{j-2} \dots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}$$

and due to these inclusions, we can identify the pure braid D_n with Δ_n^2 , the square of the fundamental Garside braid ([G]). In a recent paper ([BS]) Berceanu and the second author introduced new configuration spaces. They stratify the classical configuration spaces $\mathcal{F}_k(\mathbb{C}P^n)$ (resp. $\mathcal{C}_k(\mathbb{C}P^n)$) with complex submanifolds $\mathcal{F}_k^i(\mathbb{C}P^n)$ (resp. $\mathcal{C}_k^i(\mathbb{C}P^n)$) defined as the ordered (resp. unordered) configuration spaces of all k points in $\mathbb{C}P^n$ generating a projective subspace of dimension i . Then they compute the fundamental groups $\pi_1(\mathcal{F}_k^i(\mathbb{C}P^n))$ and $\pi_1(\mathcal{C}_k^i(\mathbb{C}P^n))$, proving that the former are trivial and the latter are isomorphic to Σ_k except when $i = 1$ providing, in this last case, a presentation for both $\pi_1(\mathcal{F}_k^1(\mathbb{C}P^n))$ and $\pi_1(\mathcal{C}_k^1(\mathbb{C}P^n))$ similar to those of the braid groups of the sphere. In this paper we apply the same technique to the affine case, i.e. to $\mathcal{F}_k(\mathbb{C}^n)$ and $\mathcal{C}_k(\mathbb{C}^n)$, showing that the situation is similar except in one case. More precisely we prove that, if $\mathcal{F}_k^{i,n} = \mathcal{F}_k^i(\mathbb{C}^n)$ and $\mathcal{C}_k^{i,n} = \mathcal{C}_k^i(\mathbb{C}^n)$ denote, respectively, the ordered and unordered configuration spaces of all k points in \mathbb{C}^n generating an affine subspace of dimension i , then the following theorem holds:

Theorem 1.1. *The spaces $\mathcal{F}_k^{i,n}$ are simply connected except for $i = 1$ or $i = n = k - 1$. In these cases*

1. $\pi_1(\mathcal{F}_k^{1,1}) = \mathcal{PB}_k$,
2. $\pi_1(\mathcal{F}_k^{1,n}) = \mathcal{PB}_k / \langle D_k \rangle$ when $n > 1$,
3. $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$ for all $n \geq 1$.

The fundamental group of $\mathcal{C}_k^{i,n}$ is isomorphic to the symmetric group Σ_k except for $i = 1$ or $i = n = k - 1$. In these cases:

1. $\pi_1(\mathcal{C}_k^{1,1}) = \mathcal{B}_k$,
2. $\pi_1(\mathcal{C}_k^{1,n}) = \mathcal{B}_k / \langle \Delta_k^2 \rangle$ when $n > 1$,
3. $\pi_1(\mathcal{C}_{n+1}^{n,n}) = \mathcal{B}_{n+1} / \langle \sigma_1^2 = \sigma_2^2 = \dots = \sigma_n^2 \rangle$ for all $n \geq 1$.

Our paper begins by defining a geometric fibration that connects the spaces $\mathcal{F}_k^{i,n}$ to the affine grassmannian manifolds $\text{Grass}^i(\mathbb{C}^n)$. In Section 3 we compute the fundamental groups for two special cases: points on a line $\mathcal{F}_k^{1,n}$ and points in general position $\mathcal{F}_k^{k-1,n}$. Then, in Section 4, we describe an open cover of $\mathcal{F}_k^{n,n}$ and, using a Van-Kampen argument, we prove the main result for the ordered configuration spaces. In Section 5 we prove the main result for the unordered configuration spaces.

2 Geometric fibrations on the affine grassmannian manifold

We consider \mathbb{C}^n with its affine structure. If $p_1, \dots, p_k \in \mathbb{C}^n$ we write $\langle p_1, \dots, p_k \rangle$ for the affine subspace generated by p_1, \dots, p_k . We stratify the configuration spaces $\mathcal{F}_k(\mathbb{C}^n)$ with complex submanifolds as follows:

$$\mathcal{F}_k(\mathbb{C}^n) = \coprod_{i=0}^n \mathcal{F}_k^{i,n},$$

where $\mathcal{F}_k^{i,n}$ is the ordered configuration space of all k distinct points p_1, \dots, p_k in \mathbb{C}^n such that the dimension $\dim \langle p_1, \dots, p_k \rangle = i$.

Remark 2.1. *The following easy facts hold:*

1. $\mathcal{F}_k^{i,n} \neq \emptyset$ if and only if $i \leq \min(k+1, n)$; so, in order to get a non empty set, $i = 0$ forces $k = 1$, and $\mathcal{F}_1^{0,n} = \mathbb{C}^n$.
2. $\mathcal{F}_k^{1,1} = \mathcal{F}_k(\mathbb{C})$, $\mathcal{F}_2^{1,n} = \mathcal{F}_2(\mathbb{C}^n)$;
3. *the adjacency of the strata is given by*

$$\overline{\mathcal{F}_k^{i,n}} = \mathcal{F}_k^{1,n} \amalg \dots \amalg \mathcal{F}_k^{i,n}.$$

By the above remark, it follows that the case $k = 1$ is trivial, so from now on we will consider $k > 1$ (and hence $i > 0$).

For $i \leq n$, let $\text{Grass}^i(\mathbb{C}^n)$ be the affine grassmannian manifold parametrizing i -dimensional affine subspaces of \mathbb{C}^n .

We recall that the map $\text{Grass}^i(\mathbb{C}^n) \rightarrow \text{Gr}^i(\mathbb{C}^n)$ which sends an affine subspace to its direction, exhibits $\text{Grass}^i(\mathbb{C}^n)$ as a vector bundle over the ordinary grassmannian manifold $\text{Gr}^i(\mathbb{C}^n)$ with fiber of dimension $n - i$. Hence, $\dim \text{Grass}^i(\mathbb{C}^n) = (i+1)(n-i)$ and it has the same homotopy groups as $\text{Gr}^i(\mathbb{C}^n)$. In particular, affine grassmannian manifolds are simply connected and $\pi_2(\text{Grass}^i(\mathbb{C}^n)) \cong \mathbb{Z}$ if $i < n$ (and trivial if $i = n$). We can also identify a generator for $\pi_2(\text{Grass}^i(\mathbb{C}^n))$ given by the map

$$g : (D^2, S^1) \rightarrow (\text{Grass}^i(\mathbb{C}^n), L_1), \quad g(z) = L_z$$

where L_z is the linear subspace of \mathbb{C}^n given by the equations

$$(1 - |z|)X_1 - zX_2 = X_{i+2} = \dots = X_n = 0 .$$

Affine grassmannian manifolds are related to the spaces $\mathcal{F}_k^{i,n}$ through the following fibrations.

Proposition 2.2. *The projection*

$$\gamma : \mathcal{F}_k^{i,n} \rightarrow \text{Grass}^i(\mathbb{C}^n)$$

given by

$$(x_1, \dots, x_k) \mapsto \langle x_1, x_2, \dots, x_k \rangle$$

is a locally trivial fibration with fiber $\mathcal{F}_k^{i,i}$.

Proof. Take $V_0 \in \text{Graff}^i(\mathbb{C}^n)$ and choose $L_0 \in \text{Gr}^{n-i}(\mathbb{C}^n)$ such that L_0 intersects V_0 in one point and define \mathcal{U}_{L_0} , an open neighborhood of V_0 , by

$$\mathcal{U}_{L_0} = \{V \in \text{Graff}^i(\mathbb{C}^n) \mid L_0 \text{ intersects } V \text{ in one point}\}.$$

For $V \in \mathcal{U}_{L_0}$, define the affine isomorphism

$$\varphi_V : V \rightarrow V_0, \quad \varphi_V(x) = (L_0 + x) \cap V_0.$$

The local trivialization is given by the homeomorphism

$$f : \gamma^{-1}(\mathcal{U}_{L_0}) \rightarrow \mathcal{U}_{L_0} \times \mathcal{F}_k^{i,i}(V_0)$$

$$y = (y_1, \dots, y_k) \mapsto (\gamma(y), (\varphi_{\gamma(y)}(y_1), \dots, \varphi_{\gamma(y)}(y_k)))$$

making the following diagram commute (where $\mathcal{F}_k^{i,i}(V_0) = \mathcal{F}_k^{i,i}$ upon choosing a coordinate system in V_0)

$$\begin{array}{ccc} \gamma^{-1}(\mathcal{U}_{L_0}) & \xrightarrow{f} & \mathcal{U}_{L_0} \times \mathcal{F}_k^{i,i} \\ & \searrow \gamma & \swarrow pr_1 \\ & & \mathcal{U}_{L_0} \end{array}$$

□

Corollary 2.3. *The complex dimensions of the strata are given by*

$$\dim(\mathcal{F}_k^{i,n}) = \dim(\mathcal{F}_k^{i,i}) + \dim(\text{Graff}^i(\mathbb{C}^n)) = ki + (i+1)(n-i).$$

Proof. $\mathcal{F}_k^{i,i}$ is a Zariski open subset in $(\mathbb{C}^i)^k$ for $k \geq i+1$. □

The canonical embedding

$$\mathbb{C}^m \longrightarrow \mathbb{C}^n, \quad \{z_0, \dots, z_m\} \mapsto \{z_0, \dots, z_m, 0, \dots, 0\}$$

induces, for $i \leq m$, the following commutative diagram of fibrations

$$\begin{array}{ccccc} \mathcal{F}_k^{i,i} & \longrightarrow & \mathcal{F}_k^{i,m} & \longrightarrow & \text{Graff}^i(\mathbb{C}^m) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_k^{i,i} & \longrightarrow & \mathcal{F}_k^{i,n} & \longrightarrow & \text{Graff}^i(\mathbb{C}^n) \end{array}$$

which gives rise, for $i < m$, to the commutative diagram of homotopy groups

$$\begin{array}{ccccccc}
\dots & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\mathcal{F}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{F}_k^{i,m}) \longrightarrow 1 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \\
\dots & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\mathcal{F}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{F}_k^{i,n}) \longrightarrow 1
\end{array}$$

where the leftmost and central vertical homomorphisms are isomorphisms. Then, also the rightmost vertical homomorphisms are isomorphisms, and we have

$$\pi_1(\mathcal{F}_k^{i,n}) \cong \pi_1(\mathcal{F}_k^{i,m}) \cong \pi_1(\mathcal{F}_k^{i,i+1}) \text{ for } i < m \leq n. \quad (1)$$

Thus, in order to compute $\pi_1(\mathcal{F}_k^{i,n})$ we can restrict to the case $k \geq n$ (note that $k > i$), computing the fundamental groups $\pi_1(\mathcal{F}_k^{i,i+1})$, and for this we can use the homotopy exact sequence of the fibration from Proposition 2.2, which leads us to compute the fundamental groups $\pi_1(\mathcal{F}_k^{i,i})$. This is equivalent, simplifying notations, to compute $\pi_1(\mathcal{F}_k^{n,n})$ when $k \geq n + 1$.

We begin by studying two special cases, points on a line and points in general position.

3 Special cases

The case $i = 1$, points on a line.

By remark 2.1 the space $\mathcal{F}_k^{1,1} = \mathcal{F}_k(\mathbb{C})$ for all $k \geq 2$ and the fibration in Proposition 2.2 gives rise to the exact sequence

$$\mathbb{Z} = \pi_2(\text{Graff}^1(\mathbb{C}^2)) \xrightarrow{\delta_*} \mathcal{PB}_n = \pi_1(\mathcal{F}_k(\mathbb{C})) \rightarrow \pi_1(\mathcal{F}_k^{1,2}) \rightarrow 1. \quad (2)$$

It follows that $\pi_1(\mathcal{F}_k^{1,2}) \cong \mathcal{PB}_n / \text{Im} \delta_*$. Since $\pi_2(\text{Graff}^1(\mathbb{C}^2)) = \mathbb{Z}$, we need to know the image of a generator of this group in \mathcal{PB}_n . Taking as generator the map

$$g : (D^2, S^1) \rightarrow (\text{Graff}^1(\mathbb{C}^2), L_1), \quad g(z) = L_z,$$

where L_z is the line of equation $(1 - |z|)X_1 = zX_2$, we chose the lifting

$$\tilde{g} : (D^2, S^1) \rightarrow (\mathcal{F}_k^{1,2}, \mathcal{F}_k(L_1))$$

$$\tilde{g}(z) = ((z, 1 - |z|), 2(z, 1 - |z|), \dots, k(z, 1 - |z|))$$

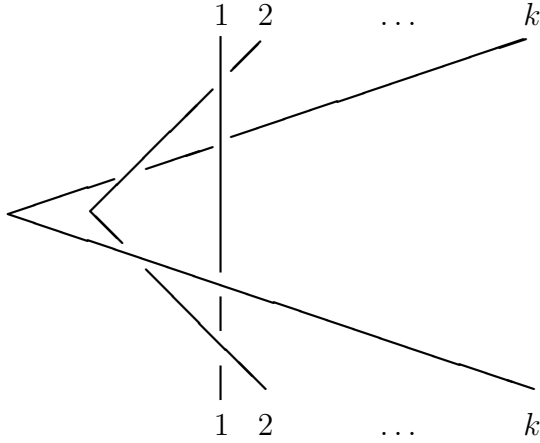
whose restriction to S^1 gives the map

$$\gamma : S^1 \longrightarrow \mathcal{F}_k(L_1) = \mathcal{F}_k(\mathbb{C})$$

$$\gamma(z) = ((z, 0), (2z, 0), \dots, (kz, 0))$$

Lemma 3.1. (see [BS]) *The homotopy class of the map γ corresponds to the following pure braid in $\pi_1(\mathcal{F}_k(\mathbb{C}))$:*

$$[\gamma] = \alpha_{12}(\alpha_{13}\alpha_{23}) \dots (\alpha_{1k}\alpha_{2k} \dots \alpha_{k-1,k}) = D_k .$$



From the above Lemma and the exact sequence in (2) we get that the image in $\pi_1(\mathcal{F}_k(\mathbb{C}))$ of the generator of $\pi_2(\text{Graf}f^1(\mathbb{C}^2))$ is D_k and the following theorem is proved.

Theorem 3.2. *For $n > 1$, the fundamental group of the configuration space of k distinct points in \mathbb{C}^n lying on a line has the following presentation (not depending on n)*

$$\pi_1(\mathcal{F}_k^{1,n}) = \langle \alpha_{ij}, 1 \leq i < j \leq k \mid (YB3)_k, (YB4)_k, D_k = 1 \rangle .$$

The case $k = i + 1$, points in general position.

Lemma 3.3. *For $1 < k \leq n + 1$, the projection*

$$p : \mathcal{F}_k^{k-1,n} \longrightarrow \mathcal{F}_{k-1}^{k-2,n}, \quad (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{k-1})$$

is a locally trivial fibration with fiber $\mathbb{C}^n \setminus \mathbb{C}^{k-2}$

Proof. Take $(x_1^0, \dots, x_{k-1}^0) \in \mathcal{F}_{k-1}^{k-2,n}$ and fix $x_k^0, \dots, x_{n+1}^0 \in \mathbb{C}^n$ such that $\langle x_1^0, \dots, x_{n+1}^0 \rangle = \mathbb{C}^n$ (that is $\langle x_k^0, \dots, x_{n+1}^0 \rangle$ and $\langle x_1^0, \dots, x_{k-1}^0 \rangle$ are skew subspaces). Define the open neighbourhood \mathcal{U} of $(x_1^0, \dots, x_{k-1}^0)$ by

$$\mathcal{U} = \{(x_1, \dots, x_{k-1}) \in \mathcal{F}_{k-1}^{k-2,n} \mid \langle x_1, \dots, x_{k-1}, x_k^0, \dots, x_{n+1}^0 \rangle = \mathbb{C}^n\}.$$

For $(x_1, \dots, x_{k-1}) \in \mathcal{U}$, there exists a unique affine isomorphism $T_{(x_1, \dots, x_{k-1})} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, which depends continuously on (x_1, \dots, x_{k-1}) , such that

$$T_{(x_1, \dots, x_{k-1})}(x_i^0) = (x_i) \text{ for } i = 1, \dots, k-1$$

and

$$T_{(x_1, \dots, x_{k-1})}(x_i^0) = (x_i^0) \text{ for } i = k, \dots, n+1 .$$

We can define the homeomorphisms φ, ψ by :

$$p^{-1}(\mathcal{U}) \begin{array}{c} \xleftarrow{\varphi} \\ \xrightarrow{\psi} \end{array} \mathcal{U} \times (\mathbb{C}^n \setminus \langle x_1^0, \dots, x_{k-1}^0 \rangle)$$

$$\varphi(x_1, \dots, x_{k-1}, x) = ((x_1, \dots, x_{k-1}), T_{(x_1, \dots, x_{k-1})}^{-1}(x))$$

$$\psi((x_1, \dots, x_{k-1}), y) = (x_1, \dots, x_{k-1}, T_{(x_1, \dots, x_{k-1})}(y))$$

satisfying $pr_1 \circ \varphi = p$.

$$\begin{array}{ccc} p^{-1}(\mathcal{U}) & \begin{array}{c} \xrightarrow{\varphi} \\ \xleftarrow{\psi} \end{array} & \mathcal{U} \times (\mathbb{C}^n \setminus \langle x_1^0, \dots, x_{k-1}^0 \rangle) \\ & \searrow p & \swarrow pr_1 \\ & \mathcal{U} & \end{array}$$

□

As $\mathbb{C}^n \setminus \mathbb{C}^{k-2}$ is simply connected when $n > k - 1$ and $k > 1$, we have

$$\pi_1(\mathcal{F}_k^{k-1,n}) \cong \pi_1(\mathcal{F}_{k-1}^{k-2,n}) \cong \pi_1(\mathcal{F}_2^{1,n}) = \pi_1(\mathcal{F}_2(\mathbb{C}^n)) \cong \pi_1(\mathcal{F}_1^{0,n}) = \pi_1(\mathbb{C}^n) = 0,$$

in particular $\pi_1(\mathcal{F}_n^{n-1,n}) = 0$. Moreover, since $\mathbb{C}^n \setminus \mathbb{C}^{k-2}$ is homotopically equivalent to an odd dimensional (real) sphere $S^{2(n-k)-1}$, its second homotopy group vanish and we have

$$\pi_2(\mathcal{F}_{k+1}^{k,n}) \cong \pi_2(\mathcal{F}_k^{k-1,n}) \cong \pi_2(\mathcal{F}_1^{0,n}) = \pi_2(\mathbb{C}^n) = 0.$$

in particular $\pi_2(\mathcal{F}_n^{n-1,n}) = 0$.

In the case $k = n + 1$, $\mathbb{C}^n \setminus \mathbb{C}^{n-1}$ is homotopically equivalent to \mathbb{C}^* , and we obtain the exact sequence:

$$\pi_2(\mathcal{F}_n^{n-1,n}) \rightarrow \mathbb{Z} \rightarrow \pi_1(\mathcal{F}_{n+1}^{n,n}) \rightarrow \pi_1(\mathcal{F}_n^{n-1,n}) \rightarrow 0.$$

By the above remarks, the leftmost and rightmost groups are trivial, so we have that $\pi_1(\mathcal{F}_{n+1}^{n,n})$ is infinite cyclic.

We have proven the following

Theorem 3.4. *For $n \geq 1$, the configuration space of k distinct points in \mathbb{C}^n in general position $\mathcal{F}_k^{k-1,n}$ is simply connected except for $k = n + 1$ in which case $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$.*

We can also identify a generator for $\pi_1(\mathcal{F}_{n+1}^{n,n})$ via the map

$$h : S^1 \rightarrow \mathcal{F}_{n+1}^{n,n} \quad h(z) = (0, e_1, \dots, e_{n-1}, ze_n), \quad (3)$$

where e_1, \dots, e_n is the canonical basis for \mathbb{C}^n (i.e. a loop that goes around the hyperplane $\langle 0, e_1, \dots, e_{n-1} \rangle$).

4 The general case

From now on we will consider $n, i > 1$.

Let us recall that, by Proposition 2.2 and equation (1), in order to compute the fundamental group of the general case $\mathcal{F}_k^{i,n}$, we need to compute $\pi_1(\mathcal{F}_k^{n,n})$ when $k \geq n + 1$. To do this, we will cover $\mathcal{F}_k^{n,n}$ by open sets with an infinite cyclic fundamental group and then we will apply the Van-Kampen theorem to them.

4.1 A good cover

Let $\mathcal{A} = (A_1, \dots, A_p)$ be a sequence of subsets of $\{1, \dots, k\}$ and the integers d_1, \dots, d_p given by $d_j = |A_j| - 1$, $j = 1, \dots, p$. Let us define

$$\mathcal{F}_k^{\mathcal{A}, n} = \{(x_1, \dots, x_k) \in \mathcal{F}_k(\mathbb{C}^n) \mid \dim \langle x_i \rangle_{i \in A_j} = d_j \text{ for } j = 1, \dots, p\}.$$

Example 4.1. *The following easy facts hold:*

1. If $\mathcal{A} = \{A_1\}$, $A_1 = \{1, \dots, k\}$, then $\mathcal{F}_k^{\mathcal{A}, n} = \mathcal{F}_k^{k-1, n}$;
2. if all A_i have cardinality $|A_i| \leq 2$, then $\mathcal{F}_k^{\mathcal{A}, n} = \mathcal{F}_k(\mathbb{C}^n)$;
3. if $p \geq 2$ and $|A_p| \leq 2$, then $\mathcal{F}_k^{(A_1, \dots, A_p), n} = \mathcal{F}_k^{(A_1, \dots, A_{p-1}), n}$;
4. if $p \geq 2$ and $A_p \subseteq A_1$, then $\mathcal{F}_k^{(A_1, \dots, A_p), n} = \mathcal{F}_k^{(A_1, \dots, A_{p-1}), n}$;
5. $\bigcup_{j \geq i} \mathcal{F}_k^{j, n} = \bigcup_{\mathcal{A}=\{A\}, A \in \binom{\{1, \dots, k\}}{i+1}} \mathcal{F}_k^{\mathcal{A}, n}$.

Lemma 4.2. *For $A = \{1, \dots, j+1\}$, $j \leq n$, and $k > j$ the map*

$$P_A : \mathcal{F}_k^{(A), n} \rightarrow \mathcal{F}_{j+1}^{j, n}, \quad (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{j+1})$$

is a locally trivial fibration with fiber $\mathcal{F}_{k-j-1}(\mathbb{C}^n \setminus \{0, e_1, \dots, e_j\})$.

Proof. Fix $(x_1, \dots, x_{j+1}) \in \mathcal{F}_{j+1}^{j, n}$ and choose $z_{j+2}, \dots, z_{n+1} \in \mathbb{C}^n$ such that $\langle x_1, \dots, x_{j+1}, z_{j+2}, \dots, z_{n+1} \rangle = \mathbb{C}^n$.

Define the neighborhood \mathcal{U} of (x_1, \dots, x_{j+1}) by

$$\mathcal{U} = \{(y_1, \dots, y_{j+1}) \in \mathcal{F}_{j+1}^{j, n} \mid \langle y_1, \dots, y_{j+1}, z_{j+2}, \dots, z_{n+1} \rangle = \mathbb{C}^n\}.$$

There exists a unique affine isomorphism $F_y : \mathbb{C}^n \rightarrow \mathbb{C}^n$, which depends continuously on $y = (y_1, \dots, y_{j+1})$, such that

$$\begin{aligned} F_y(x_i) &= y_i, \quad i = 1, \dots, j+1 \\ F_y(z_i) &= z_i, \quad i = j+2, \dots, n+1 \end{aligned}$$

and this gives a local trivialization

$$\begin{aligned} f : P_A^{-1}(\mathcal{U}) &\rightarrow \mathcal{U} \times \mathcal{F}_{k-j-1}(\mathbb{C}^n \setminus \{x_1, \dots, x_{j+1}\}) \\ (y_1, \dots, y_k) &\mapsto ((y_1, \dots, y_{j+1}), F_y^{-1}(y_{j+2}), \dots, F_y^{-1}(y_k)) \end{aligned}$$

which satisfies $pr_1 \circ f = P_A$. □

Let us remark that P_A is the identity map if $k = j + 1$ and the fibration is (globally) trivial if $j = n$ since $\mathcal{U} = \mathcal{F}_{n+1}^{n,n}$; in this last case $\pi_1(\mathcal{F}_k^{(A),n}) = \mathbb{Z}$ (recall that we are considering $n > 1$).

Let $\mathcal{A} = (A_1, \dots, A_p)$ be a p -uple of subsets of cardinalities $|A_j| = d_j + 1$, $j = 1, \dots, p$. For any given integer $h \in \{1, \dots, k\}$, we define a new p -uple $\mathcal{A}' = (A'_1, \dots, A'_p)$ and integers d'_1, \dots, d'_p as follows:

$$A'_j = \begin{cases} A_j, & \text{if } h \notin A_j \\ A_j \setminus \{h\}, & \text{if } h \in A_j \end{cases}, \quad d'_j = \begin{cases} d_j, & \text{if } h \notin A_j \\ d_j - 1, & \text{if } h \in A_j \end{cases}.$$

The following Lemma holds.

Lemma 4.3. *The map*

$$p_h : \mathcal{F}_k^{A,n} \rightarrow \mathcal{F}_{k-1}^{A',n}, \quad (x_1, \dots, x_k) \mapsto (x_1, \dots, \widehat{x_h}, \dots, x_k)$$

has local sections with path-connected fibers.

Proof. Let us suppose that $h = k$ and $k \in (A_1 \cap \dots \cap A_l) \setminus (A_{l+1} \cup \dots \cup A_p)$. Then the fiber of the map $p_k : \mathcal{F}_k^{A,n} \rightarrow \mathcal{F}_{k-1}^{A',n}$ is

$$p_k^{-1}(x_1, \dots, x_{k-1}) \approx \mathbb{C}^n \setminus (L'_1 \cup \dots \cup L'_l \cup \{x_1, \dots, x_{k-1}\})$$

where $L'_j = \langle x_i \rangle_{i \in A'_j}$. Even in the case when $\dim L_j = n$, we have $\dim L'_j < n$, hence the fiber is path-connected and nonempty. Fix a base point $x = (x_1, \dots, x_{k-1}) \in \mathcal{F}_{k-1}^{A',n}$ and choose $x_k \in \mathbb{C}^n \setminus (L'_1 \cup \dots \cup L'_l \cup \{x_1, \dots, x_{k-1}\})$. There are neighborhoods $W_j \subset \text{Graff}^{d'_j}(\mathbb{C}^n)$ of L'_j ($j = 1, \dots, l$) such that $x_k \notin L''_j$ if $L''_j \in W_j$; we take a constant local section

$$s : W = g^{-1}((\mathbb{C}^n \setminus \{x_k\})^{k-1} \times \prod_{i=1}^l W_i) \rightarrow \mathcal{F}_k^{A,n}$$

$$(y_1, \dots, y_{k-1}) \mapsto (y_1, \dots, y_{k-1}, x_k),$$

where the continuous map g is given by:

$$g : \mathcal{F}_{k-1}^{A',n} \rightarrow (\mathbb{C}^n)^{k-1} \times \text{Graff}^{d'_1}(\mathbb{C}^n) \times \dots \times \text{Graff}^{d'_l}(\mathbb{C}^n)$$

$$(y_1, \dots, y_{k-1}) \mapsto (y_1, \dots, y_{k-1}, L''_1, \dots, L''_l),$$

and $L''_j = \langle y_i \rangle_{i \in A'_j}$ for $j = 1, \dots, l$. □

Proposition 4.4. *The space $\mathcal{F}_k^{A,n}$ is path-connected.*

Proof. Use induction on p and $d_1 + d_2 + \dots + d_p$. If $p = 1$, use Lemma 4.2 and the space $\mathcal{F}_{j+1}^{j,n}$ which is path-connected. If A_p is not included in A_1 and $d_p \geq 3$, delete a point in $A_p \setminus A_1$ and use Lemma 4.3 and the fact that if C is not empty and path-connected and $p : B \rightarrow C$ is a surjective continuous map with local sections such that $p^{-1}(y)$ is path-connected for all $y \in C$, then B is path-connected (see [BS]). If $A_p \subset A_1$ or $d_p \leq 2$, use Example 4.1, (3) and (4). \square

Let e_1, \dots, e_n be the canonical basis of \mathbb{C}^n and

$$M_h = \{(x_1, \dots, x_h) \in \mathcal{F}_h(\mathbb{C}^n \setminus \{0, e_1, \dots, e_n\}) \mid x_1 \notin \langle e_1, \dots, e_n \rangle\},$$

the following Lemma holds.

Lemma 4.5. *The map*

$$p_h : M_h \rightarrow (\mathbb{C}^n)^* \setminus \langle e_1, \dots, e_n \rangle$$

sending $(x_1, \dots, x_h) \mapsto x_1$, is a locally trivial fibration with fiber $\mathcal{F}_{h-1}(\mathbb{C}^n \setminus \{0, e_1, \dots, e_n, e_1 + \dots + e_n\})$.

Proof. Let $G : B^m \rightarrow \mathbb{R}^m$ be the homeomorphism from the open unit m -ball to \mathbb{R}^m given by $G(x) = \frac{x}{1-|x|}$, (whose inverse is the map $G^{-1}(y) = \frac{y}{1+|y|}$). For $x \in B^m$ let $\tilde{G}_x = G^{-1} \circ \tau_{-G(x)} \circ G$ be an homeomorphism of B^m , where $\tau_v : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the translation by v . \tilde{G}_x sends x to 0 and can be extended to a homeomorphism of the closure $\overline{B^m}$, by requiring it to be the identity on the $m-1$ -sphere (the exact formula for $\tilde{G}_x(y)$ is $\frac{(1-|x|)y - (1-|y|x)}{(1-|x|)(1-|y|) + |(1-|x|)y - (1-|y|x)|}$). We can further extend it to an homeomorphism G_x of \mathbb{R}^m by setting $G_x(y) = y$ if $|y| > 1$. Notice that G_x depends continuously on x .

Let $\bar{x} \in (\mathbb{C}^n)^* \setminus \langle e_1, \dots, e_n \rangle$, fix an open complex ball B in $(\mathbb{C}^n)^* \setminus \langle e_1, \dots, e_n \rangle$ centered at \bar{x} and an affine isomorphism H of \mathbb{C}^n sending B to the open real $2n$ -ball B^{2n} . For $x \in B$, define the homeomorphism F_x of \mathbb{C}^n $F_x = H^{-1} \circ G_{H(x)} \circ H$ which sends x to \bar{x} , is the identity outside of B and depends continuously on x . The result follows from the continuous map

$$F : p_h^{-1}(B) \rightarrow B \times p_h^{-1}(\bar{x})$$

$$F(x, x_2, \dots, x_h) = (x, (\bar{x}, F_x(x_2), \dots, F_x(x_h)))$$

(whose inverse is the map $F^{-1} : B \times p_h^{-1}(\bar{x}) \rightarrow p_h^{-1}(B)$, $F^{-1}(x, (\bar{x}, x_2, \dots, x_h)) = (x, F_x^{-1}(x_2), \dots, F_x^{-1}(x_h))$).

The fiber $p_h^{-1}(\bar{x})$ is homeomorphic to $\mathcal{F}_{h-1}(\mathbb{C}^n \setminus \{0, e_1, \dots, e_n, e_1 + \dots + e_n\})$ via an homeomorphism of \mathbb{C}^n which fixes $0, e_1, \dots, e_n$ and sends \bar{x} to the sum $e_1 + \dots + e_n$. \square

Thus we have, since $n \geq 2$, $\pi_1(M_h) = \mathbb{Z}$, and we can choose as generator the map $S^1 \rightarrow M_h$ sending $z \mapsto (z(e_1 + \dots + e_n), x_2, \dots, x_h)$ with x_2, \dots, x_h of sufficient high norm (i.e. a loop that goes round the hyperplane $\langle e_1, \dots, e_n \rangle$).

Lemma 4.6. *For $A = \{1, \dots, n+1\}$, $B = \{2, \dots, n+2\}$, and $k > n+1$ the map*

$$P_{A,B} : \mathcal{F}_k^{(A,B),n} \rightarrow \mathcal{F}_{n+1}^{n,n}, \quad (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{n+1})$$

is a trivial fibration with fiber M_{k-n-1}

Proof. For $x = (x_1, \dots, x_{n+1}) \in \mathcal{F}_{n+1}^{n,n}$ let F_x be the affine isomorphism of \mathbb{C}^n such that $F_x(0) = x_1$, $F_x(e_i) = x_{i+1}$, for $i = 1, \dots, n$. The map

$$\mathcal{F}_{n+1}^{n,n} \times M_{k-n-1} \rightarrow \mathcal{F}_k^{(A,B),n}$$

sending

$$((x_1, \dots, x_{n+1}), (x_{n+2}, \dots, x_k)) \mapsto (x_1, \dots, x_{n+1}, F_x(x_{n+2}), \dots, F_x(x_k))$$

gives the result. \square

4.2 Computation of the fundamental group

From Lemma 4.6 it follows that $\pi_1(\mathcal{F}_k^{(A,B),n}) = \mathbb{Z} \times \mathbb{Z}$ and that it has two generators: $((z+1)(e_1 + \dots + e_n), e_1, \dots, e_n, e_1 + \dots + e_n, x_{n+3}, \dots, x_k)$ and $(0, e_1, \dots, e_n, z(e_1 + \dots + e_n), x_{n+3}, \dots, x_k)$, where x_{n+3}, \dots, x_k are chosen *far enough* to be different from the first $n+2$ points. The first generator is the one coming from the base, the second is the one from the fiber of the fibration $P_{A,B}$.

Note that using the map

$$P'_{A,B} : \mathcal{F}_k^{(A,B),n} \rightarrow \mathcal{F}_{n+1}^{n,n}, \quad (x_1, \dots, x_k) \mapsto (x_2, \dots, x_{n+2})$$

we obtain the same result and the generator coming from the base here is the one coming from the fiber above and vice versa.

The map $P_{A,B}$ factors through the inclusion $i_A : \mathcal{F}_k^{(A,B),n} \hookrightarrow \mathcal{F}_k^{(A),n}$ followed by the map

$$P_A : \mathcal{F}_k^{(A),n} \rightarrow \mathcal{F}_{n+1}^{n,n}, \quad (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{n+1})$$

and we get the following commutative diagram of fundamental groups:

$$\begin{array}{ccc} \pi_1(\mathcal{F}_k^{(A,B),n}) & \xrightarrow{P_{A,B*}} & \pi_1(\mathcal{F}_{n+1}^{n,n}) \\ & \searrow^{i_{A*}} & \nearrow^{P_{A*}} \\ & \pi_1(\mathcal{F}_k^{(A),n}) & \end{array}$$

Since P_A induces an isomorphism on the fundamental groups, this means that i_{A*} sends the generator of $\pi_1(\mathcal{F}_k^{(A,B),n})$ coming from the fiber to 0 in $\pi_1(\mathcal{F}_{n+1}^{n,n})$. That is, the generator of $\pi_1(\mathcal{F}_k^{(B),n})$ (which is homotopically equivalent to the generator of $\pi_1(\mathcal{F}_k^{(A,B),n})$ coming from the fiber) is trivial in $\pi_1(\mathcal{F}_k^{(A),n})$ and (given the symmetry of the matter) vice versa.

Applying Van Kampen theorem, we have that $\mathcal{F}_k^{(A),n} \cup \mathcal{F}_k^{(B),n}$ is simply connected. Moreover the intersection of any number of $\mathcal{F}_k^{(A),n}$'s is path connected and the same is true for the intersection of two unions of $\mathcal{F}_k^{(A),n}$'s since the intersection $\bigcap_{A \in \{\{1, \dots, k\}_{n+1}\}} \mathcal{F}_k^{(A),n}$ is not empty.

From the last example in 4.1 with $i = n$ we have $\mathcal{F}_k^{n,n} = \bigcup_{A \in \{\{1, \dots, k\}_{n+1}\}} \mathcal{F}_k^{(A),n}$, and when $k > n+1$, we can cover it with a finite number of simply connected open sets with path connected intersections, so it is simply connected by the following

Lemma 4.7. *Let X be a topological space which has a finite open cover U_1, \dots, U_n such that each U_i is simply connected, $U_i \cap U_j$ is connected for all $i, j = 1, \dots, n$ and $\bigcap_{i=1}^n U_i \neq \emptyset$. Then X is simply connected.*

Proof. By induction, let's suppose $\bigcup_{i=1}^{k-1} U_i$ is simply connected. Then, applying Van Kampen theorem to U_k and $\bigcup_{i=1}^{k-1} U_i$, we get that $\bigcup_{i=1}^k U_i$ is simply connected if $U_k \cap (\bigcup_{i=1}^{k-1} U_i)$ is connected. But $U_k \cap (\bigcup_{i=1}^{k-1} U_i) = \bigcup_{i=1}^{k-1} (U_k \cap U_i)$ is the union of connected sets with non empty intersection, and therefore is connected. \square

Now, using the fibration in Proposition 2.2 with $n = i + 1$, we obtain that $\mathcal{F}_k^{n-1,n}$ is simply connected when $k > n$. Summing up the results for the ordered case, the following main theorem is proved

Theorem 4.8. *The spaces $\mathcal{F}_k^{i,n}$ are simply connected except*

1. $\pi_1(\mathcal{F}_k^{1,1}) = \mathcal{PB}_k$,
2. $\pi_1(\mathcal{F}_k^{1,n}) = \langle \alpha_{ij}, 1 \leq i < j \leq k \mid (YB3)_k, (YB4)_k, D_k = 1 \rangle$ when $n > 1$,
3. $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$ for all $n \geq 1$, with generator described in (3).

5 The unordered case: $\mathcal{C}_k^{i,n}$

Let $\mathcal{C}_k^{i,n}$ be the unordered configuration space of all k distinct points p_1, \dots, p_k in \mathbb{C}^n which generate an i -dimensional space. Then $\mathcal{C}_k^{i,n}$ is obtained quotienting $\mathcal{F}_k^{i,n}$ by the action of the symmetric group Σ_k . The map $p : \mathcal{F}_k^{i,n} \rightarrow \mathcal{C}_k^{i,n}$ is a regular covering with Σ_k as deck transformation group, so we have the exact sequence:

$$1 \rightarrow \pi_1(\mathcal{F}_k^{i,n}) \xrightarrow{p_*} \pi_1(\mathcal{C}_k^{i,n}) \xrightarrow{\tau} \Sigma_k \rightarrow 1$$

which gives immediately $\pi_1(\mathcal{C}_k^{i,n}) = \Sigma_k$ in case $\mathcal{F}_k^{i,n}$ is simply connected. Observe that the fibration in Proposition 2.2 may be quotiented obtaining a locally trivial fibration $\mathcal{C}_k^{i,n} \rightarrow \text{Grass}^i(\mathbb{C}^n)$ with fiber $\mathcal{C}_k^{i,i}$. This gives an exact sequence of homotopy groups which, together with the one from Proposition 2.2 and those coming from regular coverings, gives the following commutative diagram for $i < n$:

$$\begin{array}{ccccccc}
& & & 1 & & 1 & \\
& & & \downarrow & & \downarrow & \\
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{\delta_*} & \pi_1(\mathcal{F}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{F}_k^{i,n}) & \longrightarrow & 1 \\
& & \downarrow \cong & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{\delta'_*} & \pi_1(\mathcal{C}_k^{i,i}) & \longrightarrow & \pi_1(\mathcal{C}_k^{i,n}) & \longrightarrow & 1 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \Sigma_k & \longrightarrow & \Sigma_k & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 1 & & 1 & &
\end{array}$$

In case $i = 1$, $\mathcal{F}_k^{1,1} = \mathcal{F}_k(\mathbb{C})$ and $\mathcal{C}_k^{1,1} = \mathcal{C}_k(\mathbb{C})$, so $\pi_1(\mathcal{F}_k^{1,1}) = \mathcal{PB}_k$ and $\pi_1(\mathcal{C}_k^{1,1}) = \mathcal{B}_k$, and since $\text{Im}\delta_* = \langle D_k \rangle \subset \mathcal{PB}_k$, the left square gives $\text{Im}\delta'_* = \langle \Delta_k^2 \rangle \subset \mathcal{B}_k$, therefore $\pi_1(\mathcal{C}_k^{1,n}) = \mathcal{B}_k / \langle \Delta_k^2 \rangle$.

For $i = n = k - 1$, we have $\pi_1(\mathcal{F}_{n+1}^{n,n}) = \mathbb{Z}$, and we can use the exact sequence of the regular covering $p : \mathcal{F}_{n+1}^{n,n} \rightarrow \mathcal{C}_{n+1}^{n,n}$ to get a presentation of $\pi_1(\mathcal{C}_{n+1}^{n,n})$.

Let's fix $Q = (0, e_1, \dots, e_n) \in \mathcal{F}_{n+1}^{n,n}$ and $p(Q) \in \mathcal{C}_{n+1}^{n,n}$ as base points and for $i = 1, \dots, n$ define $\gamma_i : [0, \pi] \rightarrow \mathcal{F}_{n+1}^{n,n}$ to be the (open) path

$$\gamma_i(t) = \left(\frac{1}{2}(e^{i(t+\pi)} + 1)e_i, e_1, \dots, e_{i-1}, \frac{1}{2}(e^{it} + 1)e_i, e_{i+1}, \dots, e_n\right)$$

(which fixes all entries except the first and the $(i + 1)$ -th and exchanges 0 and e_i by a half rotation in the line $\langle 0, e_i \rangle$).

Then $p \circ \gamma_i$ is a closed path in $\mathcal{C}_{n+1}^{n,n}$ and we denote it's homotopy class in $\pi_1(\mathcal{C}_{n+1}^{n,n})$ by σ_i . Hence $\tau_i = \tau(\sigma_i)$ is the deck transformation corresponding to the transposition $(0, i)$ (we take Σ_{n+1} as acting on $\{0, 1, \dots, n\}$) and we get a set of generators for Σ_{n+1} satisfying the following relations

$$\tau_i^2 = \tau_i \tau_j \tau_i \tau_j^{-1} \tau_i^{-1} \tau_j^{-1} = 1 \text{ for } i, j = 1, \dots, n,$$

$$[\tau_1 \tau_2 \cdots \tau_{i-1} \tau_i \tau_{i-1}^{-1} \cdots \tau_1^{-1}, \tau_1 \tau_2 \cdots \tau_{j-1} \tau_j \tau_{j-1}^{-1} \cdots \tau_1^{-1}] = 1 \text{ for } |i - j| > 2.$$

If we take T , the (closed) path in $\mathcal{F}_{n+1}^{n,n}$ in which all entries are fixed except for one which goes round the hyperplane generated by the others counter-clockwise, as generator of $\pi_1(\mathcal{F}_{n+1}^{n,n})$, then $\pi_1(\mathcal{C}_{n+1}^{n,n})$ is generated by T and the

$\sigma_1, \dots, \sigma_n$.

In order to get the relations, we must write the words σ_i^2 , $\sigma_i\sigma_j\sigma_i\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1}$ and $[\sigma_1\sigma_2\cdots\sigma_{i-1}\sigma_i\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}, \sigma_1\sigma_2\cdots\sigma_{j-1}\sigma_j\sigma_{j-1}^{-1}\cdots\sigma_1^{-1}]$ as well as $\sigma_i T \sigma_i^{-1}$ as elements of $\text{Ker } \tau = \text{Im } p_*$ for all appropriate i, j .

Observe that the path $\gamma'_i : [\pi, 2\pi] \rightarrow \mathcal{F}_{n+1}^{n,n}$, defined by the same formula as γ_i , is a lifting of σ_i with starting point $(e_i, e_1, e_2, \dots, e_{i-1}, 0, e_{i-1}, \dots, e_n)$ and that $\gamma_i\gamma'_i$ is a closed path in $\mathcal{F}_{n+1}^{n,n}$ which is the generator T of $\pi_1(\mathcal{F}_{n+1}^{n,n})$ (as you can see by the homotopy $(\frac{\epsilon}{2}(e^{i(t+\pi)}+1)e_i, e_1, \dots, e_{i-1}, \frac{2-\epsilon}{2}(e^{it}+\frac{\epsilon}{2-\epsilon}))e_i, e_{i+1}, \dots, e_n)$, $\epsilon \in [0, 1]$, where for $\epsilon = 0$ we have the point e_i going round the hyperplane $\langle 0, e_1, e_2, \dots, e_{i-1}, e_{i+1}, \dots, e_n \rangle$ counterclockwise).

Thus we have $p_*(T) = \sigma_i^2$ for all $i = 1, \dots, n$ (and that Imp_* is the center of $\pi_1(\mathcal{C}_{n+1}^{n,n})$).

Moreover, it's easy to see, by lifting to $\mathcal{F}_{n+1}^{n,n}$, that the σ_i satisfy the relations

$$\sigma_i\sigma_j\sigma_i\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1} = 1 \text{ for } i, j = 1, \dots, n$$

and

$$[\sigma_1\sigma_2\cdots\sigma_{i-1}\sigma_i\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}, \sigma_1\sigma_2\cdots\sigma_{j-1}\sigma_j\sigma_{j-1}^{-1}\cdots\sigma_1^{-1}] = 1 \text{ for } |i - j| > 2 .$$

We can represent a lifting of $\sigma'_i = \sigma_1\sigma_2\cdots\sigma_{i-1}\sigma_i\sigma_{i-1}^{-1}\cdots\sigma_1^{-1}$ (which gives the deck transformation corresponding to the transposition $(i, i+1)$) by a path which fixes all entries except the i -th and the $(i+1)$ -th and exchanges e_i and e_{i+1} by a half rotation in the line $\langle e_i, e_{i+1} \rangle$.

We can now change the set of generators by first deleting T and introducing the relations

$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$$

and then by choosing the σ'_i 's instead of the σ_i 's. Then we get that the generators σ'_i 's satisfy the relations

$$\sigma'_i\sigma'_{i+1}\sigma'_i = \sigma'_{i+1}\sigma'_i\sigma'_{i+1} \text{ for } i = 1, \dots, n-1,$$

$$[\sigma'_i, \sigma'_j] = 1 \text{ for } |i - j| > 2$$

and

$$\sigma_1'^2 = \sigma_2'^2 = \cdots = \sigma_n'^2. \quad (4)$$

Namely, $\pi_1(\mathcal{C}_{n+1}^{n,n})$ is the quotient of the braid group \mathcal{B}_{n+1} on $n+1$ strings by relations (4) and the following main theorem is proved.

Theorem 5.1. *The fundamental groups $\pi_1(\mathcal{C}_k^{i,n})$ are isomorphic to the symmetric group Σ_k except*

1. $\pi_1(\mathcal{C}_k^{1,1}) = \mathcal{B}_k$,
2. $\pi_1(\mathcal{C}_k^{1,n}) = \mathcal{B}_k / \langle \Delta_k^2 \rangle$ when $n > 1$,
3. $\pi_1(\mathcal{C}_{n+1}^{n,n}) = \mathcal{B}_{n+1} / \langle \sigma_1^2 = \sigma_2^2 = \dots = \sigma_n^2 \rangle$ for all $n \geq 1$.

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