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Wealth-driven Selection in a Financial Market with Heterogeneous Agents

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Wealth-driven selection in a financial market with heterogeneous agents

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Abstract

We study the co-evolution of asset prices and individual wealth in a financial market populated by an arbitrary number of heterogeneous, boundedly rational agents. Using wealth dynamics as a selection device we are able to characterize the long run market outcomes, i.e. asset returns and wealth distributions, for a general class of investment behaviors. Our investigation illustrates that market interaction and wealth dynamics pose certain limits on the outcome of agents' interactions even within the "wilderness of bounded rationality". As an application we consider the case of heterogeneous mean-variance optimizers and provide insights into the results of the simulation model introduced by Levy, Levy and Solomon (1994).

JEL codes: G12, D84, C62.

Keywords: Heterogeneous agents, Asset pricing model, Bounded rationality, CRRA framework, Levy-Levy-Solomon model, Evolutionary Finance.

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1 Introduction

Consider a financial market where a group of heterogeneous investors, each following a different strategy to gain superior returns, is active. The open question is to specify who will survive in the long run and how their interaction affects market returns. This paper seeks to give a contribution on this issue. At this purpose, we investigate the co-evolution of asset prices and agents' wealth in a stylized market for a long-lived financial asset populated by an arbitrary number of heterogeneous agents. We do so under three main assumptions. First, asset demands are proportional to agents' wealth. Second, each individual investment behavior can be formalized as a function of past dividends and prices. Third, the asset pays a risky dividend which follows an exogenous process.

By focusing on asset price dynamics in a market with heterogeneous agents, our paper clearly belongs to the growing field of Heterogeneous Agent Models (HAMs), see Hommes (2006) for a recent survey. We share the standard set-up of this literature and assume that agents decide whether to invest in a risk-free bond or in a risky financial asset.[1] In the spirit of Brock and Hommes (1997) and Grandmont (1998) we consider stochastic dynamical system and analyze the sequence of temporary equilibria of its deterministic skeleton.

As opposed to the majority of HAMs which consider only a few types of investors and concentrate on heterogeneity in expectations we consider a general framework which can be applied to a quite large set of investment strategies so that heterogeneity with respect to risk attitude, expectations, memory and optimization task can be accommodated. We develop the tools to characterize long-run behavior of asset prices and agents' wealth for a general set of competing investment strategies. In particular, we can encompass any investment behavior that can be specified as a smooth functions of past realizations of prices and dividends.

An important feature of our model concerns the demand specification. In contrast to many HAMs (see e.g. Brock and Hommes (1998); Gaunersdorfer (2000); Brock et al. (2005)), which employ the setting where agents' demand exhibits constant absolute risk aversion (CARA), we assume that demand increases linearly with agents' wealth, that is, it exhibits constant *relative* risk aversion (CRRA). In such a setting agents affect market price proportionally to their relative wealth. As a consequence, relative wealth represents a natural measure of performance of different investment behaviors. On the contrary, in CARA models the wealth dynamics does not affect agents' demand, implying that the performance measure has to be introduced *ad hoc* time by time. Furthermore, experimental literature seems to lean in favor of CRRA rather than CARA (see e.g. Kroll et al. (1988) and Chapter 3 in Levy et al. (2000)).

The analytical exploration of the CRRA framework with heterogeneous agents is difficult because the wealth dynamics of every agent has to be taken into account. Despite this obstacle, the recent papers of Chiarella and He (2001, 2002), Chiarella et al. (2006), Anufriev et al. (2006), Anufriev (2008) and Anufriev and Bottazzi (2006), to which our work is particularly related, have made some progress and are able to characterize the long run market outcome. All these studies, however, are based on the assumption that the price-dividend ratio is exogenous. This seems at odd with the standard approach, where the dividend process is exogenously set, while the asset prices are endogenously determined. In our paper, to overcome this problem, we analyze a market for a financial asset whose dividend process is exogenous, so that the price-dividend ratio is a dynamic variable. Our paper can be seen as an extension of these works to the case of exogenous dividends.

[1] Recently, also some models with heterogeneous agents operating in markets with multiple assets (Chiarella et al., 2007) and with derivatives (Brock et al., 2006) have been developed.

As a result we show that depending on the difference between the growth rate of dividends and the risk-free rate, which are the exogenous parameters of our model, the dynamics can converge to two types of the equilibrium steady-states. When the growth rate of dividend is higher that the risk-free rate, the equilibrium dividend yield is positive, asset gives a higher expected return than the risk-free bond, and only one or few investors have positive wealth share. Only such "survivors" can affect the price, and in this sence the wealth dynamics acts as a selection device in a given steady-state. However, multiple steady-states with different survivors and different levels of the dividend yield are possible and the range of possibilities depends on the whole ecology of traders. Otherwise, when dividends grow less than the risk-free rate, the dividend yield goes to zero, both the risky asset and the risk-free bond give the same expected return, and the wealth of all agents grows at the same rate as asset prices. We also show how both type of behaviors simply follow from the process of wealth accumulation in the economy.

An important reason for departing from previous works with CRRA demands, is that it allows for a direct application to a well known simulation model. In fact, our CRRA setup with exogenous dividend process is the set-up of one of the first agent-based simulation model of a financial market introduced by Levy, Levy and Solomon (LLS model, henceforth), see e.g. Levy et al. (1994). Their work investigates whether some stylized empirical findings in finance, such as excess volatility or long periods of overvaluation of asset, can be explained by relaxing the assumption of a fully-informed, rational representative agent. Despite some success of the LLS model in reproducing the financial "stylized facts", all its results are based on simulations. Our general setup can be applied to the specific demand schedules used in the LLS model, and, thus, provides an analytical support to its simulations.

As we are looking at agents' survival in a financial market ecology, our work can be classified within the realm of *evolutionary finance*. The seminal work of Blume and Easley (1992), as well as more recent papers of Sandroni (2000), Hens and Schenk-Hoppé (2005), Blume and Easley (2006) and Evstigneev et al. (2006), investigate how beliefs about the dividend process affect agents' dominance in the market. There are two main differences of our work with this literature. First, whereas their work focuses on portfolio selection, we focus on relating asset returns to the risk-free rate. Indeed, in their model an investor can choose among a number of different risky assets, while in our model agents can invest either in a risky or in a risk-less asset. Second, although these contributions assume that each investment strategy depends on the realization of exogenous variables, i.e. dividends, our investors can also condition on past values of endogenous variables such as past prices. As a consequence in our framework prices today influence prices tomorrow through their impact on the agents' demands. This is important especially when the stability of the surviving strategy is investigated. For instance, when the investment strategy is too responsive to price movements, fluctuations are typically amplified and unstable price dynamics are produced. Indeed, we show that local stability is related to how far agents look in the past.

This paper is organized as follows. Section 2 presents the model and leads to the definition of the stochastic dynamical system where prices and wealths co-evolve. The equilibria of the deterministic version of the system are studied in Section 3 for two simple motivating cases and in Section 4 for the general case. Section 5 applies the general model to the special case where agents are mean-variance optimizers and explains the simulation results of the LLS model. Section 6 summarizes our main results and concludes. Proofs are collected in Appendices at the end of the paper.

2 The model

Let us consider a group of N agents trading in discrete time in a market for a long-lived risky asset. Assume that the asset is in constant supply which, without loss of generality, can be normalized to 1. Alternatively, agents can buy a risk-less asset which returns a constant interest rate $r_f > 0$. The risk-less asset serves as numéraire with price normalized to 1 in every period. At time t the risky asset pays a dividend D_t in units of the numéraire, while its price P_t is fixed through market clearing.

Let $W_{n,t}$ stand for the wealth of agent n at time t. It is convenient to express agents' demand for the risky asset in terms of the fraction $x_{n,t}$ of wealth invested in this asset, so that agent n invests an amount $x_{n,t}W_{n,t}$ in the risky asset at time t. The dividend is paid before trade takes place, and the wealth of agent n evolves as

$$W_{n,t+1} = (1 - x_{n,t})\, W_{n,t}\, (1 + r_f) + x_{n,t}\, W_{n,t}\, \frac{P_{t+1} + D_{t+1}}{P_t}\,, \tag{2.1}$$

The asset price at time $t+1$ is fixed through the market clearing condition

$$\sum_{n=1}^{N} \frac{x_{n,t+1}\, W_{n,t+1}}{P_{t+1}} = 1\,. \tag{2.2}$$

Prices and wealth co-evolve because prices depend on current agent wealth via (2.2) and, at the same time, the wealth of every agent depends on the contemporaneous price via (2.1). In what follows, first, we solve these equations and make the dynamics of prices and wealth explicit, assuming that dividend process and individual investment decisions are given. Then, we give a precise specification of both investment decisions and dividend process.

2.1 Co-evolution of Wealth and Prices

Given asset prices and agents' wealth at time t, as well as agents' investment strategies and the dividend at time $t+1$, prices and wealth at $t+1$ are simultaneously determined by wealth evolution (2.1) and market clearing condition (2.2). To obtain an explicit solution for P_{t+1}, and thereafter W_{t+1}, one can use (2.1) to rewrite the market clearing equation (2.2) as

$$P_{t+1} = \sum_{n=1}^{N} x_{n,t+1} W_{n,t} \left((1 - x_{n,t})(1 + r_f) + x_{n,t} \Big(\frac{P_{t+1}}{P_t} + \frac{D_{t+1}}{P_t} \Big) \right)\,.$$

The solution of this equation with respect to P_{t+1} gives

$$P_{t+1} = \frac{\sum_{n=1}^{N} x_{n,t+1} W_{n,t} \left((1 - x_{n,t})(1 + r_f) + x_{n,t} \frac{D_{t+1}}{P_t} \right)}{1 - \frac{1}{P_t} \sum_{n=1}^{N} W_{n,t}\, x_{n,t+1}\, x_{n,t}}\,, \tag{2.3}$$

which using (2.1) fixes the level of individual wealth $W_{n,t+1}$ for every agent n. The resulting expressions can be conveniently written in terms of price return, dividend yield, and each agent's relative wealth,[2] defined respectively as

$$k_{t+1} = \frac{P_{t+1}}{P_t} - 1\,, \qquad y_{t+1} = \frac{D_{t+1}}{P_t} \qquad \text{and} \qquad \varphi_{n,t} = \frac{W_{n,t}}{\sum_m W_{m,t}}\,.$$

[2]Notice that this change of variables changes the nature of steady-states equilibria, from constant levels to constant changes. The latter is more appropriate in an economy which is possibly growing, like ours. For the same reason, other works in the literature, such as Chiarella and He (2001) and Anufriev et al. (2006), take the same approach.

Dividing both sides of (2.3) by P_t and using that $P_t = \sum x_{n,t} W_{n,t}$, one can rewrite the dynamics in terms of price returns. This, together with the resulting expression for the evolution of wealth shares, gives the following system

$$\begin{cases} k_{t+1} = r_f + \dfrac{\sum_n \big((1+r_f)\,(x_{n,t+1} - x_{n,t}) + y_{t+1}\, x_{n,t}\, x_{n,t+1}\big)\varphi_{n,t}}{\sum_n x_{n,t}\,(1 - x_{n,t+1})\,\varphi_{n,t}}\,, \\ \varphi_{n,t+1} = \varphi_{n,t}\, \dfrac{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\, x_{n,t}}{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\, \sum_m x_{m,t}\varphi_{m,t}}\,, \qquad \forall n \in \{1, \dots, N\}\,. \end{cases} \tag{2.4}$$

According to the first equation, returns depend on the totality of agents' investment decisions for two consequent periods. High investment fractions for the current period tend to increase current prices and, hence, returns. This effect is due to an increase of current demand. Moreover, the effect of agents' decisions on price returns is proportional to their relative wealth. The second equation shows that the relative wealth of every agent changes according to the *relative* investor performances, where the return of each individual wealth should be taken as performance measure.

2.2 Investment functions

We intend to study the evolution of the asset prices and agents' wealth while keeping the investment strategies as general as possible. Therefore we avoid any explicit formulation of the demand and suppose that the fraction of wealth invested in the risky asset, $x_{n,t}$, are general functions of past realizations of prices and dividends. Following Anufriev and Bottazzi (2006) we formalize this concept of investment strategy as follows.

Assumption 1. For each agent $n = 1, \dots, N$ there exists an *investment function* f_n which maps the information set into an investment share:

$$x_{n,t} = f_n(k_{t-1}, k_{t-2}, \dots, k_{t-L}; y_t, y_{t-1}, \dots, y_{t-L})\,. \tag{2.5}$$

Agents' investment decisions evolve following individual prescriptions and depend in a general way on the available information set.[3] The investment choices of period t should be made before trade starts, i.e. when price P_t is still unknown. Thus, the information set contains past price returns up to k_{t-1} and past dividend yields up to y_t.

Assumption 1 leaves a high freedom in the demand specification. The only essential restrictions are stationarity, i.e. the same information observed in different periods is mapped to the same investment decision, and that the investment share does not depend on the contemporaneous wealth of trader. This implies that the demand for the risky asset is linearly increasing with traders' wealth. In other words, *ceteris paribus* investors maintain a constant proportion of their invested wealth as their wealth level changes. Such behavior can be referred as of a constant relative risk aversion (CRRA) type.[4]

A number of standard demand specifications are consistent with Assumption 1. In Section 5.1 we will consider agents who maximize mean-variance utility of their next period expected return. Alternatively, one can consider agents behaving in accordance with the

[3] In order to deal with a finite dimensional dynamical system, we restrict the memory span of each agent to a finite L. Notice however that L can be arbitrarily large.

[4] The distinction between constant relative and constant absolute risk aversion (CARA) behavior was introduced in Arrow (1965) and Pratt (1964), who also relates these concepts with utility maximization. Under CARA framework agents maintain a constant demand for the risky asset as their wealth changes.

prospect theory of Kahneman and Tversky (1979).[5] The generality of the investment functions allows modeling *forecasting* behavior with a big flexibility too. Formulation (2.5) includes as special cases both technical trading, e.g., when agents' decisions are driven by the observed price fluctuations, and more fundamental attitudes, e.g., when the decisions are made on the basis of the price-dividend ratio. It also includes the case of a constant investment strategy, which corresponds to agents assuming a stationary *ex-ante* return distribution.

Despite its high generality, our setup does not include a number of important investment behaviors. Since current wealth is not included as an argument of the investment function, all the demand functions of CARA type are not covered by our framework. Also the current price is not among the arguments of (2.5). Therefore, investors deriving their investment share conditional on the current price cannot be reconciled with our setup.

2.3 Dividend process

The last ingredient of the model is the dividend process. The previous analytical models built in a similar framework, such as Chiarella and He (2001), Anufriev et al. (2006), Anufriev and Bottazzi (2006) and Anufriev (2008), assume that the dividend yield is an i.i.d. process. This implies that any change in the level of price causes an immediate change in the level of dividends. In reality, however, the dividend policy of firms is hardly so fast responsive to the performance of firm assets, especially when prices are driven up by speculative bubbles. Both for the sake of reality and for comparison to previous works, we find it interesting to investigate what happens when the dividend process is an exogenous process. This is the approach we take in this paper.

Assumption 2. The dividend realization follows a geometric random walk,

$$D_t = D_{t-1}\left(1 + g_t\right),$$

where the growth rate, g_t, is an i.i.d. random variable.

Rewriting this assumption in terms of dividend yield and price return we get

$$y_{t+1} = y_t \frac{1 + g_{t+1}}{1 + k_t}. \tag{2.6}$$

Equations (2.4), (2.5) and (2.6) specify the evolution of the asset market with N heterogeneous agents. The dynamics of the model is stochastic due to the fluctuations of the dividend process. Following the typical route in the literature (cf. Brock and Hommes (1997), Grandmont (1998)), in the next two sections we analyze the *deterministic skeleton* of this stochastic dynamics. The skeleton is obtained by fixing the growth rate of dividends g_t at the constant level g. Later in this paper we show that the local stability analysis of the deterministic skeleton gives a considerable insight into the analysis of stochastic simulations, performed with g_t as a random variable.[6]

[5] This is shown, for instance, in Chapter 9 of Levy et al. (2000).

[6] It is important to keep in mind that the agents do take the risk due to randomness into account, when deriving their investment functions. Given agents' behavior, we, as the modelers, set the noise level to zero and analyze the resulting deterministic dynamics.

3 Two special cases

Before analyzing the general model, it is informative to consider some simple examples. In this section we assume that agent investment decisions are affected only by the last observation of price return and dividend yield. For such specification we study and compare the situations with one or two agents trading in the market. The main purpose of these simplifications is to get an overview of the different possibilities which our model can generate and get an insight into its dynamics and underlying economic mechanisms. A careful mathematical analysis, with all computations and proofs, is postponed to Section 4, where we investigate the general case.

3.1 Single agent

The simplest situation is the single agent case, when wealth evolution can be ignored. Equations (2.4)–(2.6) give altogether the market dynamics which, omitting the agent-specific subindex, can be written as

$$\begin{cases} x_{t+1} = f(k_t, y_{t+1})\,, \\ k_{t+1} = r_f + \dfrac{(1+r_f)\,(x_{t+1}-x_t) + y_{t+1}\,x_t\,x_{t+1}}{x_t\,(1-x_{t+1})}\,, \\ y_{t+1} = y_t\,\dfrac{1+g}{1+k_t}\,. \end{cases} \tag{3.1}$$

The steady-states of the dynamics correspond to the constant values x^*, k^* and y^*.

Assume, first, that the dividend yield y^* is positive. To sustain constant yield, the prices should grow with the same rate as the dividend. Indeed, $k^* = g$ from the last equation of (3.1). Substitution to the first two equations leads to

$$x^* = f(g; y^*) \qquad \text{and} \qquad x^* = \frac{g - r_f}{y^* + g - r_f}\,, \tag{3.2}$$

which, g and r_f being exogenous, represents a system of two equations in two variables, the investment share x^* and the dividend yield y^*. When the investment function is specified any solution of this system completes the computation of the steady-states. Both equations in (3.2) define a one-dimensional curve in coordinates (y, x), so that solutions can be found graphically as the intersections of the two curves. The first curve, which will be referred as the Equilibrium Investment Function (EIF), is a cross-section of the investment function $f(k, y)$ by the plane $k = g$. For any value of the dividend yield y, the EIF gives the agent's investment share corresponding to this yield, *and* consistent with the condition of constant yield (i.e. with prices growing at rate g). The second curve,

$$l(y) = \frac{g - r_f}{y + g - r_f}\,, \tag{3.3}$$

defined for $y > 0$, is called the Equilibrium Market Curve (EMC). For any y, it determines an investment share necessary for generating the endogenous prices growing at rate g. At the steady-state, where (3.2) holds and the curves intersect, the agent invests as much as needed to generate the constant yield economy (since the point belongs to the EMC) and his investment is consistent with his own investment function (since the point belongs to the EIF).

In the left panel of Fig. 1 the EMC is drawn as a thick curve, with one example of EIF shown as a thin curve. From our discussion it follows that the steady-states of (3.1) are the intersections of the EMC with the EIF, in this case points A and B. Only here the investment behavior is consistent with market clearing, constant returns, and constant dividend yields. In point A the agent invests a high proportion of his wealth in the risky asset. Such investment behavior pushes the price up and leads to a low yield. In point B the investment share is lower, leading to a higher yield.

FIGURE 1 IS ABOUT HERE.

The previous discussion concerned the case with positive steady-state yield. When the dividend yield $y^* = 0$, the second equation of (3.1) implies that $k^* = r_f$, and the investment share is unambiguously determined as $x^* = f(r_f, 0)$. In this case we have a unique steady-state equilibrium.[7]

To summarize, the dynamics of (3.1) may generate two types of steady states. The first type has $k^* = g$, while the dividend yield and investment share are determined simultaneously. From the EMC-plot it is clear that for a given trader's behavior, there could be none, one or multiple steady-states. The second type has always a unique steady-state with zero dividend yield, $k^* = r_f$, and investment share $x^* = f(r_f, 0)$.

Local Stability. Since multiple steady-states can exist in the model, the question of their local stability becomes very important. The first conclusion is that for given parameters g and r_f the steady-states of only one type can be stable, and, therefore, observable. In fact, in the steady-state of the first type one of the eigenvalues is equal to $(1 + r_f)/(1 + g)$, while in all the steady-states of the second type one of the eigenvalues is equal to $(1 + g)/(1 + r_f)$, which is exactly the inverse. For instance, if $g > r_f$ only the steady-states with positive yield can be stable. To understand this result we note that it is equivalent to the fact that in the stable steady-states the price return k^* is equal to *maximum* of g and r_f. In our economy new money (and therefore wealth) is continuously arriving both through dividends and interest payment. When the economy converges to a steady state, the long-run rate of growth of the total wealth is determined by the fastest among these two sources and the price return of the risky asset simply reflect this growth of the numéraire.

The dynamics have two other eigenvalues, which has to be identified in order to complete the stability analysis. In the case of type one steady-states these other two eigenvalues are the roots of

$$\mu^2 + \mu \left(\frac{f^k}{l'} \frac{1+g}{y^*} - \frac{f^y}{l'} \right) - \frac{f^k}{l'} \frac{1+g}{y^*}, \tag{3.4}$$

where l' denotes the slope of the EMC, and f^k and f^y stand for the partial derivatives of the investment function with respect to the past price return and the past dividend yield, respectively, all computed in the corresponding steady-state. Thus, the steady-state with positive yield is locally stable if $g > r_f$ and the two roots of polynomial (3.4) lie inside the unit circle. In particular, it is the relative slope of the investment function with respect to the

[7] Notice that the steady-state with zero dividend yield can be observed only asymptotically, since dividends are positive. The reader can find more on this point in Section 4.

EMC at the steady-state which is crucial for stability. For example, when multiple steady-states are present (as for agent I in the left panel of Fig. 1), some of the steady-state can be stable and some not.

When $g < r_f$ the only candidate for stability is the zero yield steady-state. This steady-state is locally stable if both roots of polynomial

$$\mu^2 - \mu \frac{1+r_f}{x^*(1-x^*)} f^k + \frac{1+r_f}{x^*(1-x^*)} f^k \tag{3.5}$$

lie inside the unit circle. The slope of the investment function as well as the level of investment share determine stability of this steady-state.

3.2 Two agents

The case of two co-existing agents is more interesting because the relative wealth dynamics starts to play a role. Using (2.4)–(2.6) we obtain

$$\begin{cases} x_{1,t+1} = f_1\big(k_t; y_{t+1}\big)\,, \\ x_{2,t+1} = f_2\big(k_t; y_{t+1}\big)\,, \\ \varphi_{1,t+1} = \varphi_{1,t}\, \dfrac{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\, x_{1,t}}{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\,(\varphi_{1,t}\, x_{1,t} + \varphi_{2,t}\, x_{2,t})}\,, \\ k_{t+1} = r_f + \dfrac{\sum_{n=1}^{2} \big((1+r_f)\,(x_{n,t+1} - x_{n,t}) + y_{t+1}\, x_{n,t}\, x_{n,t+1}\big)\varphi_{n,t}}{x_{1,t}\,(1-x_{1,t+1})\,\varphi_{1,t} + x_{2,t}\,(1-x_{2,t+1})\,\varphi_{2,t}}\,, \\ y_{t+1} = y_t\, \dfrac{1+g}{1+k_t}\,. \end{cases} \tag{3.6}$$

A steady state of the market dynamics consists of constant values of the two investment shares (x_1^* and x_2^*), constant wealth distribution (φ_1^* and $\varphi_2^* = 1 - \varphi_1^*$), and constant levels of price returns and dividend yields (k^* and y^*).

We start with the first type of steady-states, those with positive y^*. Again this leads to $k^* = g$ and, from the wealth dynamics, we obtain that the investment share of any agent with positive wealth share is equal to $\varphi_1^* x_1^* + \varphi_2^* x_2^*$. Let us call "survivor" an agent whose wealth share is positive in a given steady-state. Then we have just established the fact that *all survivors have the same investment share in a steady-state*. Given this, two possibilities can arise. If only one agent, say the first, survives, which implies $\varphi_1^* = 1$, then his investment share and the dividend yield should simultaneously satisfy

$$x_1^* = f_1(g; y^*) \qquad \text{and} \qquad x_1^* = \frac{g - r_f}{y^* + g - r_f}\,, \tag{3.7}$$

which is the same system of equations as (3.2). Thus, again, the EMC-plot provides the geometric characterization, with steady-state values (y^*, x_1^*) given by the coordinates of intersection of the EMC with the EIF of the first agent. The second agent has zero wealth share and invests according to his investment function $x_2^* = f_2(g, y^*)$. Another possibility arises when both agents survive, i.e., $\varphi_1^* \in (0,1)$. Then it must be

$$x_1^* = f_1(g; y^*)\,, \qquad x_2^* = f_2(g; y^*) \qquad \text{and} \qquad x_1^* = x_2^* = \frac{g - r_f}{y^* + g - r_f}\,, \tag{3.8}$$

that is, both agents have to invest the same fraction of wealth in the risky asset. Since the wealth share φ_1^* is free of choice, in this case there exists an infinite number of the steady-states, a continuum manifold of them to be precise. Indeed, any wealth distribution corresponds to a different steady-state, though the levels of k^* and y^* are the same in all of them.

The right panel of Fig. 1 gives a specific example of positive yield equilibria. The EIF of agents I and II are plotted as the thin lines I and II respectively. All possible steady-states of the dynamics are the intersections between these curves and the EMC. There exist three steady-states marked by points A, B and C. Point A corresponds to the case where (3.7) is satisfied for agent I. Therefore, in this steady-state he survives and, being alone, takes all available wealth, $\varphi_I^* = 1$. As in the single agent case, the equilibrium dividend yield y^* is the abscissa of point A, while the investment share of the survivor, x_I^*, is the ordinate of A. The equilibrium investment share of the second agent, x_{II}^*, can be found as a value of his investment function at y^*. In the other two steady-states the variables are determined in a similar way. In particular, agent I is the only survivor at B, while at C only the second agent survives. To illustrate a situation with infinity of steady-states, imagine to shift curve II upward so that points C and A coincide. This would be a situation with two survivors. In fact when the two EIF intersect the EMC in the same point, system (3.8) is satisfied. Any different wealth distribution (i.e. all possible combinations of φ_I and φ_{II} satisfying to $\varphi_I + \varphi_{II} = 1$) defines a different steady-state. However, every steady-state must have the same yield and the same investment share for both agents as determined by the coordinates of point $\mathrm{A} \equiv \mathrm{C}$. Since it is unlikely that two generic investment functions intersect the EMC at the same point, we refer to such case of coexisting survivors as non-generic.

The second type of steady-states have zero yield. If $y^* = 0$ we derive from (3.6) that $k^* = r_f$ and the investment shares are uniquely determined from the investment functions. Now, however, the zero yield steady-state is not unique, because any wealth distribution determines steady-state. In all of them, however, the aggregate economic variables (the price return and dividend yield) are the same. Therefore, such steady-states are indistinguishable when looking at the aggregate time series.

To summarize, as in a single agent case there are two types of the steady-states. In the steady-states of the first type $k^* = g$, the yield is positive, and all the agents with positive wealth shares ("survivors") have the same investment share. This share is determined simultaneously with the dividend yield as the coordinates of an intersection of the EMC with the EIF of the survivor(s). Generically there is one survivor in each steady-state, but if the EIF of the two agents intersect in a point which also belongs to the EMC, there exist an infinite number of the steady-states, corresponding to any wealth distribution between the two agents. In the steady-states of the second type the yield is zero, $k^* = r_f$, and, the agents invest according to their investment functions. Any wealth distribution of agents is allowed, so that there exists an infinite number of such steady-states.

Local Stability. Consider, first, the steady-states with positive yield. When only one agent (say, the first) survives, the Jacobian matrix of system (3.6) has one zero eigenvalue, one eigenvalue equal to

$$\frac{1 + r_f + (g + y^* - r_f)x_2^*}{1 + r_f + (g + y^* - r_f)x_1^*}, \tag{3.9}$$

and three other eigenvalues which are the same as for the single agent system (3.1) with investment function f_1. Thus, the stability conditions can be divided into two parts. First, the

eigenvalue (3.9) should be inside the unit circle, which is equivalent to condition $x_1^* > x_2^*$. Second, the same steady-state should be stable for a lower-dimensional, "reduced" system, which is obtained from the original one by eliminating the agent who does not survive. According to Section 3.1 the reduced system is locally stable when (i) $g > r_f$ and (ii) both roots of polynomial

$$\mu^2 + \mu\left(\frac{f_1^k}{l'}\frac{1+g}{y^*} - \frac{f_1^y}{l'}\right) - \frac{f_1^k}{l'}\frac{1+g}{y^*}\,, \tag{3.10}$$

lie inside the unit circle. In this polynomial f_1^k and f_1^y are the partial derivatives of the investment function of the *survivor* at the steady-state. As for the condition $x_1^* > x_2^*$, it means that the survivor should behave more "aggressively" in the stable steady-state, i.e. invest more in the risky asset, than the agent who does not survive. This result is a consequence of wealth-based selection inherent to the dynamics of our model. As long as the risky asset yields a higher average return than the riskless asset, the most aggressive agent has also the highest individual return. Consider the steady-state where the survivor with $\varphi_1^* = 1$ invests smaller share than the second agent. Consider now a deviation from such steady-state when small relative wealth redistributed in favor of the second agent. As soon as the second agent has positive wealth, he will get higher return and faster growing wealth. Thus, the dynamics will not return to the initial steady-state.[8] With respect to the example in the right hand panel of Fig. 1, the stability analysis implies that steady-states B and C are unstable. At B investor I is the survivor, but in this steady-state investor II invests more. The same is in C, where the survivor (agent II) is less aggressive than the other agent.

In the non-generic situation, when both agents survive, there exists a manifold of the steady-states with positive yield. Any of them is identified by some wealth distributions $(\varphi_1^*, \varphi_2^*)$. The conditions for local stability of such steady-state are (i) $g > r_f$ and (ii) all the roots of polynomial

$$\mu^2 + \mu\left(\frac{\varphi_1^* f_1^k + \varphi_2^* f_2^k}{l'}\,\cdot\,\frac{1+g}{y^*} - \frac{\varphi_1^* f_1^y + \varphi_2^* f_2^y}{l'}\right) - \frac{\varphi_1^* f_1^k + \varphi_2^* f_2^k}{l'}\,\cdot\,\frac{1+g}{y^*} \tag{3.11}$$

lie within the unit circle. This polynomial explicitly depends on the wealth distribution at the steady-state. As a result it can happen that steady-states with the same values of k^*, y^* and x^*, are stable for some wealth distributions and unstable for others. Notice that (3.11) coincides with (3.10) at the steady-state where only the first agent survives.

Finally, we turn to the steady-states of the second type, with zero yield. There exists a manifold of them, but in all these steady-states there are eigenvalues 0, 1 and $(1+g)/(1+r_f)$. While the unit eigenvalue corresponds to the directions of the change in wealth distribution, the last eigenvalues provides condition $g < r_f$ for stability. Finally, the remaining eigenvalues are the roots of polynomial

$$\mu^2 - \mu\frac{(1+r_f)(f_1^k\varphi_1^* + f_2^k\varphi_2^*)}{x_1^*(1-x_1^*)\varphi_1^* + x_2^*(1-x_2^*)\varphi_2^*} + \frac{(1+r_f)(f_1^k\varphi_1^* + f_2^k\varphi_2^*)}{x_1^*(1-x_1^*)\varphi_1^* + x_2^*(1-x_2^*)\varphi_2^*}\,, \tag{3.12}$$

where f_1^k and f_2^k stay for the partial derivatives of the two investment functions in the steady-states. When there is only one survivor (e.g., $\varphi_1^* = 1$), this polynomial coincides with (3.5). Thus, the stability conditions in the case of the two agents generalize those of the single

[8] We have defined a survivor as an agent with positive wealth share at the steady-state. Obviously, if this steady-state is unstable, one cannot observe asymptotically the survivance of survivor.

agent case. Notice that the characteristic polynomial (3.12) explicitly depends on the wealth distribution.

Let us summarize our results and intuition behind them. One feature of our model, the inflow of the numéraire from dividends and risk-less return, implies that the steady-states with constant values of return and wealth shares can only be stable when prices grow with a rate equal to $\max(g, r_f)$. When $g > r_f$ the dynamics are consistent with a positive dividend yield, implying positive excess return of the risky asset. In this case the second feature of our model starts to play its role. Namely, in our CRRA framework, wealth dynamics rewards more aggressive agents having higher investment shares. Such wealth-driven selection of agents is responsible for the fact that only few agents (behaving identically) can survive in the steady-state. On the other hand, when $g < r_f$ one has zero excess return, and the wealth selection is not "activated". In fact, every investment strategy gives the same return. As a result, a variety of investment behaviors can be observed at the steady-state. In addition to these two forces, i.e. monetary expansion and wealth selection, the investment behavior itself affects the stability of the different steady-states. In line with the HAM literature, we find that investment functions should be flat enough in a neighborhood of steady-states (implicitly this is implied by conditions on the roots of polynomials (3.10)–(3.12)). Otherwise, small deviations of price returns or dividend yield will be amplified by agents behavior.

It turns out that, generally speaking, all these results do not change much in the case of arbitrary number of agents with more general behavior. This is demonstrated in the next section.

4 Equilibrium returns and agent survival

In this section we address the equilibrium and stability analysis of our market dynamics in full generality, that is, we consider a market populated by N investors each trading according to a different investment function.

Before we start, let us define the investment decision weighted with relative wealth as

$$\left\langle x_t \right\rangle_s = \sum_{n=1}^{N} x_{n,t} \varphi_{n,s} \,, \tag{4.1}$$

where the time of the decision, t, and the time of the weighting wealth distribution, s, can be different. Applying this notation to (2.4)–(2.6), the deterministic skeleton of the market dynamics with N investors can be written as

$$\begin{cases} x_{n,t+1} = f_n\big(k_t, k_{t-1}, \ldots, k_{t-L+1}; y_{t+1}, y_t, \ldots, y_{t-L+1}\big) \,, & \forall n \in \{1, \ldots, N\} \\ \varphi_{n,t+1} = \varphi_{n,t} \dfrac{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\, x_{n,t}}{(1+r_f) + (k_{t+1} + y_{t+1} - r_f) \left\langle x_t \right\rangle_t} \,, & \forall n \in \{1, \ldots, N\} \\ k_{t+1} = r_f + \dfrac{(1+r_f)\left\langle x_{t+1} - x_t \right\rangle_t + y_{t+1} \left\langle x_t\, x_{t+1} \right\rangle_t}{\left\langle x_t\,(1 - x_{t+1}) \right\rangle_t} \,, \\ y_{t+1} = y_t \dfrac{1+g}{1+k_t} \,. \end{cases} \tag{4.2}$$

Given the arbitrariness of the population size N, of the memory span L, and the absence of any specification for the investment functions, the analysis of the dynamics generated by (4.2)

is highly non-trivial in its general formulation. However, the constraints on the dynamics set by the dividend process, the market clearing equation and the wealth evolution are sufficient to (i) uniquely characterize the steady-state equilibrium level of price returns, (ii) describe the corresponding possible distributions of wealth among agents, and (iii) restrict the possible values of steady-state equilibria dividend yields to a low-dimensional set. Moreover it is possible to derive general conditions under which convergence to these equilibria is guaranteed.

The primary issue is whether restricting the dynamics to the set of economically relevant values delivers a well-defined dynamical system. In particular, positivity of prices and dividends imply that price returns should always exceed -1 and dividend yields should always be larger than 0. The following result shows that at this purpose it is sufficient to forbid investors to take short positions.

Proposition 4.1. *Assume that agents do not take short positions in both the risk-less and the risky assets, or using an investment functions f_n as defined in Ass. 1, assume that*

$$\text{the image of } f_n \text{ belongs to } (0,1) \text{ for every } n. \tag{4.3}$$

Then the system (4.2) *defines a $2N+2L$-dimensional dynamical system of first-order equations. The evolution operator associated with this system*

$$\mathcal{T}(x_1,\dots,x_N;\varphi_1,\dots,\varphi_N;k_1,\dots,k_L;y_1,\dots,y_L) \tag{4.4}$$

is well-defined on the set

$$\mathcal{D} = (0,1)^N \times \Delta_N \times (-1,\infty)^L \times (0,\infty)^L\,, \tag{4.5}$$

consisting respectively of the investment shares, the wealth shares, the (lagged) price returns and the (lagged) dividend yields, and Δ_N denotes the unit simplex in N-dimensional space

$$\Delta_N = \Big\{(\varphi_1,\dots,\varphi_N) : \sum\nolimits_{m=1}^{N} \varphi_m = 1,\ \varphi_m \geq 0 \quad \forall m\Big\}.$$

Proof. We prove that the dynamics from $\mathcal{D}$ to $\mathcal{D}$ is well-defined. The explicit evolution operator $\mathcal{T}$, which is used in the stability analysis, is provided in Appendix A.

Let us start with period-t variables belonging to the domain $\mathcal{D}$ and apply the dynamics described by (4.2) to them. Since $k_t > -1$, the fourth equation is well defined and y_{t+1} is positive. As a result, the first equation defines the new investment shares belonging to $(0,1)$ in accordance with the assumption in (4.3). It, in turn, implies that in the right-hand side of the third equation all the variables are defined, and the denominator is positive. Thus, k_{t+1} can be computed. Moreover, the denominator does not exceed 1, as a convex combination of numbers non-exceeding 1. Then, a simple computation gives

$$k_{t+1} > r_f + \sum\nolimits_m \big((1+r_f)(-1)+0\big)\varphi_{m,t} = -1\,.$$

Finally, it is easy to see that both the numerator and the denominator of the second equation are positive and that $\sum_m \varphi_{m,t+1} = 1$. Therefore, the dynamics of the wealth shares is well-defined and takes place within the unit simplex Δ_N. □

The proposition shows that given any initial conditions for (4.2), the dynamics of price returns, dividend yields and relative wealth shares are completely specified. One can now easily derive the dynamics of price and wealth levels as well, but only if the initial price and the initial

wealth of agents are given. In addition to proving the existence of a well-defined map,[9] Proposition 4.1 also shows that when short positions are forbidden, agents' wealth shares are bounded between 0 and 1. This makes sense because only short investment positions can give rise to a negative wealth for some agents. Throughout the remaining part of this paper we impose the no-short-selling condition (4.3).

4.1 Location of steady-state equilibria

In a steady-state, aggregate economic variables, such as price returns and dividend yields, are constant and will be denoted by k^* and y^*, respectively.[10] Every steady-state has also constant investment shares $(x_1^*, \ldots, x_n^*)$, and wealth distribution $(\varphi_1^*, \ldots, \varphi_n^*)$. Concerning the latter we introduce the following definition.

Definition 4.1. In a steady-state equilibrium $(x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_N^*; k^*; y^*)$ an agent n is said to *survive* if his wealth share is strictly positive, $\varphi_n^* > 0$.

In every state of the economy there exists at least one survivor. In a steady-state with M surviving agents ($1 \leq M \leq N$) we will always assume that the first M agents are those who survive.

The characterization of all possible steady-states of the dynamical system defined on the set $\mathcal{D}$ is given below.

Proposition 4.2. *Steady-state equilibria of the dynamical system* (4.2) *evolving on the set* $\mathcal{D}$ *exist only when the dividend growth rate* g *is larger than the interest rate* r_f.

Let $g > r_f$ *and let* $(x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_N^*; k^*; y^*)$ *be a steady-state of* (4.2). *Then:*

- *The steady-state price return is equal to the growth rate of dividends,* $k^* = g$;
- *All surviving agents have the same investment share* $x_\diamond^*$, *which together with the steady-state dividend yield* y^* *satisfy*

$$x_\diamond^* = \frac{g - r_f}{y^* + g - r_f}\,. \tag{4.6}$$

- *The steady-state wealth shares satisfy*

$$\begin{cases} \varphi_m^* \in (0,1] & \text{if } m \leq M \\ \varphi_m^* = 0 & \text{if } m > M \end{cases} \qquad \text{and} \qquad \sum_{m=1}^{M} \varphi_m^* = 1\,. \tag{4.7}$$

Proof. See Appendix B. □

We have established that a steady-state can only exist when $g > r_f$. This result begs a question of what happens in the opposite case, when the dividend growth rate is smaller than r_f. It turns out that when $g < r_f$ the dynamics converges to a point where $y^* = 0$ which, formally, does not belong to the domain $\mathcal{D}$ defined in (4.5). We postpone the formal analysis of these situations, which we have already encountered in the examples of

[9] With some abuse of language we usually refer to (4.2) and not to the first order map $\mathcal{T}$ in (4.4) as "the dynamical system". The explicit map $\mathcal{T}$ will be used only in the proofs.

[10] Notice that in the steady-state all L lagged values of the return and yield, which were introduced to define the evolution operator $\mathcal{T}$, are equal to their contemporaneous values, i.e. to k^* and y^*.

Section 3, to Section 4.3. For the moment just assume that $g > r_f$. Then many situations are possible, including the cases with no steady-state, with multiple steady-states, and with different number of survivors in the same steady-state.

Above all notice that in all the steady-state equilibria the price grows at the same rate as the dividend, which is a consequence of requiring a constant dividend yield. In contrast, the steady-state level of dividend yield depends on agent behaviors. Proposition 4.2 implies that the dividend yield, y^*, and the investment share of survivors, $x^*_\diamond$, are determined simultaneously by (4.6). At the same time, the investment share $x^*_\diamond$ is given by the value of the agent's investment function at the steady-state. Using the Equilibrium Market Curve (EMC) $l(y)$ defined in (3.3), equation (4.6) can be rewritten as the following system of M equations

$$l(y^*) = f_m\big(g, \dots, g; y^*, \dots, y^*\big) \qquad \forall\, 1 \le m \le M\,. \tag{4.8}$$

The right hand-side of this relation represents the Equilibrium Investment Function (EIF) of survivor m, i.e. the cross-section of his investment function by the set

$$\big\{k_t = k_{t-1} = \cdots = k_{t-L+1} = g; \quad y_{t+1} = y_t = \cdots = y_{t-L+1} = y\big\}\,. \tag{4.9}$$

In Fig. 1 we have already shown how the condition (4.8) can be expressed graphically. Namely, all possible pairs $(y^*, x^*_\diamond)$ can be found as the intersections of the EMC with each EIF. The EMC-plot is an useful graphical tool that can be used to illustrate different long-run market dynamics. It is possible that the dynamics do not possess any steady-state (if no investment function intersects the EMC). It is also possible that the system possesses multiple steady-states with different levels of dividend yield, as well as multiple steady-states with the same level of yield. Despite encompassing quite a rich range of possibilities, the EMC-plot also shows that the often heard conjecture that, in the world of heterogeneous agents, "anything goes" is not necessarily valid. Even when the strategies of agents are generic, as in our framework, the market plays its role in shaping the aggregate outcome. The steady-states of system (4.2) can lie only on the EMC, which is a small subset of the original domain. The shape of the EMC is entirely determined by the exogenous parameters of the model, as g and r_f, and does not depend on agents' behaviors.

To complete the discussion of the steady-states on the EMC curve, notice that "more aggressive" behavior in the steady-state (i.e. investment of larger wealth fraction to the risky asset) implies smaller dividend yield. More aggressive behavior pushes the demand up, and leads to larger price level, which decreases the dividend yield. Does it imply that aggressive behavior is harmful for the investor's return? Rather surprisingly, the answer is no. From (4.6) the steady-state excess return is given by

$$g + y^* - r_f = \frac{g - r_f}{x^*_\diamond}\,.$$

Even if it decreases in the investment level, the agent's return is given by multiple x^* of it. This implies

Corollary 4.1. *At any steady-state equilibrium described in Proposition 4.2, the wealth return of each surviving agent is equal to g.*

Thus, even if different steady-states have different survivors and different yields, all of them are welfare-equivalent. Corollary 4.1 sheds light on the nature of the EMC. It is nothing but the locus of yield-investment share points where wealth returns, which are a convex combination of risky and risk-less returns, are all equal to g. Were steady-states not located on the EMC then total wealth would grow at an arbitrary different rate, which is not possible since the inflow of money in the economy is exogenously fixed and equal to $g = \max(g, r_f)$.

4.2 Stability analysis

We turn now to the characterization of the local stability conditions for the steady-states found in Proposition 4.2. At this purpose we assume that all the investment functions entering into the dynamics (4.2) are differentiable at their steady-states.

4.2.1 Wealth-driven selection

The steady-states identified in Proposition 4.2 are characterized by a positive excess return. It allows the market to play the role of a natural selecting force. In fact, trading rewards some agents at the expense of others, shaping in this way the long-run wealth distribution. The first part of our stability analysis focuses on this "natural selection" mechanism. The following general result holds.

Proposition 4.3. *Consider the steady-state equilibrium* $\left(x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_N^*; k^*; y^*\right)$ *described in Proposition 4.2, where the first* M *agents survive and invest* $x_\diamond^*$*. It is (locally) stable if the following two conditions are met:*

1) *the investment shares of the non-surviving agents are such that*

$$x_m^* < x_\diamond^* \qquad \forall m \in \{M+1, \ldots, N\}\,. \tag{4.10}$$

2) *the steady-state* $\left(x_1^* \equiv x_\diamond^*, \ldots, x_M^* \equiv x_\diamond^*; \varphi_1^*, \ldots, \varphi_M^*; k^*; y^*\right)$ *of the reduced system, obtained by elimination of all the non-surviving agents from the economy, is locally stable.*

The system generically exhibits a fold *bifurcation when the rightmost inequality in* (4.10) *becomes an equality, and it exhibits a* flip *bifurcation if the leftmost inequality in* (4.10) *becomes an equality.*

Proof. To derive the stability conditions, the $(2N + 2L) \times (2N + 2L)$ Jacobian matrix of the system has to be computed and evaluated at the steady-state[11]. For the stability of the system, the eigenvalues of this Jacobian should be inside the unit circle. In Appendix C we show that condition ***1)*** is necessary and sufficient to guarantee that M eigenvalues of the Jacobian matrix lie inside the unit circle. Among the other eigenvalues there will be M zeros. Finally, all the remaining eigenvalues can be derived from the Jacobian associated with the "reduced" dynamical system, i.e. without non-surviving agents, evaluated in the steady-state. This implies condition ***2)***. □

This proposition gives an important necessary condition for stability. Namely, the investment shares of the non-surviving agents must satisfy (4.10). In other words, the survivors should invest more in the risky asset than those who do not survive. We have illustrated this fact in the left panel of Fig. 2. At the stable equilibrium the investment shares of the non-surviving agents should belong to the gray area, i.e. lie below the investment shares of survivors. Given the investment function of I and II, only A could possibly be stable.

FIGURE 2 IS ABOUT HERE.

[11] General references on the modern treatment of stability and bifurcation theory in discrete-time dynamical systems are Medio and Lines (2001) and Kuznetsov (2004).

4.2.2 Stability of equilibria with survivors

According to condition **2)** in Proposition 4.3, when (4.10) is satisfied, the non-survivors can be eliminated from the market. The dynamics can then be described by the reduced system, that is, the same system (4.2) but with only M agents, all investing the same, as if there is a representative investor. When is the corresponding steady-state equilibrium stable? The general answer to this question is quite complicated, because the stability depends upon the behavior of the survivors in a small neighborhood of the steady-state, i.e. on the slopes of their investment functions. More precisely, we shall see that the stability is determined by the average values of partial derivatives of agent investment functions weighted by the agent equilibrium wealth shares.

Consider the steady-state $\left(x_1^*, \ldots, x_M^*; \varphi_1^*, \ldots, \varphi_M^*; k^*; y^*\right)$ and denote the vector of lagged returns and yields as $\boldsymbol{e^*} = (k^*, \ldots, k^*; y^*, \ldots, y^*)$. In the notation below index $m = 1, \ldots, N$ and index $l = 0, \ldots, L-1$. Denote the derivative of the investment function f_m with respect to the contemporaneous dividend yield as f_m^Y, the derivative with respect to the dividend yield of lag $l+1$ as f^{y_l}, and the derivative with respect to the price return of lag $l+1$ as f^{k_l}. Furthermore, consistently with our previous notation, name

$$\left\langle f^Y \right\rangle = \sum_{m=1}^{M} \varphi_m^* f_m^Y(\boldsymbol{e^*}), \quad \left\langle f^{y_l} \right\rangle = \sum_{m=1}^{M} \varphi_m^* f_m^{y_l}(\boldsymbol{e^*}), \quad \left\langle f^{k_l} \right\rangle = \sum_{m=1}^{M} \varphi_m^* f_m^{k_l}(\boldsymbol{e^*}),$$

which are the weighted derivatives of the investment functions evaluated at the steady-state. Finally, $l'(y^*)$ is the slope of the EMC at the steady-state equilibrium y^*. The next result reduces the stability problem to the exploration of the roots of a certain polynomial.

Proposition 4.4. *The steady-state $\left(x_1^*, \ldots, x_M^*; \varphi_1^*, \ldots, \varphi_M^*; k^*; y^*\right)$, described in Proposition 4.2, with M survivors is locally stable if all the roots of the polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{1}{l'(y^*)}\left(\left\langle f^Y \right\rangle \mu^L + \sum_{l=0}^{L-1} \left\langle f^{y_l} \right\rangle \mu^{L-1-l} + (1-\mu)\frac{1+g}{y^*} \sum_{l=0}^{L-1} \left\langle f^{k_l} \right\rangle \mu^{L-1-l}\right) \quad (4.11)$$

lie inside the unit circle. If, in addition, only one agent survives, then the steady-state is locally asymptotically stable.

The steady-state is unstable if at least one of the roots of polynomial $Q(\mu)$ is outside the unit circle.

Proof. See Appendix D. □

When the investment functions are specified, this proposition provides a definite answer to the question about stability of a given steady-state. One has to evaluate the polynomial (4.11) in this steady-state and compute (e.g. numerically) all its $L+1$ roots. Despite not providing explicit stability conditions, Proposition 4.4 reduces the complexity of the problem. In fact, whereas the characteristic polynomial of the dynamical system (4.2) has dimension $2N + 2L$, we are left to the analysis of a polynomial of degree $L+1$. Moreover, Proposition 4.4 allows us to get some insights about the determinants of stability. For example, when the investment functions of all survivors are neither responsive to a change in the yield nor to a change in the return (i.e. all the derivatives of the investment functions are 0 in the steady-state), the expression in parenthesis becomes zero and the stability condition is obviously satisfied. Using a continuity argument, this also implies that the steady-state is stable if the relative average slopes of the investment functions are small enough with respect to the slope of the EMC.

Recall that in the case with many survivors, there exists a set of steady-states corresponding to different distributions of wealth among survivors. Since the stability conditions depend on the partial derivatives of the investment functions weighted with the equilibrium relative wealth shares, some of the steady-states on the same manifold (i.e. with the same dividend yield and investment share) can be stable, while other can be unstable. Finally, notice that both polynomials (3.4) and (3.11) are the special cases of $Q(\mu)$.

4.2.3 An example with investment conditioned on the total return

Let us characterize the stability for a special case that is important for the applications developed in Section 5. Suppose that the investment functions depend on the average of past L total returns, given by the sum of price returns and dividend yields. For example investors might use this average as a forecast of future returns and invest a generic function of this forecast. Formally, assume that each individual investment share is given by

$$x_{n,t} = f_n\left(\frac{1}{L}\sum_{\tau=1}^{L}\left(y_{t-\tau}+k_{t-\tau}\right)\right). \tag{4.12}$$

Plugging in (4.11) and simplifying, it leads to

$$\tilde{Q}(\mu) = \mu^{L+1} - \frac{1+\mu+\cdots+\mu^{L-1}}{L}\Big(1+(1-\mu)\frac{1+g}{y^*}\Big)\frac{\sum_{m=1}^{M} f_m'(y^*+g)\varphi_m^*}{l'(y^*)}. \tag{4.13}$$

The only unknown part of the polynomial $\tilde{Q}(\mu)$ is its last fraction, which is relative slope between the survivor "average" investment function and the EMC, both evaluated at the at the steady-state. Its role in determining stability is still implicit. When $L=1$ this requirement can be made explicit.

Corollary 4.2. *Consider a steady-state of the system* (4.2) *with investment functions* (4.12) *and lag $L=1$, where all the non-survivors have been eliminated. The steady-state is locally stable if*

$$\frac{-y^*}{1+g+y^*} < \frac{\left\langle f'(y^*+g)\right\rangle}{l'(y^*)} < \frac{y^*}{y^*+2(1+g)}. \tag{4.14}$$

The steady-state generically exhibits flip *or* Neimark-Sacker *bifurcation if the right- or left-most inequality in* (4.14) *turns to equality, respectively.*

Proof. This follows from standard conditions for the roots of second-degree polynomial to be inside the unit circle. See appendix E for the details. □

Conditions (4.14) are illustrated in the right panel of Fig. 2 in the coordinates $(y^*, \langle f'\rangle/l')$. The steady-state is stable if the corresponding point belongs to the dark-grey area. From the diagram it is clear that the dynamics are stable for a low (in absolute value) relative slope $\langle f'\rangle/l'$ at the steady-state.

How does the stability depend on the memory span L? A mixture of analytic and numeric tools help to reveal the behavior of the roots of polynomial (4.13) with higher L. The stability conditions for $L=2$, derived in Appendix E, can be confronted with the $L=1$ case, see the right panel of Fig. 2. An increase of the memory span L from 1 to 2 enlarges the stability region. This result comes at no surprise, since as agents look further back, any recent shock in price or return gets smaller impact on their behavior. For further increase of the memory span the following result holds

Corollary 4.3. *Consider a steady-state of the system* (4.2) *with investment functions* (4.12)*, where all the non-survivors have been eliminated. Provided that*

$$\frac{\big\langle f'(y^*+g)\big\rangle}{l'(y^*)} < 1\,, \tag{4.15}$$

the corresponding steady-state is locally stable for high enough L.

To summarize, if L is finite and low, the system can be stabilized by decreasing the average slope of survivor investment functions with respect to the slope of the EMC. Furthermore, if inequality (4.15) holds, an increase of memory span always stabilizes the system.

4.3 Zero-yield equilibria

So far we have dealt with economies where dividends grow faster than the risk-free rate r_f. In fact, according to Proposition 4.2, only in this case there exist steady-states equilibria. One may wonder what happens when dividends grow, on average, more slowly than r_f. We shall see that in this case prices tend to grow faster than dividends, so that the dividend yield goes to zero. Formally however $y = 0$ cannot be a point of our domain, because dividends and prices are always positive. It is now clear that the reason why in Proposition 4.2 we did not find any steady-state when $g < r_f$ is very simple. Since the domain $\mathcal{D}$ given in (4.5) is not a closed set, the dynamics can easily converge to a point which at its boundary, as $y = 0$.

Let us, therefore, extend our formal analysis of the dynamics on the set

$$\mathcal{D}' = (0,1)^N \times \Delta_N \times [-1,\infty)^L \times [0,\infty)^L\,. \tag{4.16}$$

It turns out that (4.2) has a well defined dynamics also on $\mathcal{D}'$. In this way we are able to characterize possible *asymptotical* converge to a steady-state equilibrium with zero dividend yield. The next result applies.

Proposition 4.5. *Consider the dynamical system* (4.2) *evolving on the set $\mathcal{D}'$ introduced in* (4.16) *and assume that the no-short selling constraint* (4.3) *is satisfied. Apart from the steady-state equilibria described in Proposition 4.2, the system has other steady-states equilibria $(x_1^*,\dots,x_N^*;\varphi_1^*,\dots,\varphi_N^*;k^*;y^*)$ where:*

- *The price return is equal to the risk-free rate, $k^* = r_f$.*
- *The dividend yield is zero, $y^* = 0$.*
- *The wealth shares satisfy*

$$\begin{cases} \varphi_m^* \in (0,1] & \text{if } m \le M \\ \varphi_m^* = 0 & \text{if } m > M \end{cases} \qquad \text{and} \qquad \sum_{m=1}^{N} \varphi_m^* = 1\,. \tag{4.17}$$

Proof. See Appendix F. □

Contrary to the steady-state equilibria with positive dividend yield, in the steady-states derived in Proposition 4.5, the total return of the asset, $k^* + y^*$, coincides with r_f.

Corollary 4.4. *At any steady-state equilibrium with $y^* = 0$ as found in Proposition 4.5, the wealth return of each agent is equal to r_f.*

At these steady-states, therefore, there is no difference between the return on investment of the risky and risk-less asset. As a result all investment strategies are equivalent and any wealth distribution can be a steady-state. As opposed to the steady-states with positive yield survivors may behave differently, i.e. homogeneous behavior is not necessary but, similarly, the dividend and price dynamics is set by the representative investor.

The local stability of the steady-state equilibria with zero dividend yield can be analyzed along the same lines of Proposition 4.3 and leads to

Proposition 4.6. *Steady states of the dynamical system* (4.2) *evolving on the set* $\mathcal{D}'$ *can be stable only if* $g < r_f$.

Let $g < r_f$ *and let* $\left(x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_N^*; k^*; y^*\right)$ *be a fixed point of* (4.2) *with* $y^* = 0$. *This point is locally stable if all the roots of the polynomial*

$$Q_0(\mu) = \mu^{L+1} + \frac{1+r_f}{\left\langle x^*(1-x^*)\right\rangle}(1-\mu)\sum\nolimits_{l=0}^{L-1}\left\langle f^{k_l}\right\rangle \mu^{L-1-l} \tag{4.18}$$

lie inside the unit circle.

The steady-state is unstable if at least one of the roots of the polynomial $Q_0(\mu)$ *is outside the unit circle.*

Proof. See Appendix G. □

In the special cases of a one or two agents, the polynomial $Q_0(\mu)$ simplifies to (3.5) and (3.12), respectively.

5 An example with mean-variance optimizers

In this section we analyze the co-evolution of price returns, dividend yields and wealth shares for a specific set of agents' investment behaviors. We consider agents who are allocating wealth between the risky and the risk-less asset as to maximize a one period ahead mean-variance utility. We assume that agents' expectations of future variables are computed using averages of past observations, as in Section 4.2.3. Agents are heterogeneous with respect to memory span and risk aversion.

We proceed along two lines. First, we perform simulations of the system when the growth rate of dividends is stochastic and explain them using results from 4. Second, we apply the same tools to the analysis of a related model, the LLS model introduced in Levy et al. (1994). As a result, we are able to resolve some puzzles put forward by Zschischang and Lux (2001) regarding the interplay between risk aversion and memory span in the simulations of the LLS model.

5.1 Mean-variance optimizers

Throughout this section it is assumed that the growth rates of the dividends is

$$g_t = (1+g)\eta_t - 1\,,$$

where $\log(\eta_t)$ are i.i.d. normal random variables with mean 0 and variance σ_g^2. Assumption 2 on the dividend process is satisfied, and in the deterministic skeleton the dividend grows with rate g. For the moment let us further assume that $g > r_f$.

Agents maximize the mean-variance utility of the next period total return

$$U = E_t[x_t(k_{t+1} + y_{t+1}) + (1 - x_t)r_f] - \frac{\gamma}{2} V_t[x_t(k_{t+1} + y_{t+1})]\,,$$

where E_t and V_t denote, respectively, the mean and the variance conditional on the information available at time t, and γ is the coefficient of risk aversion. Assuming constant expected variance $V_t = \sigma^2$, the investment fraction which maximizes U is

$$x_t = \frac{E_t[k_{t+1} + y_{t+1} - r_f]}{\gamma\sigma^2}\,. \tag{5.1}$$

Agents estimate the next period return as the average of L past realized returns. Following condition (4.3), i.e. forbidding short positions, we bound the investment shares in the interval $[0.01, 0.99]$. Summing up, an investment function with parameters $\alpha = \gamma\sigma^2$ and L is

$$f_{\alpha,L} = \min\left\{0.99, \max\left\{0.01, \frac{1}{\alpha}\Big(\frac{1}{L}\sum\nolimits_{\tau=1}^{L}(k_{t-\tau} + y_{t-\tau}) - r_f\Big)\right\}\right\}\,, \tag{5.2}$$

where α is the "normalized" risk aversion and L is the memory span. These are the two parameters that differ across agents.

FIGURE 3 IS ABOUT HERE.

To derive all possible equilibria we fix $\sigma_g^2 = 0$ and turn to the deterministic skeleton. Let the price return k^* be equal to g, and consider the EIF derived from (5.2)

$$\tilde{f}_\alpha(y) = \min\left\{0.99, \max\left\{0.01, \frac{y + g - r_f}{\alpha}\right\}\right\}\,. \tag{5.3}$$

This is shown by the thin curve on the bottom-right panel of Fig. 3. All the steady-state equilibria can be found as the intersections of the EIF with the EMC. Notice that (5.3) does not depend on L, that is the memory span does not influence the location of the steady-states. Geometrically, all the multi-dimensional investment functions differed only in L collapse onto the same EIF. Analytically the steady-state equilibria with a single investor can be derived from Proposition 4.2. We obtain

Corollary 5.1. *Consider the system* (4.2) *with* $g > r_f$ *and with a single agent investing according to* (5.2)*. There exists a unique steady-state equilibrium* (x^*, k^*, y^*) *and it is characterized by* $k^* = g$ *and* $A_\alpha = (y^*, x^*)$ *with:*

$$y^* = \sqrt{\alpha\,(g - r_f)} - (g - r_f), \quad x^* = \sqrt{\frac{g - r_f}{\alpha}}\,. \tag{5.4}$$

The bottom-right panel of Fig. 3 illustrates this result. The market has a unique steady-state, A_α, whose abscissa, y^*, is the dividend yield, and whose ordinate, x^*, is the investment share. The position of this steady-state depends on the (normalized) risk aversion coefficient α. It is immediate to see that when α increases, the line $x = (y+g-r_f)/\alpha$ rotates clockwise, so that the steady-state dividend yield increases, while the investment share decreases. Eq. (5.4) confirms it, as $\partial y^*/\partial\alpha > 0$ and $\partial x^*/\partial\alpha < 0$.

What are the determinants of stability of the steady-state equilibrium A_α? First of all, notice that we can apply the stability analysis of Section 4.2.3, because the investment function $f_{\alpha,L}$ is of the type specified in (4.12). The stability, therefore, is determined both by the memory span L and by the ratio of the slopes of the function $\tilde{f}_\alpha$ and the EMC in the point A_α. Straight-forward computations show that this ratio does not depend on α and it is always equal to -1. Corollary 4.3 then implies that for any given normalized risk aversion α the dynamics stabilizes with high enough memory span L.

To confirm that these results are applicable also to a stochastic system, we simulate the model with investment function $\tilde{f}_\alpha$ and a stochastic dividend process. We plot the resulting dynamics in Fig. 3. The top-right panel shows the realization of the exogenous dividend process. Given this process, simulations are performed for investment strategies with the same level of the risk aversion $\alpha = 2$ and two different memory spans. The left panels show the price dynamics (top) and investment shares (bottom). When the memory span is $L = 10$ (solid line), the steady-state is unstable and prices fluctuate. These endogenous fluctuations are determined by the upper and lower bounds of the investment function and are much more pronounced than the fluctuations of the exogenous dividend process. When the memory span is increased to $L = 20$ (dotted line), the system converges to the stable steady-state equilibrium and observed fluctuations are only due to exogenous noise affecting the dividend growth rate.

FIGURE 4 IS ABOUT HERE.

Being particularly interested in assessing the effect of wealth-driven selection, we turn now to the analysis of a market with many agents. The bottom-right panel of Fig. 4 shows investment functions (5.3) for two different values of risk aversion, α and $\alpha' < \alpha$. According to Proposition 4.3, the survivor should have the highest investment share at the steady state. Since at y^*_α the agent with high risk aversion, α, invests less than the agent with low risk aversion, α', he cannot dominate the market. As a result the steady-state A_α is unstable. Whether the less risk averse agent can dominate the market depends on the stability of the second steady-state, $A_{\alpha'}$, that is on his memory span. If the memory span is high enough, the steady-state $A_{\alpha'}$ is stable and the less risk averse agent dominates the market.

Fig. 4 shows the market dynamics when one agent has risk aversion $\alpha = 2$ and memory $L = 20$ (which produces a stable dynamics in a single agent case, cf. Fig. 3), and the other agent has risk aversion $\alpha' = 1$ and memory L'. Simulations for two different values of the memory span L' are compared. When the memory span of the less risk averse agent is low, $L' = 20$, the steady-state $A_{\alpha'}$ is unstable (see dotted lines). Wealth share of both agents keep fluctuating between zero and one. However, when the memory of the less risk averse agent increases to $L' = 30$, the new steady-state $A_{\alpha'}$ is stabilized and he ultimately dominates the market (solid lines). The steady-state return now converges, on average, to $g + y^*_{\alpha'} < g + y^*_\alpha$. Interestingly, in our framework low risk aversion leads to market dominance at the cost of lowering the market return. In fact, the agent with a lower risk aversion dominates the market, but produces lower equilibrium yield by investing a higher wealth share in the risky asset. However, as stated in Corollary 4.1, the total wealth return is g, independently from the survivor investment strategy.

FIGURE 5 IS ABOUT HERE.

So far we have considered the case of $g > r_f$. What happens when $g < r_f$? We repeat the simulations we have just performed with an increase of r_f such that $g < r_f$. Fig. 5 shows market dynamics for a single agent with memory span either $L = 10$ or $L = 20$. It should be compared with Fig. 3. Whereas with $g > r_f$ the market is stable with long memory and unstable with high memory, with $g < r_f$ the market dynamics stabilizes no matter the value of L.[12] Moreover, the price grows at the constant rate r_f (top-left panel), no matter the exogenous fluctuations of the dividend process (top-right panel). Since the price grows faster than the dividend, the dividend yield converges to 0 (bottom-right panel). Notice also that at the steady-state the agent is investing a constant fraction of wealth equal to the lower bound of (5.2), i.e. $x^* = 0.01$ in this case (bottom-left panel).

FIGURE 6 IS ABOUT HERE.

Fig. 6 shows the market dynamics when two mean-variance optimizers, with different values of risk aversions, are active, and should be compared with Fig. 4. No matter the memory of the most aggressive agent, the price dynamics stabilizes (top-left panel). Prices grow at the constant rate r_f, despite the exogenous fluctuations of the dividend process. Since prices are growing faster than dividends, the dividend yield converges to 0 (top-right panel). At the steady-state both agents survive having positive wealth shares (bottom-left panel), and they both invest $x^* = 0.01$ (bottom-right panel).

Summarizing, when $g < r_f$, in accordance with Proposition 4.5, market returns are equal to the risk-free rate, independently from the noise of the dividend process and from the initial set of investment strategies. As a result, all the investor are gaining the same returns and the market is not selecting among them.

5.2 The LLS model revisited

The insights developed so far can be used to evaluate various simulations of the LLS model performed in Levy et al. (1994); Levy and Levy (1996); Levy et al. (2000); Zschischang and Lux (2001). In fact, as far as the co-evolution of prices and wealth, the demand specification, and the dividend process are concerned, the LLS model is built on the same framework as we consider in this paper.

In the LLS model, at period t investors maximize a power utility function $U(W_{t+1}, \gamma) = W_{t+1}^{1-\gamma}/(1-\gamma)$ with relative risk aversion $\gamma > 0$. Furthermore, to forecast the next period total return $z_{t+1} = k_{t+1} + y_{t+1}$, agents assume that any of the last L returns can occur with equal probability. Solutions of the maximization of a power utility are not available analytically but they have been shown to give a wealth independent investment shares. This property holds for any perceived distribution, $g(z)$, of the next period total return, which is discrete uniform in this case. Let us denote the corresponding investment function as $f^{EP}\big(\gamma, g(z)\big)$, where EP stands for Expected Power. As this investment function is unavailable in explicit form, the analysis of the LLS model relies on numeric solutions.

Since our results in Section 4 are valid for any functional form of the investment function, we are able to give an analytic support to the LLS model. In the example with mean-variance maximizers, we have seen that the risk aversion determines the capability of agents

[12] By applying Proposition 4.6 to the investment function f in (5.2), and noticing that, due to the lower bound, the investment function is always flat at $y^* = 0$, it can be shown that the steady-state $y^* = 0$ is stable for any value of L.

to invade the market, whereas the memory span influences the stability of the dynamics. These properties hold as long as the EIF on the EMC-plot shifts upward following a decrease in the risk aversion. As the following result shows (see Anufriev, 2008 for a proof), the function $f^{EP}\big(\gamma, g(z)\big)$ has this property.

Proposition 5.1. *Let f^{EP}_{γ} stand for the partial derivative of the investment function f^{EP} with respect to the risk aversion coefficient γ, and $\bar{z}$ for the expected value of the total return. Then the following result holds:*

$$\textit{If} \quad \bar{z} \gtreqless 0\,, \quad \textit{then} \quad f^{EP} \gtreqless 0 \quad \textit{and} \quad f^{EP}_{\gamma} \lesseqgtr 0 \quad .$$

In our setting, when a positive return is expected, agents with lower risk aversion invest higher share. Having said this we can use Propositions 4.3 and 4.4 to provide rigorous analytic support to the simulation results of the LLS model.

In Levy and Levy (1996) the focus is on the role of memory. The authors show that with a small memory span the log-price dynamics is characterized by crashes and booms. Our analysis shows that this result is due to the presence of an unstable steady-state and to the upper and lower bounds of the investment shares. We also show that this steady-state becomes stable if the memory is high enough. Simulations in Levy and Levy (1996) confirm this statement; when agents with higher memory are introduced, booms and crashes disappear and price fluctuations become erratic. But these are the fluctuations which are mainly due to the exogenous noise of the dividend process, and not to agents' interactions.

Some other simulations in Levy and Levy (1996) are performed with positive risk-free rate and zero dividend growth rate (i.e. $g = 0$). These simulations do converge, irrespectively of the noisy dividend. To understand why and where they converge, recall our analysis for the case $g < r_f$. We have shown that prices are always growing at the rate r_f, no matter the initial set of investment strategies, and the dividend yield converges to $y^* = 0$. As a result wealth return is r_f for any investment strategy, and no selection on the set of investment strategies occurs. This is exactly what happens in the simulations.

In Zschischang and Lux (2001) the focus is on the interplay between the length of the memory span and the risk aversion. Their simulations suggest that the risk aversion is more important than the memory span in the determination of the dominating agents, providing that the memory is not too short. The argument has not been put forward in a decisive way though, as the following quote from Zschischang and Lux (2001) (p. 568, 569) shows:

> "Looking more systematically at the interplay of risk aversion and memory span, it seems to us that the former is the more relevant factor, as with different [risk aversion coefficients] we frequently found a reversal in the dominance pattern: groups which were fading away before became dominant when we reduced their degree of risk aversion. [...] It also appears that when adding different degrees of risk aversion, the differences of time horizons are not decisive any more, provided the time horizon is not too short."

Our analytic results make clear how and why this is the case. Agents with low risk aversion are indeed able to destabilize the market populated by agents with high risk aversion. However, this "invasion" leads to an ultimate domination only if the invading agents have sufficiently long memory. Otherwise, and this complements the conclusions of Zschischang and Lux (2001) and related works, agents with different risk aversion coefficients will coexist.

Another new result concerns the case of agents investing a constant fraction of wealth. In Zschischang and Lux (2001) the authors claim that such agents always dominate the market and add (p. 571):

> "Hence, the survival of such strategies in real-life markets remains a puzzle within the Levy, Levy and Solomon microscopic simulation framework as it does within the Efficient Market Theory."

Our analysis allows to make this statement more precise. The agents with constant investment fraction are characterized by the horizontal investment functions, for which Proposition 4.4 guarantees stability, independently of L. If these agents are able to invade the market successfully, they will ultimately dominate. However, their market invasion will fail, as soon as other agents are more aggressive in the steady-states created by invaders.

6 Conclusion

In his recent survey, LeBaron (2006) stresses that agent-based models do not require analytical tractability (as opposed to Heterogeneous Agents Models) and, therefore, are more flexible and realistic for what concerns their assumptions. In this paper we show that flexibility can be achieved in an analytically-tractable heterogeneous agent framework too. In fact, we have performed an analytic investigation of a stylized model of a financial market where an arbitrary set of investors is trading. Under the assumption that the impact of different agents on the market depends on their wealth shares, we have derived existence and stability results for a general set of investment functions and an arbitrary number of agents. Due to the selecting role of wealth dynamics, the steady-state equilibrium asset return and agent survival can be characterized by looking at the intersection of each agent Equilibrium Investment Function with the so called Equilibrium Market Curve. We have also shown that our analysis of a deterministic market dynamics is helpful for the understanding of its simulations with a stochastic dividend process.

Our approach of dealing with general CRRA behavior owes to the work of Anufriev and Bottazzi (2006). Assuming a more realistic dividend process, namely exogenous growth, we have reached two research objectives. First, we have been able to investigate which features of their results are due to their constant dividend yield assumption and which are of a more general nature. Second, we have provided an analytical support of the LLS simulations, which would have not be possible within the framework of Anufriev and Bottazzi. As for our first objective, we have shown that the specification of the dividend process does play a role in shaping the price and dividend dynamics. In our framework, the prices grow at a rate that is derived from the exogenous parameters, while the ecology of trading behaviors can affect only the dividend yield. In Anufriev and Bottazzi (2006), on the other hand, the dividend yield is fixed and the ecology of behaviors affects the growth rate of prices and, consequently, dividends. At the same time the common CRRA framework, with a wealth-driven selection and a coupled price-wealth dynamics, is responsible for similarities such as the existence of a low-dimensional locus of possible steady-states, i.e. Equilibrium Market Curve (though of different shapes).

On the way to our second objective, we have considered an example with mean variance maximizers characterized by two parameters: degree of risk aversion and memory span used to estimate future returns. When the growth rate of dividends is bigger than the risk-free rate, the agents with the lowest risk aversion dominate the market, provided that their memory

spans are big enough. If it is so, the market dynamics converge to the stable steady-state equilibrium, where prices are growing as fast as the dividends and the dividend yield increases with the risk aversion. In this case price fluctuations are due to the to the exogenous fluctuations of dividends. Otherwise, when the memory is not high enough, agents with different investment strategies coexist and the price fluctuations are endogenously determined. When, instead, the growth rate of dividends is smaller than the risk-free rate, steady-state equilibrium asset returns are equal to risk-free returns and the dividend yield converges to zero, no matter the ecology of agents. As a result wealths returns are equal for all investment functions and there is no selection. We have also explained that these differences in the market dynamics and selecting regime are due to the exogenous inflow of wealth in the economy, both through dividends and risk-less returns. Finally, due to the generality of our approach, we have extended these results to the simulation study of Levy et al. (1994, 2000), and have resolved a number of issues concerning the interplay between risk aversion and memory span as reported in Zschischang and Lux (2001).

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Appendix

A Dynamical System defined in Proposition 4.1

After Proposition 4.1 we have shown that the system of equations in (4.2) leads to the well-defined map from the domain $\mathcal{D}$, specified in (4.5), to itself. Here we explicitly provide the evolution operator of the first-order dynamical system of $2N + 2L$ variables. We use the following notation for time t variables

$$x_{n,t}\,,\ \varphi_{n,t} \quad \forall n \in \{1,\dots,N\} \qquad \text{and} \qquad k_{t,l}\,,\ y_{t,l} \quad \forall l \in \{0,\dots,L-1\}\,, \tag{A.1}$$

where $k_{t,l}$ and $y_{t,l}$ denote the price return and the dividend yield at time $t-l$, respectively. We order the equations in four separated blocks: $\mathcal{X}$, $\mathcal{W}$, $\mathcal{K}$ and $\mathcal{Y}$. They define, respectively, N investment choices, N wealth shares, L price returns and L dividend yields. The last two blocks are needed to update the lagged variables. The map $\mathcal{T}$ referred in (4.4) is given by

$$\begin{aligned}
\mathcal{X}: &\left[\begin{array}{rcl}
x_{1,t+1} &=& f_1\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big)\\
\vdots &\vdots& \vdots\\
x_{N,t+1} &=& f_N\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big)
\end{array}\right.\\
\mathcal{W}: &\left[\begin{array}{rcl}
\varphi_{1,t+1} &=& \Phi_1\Big(x_{1,t},\dots,x_{N,t};\varphi_{1,t},\dots,\varphi_{N,t};Y(y_{t,0},k_{t,0});\\
&& \quad K\Big[f_1\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big),\dots,\\
&& \quad f_N\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big);\\
&& \quad x_{1,t},\dots,x_{N,t};\varphi_{1,t},\dots,\varphi_{N,t};Y(y_{t,0},k_{t,0})\Big]\Big)\\
\vdots &\vdots& \vdots\\
\varphi_{N,t+1} &=& \Phi_N\Big(x_{1,t},\dots,x_{N,t};\varphi_{1,t},\dots,\varphi_{N,t};Y(y_{t,0},k_{t,0});\\
&& \quad K\Big[f_1\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big),\dots,\\
&& \quad f_N\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big);\\
&& \quad x_{1,t},\dots,x_{N,t};\varphi_{1,t},\dots,\varphi_{N,t};Y(y_{t,0},k_{t,0})\Big]\Big)
\end{array}\right.\\
\mathcal{K}: &\left[\begin{array}{rcl}
k_{t+1,0} &=& K\Big[f_1\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big),\dots,\\
&& \quad f_N\big(k_{t,0},\dots,k_{t,L-1};Y(y_{t,0},k_{t,0}),y_{t,0},\dots,y_{t,L-1}\big);\\
&& \quad x_{1,t},\dots,x_{N,t};\varphi_{1,t},\dots,\varphi_{N,t};Y(y_{t,0},k_{t,0})\Big]\\
k_{t+1,1} &=& k_{t,0}\\
\vdots &\vdots& \vdots\\
k_{t+1,L-1} &=& k_{t,L-2}
\end{array}\right.\\
\mathcal{Y}: &\left[\begin{array}{rcl}
y_{t+1,0} &=& Y(y_{t,0},k_{t,0})\\
y_{t+1,1} &=& y_{t,0}\\
\vdots &\vdots& \vdots\\
y_{t+1,L-1} &=& y_{t,L-2}\,,
\end{array}\right.
\end{aligned} \tag{A.2}$$

where the following three functions Y, K, and Φ_n have been introduced. The function

$$Y(y,k) = y\,\frac{1+g}{1+k} \tag{A.3}$$

gives the dividend yield as a function of past realization of the yield and return, as in the right-hand side of the fourth equation in (4.2). The function

$$K\Big[z_1,\dots,z_N;x_1,\dots,x_N;\varphi_1,\dots,\varphi_N;y\Big] = r_f + \frac{(1+r_f)\sum_{m=1}^N (z_m-x_m)\varphi_m + y\sum_{m=1}^N x_m z_m \varphi_m}{\sum_{m=1}^N x_m(1-z_m)\varphi_m} \tag{A.4}$$

gives the price return as a function of the investment choices, wealth shares and the dividend yield as in the right-hand side of the third equation in (4.2). Finally,

$$\Phi_n\big(x_1,\dots,x_N;\varphi_1,\dots,\varphi_N;y;k\big) = \varphi_n\,\frac{1+r_f+(k+y-r_f)x_n}{1+r_f+(k+y-r_f)\sum_{m=1}^N x_m\varphi_m} \qquad \forall n\in\{1,\dots,N\} \tag{A.5}$$

specifies the wealth share of agent n as a function of the investment choices, wealth shares, the dividend yield and price return[13] as in the right-hand side of the second equation in (4.2). □

B Proof of Proposition 4.2

To solve for the equilibrium of the system (4.2), one can substitute the time variables with equilibrium values and solve the resulting system for $(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; k^*; y^*)$. The system to be solved is as follows

$$\begin{cases} x_n^* = f_n\left(k^*, \dots, k^*; y^*, \dots, y^*\right), & \forall n \in \{1, \dots, N\}, \\ \varphi_n^* = \varphi_n^* \dfrac{(1+r_f) + (k^* + y^* - r_f)\, x_n^*}{(1+r_f) + (k^* + y^* - r_f)\left\langle x^* \right\rangle}, & \forall n \in \{1, \dots, N\}, \\ k^* = r_f + \dfrac{y^* \left\langle x^{*2} \right\rangle}{\left\langle x^* (1 - x^*) \right\rangle}, \\ y^* = y^* \dfrac{1+g}{1+k^*}. \end{cases} \tag{B.1}$$

Since y^* and investment shares are positive, from the third equation $k^* > r_f$, while the fourth equation fixes k^* to g. Thus, equilibria exist only when $g > r_f$. In particular, it means that $k^* + y^* - r_f > 0$. The equations for the wealth shares imply that every surviving agent invests $x_n^* = \left\langle x^* \right\rangle$, which is independent of n. Therefore, all the survivors invest the same share, $x_\diamond^*$. Plugging this share into the third equation, one gets (4.6). □

C Proof of Proposition 4.3

A fixed point of the system (A.2) will be denoted as

$$\boldsymbol{x^*} = \big(x_1^*, \dots, x_N^*; \varphi_1^*, \dots, \varphi_N^*; \underbrace{k^*, \dots, k^*}_{L}; \underbrace{y^*, \dots, y^*}_{L}\big).$$

To derive the stability conditions in different equilibria, the Jacobian matrix has to be computed. The Jacobian depends on the derivatives of the functions Y, K and Φ_n introduced in Appendix A. We compute now the derivatives of these functions with respect to different arguments and perform the straightforward computation in the fixed point $\boldsymbol{x^*}$. For the function Y introduced in (A.3) the derivatives are given by

$$Y^y = \frac{\partial Y}{\partial y} = \frac{1+g}{1+k^*}, \qquad Y^k = \frac{\partial Y}{\partial k} = -y^* \frac{1+g}{(1+k^*)^2}. \tag{C.1}$$

For the function K introduced in (A.4), for all $1 \le m \le N$, we have

$$\begin{aligned} K^{z_m} &= \frac{\partial K}{\partial z_m} = \varphi_m^* \frac{1 + r_f + (k^* + y^* - r_f) x_m^*}{\left\langle x^*(1 - x^*) \right\rangle}, \\ K^{x_m} &= \frac{\partial K}{\partial x_m} = \varphi_m^* \frac{-1 - k^* + (k^* + y^* - r_f) x_m^*}{\left\langle x^*(1 - x^*) \right\rangle}, \\ K^{\varphi_m} &= \frac{\partial K}{\partial \varphi_m} = x_m^* \frac{r_f - k^* + (k^* + y^* - r_f) x_m^*}{\left\langle x^*(1 - x^*) \right\rangle}, \\ K^y &= \frac{\partial K}{\partial y} = \frac{\left\langle x^{*2} \right\rangle}{\left\langle x^*(1 - x^*) \right\rangle}. \end{aligned} \tag{C.2}$$

[13] Notice that since the sum of the wealth shares is equal to 1 at any period, one of the equations in the system (e.g. the last equation of the block $\mathcal{W}$) is redundant and the dynamics can be fully described by the system of dimension $2N + 2L - 1$. However, the computations are more symmetric when the relation $\varphi_{N,t} = 1 - \sum_{m=1}^{N-1} \varphi_{m,t}$ is not taken into account explicitly.

Finally, for the function Φ_n introduced in (A.5) and for all $1 \le m \le N$, we have

$$\begin{aligned}
\Phi_n^{x_m} &= \frac{\partial \Phi_n}{\partial x_m} = (k^* + y^* - r_f)\, \frac{(\delta_n^m - \varphi_m^*)\varphi_n^*}{1 + r_f + (k^* + y^* - r_f)\langle x^* \rangle}\,, \\
\Phi_n^{\varphi_m} &= \frac{\partial \Phi_n}{\partial \varphi_m} = \frac{\delta_n^m (1 + r_f) + (k^* + y^* - r_f)(\delta_n^m x_n^* - \varphi_n^* x_m^*)}{1 + r_f + (k^* + y^* - r_f)\langle x^* \rangle}\,, \\
\Phi_n^{y} &= \frac{\partial \Phi_n}{\partial y} = \varphi_n^*\, \frac{x_n^* - \langle x^* \rangle}{1 + r_f + (k^* + y^* - r_f)\langle x^* \rangle}\,, \\
\Phi_n^{k} &= \frac{\partial \Phi_n}{\partial k} = \varphi_n^*\, \frac{x_n^* - \langle x^* \rangle}{1 + r_f + (k^* + y^* - r_f)\langle x^* \rangle}\,,
\end{aligned} \tag{C.3}$$

where δ_n^m is the Kronecker's delta. Using the block structure introduced in Appendix A, the Jacobian can be written in general form as:

$$\boldsymbol{J} = \begin{Vmatrix} \frac{\partial \mathcal{X}}{\partial \mathcal{X}} & \frac{\partial \mathcal{X}}{\partial \mathcal{W}} & \frac{\partial \mathcal{X}}{\partial \mathcal{K}} & \frac{\partial \mathcal{X}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{W}}{\partial \mathcal{X}} & \frac{\partial \mathcal{W}}{\partial \mathcal{W}} & \frac{\partial \mathcal{W}}{\partial \mathcal{K}} & \frac{\partial \mathcal{W}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{K}}{\partial \mathcal{X}} & \frac{\partial \mathcal{K}}{\partial \mathcal{W}} & \frac{\partial \mathcal{K}}{\partial \mathcal{K}} & \frac{\partial \mathcal{K}}{\partial \mathcal{Y}} \\ \frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{W}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{K}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Y}} \end{Vmatrix}\,. \tag{C.4}$$

The block $\partial\mathcal{X}/\partial\mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of the agents' present investment choices with respect to the agents' past investment choices. Since the investment choice of any agent does not explicitly depend on the investment choices in the previous period

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{X}}\right]_{n,m} = 0\,, \qquad 1 \le n, m \le N\,,$$

and this block is a zero matrix. The block $\partial\mathcal{X}/\partial\mathcal{W}$ is a $N \times N$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' wealth shares. This is also a zero matrix and

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{W}}\right]_{n,m} = 0\,, \qquad 1 \le n \le N\,, \quad 1 \le m \le N\,.$$

The block $\partial\mathcal{X}/\partial\mathcal{K}$ is a $N \times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past price returns. Let us introduce a special notation for partial derivatives of the investment functions:

$$\frac{\partial f_n}{\partial k_{t-l}} = f_n^{k_l}\,, \qquad \frac{\partial f_n}{\partial y_{t+1}} = f_n^{Y}\,, \qquad \frac{\partial f_n}{\partial y_{t-l}} = f_n^{y_l}\,, \qquad 1 \le n \le N\,, \quad 0 \le l \le L-1\,.$$

Then

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{K}}\right]_{n,l} = \begin{cases} f_n^{k_0} + f_n^{Y} \cdot Y^k & \text{for } l = 0 \text{ (the first column)} \\ f_n^{k_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{X}/\partial\mathcal{Y}$ is a $N \times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past dividend yield. This block is given by

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{Y}}\right]_{n,l} = \begin{cases} f_n^{y_0} + f_n^{Y} \cdot Y^y & \text{for } l = 0 \text{ (the first column)} \\ f_n^{y_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{W}/\partial\mathcal{X}$ is $N \times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' investment choices. It holds

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}}\right]_{n,m} = \Phi_n^{x_m} + \Phi_n^{k} \cdot K^{x_m}\,, \qquad 1 \le n, m \le N\,.$$

The block $\partial\mathcal{W}/\partial\mathcal{W}$ is a $N \times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' wealth shares. It holds

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} = \Phi_n^{\varphi_m} + \Phi_n^{k} \cdot K^{\varphi_m}\,, \qquad 1 \le n, m \le N\,.$$

The block $\partial\mathcal{W}/\partial\mathcal{K}$ is a $N \times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged returns. For $1 \le n \le N$ and $0 \le l \le L-1$, it reads

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{K}}\right]_{n,l} = \begin{cases} \Phi_n^k \cdot \left(\sum_m K^{z_m}(f_m^{k_0} + f_m^Y Y^k) + K^y Y^k\right) + \Phi_n^y \cdot Y^k\,, & \text{for } l = 0 \\ \Phi_n^k \cdot \sum_m K^{z_m} f_m^{k_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{W}/\partial\mathcal{Y}$ is a $N \times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged dividend yields. For $1 \le n \le N$ and $0 \le l \le L-1$, it reads

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{Y}}\right]_{n,l} = \begin{cases} \Phi_n^k \cdot \left(\sum_m K^{z_m}(f_m^{y_0} + f_m^Y Y^y) + K^y Y^y\right) + \Phi_n^y \cdot Y^y\,, & \text{for } l = 0 \\ \Phi_n^k \cdot \sum_m K^{z_m} f_m^{y_l} & \text{otherwise}\,. \end{cases}$$

The block $\partial\mathcal{K}/\partial\mathcal{X}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' investment choices. Its structure is simple, since only the first line can contain non-zero elements. It reads

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{X}}\right]_{l,n} = \begin{cases} K^{x_n} & \text{for } l = 0 \text{ (the first row)} \\ 0 & \text{otherwise} \end{cases}\,, \qquad 0 \le l \le L-1\,, \quad 1 \le n \le N\,.$$

The block $\partial\mathcal{K}/\partial\mathcal{W}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' wealth shares. It also has $L-1$ zero rows and reads

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{W}}\right]_{l,n} = \begin{cases} K^{\varphi_n} & \text{for } l = 0 \text{ (the first row)} \\ 0 & \text{otherwise} \end{cases}\,, \qquad 0 \le l \le L-1\,, \quad 1 \le n \le N\,.$$

The block $\partial\mathcal{K}/\partial\mathcal{K}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to themselves. It has a typical structure for such matrix with 1's under the main diagonal

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{K}}\right] = \begin{Vmatrix} \sum K^{z_m}(f_m^{k_0} + f_m^Y Y^k) + K^y Y^k & \sum K^{z_m} f_m^{k_1} & \dots & \sum K^{z_m} f_m^{k_{L-2}} & \sum K^{z_m} f_m^{k_{L-1}} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{Vmatrix}\,.$$

The block $\partial\mathcal{K}/\partial\mathcal{Y}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to the lagged dividend yields. It is given by

$$\left[\frac{\partial\mathcal{K}}{\partial\mathcal{Y}}\right] = \begin{Vmatrix} \sum K^{z_m}(f_m^{y_0} + f_m^Y Y^y) + K^y Y^y & \sum K^{z_m} f_m^{y_1} & \dots & \sum K^{z_m} f_m^{y_{L-2}} & \sum K^{z_m} f_m^{y_{L-1}} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{Vmatrix}\,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{X}$ is a $L \times N$ matrix containing the partial derivatives of the lagged dividend yields with respect to the agents' investment choices. This block is a zero matrix

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{X}}\right]_{l,m} = 0\,, \qquad 0 \le l \le L-1\,, \quad 1 \le m \le N\,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{W}$ is a $L \times N$ matrix containing the partial derivatives of the lagged dividend yields with respect to the agents' wealth shares. This is also a zero matrix and

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{W}}\right]_{l,m} = 0\,, \qquad 0 \le l \le L-1\,, \quad 1 \le m \le N\,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{K}$ is a $L \times L$ matrix containing the partial derivatives of the lagged dividend yields with respect to the past price returns. The only non-zero element of this matrix is in the upper left corner, i.e.

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{K}}\right]_{l,j} = \begin{cases} Y^k & \text{for } l = j = 0 \text{ (the first row, the first column)} \\ 0 & \text{otherwise}\,. \end{cases}$$

Finally, the block $\partial\mathcal{Y}/\partial\mathcal{Y}$ is a $L\times L$ matrix containing the partial derivatives of the lagged dividend yields with respect to themselves. This matrix is given by

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{Y}}\right] = \begin{Vmatrix} Y^y & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{Vmatrix} .$$

With all these definitions, one obtains the following statement about the Jacobian in the equilibria of the system.

Lemma C.1. *Let $\boldsymbol{x}^*$ be an equilibrium of system* (4.2) *described in Prop. 4.2 and let first M agents survive in this equilibrium. The corresponding Jacobian matrix, $\boldsymbol{J}(\boldsymbol{x}^*)$, has the following structure, where the actual values of non-zero elements, denoted by the symbols $\star$, are varying.*

$$\left\|\begin{array}{cccccc|cccccc|cccccc}
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & [& \frac{\partial\mathcal{X}}{\partial\mathcal{K}} & &] & [& & \frac{\partial\mathcal{X}}{\partial\mathcal{Y}} & &] \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
\hline
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
[& \Phi^x &] & \vdots & \ddots & \vdots & [& \Phi^{\varphi}_S &] & [& \Phi^{\varphi}_{NS} &] & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
\star & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{-y^*}{1+g} & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{array}\right\|$$

The solid lines divide the matrix in the same 16 *blocks as in* (C.4)*. In addition, the first and the second column-blocks as well as the second row-block are split into the two parts of sizes M and $N-M$, corresponding to the survivors and the non-survivors, respectively.*

Proof. Let us start with the first row-block having N rows. The first two blocks of columns in this block, $\partial\mathcal{X}/\partial\mathcal{X}$ and $\partial\mathcal{X}/\partial\mathcal{W}$, are always zero. Two other blocks, $\partial\mathcal{X}/\partial\mathcal{K}$ and $\partial\mathcal{X}/\partial\mathcal{Y}$, in general contain non-zero elements and simplified, because in the equilibrium $k^*=g$ and therefore $Y^y=1$ and $Y^k=-y^*/(1+g)$.

To simplify the second row-block, notice that $\Phi^k_n=\Phi^y_n=0$ in this equilibrium. Indeed, the numerators of the corresponding general expressions in (C.3) are 0, because all the survivors invest the same share in the equilibrium (i.e. $x^*_n=\langle x^*\rangle = x^*_\diamond$ for all $n\le M$), while for the non-survivors $\varphi^*_n=0$. This immediately implies that the two last blocks, $\partial\mathcal{W}/\partial\mathcal{K}$ and $\partial\mathcal{W}/\partial\mathcal{Y}$, contain only zero elements. Furthermore, from the Equilibrium Market Curve relation (4.6) we get in the equilibrium

$$1+r_f+(g+y^*-r_f)x^*_\diamond = 1+g\,. \tag{C.5}$$

Thus, in the equilibrium

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \Phi^{x_m}_n = \begin{cases} \varphi^*_n(\delta^m_n-\varphi^*_m)(y^*+g-r_f)/(1+g) & \text{for } n\le M \text{ (agent } n \text{ survives)} \\ 0 & \text{otherwise}\,, \end{cases}$$

and all the rows corresponding to the non-survivors in this block are zero rows. Moreover, all the columns corresponding to the non-survivors contain only zero elements as well, since then $\delta^m_n=\varphi^*_m=0$. We denote as Φ^x the remaining (non-zero) part of the block $\partial\mathcal{W}/\partial\mathcal{X}$.

The simplifications in the next block lead to

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} = \Phi_n^{\varphi_m} = \begin{cases} \delta_n^m - \varphi_n^*(g - r_f)/(1+g) & \text{for } n, m \le M \\ -\varphi_n^* x_m^*(y^* + g - r_f)/(1+g) & \text{for } n \le M, m > M \\ \delta_n^m\big(1 + r_f + x_n^*(y^* + g - r_f)\big)/(1+g) & \text{for } n > M\,. \end{cases} \tag{C.6}$$

The block of the elements from the first line of the previous expression is denoted as Φ_S^{φ}; the block of the elements of the second line is denoted as Φ_{NS}^{φ}; while the block of the elements from the third line (i.e. when $n > M$) only the diagonal elements are non-zero.

It is obvious that in the next row-block with L rows the elements are zero in all the lines but the first. The only exception from this rule are the elements below the main diagonal in the block $\partial\mathcal{K}/\partial\mathcal{K}$ which are all equal to 1. To compute the elements in the first row consider the derivatives of function K derived in (C.2). For the first block, $\partial\mathcal{K}/\partial\mathcal{X}$, notice that for the non-surviving agents $K^{x_m} = 0$, while for the survivors, i.e. for $m \le M$

$$K^{x_m} = -\varphi_m^* \frac{1 + r_f}{(1 - x_\diamond^*)x_\diamond^*}\,. \tag{C.7}$$

Analogously, in the next block, $\partial\mathcal{K}/\partial\mathcal{W}$, for all the survivors $K^{\varphi_m} = 0$, while for all other agents $m > M$ the elements are given by

$$K^{\varphi_m} = x_m^* \frac{r_f - g + x_m^*(y^* + g - r_f)}{(1 - x_\diamond^*)x_\diamond^*} = x_m^* \frac{(x_m^* - x_\diamond^*)(y^* + g - r_f)}{(1 - x_\diamond^*)x_\diamond^*}\,, \tag{C.8}$$

where (C.5) was used to derive the last equality.

The simplifications in the blocks $\partial\mathcal{K}/\partial\mathcal{K}$ and $\partial\mathcal{K}/\partial\mathcal{Y}$ are minor. Notice from (C.2) that the derivatives K^{z_m} for all the non-survivors are zeros, while for the survivors ($m \le M$) they are given by

$$K^{z_m} = \varphi_m^* \frac{1 + g}{(1 - x_\diamond^*)x_\diamond^*}\,. \tag{C.9}$$

Thus, all the sums in the first row of this block have to be taken only with respect to the surviving agents.

Finally, in the last row-block the simplifications are straight-forward. □

The rest of the proof of the Proposition is now clear. Consider the Jacobian matrix derived in Lemma C.1. The last $N - M$ columns of the left column-block contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N - M$. This eigenvalue does not affect stability. Moreover, these columns and the corresponding rows can be eliminated from the Jacobian. Analogously, in each of the last $N - M$ rows in the second row-block the only non-zero entries belong to the main diagonal. Consequently, $\Phi_n^{\varphi_n}$ for $n > M$ are the eigenvalues of the matrix, with multiplicity (at least) one, and the rows (together with the corresponding columns) can be eliminated from the Jacobian. Using the third line of (C.6) we get the following $N - M$ eigenvalues

$$\mu_n = \frac{1 + r_f + x_n^*(y^* + g - r_f)}{1 + g} = \frac{1 + r_f + (g - r_f)(x_n^*/x_\diamond^*)}{1 + g}$$

where the last equality follows from (C.5). Recall that the equilibria we consider, exist only when $g > r_f$. Then, with a bit of algebra, the stability conditions $-1 < \mu_n < 1$ can be simplified to conditions (4.10).

Finally, notice that the elimination of the rows and columns which we have performed reduce the Jacobian to the shape which correspond to the Jacobian of the same system in the same equilibrium but without non-surviving agents. □

D Proof of Proposition 4.4

Let us proceed with a reduced Jacobian obtained from the matrix in Lemma C.1 after eliminating the rows and columns corresponding to the survivors. We denote this Jacobian as $\boldsymbol{L}$, and an identity matrix of the same dimension $(2M + 2L) \times (2M + 2L)$ as $\boldsymbol{I}$. Then the characteristic polynomial whose roots are the eigenvalues of $\boldsymbol{L}$ is the determinant $\det(\boldsymbol{L} - \mu\boldsymbol{I})$. First, we analyze it and then we identify new eigenvalues.

Let us look at the second column block of the size M in this determinant. The only non-zero elements in this block lie in the rows of the second row block, in the part which was called $[\Phi_S^\varphi]$. The elements of this part have been computed in the first line in (C.6). Thus, this column block can be represented as $\| \, v\,\boldsymbol{b}+\boldsymbol{b}_1 \mid \ldots \mid v\,\boldsymbol{b}+\boldsymbol{b}_k \, \|$, where $v=(g-r_f)/(1+g)$ and the following column vectors have been introduced

$$
\begin{array}{rcl}
\boldsymbol{b} &=& \| \;\; 0 \;\; \ldots \;\; 0 \;\; \mid \;\; -\varphi_1^* \;\; \ldots \;\; -\varphi_M^* \;\; \mid \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \mid \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \|, \\
\boldsymbol{b}_1 &=& \| \;\; 0 \;\; \ldots \;\; 0 \;\; \mid \;\; 1-\mu \;\; \ldots \;\; 0 \;\; \mid \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \mid \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \|, \\
& \vdots & \\
\boldsymbol{b}_M &=& \| \;\; 0 \;\; \ldots \;\; 0 \;\; \mid \;\; 0 \;\; \ldots \;\; 1-\mu \;\; \mid \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \mid \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; \|.
\end{array}
$$

We consider each of the columns in the central block as a sum of two terms and, applying the multilinear property of the discriminant to the whole matrix $\boldsymbol{L}-\mu\boldsymbol{I}$, end up with a sum of 2^M determinants. Many of them are zeros, since they contain two or more columns proportional to the same vector $\boldsymbol{b}$. Actually, there are only $M+1$ non-zero elements in the expansion. The simplest has the structure $\| \, \boldsymbol{b}_1 \mid \ldots \mid \boldsymbol{b}_M \, \|$ in the second column block. Since the only non-zero elements in this block are the terms $1-\mu$ belonging to the main diagonal, the determinant of that part is equal to $(1-\mu)^M \det \boldsymbol{N}$, where the matrix $\boldsymbol{N}$ is defined as follows

$$
\left\|\begin{array}{ccc|ccc|ccc|ccc}
-\mu & \ldots & 0 & f_1^{k_0}+c_1 f_1^Y & f_1^{k_1} & \ldots & f_1^{k_{L-1}} & f_1^{y_0}+f_1^Y & f_1^{y_1} & \ldots & f_1^{y_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -\mu & f_M^{k_0}+c_1 f_M^Y & f_M^{k_1} & \ldots & f_M^{k_{L-1}} & f_M^{y_0}+f_M^Y & f_M^{y_1} & \ldots & f_M^{y_{L-1}} \\
\hline
K^{x_1} & \ldots & K^{x_M} & c_2(\langle f^{k_0}\rangle + c_1\langle f^Y\rangle) + c_1 K^y - \mu & c_2\langle f^{k_1}\rangle & \ldots & c_2\langle f^{k_{L-1}}\rangle & c_2(\langle f^{y_0}\rangle + \langle f^Y\rangle) + K^y & c_2\langle f^{y_1}\rangle & \ldots & c_2\langle f^{y_{L-1}}\rangle \\
0 & \ldots & 0 & 1 & -\mu & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & -\mu & 0 & 0 & \ldots & 0 \\
\hline
0 & \ldots & 0 & c_1 & 0 & \ldots & 0 & 1-\mu & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & -\mu & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -\mu
\end{array}\right\|.
$$

The non-zero elements in this matrix have been computed during the proof of Lemma C.1. Namely, the constant $c_1 = Y^k = -y^*/(1+g)$, the values of K^{x_m} are given in (C.7), $c_2 = (1+g)/(x_\diamond^*(1-x_\diamond^*))$ comes from (C.9), and the derivative $K^y = x_\diamond^*/(1-x_\diamond^*)$ is computed from (C.2). Finally, by $\langle f^{k_l}\rangle$ and $\langle f^{y_l}\rangle$ for $l=1,\ldots,L$, as well as $\langle f^Y\rangle$ we mean the averages of the corresponding derivatives of the survivors' investment functions weighted by their equilibrium wealth shares.

Coming back to the computation of $\det(\boldsymbol{L}-\mu\boldsymbol{I})$, recall that there are other M non-zero blocks in the sum for this determinant. They are obtained when in the second column block all the vectors are $\boldsymbol{b}_i$ apart from the one column $v\,\boldsymbol{b}$. But all these determinants can be simplified since $M-1$ of their columns have only one non-zero element $1-\mu$ on the diagonal, and after eliminating the corresponding columns and rows the remaining column in the second block will contain the element $-v\,\varphi_\nu^*$ in the diagonal and zero elements in other positions. Therefore

$$
\det(\boldsymbol{L}-\mu\boldsymbol{I}) = (1-\mu)^M \det\boldsymbol{N} - (1-\mu)^{M-1}\cdot\frac{g-r_f}{1+g}\sum_{\nu=1}^{M}\varphi_\nu^* \det\boldsymbol{N} = (1-\mu)^{M-1}\Big(1-\mu-\frac{g-r_f}{1+g}\Big)\det\boldsymbol{N}. \quad \text{(D.1)}
$$

¿From this expression we obtain the eigenvalue equal to 1 of multiplicity $M-1$. Notice that when $M=1$ there are no such eigenvalues. That is why the system with one survivor is asymptotically stable (of course if all the roots of polynomial (4.13) are inside the unit circle.) When $M>1$ the eigenvalue 1 obviously corresponds to the movement of the system along the manifold of equilibria. Therefore, it is only the wealth distribution which is changing in the equilibria but not the other quantities.

Another eigenvalue obtained in the expansion (D.1) is $(1+r_f)/(1+g)$. It does not affect the stability, since $r_f<g$. All the remaining eigenvalues can be obtained from $(M+2L)\times(M+2L)$ matrix $\boldsymbol{N}$. We expand this matrix on the minors of the elements of the first row in the last block. Simplifying the resulting minors, we get

$$
\det\boldsymbol{N} = (-1)^L c_1 \mu^{L-1}\det\boldsymbol{N}_1(M) + (1-\mu)(-\mu)^{L-1}\det\boldsymbol{N}_2(M)\,, \quad \text{(D.2)}
$$

where

$$\boldsymbol{N}_1(M) = \left\| \begin{array}{ccc|ccccc} -\mu & \dots & 0 & f_1^{y_0} + f_1^Y & f_1^{y_1} & \dots & f_1^{y_{L-2}} & f_1^{y_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_M^{y_0} + f_M^Y & f_M^{y_1} & \dots & f_M^{y_{L-2}} & f_M^{y_{L-1}} \\ \hline K^{x_1} & \dots & K^{x_M} & c_2\big(\langle f^{y_0}\rangle + \langle f^Y\rangle\big) + K^y & c_2\langle f^{y_1}\rangle & \dots & c_2\langle f^{y_{L-2}}\rangle & c_2\langle f^{y_{L-1}}\rangle \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{array} \right\| ,$$

and

$$\boldsymbol{N}_2(M) = \left\| \begin{array}{ccc|ccccc} -\mu & \dots & 0 & f_1^{k_0} + c_1 f_1^Y & f_1^{k_1} & \dots & f_1^{k_{L-2}} & f_1^{k_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_M^{k_0} + c_1 f_M^Y & f_M^{k_1} & \dots & f_M^{k_{L-2}} & f_M^{k_{L-1}} \\ \hline K^{x_1} & \dots & K^{x_M} & c_2\big(\langle f^{k_0}\rangle + c_1\langle f^Y\rangle\big) + c_1 K^y - \mu & c_2\langle f^{k_1}\rangle & \dots & c_2\langle f^{k_{L-2}}\rangle & c_2\langle f^{k_{L-1}}\rangle \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{array} \right\| ,$$

The determinants of these two matrices of similar structure are computed in a recursive way. The following lemma is used.

Lemma D.1.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ 1 & -\mu & 0 & \dots & 0 & 0 \\ 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\mu & 0 \\ 0 & 0 & 0 & \dots & 1 & -\mu \end{vmatrix} = (-1)^{n+1} \sum_{k=1}^{n} x_k \, \mu^{n-k} \quad , \tag{D.3}$$

Proof. Consider this determinant as a sum of elements from the first row multiplied on the corresponding minor. The minor of element x_k, whose corresponding sign is $(-1)^{k+1}$, is a block-diagonal matrix consisting of two blocks. The upper-left block is an upper-diagonal matrix with 1's on the diagonal. The lower-right block is a lower-diagonal matrix with $-\mu$'s on the diagonal. The determinant of this minor is equal to $(-\mu)^{n-k}$ and the relation to be proved immediately follows. □

Consider now the expansion of the matrix $\boldsymbol{N}_1(M)$ by the minors of the elements from the first column. The minor of the first element $-\mu$ is a matrix with a structure similar to $\boldsymbol{N}_1(M)$, which we denote as $\boldsymbol{N}_1(M-1)$. The minor associated with K^{x_1} has a left upper block with $M-1$ entries equal to $-\mu$ below the main diagonal. This block generates a contribution μ^{M-1} to the determinant and once its columns and rows are eliminated, one remains with a matrix of the type (D.3). Applying Lemma D.1 one then has

$$\det \boldsymbol{N}_1(M) = (-\mu)\, \det \boldsymbol{N}_1(M-1) + (-1)^M K^{x_1} \mu^{M-1} (-1)^{L+1} \left(f_1^Y \mu^{L-1} + \sum_{l=0}^{L-1} f_1^{y_l} \mu^{L-1-l} \right).$$

Applying recursively the relation above, the dimension of the determinant is progressively reduced. At the end the lower right block of the original matrix remains, which is again a matrix similar to (D.3). Applying once more Lemma D.1 one has for $\boldsymbol{N}_1(M)$ the following

$$\begin{aligned} \det \boldsymbol{N}_1(M) &= (-1)^{M+L+1}\, \mu^{M-1} \sum_{m=1}^{M} K^{x_m} \left(f_m^Y \mu^{L-1} + \sum_{l=0}^{L-1} f_m^{y_l} \mu^{L-1-l} \right) + \\ &\quad + (-1)^{M+L+1}\, \mu^M \left(\big(K^Y + c_2\langle f^Y\rangle\big)\mu^{L-1} + c_2 \sum_{l=0}^{L-1} \langle f^{y_l}\rangle \mu^{L-1-l} \right) = \\ &= (-1)^{M+L+1}\, \mu^{M-1} \left[\mu^L K^Y + \left(\langle f^Y\rangle \mu^{L-1} + \sum_{l=0}^{L-1} \langle f^{y_l}\rangle \mu^{L-1-l} \right) \cdot \left(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2 \right) \right]. \end{aligned}$$

The determinant of matrix $\boldsymbol{N}_2(M)$ can be computed analogously and we obtain

$$\begin{aligned}\det \boldsymbol{N}_2(M) &= (-1)^{M+L+1}\,\mu^{M-1}\sum_{m=1}^{M} K^{x_m}\Big(c_1 f_m^Y \mu^{L-1} + \sum_{l=0}^{L-1} f_m^{k_l}\mu^{L-1-l}\Big) + \\ &\quad + (-1)^{M+L+1}\,\mu^{M}\Big(-\mu^L + c_1\big(K^Y + c_2\langle f^Y\rangle\big)\mu^{L-1} + c_2\sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\Big) = \\ &= (-1)^{M+L+1}\,\mu^{M-1}\Big[-\mu^{L+1} + c_1\mu^L K^Y + \Big(c_1\langle f^Y\rangle\mu^{L-1} + \sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\Big)\cdot\Big(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\Big)\Big].\end{aligned}$$

Plugging the two last expressions in (D.2) we finally obtain

$$\begin{aligned}\det \boldsymbol{N} &= (-1)^{M+1}\mu^{M+L-2}c_1\Big[\mu^L K^Y + \Big(\langle f^Y\rangle\mu^{L-1} + \sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l}\Big)\cdot\Big(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\Big)\Big] + \\ &\quad + (1-\mu)(-1)^{M}\mu^{M+L-2}\Big[-\mu^{L+1} + c_1\mu^L K^Y + \Big(c_1\langle f^Y\rangle\mu^{L-1} + \sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\Big)\cdot\Big(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\Big)\Big] = \\ &= (-1)^{M+1}\mu^{M+L-2}\Big[(1-\mu)\mu^{L+1} + c_1\mu^L K^Y\big(1-(1-\mu)\big) + \\ &\quad + \Big(-\frac{1+r_f}{(1-x_\diamond^*)x_\diamond^*} + \mu c_2\Big)\Big(c_1\langle f^Y\rangle\mu^{L-1}\big(1-(1-\mu)\big) + c_1\sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l} - (1-\mu)\sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\Big)\Big] = \\ &= (-1)^{M+1}\mu^{M+L-2}\Big(1 + c_1K^y - \mu\Big)\times \\ &\quad \times\Big[\mu^{L+1} - c_2\Big(c_1\langle f^Y\rangle\mu^L + c_1\sum_{l=0}^{L-1}\langle f^{y_l}\rangle\mu^{L-1-l} - (1-\mu)\sum_{l=0}^{L-1}\langle f^{k_l}\rangle\mu^{L-1-l}\Big)\Big]\end{aligned}$$

where in the last equality we used the relation $-c_2(1+c_1K^y) = -(1+r_f)/(x_\diamond^*(1-x_\diamond^*))$, which can be easily checked using the definitions of the constants c_2, c_1 and K^y.

Thus, we have found another zero eigenvalue of multiplicity $M+L-2$ and yet another eigenvalue $1+c_1K^y = (1+r_f)/(1+g)$ which lies inside the unit circle since $r_f < g$. The stability will depend only on the roots of the polynomial in the squared brackets. After some simplifications and using the relation $x_\diamond^*(1-x_\diamond^*) = -y^*l'(y^*)$, which can be directly checked from the definition of the Equilibrium Market Curve, we get the polynomial (4.13).

□

E Proof of Corollaries 4.2 and 4.3

When $L = 1$ the polynomial $\tilde{Q}(\mu)$ in (4.13) can be simplified and it is given by

$$\mu^2 + \mu\, C\frac{1+g}{y^*} - C\Big(1 + \frac{1+g}{y^*}\Big),$$

where $C = \sum_{m=1}^{M} f_m'(y^*+g)\varphi_m^*/l'(y^*)$. Let us introduce two quantities, trace and determinant, as follows $\mathrm{Tr} = -C(1+g)/y^*$ and $\mathrm{Det} = -C(1+(1+g)/y^*)$. According to standard results for the second-degree polynomial (see e.g. Medio and Lines (2001)), we get the following conditions for stability, whose equality correspond to the bifurcation loci of fold, flip and Neimark-Sacker bifurcation respectively,

$$1 - \mathrm{Tr} + \mathrm{Det} > 0\,, \quad 1 + \mathrm{Tr} + \mathrm{Det} > 0\,, \quad \text{and} \quad \mathrm{Det} < 1\,.$$

Using our definitions of Tr and Det we get the conditions $C < 1$, $C < y^*/(y^*+2(1+g))$ and $C > -y^*/(1+g+y^*)$ respectively. The first condition is redundant, while the last two give (4.14).

For larger L the results on stability are limited. First, we can derive the loci of fold and flip bifurcations substituting, respectively, $\mu = 1$ and $\mu = -1$ into polynomial $\tilde{Q}(\mu)$ in (4.13). Straight-forward computations show that the line $C = 1$ is a locus of fold bifurcation for any L, while the curve $C = y^*/(y^*+2(1+g))$ is a locus of flip bifurcation for any odd L (and there is no flip bifurcation, when L is even).

Second, plugging $\mu = e^{i\psi}$, where ψ is arbitrary angle and i is the imaginary unit, into equation $\tilde{Q}(\mu) = 0$ we can derive the locus of Neimark-Sacker bifurcation. In case of $L = 2$ the equation can be solved and, after tedious computations, one get the condition

$$C^2 \left(y^{*2} + 3(1+g)y^* + 2(1+g)^2\right) + 2\,C\,y^{*2} - 4\,y^{*2} = 0\,.$$

This second-order curve is depicted in the right panel of Fig. 2 in coordinates (y^*, C).

Finally, we analyze the case $L \to \infty$. Rewrite polynomial (4.13) as follows

$$\tilde{Q}(\mu) = \mu^{L-1} \left(\mu^2 - \frac{1}{L}\frac{1-(1/\mu)^L}{1-1/\mu}\left(1 + (1-\mu)\frac{1+g}{y^*}\right) C\right)\,. \tag{E.1}$$

We want to proof that all the roots of this polynomial lie inside the unit circle of the complex plane for L high enough. Consider the region outside the unit circle (including the circle itself), fix $\mu = \mu_0$ and let $L \to \infty$. Since $|\mu_0| \geq 1$, the first term in (E.1) cannot be equal to zero. Therefore, μ_0 can be a root of the characteristic polynomial only if the expression in the parenthesis cancels out. First, assume that $|\mu_0| > 1$. Then when $L \to \infty$ the expression in the parenthesis leads to $\mu_0 = 0$ which contradicts our choice of μ_0. Second, let $|\mu_0| = 1$ but $\mu_0 \neq 1$. In this case the expression $|1 - (1/\mu)^L|$ is bounded (uniformly with L), and so, again taking the limit $L \to \infty$ we obtain $\mu_0 = 0$. Thus, the only remaining possibility is $\mu_0 = 1$, which wul imply $C = 1$, that is to the locus of fold bifurcation. Since we know that when the relative slope is $C = 0$, the steady-state is stable by continuity it follows that whenever $C < 1$ the steady state is stable too. This implies (4.15), and proofs Corollary 4.3. □

F Proof of Proposition 4.5

In Proposition 4.1 we have proved that the system is well defined on $\mathcal{D}$ given in (4.5). Along the same lines it is straightforward to show that $\mathcal{T}$ is also well defined on $\mathcal{D}'$. In particular, an extension for zero dividend yield does not create any problem. Since $\mathcal{D} \subset \mathcal{D}'$, the fixed points defined in Proposition 4.2 are also the fixed points in $\mathcal{D}'$. In those points, of course, $y^* \neq 0$.

In all other fixed points $y^* = 0$, while other quantities are obtained again from (B.1). From the third equation it immediately follows that $k^* = r_f$. Thus, the investment in the risky and the riskless asset yields the same return. Therefore, the wealths of all the agents increase with the same rate, the second equations (B.1) are always satisfied, and no other restrictions on the agents' wealth shares are required. □

G Proof of Proposition 4.6

The procedure in this proof is analogous to the one we use for proving Propositions 4.3 and 4.4. In particular we use the derivatives and the general Jacobian structure which has been derived in Appendix C. The next Lemma, which is analogous to Lemma C.1 describes the Jacobian matrix for the steady-states with zero yield.

Lemma G.1. *Let $\boldsymbol{x}^*$ be a steady-state of dynamics* (4.2) *described in Proposition 4.5 and let the first M agents survive in this equilibrium. The corresponding Jacobian matrix, $\boldsymbol{J}(\boldsymbol{x}^*)$, has the following structure,*

where the actual values of non-zero elements, denoted by the symbols $\star$, are varying.

$$
\left\|\begin{array}{cccccc|cccccc|cccccc}
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & [& \frac{\partial \mathcal{X}}{\partial \mathcal{K}} & &] & [& & \frac{\partial \mathcal{X}}{\partial \mathcal{Y}} & &] \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
\hline
\star & \dots & \star & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
[& \Phi^x &] & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & [& \Phi^k & &] & [& & \Phi^y & &] \\
\star & \dots & \star & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
\star & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \star & \star & \dots & \star & \star \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \frac{1+g}{1+r_f} & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{array}\right\|
$$

The solid lines divide the matrix in the same 16 *blocks as in* (C.4)*. In addition, the first and second column-blocks and second row-block are split into the two parts of sizes M and $N-M$, corresponding to the survivors and the non-survivors, respectively.*

Proof. Let us start with the first row-block having N rows. The first two blocks of columns in this block, $\partial\mathcal{X}/\partial\mathcal{X}$ and $\partial\mathcal{X}/\partial\mathcal{W}$, are always zero. Two other blocks, $\partial\mathcal{X}/\partial\mathcal{K}$ and $\partial\mathcal{X}/\partial\mathcal{Y}$, in general contain non-zero elements and simplified, because in the equilibrium $k^* = r_f$ and $y^* = 0$, and therefore $Y^y = (1+g)/(1+r_f)$ and $Y^k = 0$.

To simplify the remaining row-blocks, notice from (C.3) that $\Phi_n^{x_m} = 0$ and $\Phi_n^{\varphi_m} = \delta_n^m$, while $\Phi_n^k = \Phi_n^y = \varphi_n^*(x_n^* - \langle x \rangle)/(1+r_f)$ in this equilibrium. This follows immediately from the relation $k^* + y^* - r_f = 0$. At the same time from (C.2) we have $K^{\varphi_m} = 0$, $K^y = \langle x^{*2} \rangle / \langle x^*(1-x^*) \rangle$, and $K^{z_m} = -K^{x_m} = \varphi_m^*(1+r_f)/\langle x^*(1-x^*) \rangle$.

Thus, in the first block of the second row-block, $\partial\mathcal{W}/\partial\mathcal{X}$, the elements are equal to $\Phi_n^k \cdot K^{x_m}$ and they are zeros as soon as either n or m is larger than M. In the next block, $\partial\mathcal{W}/\partial\mathcal{W}$, all the elements are zeros, apart from the main diagonal elements. All the elements of the next block, $\partial\mathcal{W}/\partial\mathcal{K}$, contain the multiplying term Φ_n^k, so that they are non-zeros only for the surviving agents. We denote the corresponding part of the block as $[\Phi^k]$. Similarly, in the block $\partial\mathcal{W}/\partial\mathcal{Y}$ all the elements are the sums containing either the term Φ_n^k or the term Φ_n^y, so that they are non-zeros only for the surviving agents. We denote the corresponding part of the block as $[\Phi^y]$.

In the next row-block, with L rows, the elements are zeros in all the lines but the first. The only exception from thisrule are the elements below the main diagonal in the block $\partial\mathcal{K}/\partial\mathcal{K}$ which are all equal to 1. For the elements in the first row we use the derivatives of function K derived above. Consequently, in the first block, $\partial\mathcal{K}/\partial\mathcal{X}$, for the non-surviving agents we have $K^{x_m} = 0$. Analogously, in the next block, $\partial\mathcal{K}/\partial\mathcal{W}$, all the elements are zeros. The simplifications in the blocks $\partial\mathcal{K}/\partial\mathcal{K}$ and $\partial\mathcal{K}/\partial\mathcal{Y}$ are minor. Notice from (C.2) that the derivatives K^{z_m} for all the non-survivors are zeros, therefore all the sums in the first row of this block have to be taken only with respect to the surviving agents.

Finally, in the last row-block the simplifications are straight-forward. □

In the remaining part of this proof we identify different multipliers of the matrix derived in the previous Lemma. From the first line in the fourth row-block we immediately obtain the eigenvalue $(1+g)/(1+r_f)$ and condition $g < r_f$ for stability. Elimination of this line together with the corresponding column creates a zero line in the same block. Proceeding recursively, we obtain the eigenvalue 0 with multiplicity $L-1$ and eliminate the fourth line- and column-block entirely.

From the second column-block we get the eigenvalue 1 with multiplicity N. These eigenvalues correspond to the directions of change in the wealth distribution between different agents. (Recall from Proposition 4.5

that the wealth shares are free of choice.) Consequently, there are no asymptotically stable equilibria. At the same time, it is clear that these eigenvalues, lying on the border of the unit circle, do not prevent the steady-state from stability.

From the last $N-M$ columns of the first column-block we obtain the eigenvalue 0 with multiplicity $N-M$. Eliminating corresponding columns and rows we get the following matrix

$$\boldsymbol{N}_3(M) = \left\| \begin{array}{ccc|ccccc} -\mu & \dots & 0 & f_1^{k_0} & f_1^{k_1} & \dots & f_1^{k_{L-2}} & f_1^{k_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_M^{k_0} & f_M^{k_1} & \dots & f_M^{k_{L-2}} & f_M^{k_{L-1}} \\ \hline K^{x_1} & \dots & K^{x_M} & \sum_m K^{z_m} f_m^{k_0} - \mu & \sum_m K^{z_m} f_m^{k_1} & \dots & \sum_m K^{z_m} f_m^{k_{L-2}} & \sum_m K^{z_m} f_m^{k_{L-1}} \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{array} \right\|,$$

where, as we found in Lemma G.1, the derivatives are $K^{z_m} = -K^{x_m} = \varphi_m^*(1+r_f)/\big\langle x^*(1-x^*)\big\rangle$. This matrix has the same functional form as matrices $\boldsymbol{N}_1(M)$ and $\boldsymbol{N}_2(M)$ whose discriminant we computed in Appendix D. Proceeding in analogous way, we get

$$\begin{aligned} \det \boldsymbol{N}_3(M) &= (-1)^{M+L+1}\,\mu^{M-1} \sum\nolimits_{m=1}^{M} K^{x_m} \Big(\sum\nolimits_{l=0}^{L-1} f_m^{k_l} \mu^{L-1-l} \Big) + \\ &\quad + (-1)^{M+L+1}\,\mu^{M} \Big(-\mu^L + \sum\nolimits_{l=0}^{L-1} \sum\nolimits_{m} K^{z_m} f_m^{k_l} \mu^{L-1-l} \Big) = \\ &= -(-1)^{M+L+1}\,\mu^{M-1}\,\frac{1+r_f}{\big\langle x^*(1-x^*)\big\rangle} \sum\nolimits_{l=0}^{L-1} \big\langle f^{k_l} \big\rangle \mu^{L-1-l} + \\ &\quad + (-1)^{M+L+1}\,\mu^{M} \Big(-\mu^L + \frac{1+r_f}{\big\langle x^*(1-x^*)\big\rangle} \sum\nolimits_{l=0}^{L-1} \big\langle f^{k_l} \big\rangle \mu^{L-1-l} \Big) = \\ &= (-1)^{M+L}\,\mu^{M-1} \Big(\mu^{L+1} + \frac{1+r_f}{\big\langle x^*(1-x^*)\big\rangle} (1-\mu) \sum\nolimits_{l=0}^{L-1} \big\langle f^{k_l} \big\rangle \mu^{L-1-l} \Big). \end{aligned}$$

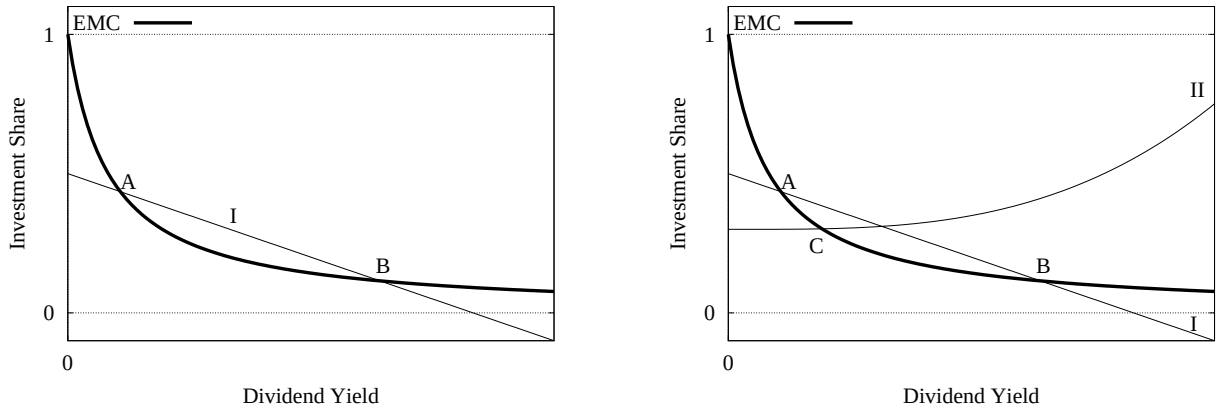

Figure 1: Location of equilibria for $g > r_f$. **Left panel:** The Equilibrium Market Curve is shown together with one Equilibrium Investment Function, curve I. In total there are two intersections with the EMC, corresponding to two different steady-states, A and B. **Right panel:** When the investment function II is added one more steady state is possible, point C. In points A and B agent I survives with $\varphi_I = 1$. In point C agent II survives with $\varphi_{II} = 1$. The abscissa of a point gives the dividend yield, while the ordinate gives the investment share of the survivor.

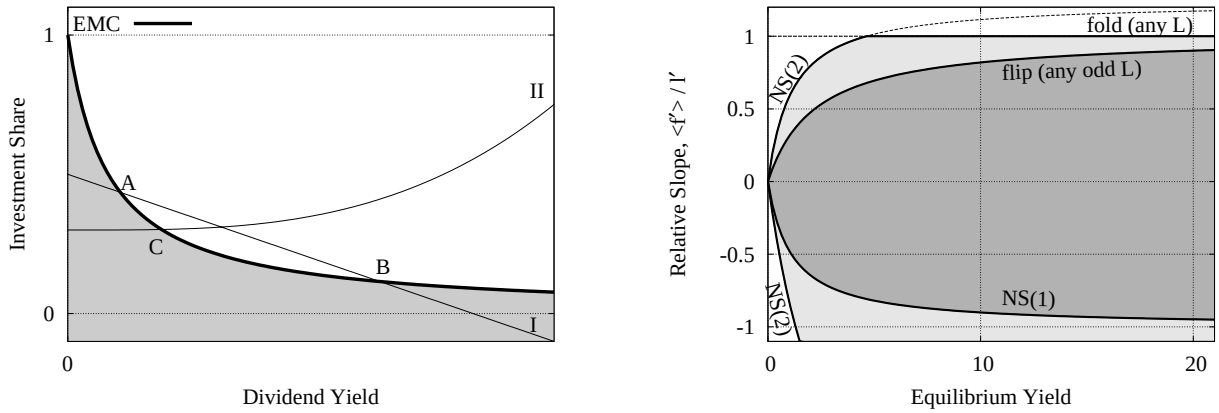

Figure 2: Stability conditions. **Left panel:** According to (4.10), the steady-state is stable (against the invasion by the non-survivors) if the non-survivors are less aggressive, i.e. their investment shares lie below the investment share of the survivors (gray area). **Right panel:** Stability of a steady-state when investment depends upon the average of past L total returns. In a stable steady-state for $L = 1$, the pair $\big(y^*, \big\langle f'(y^* + g)\big\rangle / l'(y^*)\big)$ belongs to the dark-grey area. When $L = 2$ the stability region expands and consists of the union of the dark-grey and light-grey areas. When $L \to \infty$ the stability region occupies all the space below "fold" line $\big\langle f'(y^* + g)\big\rangle / l'(y^*) = 1$. Crossing the border of the stability region causes the corresponding type of bifurcation, where NS stands for Neimark-Sacker.

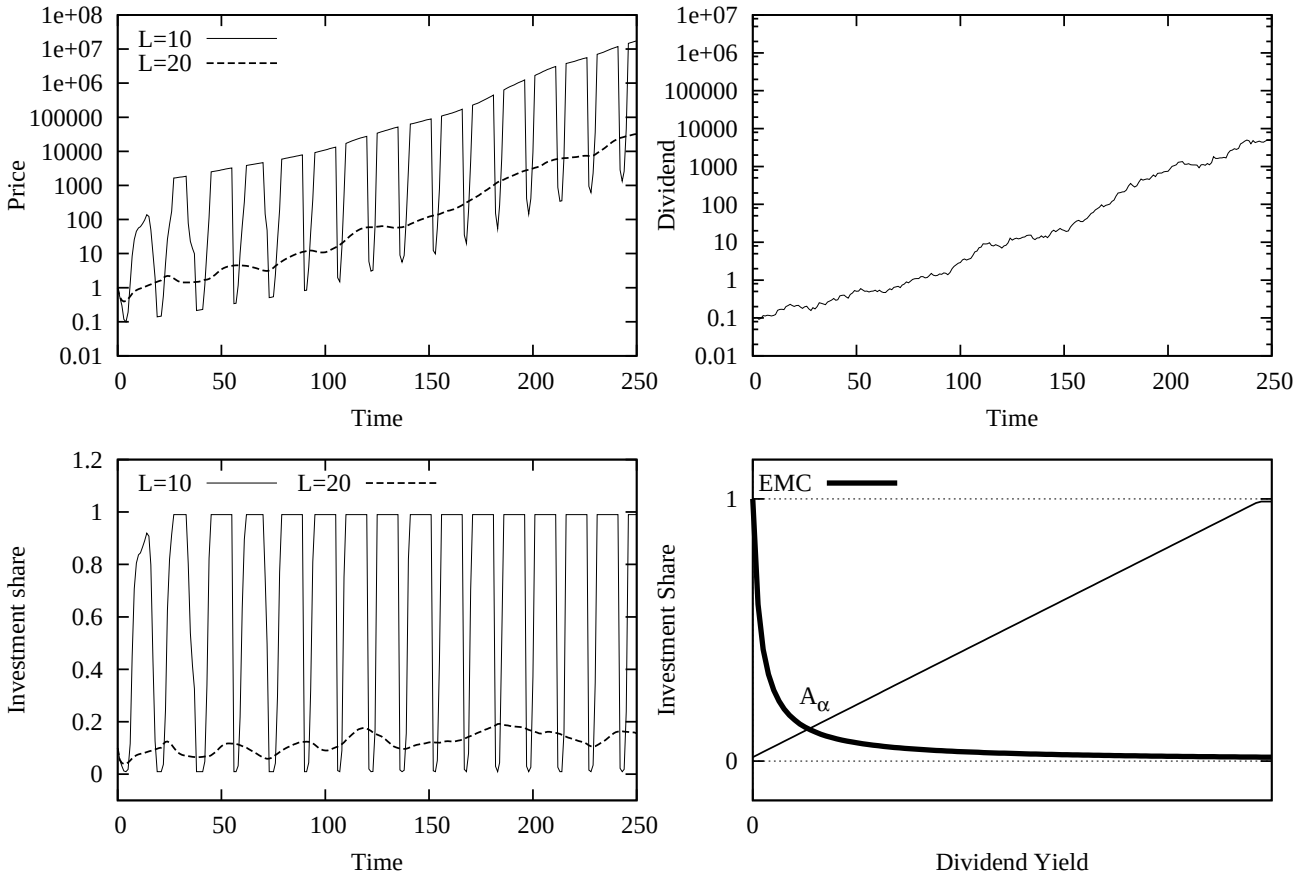

Figure 3: Dynamics with a single mean-variance maximizer in a market with $r_f = 0.01$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span are compared. **Top-left panel:** log-price. **Bottom-left panel:** investment share. **Top-right panel:** dividend process. **Bottom-right panel:** equilibrium on the EMC.

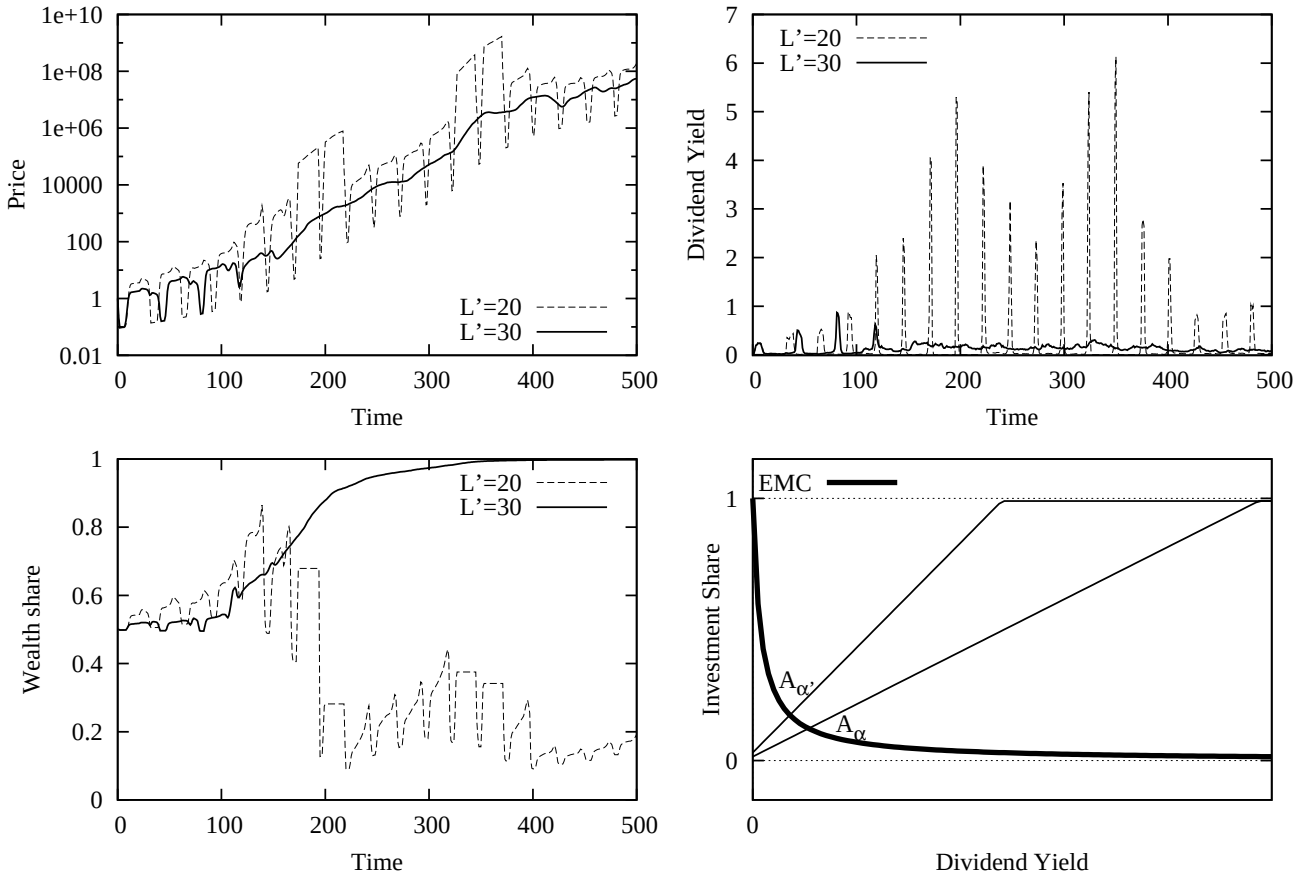

Figure 4: Dynamics with two mean-variance maximizers in a market with $r_f = 0.01$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span of the agent with lower risk aversion are compared. **Top-left panel:** log-price. **Bottom-left panel:** wealth share of the agent with lower risk aversion. **Top-right panel:** dividend yield. **Bottom-right panel:** EMC and two investment functions. The agent with lower risk aversion α' produces the steady state $A_{\alpha'}$.

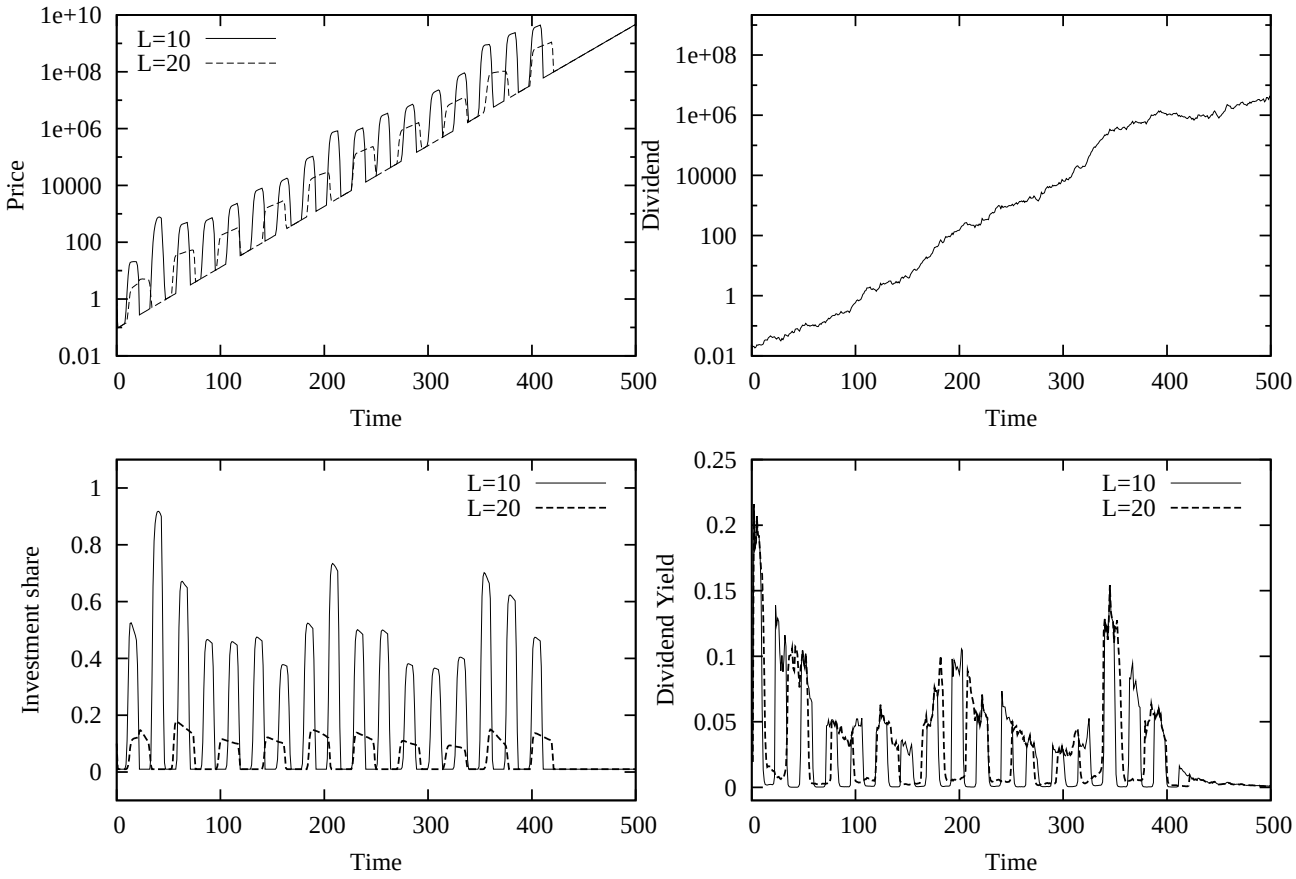

Figure 5: Dynamics with a single mean-variance maximizer in a market with $r_f = 0.05$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span are compared. **Top-left panel:** log-price. **Bottom-left panel:** investment share. **Top-right panel:** dividend process. **Bottom-right panel:** dividend yield.

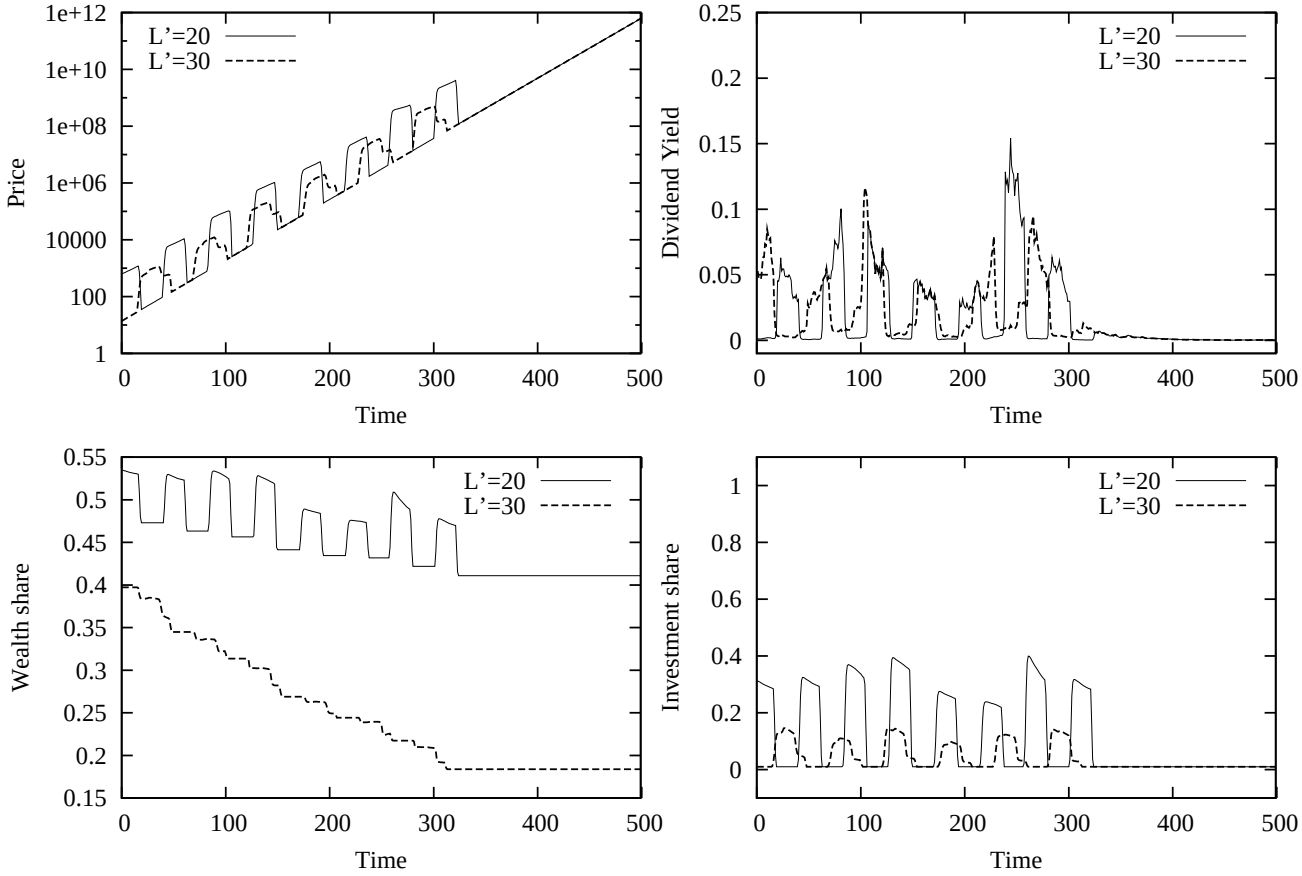

Figure 6: Dynamics with two mean-variance maximizers in a market with $r_f = 0.05$ and stochastic dividend with average growth rate $g = 0.04$ and variance $\sigma_g^2 = 0.1$. Two levels of memory span of the agent with lower risk aversion are compared. **Top-left panel:** log-price dynamics. **Bottom-left panel:** wealth share of the agent with lower risk aversion α'. **Top-right panel:** dividend yield. **Bottom-right panel:** weighted average of the agents' investment shares.