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# On the configuration spaces of grassmannian manifolds 

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# On the Configuration Spaces of Grassmannian Manifolds 

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# On the Configuration Spaces of Grassmannian Manifolds 

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#### Abstract

Let $\mathcal{F}_{h}^{i}(k, n)$ be the $i$ th ordered configuration space of all distinct points $H_{1}, \ldots, H_{h}$ in the Grassmannian $\operatorname{Gr}(k, n)$ of $k$-dimensional subspaces of $\mathbb{C}^{n}$, whose sum is a subspace of dimension $i$. We prove that $\mathcal{F}_{h}^{i}(k, n)$ is (when non empty) a complex submanifold of $\operatorname{Gr}(k, n)^{h}$ of dimension $i(n-i)+h k(i-k)$ and its fundamental group is trivial if $i=\min (n, h k), h k \neq n$ and $n>2$ and equal to the braid group of the sphere $\mathbb{C} P^{1}$ if $n=2$. Eventually we compute the fundamental group in the special case of hyperplane arrangements, i.e. $k=n-1$.


## Keywords:

complex space, configuration spaces, braid groups.

MSC (2010): 20F36, 52C35, 57M05, 51A20.

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## 1 Introduction

Let $M$ be a manifold. The ordered configuration space of $h$ distinct points in $M, \mathcal{F}_{h}(M)=\left\{\left(x_{1}, \ldots, x_{h}\right) \in M^{h} \mid x_{i} \neq x_{j}, i \neq j\right\}$, has been widely studied after it has been introduced by Fadell and Neuwirth [FaN] and Fadell [Fa] in the sixties. It is well known that for a simply connected manifold $M$ of dimension $\geq 3$, the pure braid group on $h$ strings of $M, \pi_{1}\left(\mathcal{F}_{h}(M)\right)$, is trivial, while in low dimension there are non trivial pure braids. For example, the pure braid group of the sphere $S^{2} \approx \mathbb{C} P^{1}$ is not trivial with the following presentation:

$$
\pi_{1}\left(\mathcal{F}_{h}\left(\mathbb{C} P^{1}\right)\right) \cong\left\langle\alpha_{i j}, 1 \leq i<j \leq h-1 \mid(Y B 3)_{h-1},(Y B 4)_{h-1}, D_{h-1}^{2}=1\right\rangle
$$

where $D_{k}=\alpha_{12}\left(\alpha_{13} \alpha_{23}\right)\left(\alpha_{14} \alpha_{24} \alpha_{34}\right) \cdots\left(\alpha_{1 k} \alpha_{2 k} \cdots \alpha_{k-1}\right)$ (see [Bi] and [FaH] for the Yang-Baxter relations $\left.(Y B, 3)_{h-1},(Y B, 4)_{h-1}\right)$.
In a recent paper ([BaS]) Berceanu and Parveen introduced new configuration spaces. They stratify the classical configuration spaces $\mathcal{F}_{h}\left(\mathbb{C P}^{n}\right)$ with complex submanifolds $\mathcal{F}_{h}^{i}\left(\mathbb{C P}^{n}\right)$ defined as the ordered configuration spaces of all $h$ points in $\mathbb{C P}^{n}$ generating a projective subspace of dimension $i$. Then they compute the fundamental groups $\pi_{1}\left(\mathcal{F}_{h}^{i}\left(\mathbb{C P}^{n}\right)\right)$ proving that they are trivial except when $i=1$ providing, in this last case, a presentation for $\pi_{1}\left(\mathcal{F}_{h}^{1}\left(\mathbb{C P}^{n}\right)\right)$ similar to those of the pure braid group of the sphere.
In a subsequent paper ([MPS]), the authors apply similar techniques to the affine case, i.e. to $\mathcal{F}_{h}\left(\mathbb{C}^{n}\right)$, showing that the situation is similar except in one case. More precisely they prove that, if $\mathcal{F}_{h}^{i, n}=\mathcal{F}_{h}^{i}\left(\mathbb{C}^{n}\right)$ denotes the ordered configuration space of all $h$ points in $\mathbb{C}^{n}$ generating an affine subspace of dimension $i$, then the spaces $\mathcal{F}_{h}^{i, n}$ are simply connected except for $i=1$ or $i=n=h-1$ and, in the last cases, they provide a presentation of the fundamental groups $\pi_{1}\left(\mathcal{F}_{h}^{i, n}\right)$.
In this paper we generalize the result in $[\mathrm{BaS}]$ to the Grassmannian manifold $G r(k, n)$ parametrizing $k$-dimensional subspaces of $\mathbb{C}^{n}$. We define the $i$ th ordered configuration space $\mathcal{F}_{h}^{i}(k, n)$ as the ordered configuration space of all distinct points $H_{1}, \ldots, H_{h}$ in the Grassmannian $\operatorname{Gr}(k, n)$ such that the sum $\left(H_{1}+\cdots+H_{h}\right)$ is an $i$-dimensional space.
We prove that the $i$ th ordered configuration space $\mathcal{F}_{h}^{i}(k, n)$ is (when non empty) a complex submanifold of $G r(k, n)^{h}$ and we compute its dimension. As a corollary, we prove that if $n \neq h k$ and $i=\min (n, h k)$ then the $i$ th ordered configuration space $\mathcal{F}_{h}^{i}(k, n)$ has trivial fundamental group except
when $n=2$ where we get the pure braid group of the sphere, that is

$$
\begin{align*}
& \pi_{1}\left(\mathcal{F}_{h}^{\min (n, h k)}(k, n)\right)=0 \quad \text { if }(k, n) \neq(1,2) \\
& \pi_{1}\left(\mathcal{F}_{1}^{1}(1,2)\right)=\pi_{1}\left(\mathcal{F}_{2}\left(\mathbb{C P}^{1}\right)\right) . \tag{1}
\end{align*}
$$

In the particular case of hyperplane arrangements, i.e. $k=n-1$, we remark that the fundamental group of the $i$ th ordered configuration space $\mathcal{F}_{h}^{i}(n-1, n)$ vanishes except when $n=2$. Moreover using a dual argument we get similar results for the fundamental group of the ordered configuration space of all distinct $k$-dimensional subspaces $H_{1}, \ldots, H_{h}$ in $\mathbb{C}^{n}$ such that the intersection $\left(H_{1} \cap \cdots \cap H_{h}\right)$ is an $i$-dimensional subspace.
We conjecture that a similar result to the one obtained in $[\mathrm{BaS}]$ for projective spaces holds also for Grassmannian manifolds and the fundamental group of the $i$ th ordered configuration space $\mathcal{F}_{h}^{i}(k, n)$ vanishes except for low values of $i$. This will be the object of a forthcoming paper together with a generalization of the Pappus's construction in [BaS].

## 2 Main Section

For $0<k<n$, let us consider the Grassmannian manifold $\operatorname{Gr}(k, n)$ parametrizing $k$-dimensional subspaces of the $n$-dimensional complex space $\mathbb{C}^{n}$, and its ordered configuration spaces $\mathcal{F}_{h}(G r(k, n))$.

The spaces $\mathcal{F}_{h}^{i}(k, n)$. Let's define the $i$ th ordered configuration space $\mathcal{F}_{h}^{i}(k, n)$ as the space of all distinct points $H_{1}, \ldots, H_{h}$ in the Grassmannian $G r(k, n)$ whose sum is an $i$-dimensional subspace of $\mathbb{C}^{n}$, i.e.

$$
\mathcal{F}_{h}^{i}(k, n)=\left\{\left(H_{1}, \ldots, H_{h}\right) \in \mathcal{F}_{h}(G r(k, n)) \mid \operatorname{dim}\left(H_{1}+\cdots+H_{h}\right)=i\right\} .
$$

It is an easy remark that the following results hold:

1. in order to get a non empty set, $h=1$ forces $i=k$ and we have $\mathcal{F}_{1}^{k}(k, n)=G r(k, n)$;
2. in order to get a non empty set, $i=1$ forces $k=h=1$, and we have $\mathcal{F}_{1}^{1}(1, n)=G r(1, n)=\mathbb{C} \mathbb{P}^{n-1} ;$
3. for $h \geq 2, \mathcal{F}_{h}^{i}(k, n) \neq \emptyset$ if and only if $i \geq k+1$ and $i \leq \min (h k, n)$;
4. for $i=h k \leq n$, then the $h$ subspaces giving a point of $\mathcal{F}_{h}^{h k}(k, n)$ are in direct sum;
5. for $h \geq 2, \mathcal{F}_{h}(G r(k, n))=\coprod_{i=2}^{n} \mathcal{F}_{h}^{i}(k, n)$;
6. for $h \geq 2$, the adjacency of the strata is given by

$$
\overline{\mathcal{F}_{h}^{i}(k, n)}=\mathcal{F}_{h}^{i}(k, n) \coprod \mathcal{F}_{h}^{i-1}(k, n) \coprod \ldots \coprod \mathcal{F}_{h}^{2}(k, n) .
$$

By the above remarks, it follows that the case $h=1$ is trivial, so from now on we will consider $h>1$ (and hence $i>k$ ).
We want to show that $\mathcal{F}_{h}^{i}(k, n)$ is (when non empty) a complex submanifold of $G r(k, n)^{h}$ and compute its dimension. In order to do it we need to briefly recall a few easy facts and introduce some notation.

The determinantal variety. Let's recall that the determinantal variety $D_{r}\left(m, m^{\prime}\right)$ is the variety of $m \times m^{\prime}$ matrices with complex entries of rank less than or equal to $r \leq \min \left(m, m^{\prime}\right)$. It is an analytic (algebraic, in fact) variety of dimension $r\left(m+m^{\prime}-r\right)$ whose set of singular points is given by those matrices of rank less than $r$. From now on, $D_{r}\left(m, m^{\prime}\right)^{*}$ will denote the set of non-singular points of the determinantal variety $D_{r}\left(m, m^{\prime}\right)$, that is the set of $m \times m^{\prime}$ matrices of rank equal to $r$.

A system of local coordinates for $G r(k, n)^{h}$. Let $V_{0} \subset \mathbb{C}^{n}$ be a subspace of dimension $\operatorname{dim} V_{0}=n-k$, then the set

$$
U_{V_{0}}=\left\{H \in G r(k, n) \mid H \oplus V_{0}=\mathbb{C}^{n}\right\}
$$

is an open dense subset of $\operatorname{Gr}(k, n)$.
Fix a basis $B=\left\{w_{1}, \ldots, w_{k}, v_{1}, \ldots, v_{n-k}\right\}$ of $\mathbb{C}^{n}$ such that $\left\{v_{1}, \ldots, v_{n-k}\right\}$ is a basis of $V_{0}$. Then we get a (complex) coordinate system on $U_{V_{0}}$ as follows. Let $H$ be an element in $U_{V_{0}}$, then the affine subspaces $V_{0}+w_{j}$ intersect $H$ in one point $u_{j}$ for any $j=1, \cdots, k$ and $\left\{u_{1}, \cdots, u_{k}\right\}$ will be a basis of $H$. Hence $H$ is uniquely determined by a $n \times k$ matrix of the form $\binom{I}{A}$, where $I$ is the $k \times k$ identity matrix and $A$ is the $(n-k) \times k$ matrix of the coordinates of $u_{1}-w_{1}, \ldots, u_{k}-w_{k}$ with respect to the vectors $\left\{v_{1}, \ldots, v_{n-k}\right\}$. The coefficients of $A$ give complex coordinates in $U_{V_{0}} \cong \mathbb{C}^{k(n-k)}$.

Let $\left(H_{1}, \ldots, H_{h}\right)$ be a point in $\operatorname{Gr}(k, n)^{h}$, the open sets $U_{H_{1}}, \ldots, U_{H_{h}}$ in the Grassmannian manifold $\operatorname{Gr}(n-k, n)$ have non empty intersection, that is there exists an element $V_{0} \in \operatorname{Gr}(n-k, n)$ such that $V_{0} \oplus H_{j}=\mathbb{C}^{n}$ for all $j=1, \ldots, h$. Thus, $G r(k, n)^{h}$ is covered by the open sets $U_{V_{0}}^{h}$ as $V_{0}$ varies in $G r(n-k, n)$. Taking a basis as defined above, each element in $U_{V_{0}}^{h}$ is uniquely determined by a $n \times h k$ matrix of the form $\left(\begin{array}{cccc}I & I & \cdots & I \\ A_{1} & A_{2} & \cdots & A_{h}\end{array}\right)$ and the coefficients of $\left(\begin{array}{llll}A_{1} & A_{2} & \cdots & A_{h}\end{array}\right)$ give complex coordinates in $U_{V_{0}}^{h} \cong \mathbb{C}^{h k(n-k)}$.

A system of local coordinates for $\mathcal{F}_{h}^{i}(k, n)$. In terms of the above coordinates, $\left(H_{1}, \ldots, H_{h}\right) \in U_{V_{0}}^{h}$ belongs to $\mathcal{F}_{h}^{i}(k, n)$ if and only if $A_{j} \neq A_{l}$ if $j \neq l$ and $\operatorname{rank}\left(\begin{array}{cccc}I & I & \cdots & I \\ A_{1} & A_{2} & \cdots & A_{h}\end{array}\right)=i$. Let us remark that

$$
\begin{aligned}
\operatorname{rank}\left(\begin{array}{cccc}
I & I & \cdots & I \\
A_{1} & A_{2} & \cdots & A_{h}
\end{array}\right) & =\operatorname{rank}\left(\begin{array}{cccc}
I & I & \cdots & I \\
0 & A_{2}-A_{1} & \cdots & A_{h}-A_{1}
\end{array}\right) \\
& =k+\operatorname{rank}\left(A_{2}-A_{1}\right. \\
\cdots & \left.A_{h}-A_{1}\right)
\end{aligned}
$$

We can then change coordinates taking the coefficients of $B_{j}=A_{j}-A_{1}$ instead of those of $A_{j}$ for $j=2, \ldots, h$. In these new coordinates $U_{V_{0}} \cap \mathcal{F}_{h}^{i}(k, n)$ corresponds in $\mathbb{C}^{h k(n-k)}$ to $\mathbb{C}^{k(n-k)} \times D_{i-k}(n-k, h k-k)^{*}$ minus the closed sets given by $B_{j}=0$ for $2 \leq j \leq h$ and by $B_{j}=B_{l}$ for $2 \leq j, l \leq h, j \neq l$. So far we have proved the following theorem.

Theorem 2.1. The ith ordered configuration space $\mathcal{F}_{h}^{i}(k, n)$ is a complex submanifold of the Grassmannian manifold $\operatorname{Gr}(k, n)$ of dimension

$$
d_{h}^{i}(k, n)=i(n-i)+h k(i-k) .
$$

The computation on dimension comes from the easy equality

$$
k(n-k)+(i-k)(n-k+h k-k-(i-k))=i(n-i)+h k(i-k) .
$$

Moreover, $d_{h}^{i}(k, n)=h k(n-k)$ if and only if $i=n$ or $i=h k$ and so, as a function of $i, d_{h}^{i}(k, n)$ is strictly increasing for $i \leq \min (n, h k)$.

The fundamental group of $\mathcal{F}_{h}^{\min (n, h k)}(k, n)$. The space $\mathcal{F}_{h}^{\min (n, h k)}(k, n)$ is an open subset of the ordered configuration space $\mathcal{F}_{h}(G r(k, n))$ and all other (non void) $\mathcal{F}_{h}^{j}(k, n)$ have strictly lower dimension. Moreover, if $i=n$ the
difference of dimensions $d_{h}^{i}(k, n)-d_{h}^{i-1}(k, n)$ equals $1+h k-n$ and if $i=h k$ it equals $1+n-h k$. Then if $n \neq h k$, all (non void) $\mathcal{F}_{h}^{j}(k, n)$ with $j<\min (n, h k)$ have real codimension at least 4 in $\mathcal{F}_{h}(G r(k, n))$. Then, if $n \neq h k$ and $i=\min (n, h k)$, the fundamental group of $\mathcal{F}_{h}^{i}(k, n)=\mathcal{F}_{h}(G r(k, n)) \backslash \overline{\mathcal{F}_{h}^{i-1}(k, n)}$ is the same as the fundamental group of $\mathcal{F}_{h}(\operatorname{Gr}(k, n))$ (since, by the adjacency of the strata, the closure $\overline{\mathcal{F}_{h}^{i-1}(k, n)}$ is the finite union of complex subvarieties of $\mathcal{F}_{h}(\operatorname{Gr}(k, n))$ of real codimension at least 4).
Let us recall that the complex Grassmannian manifolds $\operatorname{Gr}(k, n)$ are simply connected and have real dimension at least 4 except $G r(1,2)=\mathbb{C P} \mathbb{P}^{1}$ and that for a simply connected manifold of real dimension at least 3 the pure braid groups vanish, i.e. $\pi_{1}\left(\mathcal{F}_{h}(G r(k, n))\right)=0$ if $(k, n) \neq(1,2)$.
Then we get the following corollary:
Corollary 2.2. The fundamental group of the ith ordered configuration space $\mathcal{F}_{h}^{i}(k, n)$ vanishes if $n \neq h k$ and $i=\min (n, h k)$ except when $n=2$ for which we get the pure braid group of the sphere.

The dual case. Let $\operatorname{Gr}(k, n)^{*}$ be the Grassmannian manifold parametrizing $k$-dimensional subspaces in the dual space $\left(\mathbb{C}^{n}\right)^{*}$. Then we can define the $i$ th dual ordered configuration space $\mathcal{F}_{h}^{i}(k, n)^{*}$ as

$$
\mathcal{F}_{h}^{i}(k, n)^{*}=\left\{\left(H_{1}, \ldots, H_{h}\right) \in \mathcal{F}_{h}\left(G r(k, n)^{*}\right) \mid \operatorname{dim}\left(H_{1} \cap \cdots \cap H_{h}\right)=i\right\} .
$$

The spaces $\mathcal{F}_{h}^{i}(k, n)^{*}$ stratify the ordered configuration space $\mathcal{F}_{h}\left(\operatorname{Gr}(k, n)^{*}\right)$ of the Grassmannian manifold $\operatorname{Gr}(k, n)^{*}$.
Taking annihilators, we get homeomorphisms Ann: $G r(n-k, n) \rightarrow G r(k, n)^{*}$ which induce homeomorphisms between the $(n-i)$ th ordered configuration space $F_{h}^{n-i}(n-k, n)$ and the $i$ th dual ordered configuration space $F_{h}^{i}(k, n)^{*}$. As a consequence we get that the spaces $F_{h}^{\max (0, n-h k)}(n-k, n)^{*}$ are simply connected manifolds except when $n=2$. In this case the fundamental group is the pure braid group of the sphere.
$i$ th ordered configuration spaces of hyperplane arrangements. Let us remark that when $k=n-1$ we get an $h$-uple of hyperplanes in $\mathbb{C}^{n}$, i.e. an ordered arrangement of hyperplanes. In this case, if $h=1$ then $i=n-1$ and we get that the $i$ th ordered configuration space is simply the Grassmannian manifold, i.e. $\mathcal{F}_{1}^{n-1}(n-1, n)=G r(n-1, n)$. If $h>1$, since the sum of two
(different) hyperplanes is the whole space $\mathbb{C}^{n}$, we get that $h \geq 2$ forces $i=n$ and the following equalities hold

$$
\mathcal{F}_{h}^{n}(n-1, n)=\mathcal{F}_{h}(G r(n-1, n))=\mathcal{F}_{h}\left(\mathbb{C P}^{n-1}\right)
$$

Hence, the fundamental group of the $i$ th ordered configuration space of hyperplane arrangements $\mathcal{F}_{h}^{i}(n-1, n)$ vanishes except when $n=2$. In this case we get the fundamental group of the sphere $\mathbb{C P}^{1}$.
Taking duals, we have $\mathcal{F}_{h}^{i}(n-1, n)^{*} \cong \mathcal{F}_{h}^{n-i}(1, n)$ and the fundamental groups of the latter spaces have been computed in [BaS]. We get that the space of $h$-uples of distinct hyperplanes in $\mathbb{C}^{n}$ whose intersection has dimension equal to $i$ is simply connected except for $i=n-1$.

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