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On the Configuration Spaces of Grassmannian Manifolds

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On the Configuration Spaces of Grassmannian Manifolds

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Abstract

Let $\mathcal{F}_h^i(k, n)$ be the *i*th ordered configuration space of all distinct points H_1, \ldots, H_h in the Grassmannian Gr(k, n) of k-dimensional subspaces of \mathbb{C}^n , whose sum is a subspace of dimension *i*. We prove that $\mathcal{F}_h^i(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ of dimension i(n-i) + hk(i-k) and its fundamental group is trivial if $i = min(n, hk), hk \neq n$ and n > 2 and equal to the braid group of the sphere $\mathbb{C}P^1$ if n = 2. Eventually we compute the fundamental group in the special case of hyperplane arrangements, i.e. k = n - 1.

Keywords:

complex space, configuration spaces, braid groups.

MSC (2010): 20F36, 52C35, 57M05, 51A20.

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1 Introduction

Let M be a manifold. The ordered configuration space of h distinct points in $M, \mathcal{F}_h(M) = \{(x_1, \ldots, x_h) \in M^h | x_i \neq x_j, i \neq j\}$, has been widely studied after it has been introduced by Fadell and Neuwirth [FaN] and Fadell [Fa] in the sixties. It is well known that for a simply connected manifold M of dimension ≥ 3 , the pure braid group on h strings of $M, \pi_1(\mathcal{F}_h(M))$, is trivial, while in low dimension there are non trivial pure braids. For example, the pure braid group of the sphere $S^2 \approx \mathbb{C}P^1$ is not trivial with the following presentation:

$$\pi_1(\mathcal{F}_h(\mathbb{C}P^1)) \cong \left\langle \alpha_{ij}, 1 \le i < j \le h - 1 \left| (YB3)_{h-1}, (YB4)_{h-1}, D_{h-1}^2 = 1 \right\rangle$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{24}\alpha_{34})\cdots(\alpha_{1k}\alpha_{2k}\cdots\alpha_{k-1\ k})$ (see [Bi] and [FaH] for the Yang-Baxter relations $(YB,3)_{h-1}$, $(YB,4)_{h-1}$).

In a recent paper ([BaS]) Berceanu and Parveen introduced new configuration spaces. They stratify the classical configuration spaces $\mathcal{F}_h(\mathbb{CP}^n)$ with complex submanifolds $\mathcal{F}_h^i(\mathbb{CP}^n)$ defined as the ordered configuration spaces of all h points in \mathbb{CP}^n generating a projective subspace of dimension i. Then they compute the fundamental groups $\pi_1(\mathcal{F}_h^i(\mathbb{CP}^n))$ proving that they are trivial except when i = 1 providing, in this last case, a presentation for $\pi_1(\mathcal{F}_h^1(\mathbb{CP}^n))$ similar to those of the pure braid group of the sphere.

In a subsequent paper ([MPS]), the authors apply similar techniques to the affine case, i.e. to $\mathcal{F}_h(\mathbb{C}^n)$, showing that the situation is similar except in one case. More precisely they prove that, if $\mathcal{F}_h^{i,n} = \mathcal{F}_h^i(\mathbb{C}^n)$ denotes the ordered configuration space of all h points in \mathbb{C}^n generating an affine subspace of dimension i, then the spaces $\mathcal{F}_h^{i,n}$ are simply connected except for i = 1 or i = n = h - 1 and, in the last cases, they provide a presentation of the fundamental groups $\pi_1(\mathcal{F}_h^{i,n})$.

In this paper we generalize the result in [BaS] to the Grassmannian manifold Gr(k, n) parametrizing k-dimensional subspaces of \mathbb{C}^n . We define the *i*th ordered configuration space $\mathcal{F}_h^i(k, n)$ as the ordered configuration space of all distinct points H_1, \ldots, H_h in the Grassmannian Gr(k, n) such that the sum $(H_1 + \cdots + H_h)$ is an *i*-dimensional space.

We prove that the *i*th ordered configuration space $\mathcal{F}_{h}^{i}(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^{h}$ and we compute its dimension. As a corollary, we prove that if $n \neq hk$ and $i = \min(n,hk)$ then the *i*th ordered configuration space $\mathcal{F}_{h}^{i}(k,n)$ has trivial fundamental group except when n = 2 where we get the pure braid group of the sphere, that is

$$\pi_1(\mathcal{F}_h^{\min(n,hk)}(k,n)) = 0 \quad \text{if } (k,n) \neq (1,2) \pi_1(\mathcal{F}_1^1(1,2)) = \pi_1(\mathcal{F}_2(\mathbb{CP}^1)).$$
(1)

In the particular case of hyperplane arrangements, i.e. k = n - 1, we remark that the fundamental group of the *i*th ordered configuration space $\mathcal{F}_h^i(n-1,n)$ vanishes except when n = 2. Moreover using a dual argument we get similar results for the fundamental group of the ordered configuration space of all distinct k-dimensional subspaces H_1, \ldots, H_h in \mathbb{C}^n such that the intersection $(H_1 \cap \cdots \cap H_h)$ is an *i*-dimensional subspace.

We conjecture that a similar result to the one obtained in [BaS] for projective spaces holds also for Grassmannian manifolds and the fundamental group of the *i*th ordered configuration space $\mathcal{F}_h^i(k,n)$ vanishes except for low values of *i*. This will be the object of a forthcoming paper together with a generalization of the Pappus's construction in [BaS].

2 Main Section

For 0 < k < n, let us consider the Grassmannian manifold Gr(k, n) parametrizing k-dimensional subspaces of the n-dimensional complex space \mathbb{C}^n , and its ordered configuration spaces $\mathcal{F}_h(Gr(k, n))$.

The spaces $\mathcal{F}_h^i(k, n)$. Let's define the *i*th ordered configuration space $\mathcal{F}_h^i(k, n)$ as the space of all distinct points H_1, \ldots, H_h in the Grassmannian Gr(k, n) whose sum is an *i*-dimensional subspace of \mathbb{C}^n , i.e.

$$\mathcal{F}_h^i(k,n) = \{(H_1,\ldots,H_h) \in \mathcal{F}_h(Gr(k,n)) \mid \dim(H_1+\cdots+H_h) = i\}.$$

It is an easy remark that the following results hold:

- 1. in order to get a non empty set, h = 1 forces i = k and we have $\mathcal{F}_1^k(k,n) = Gr(k,n);$
- 2. in order to get a non empty set, i = 1 forces k = h = 1, and we have $\mathcal{F}_1^1(1, n) = Gr(1, n) = \mathbb{CP}^{n-1}$;
- 3. for $h \ge 2$, $\mathcal{F}_h^i(k, n) \neq \emptyset$ if and only if $i \ge k+1$ and $i \le \min(hk, n)$;

4. for $i = hk \leq n$, then the *h* subspaces giving a point of $\mathcal{F}_{h}^{hk}(k, n)$ are in direct sum;

5. for
$$h \ge 2$$
, $\mathcal{F}_h(Gr(k,n)) = \prod_{i=2}^n \mathcal{F}_h^i(k,n);$

6. for $h \ge 2$, the adjacency of the strata is given by

$$\overline{\mathcal{F}_h^i(k,n)} = \mathcal{F}_h^i(k,n) \coprod \mathcal{F}_h^{i-1}(k,n) \coprod \dots \coprod \mathcal{F}_h^2(k,n).$$

By the above remarks, it follows that the case h = 1 is trivial, so from now on we will consider h > 1 (and hence i > k).

We want to show that $\mathcal{F}_h^i(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ and compute its dimension. In order to do it we need to briefly recall a few easy facts and introduce some notation.

The determinantal variety. Let's recall that the determinantal variety $D_r(m, m')$ is the variety of $m \times m'$ matrices with complex entries of rank less than or equal to $r \leq \min(m, m')$. It is an analytic (algebraic, in fact) variety of dimension r(m + m' - r) whose set of singular points is given by those matrices of rank less than r. From now on, $D_r(m, m')^*$ will denote the set of non-singular points of the determinantal variety $D_r(m, m')$, that is the set of $m \times m'$ matrices of rank equal to r.

A system of local coordinates for $Gr(k, n)^h$. Let $V_0 \subset \mathbb{C}^n$ be a subspace of dimension dim $V_0 = n - k$, then the set

$$U_{V_0} = \{ H \in Gr(k, n) \mid H \oplus V_0 = \mathbb{C}^n \}$$

is an open dense subset of Gr(k, n).

Fix a basis $B = \{w_1, \ldots, w_k, v_1, \ldots, v_{n-k}\}$ of \mathbb{C}^n such that $\{v_1, \ldots, v_{n-k}\}$ is a basis of V_0 . Then we get a (complex) coordinate system on U_{V_0} as follows. Let H be an element in U_{V_0} , then the affine subspaces $V_0 + w_j$ intersect Hin one point u_j for any $j = 1, \cdots, k$ and $\{u_1, \cdots, u_k\}$ will be a basis of H. Hence H is uniquely determined by a $n \times k$ matrix of the form $\begin{pmatrix} I \\ A \end{pmatrix}$, where I is the $k \times k$ identity matrix and A is the $(n-k) \times k$ matrix of the coordinates of $u_1 - w_1, \ldots, u_k - w_k$ with respect to the vectors $\{v_1, \ldots, v_{n-k}\}$. The coefficients of A give complex coordinates in $U_{V_0} \cong \mathbb{C}^{k(n-k)}$. Let (H_1, \ldots, H_h) be a point in $Gr(k, n)^h$, the open sets U_{H_1}, \ldots, U_{H_h} in the Grassmannian manifold Gr(n-k, n) have non empty intersection, that is there exists an element $V_0 \in Gr(n-k, n)$ such that $V_0 \oplus H_j = \mathbb{C}^n$ for all $j = 1, \ldots, h$. Thus, $Gr(k, n)^h$ is covered by the open sets $U_{V_0}^h$ as V_0 varies in Gr(n-k, n). Taking a basis as defined above, each element in $U_{V_0}^h$ is uniquely determined by a $n \times hk$ matrix of the form $\begin{pmatrix} I & I & \cdots & I \\ A_1 & A_2 & \cdots & A_h \end{pmatrix}$ and the coefficients of $(A_1 \quad A_2 \quad \cdots \quad A_h)$ give complex coordinates in $U_{V_0}^h \cong \mathbb{C}^{hk(n-k)}$.

A system of local coordinates for $\mathcal{F}_{h}^{i}(k, n)$. In terms of the above coordinates, $(H_{1}, \ldots, H_{h}) \in U_{V_{0}}^{h}$ belongs to $\mathcal{F}_{h}^{i}(k, n)$ if and only if $A_{j} \neq A_{l}$ if $j \neq l$ and rank $\begin{pmatrix} I & I & \cdots & I \\ A_{1} & A_{2} & \cdots & A_{h} \end{pmatrix} = i$. Let us remark that rank $\begin{pmatrix} I & I & \cdots & I \\ A_{1} & A_{2} & \cdots & A_{h} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} I & I & \cdots & I \\ 0 & A_{2} - A_{1} & \cdots & A_{h} - A_{1} \end{pmatrix}$ $=k + \operatorname{rank} (A_{2} - A_{1} & \cdots & A_{h} - A_{1}).$

We can then change coordinates taking the coefficients of $B_j = A_j - A_1$ instead of those of A_j for j = 2, ..., h. In these new coordinates $U_{V_0} \cap \mathcal{F}_h^i(k, n)$ corresponds in $\mathbb{C}^{hk(n-k)}$ to $\mathbb{C}^{k(n-k)} \times D_{i-k}(n-k, hk-k)^*$ minus the closed sets given by $B_j = 0$ for $2 \leq j \leq h$ and by $B_j = B_l$ for $2 \leq j, l \leq h, j \neq l$. So far we have proved the following theorem.

Theorem 2.1. The *i*th ordered configuration space $\mathcal{F}_h^i(k,n)$ is a complex submanifold of the Grassmannian manifold Gr(k,n) of dimension

$$d_{h}^{i}(k,n) = i(n-i) + hk(i-k).$$

The computation on dimension comes from the easy equality

$$k(n-k) + (i-k)(n-k+hk-k-(i-k)) = i(n-i) + hk(i-k).$$

Moreover, $d_h^i(k,n) = hk(n-k)$ if and only if i = n or i = hk and so, as a function of i, $d_h^i(k,n)$ is strictly increasing for $i \leq \min(n,hk)$.

The fundamental group of $\mathcal{F}_{h}^{\min(n,hk)}(k,n)$. The space $\mathcal{F}_{h}^{\min(n,hk)}(k,n)$ is an open subset of the ordered configuration space $\mathcal{F}_{h}(Gr(k,n))$ and all other (non void) $\mathcal{F}_{h}^{j}(k,n)$ have strictly lower dimension. Moreover, if i = n the difference of dimensions $d_h^i(k,n) - d_h^{i-1}(k,n)$ equals 1 + hk - n and if i = hk it equals 1 + n - hk. Then if $n \neq hk$, all (non void) $\mathcal{F}_h^j(k,n)$ with $j < \min(n,hk)$ have real codimension at least 4 in $\mathcal{F}_h(Gr(k,n))$. Then, if $n \neq hk$ and $i = \min(n,hk)$, the fundamental group of $\mathcal{F}_h^i(k,n) = \mathcal{F}_h(Gr(k,n)) \setminus \overline{\mathcal{F}_h^{i-1}(k,n)}$ is the same as the fundamental group of $\mathcal{F}_h(Gr(k,n))$ (since, by the adjacency of the strata, the closure $\overline{\mathcal{F}_h^{i-1}(k,n)}$ is the finite union of complex subvarieties of $\mathcal{F}_h(Gr(k,n))$ of real codimension at least 4).

Let us recall that the complex Grassmannian manifolds Gr(k, n) are simply connected and have real dimension at least 4 except $Gr(1, 2) = \mathbb{CP}^1$ and that for a simply connected manifold of real dimension at least 3 the pure braid groups vanish, i.e. $\pi_1(\mathcal{F}_h(Gr(k, n))) = 0$ if $(k, n) \neq (1, 2)$. Then we get the following corollary:

Corollary 2.2. The fundamental group of the *i*th ordered configuration space $\mathcal{F}_h^i(k,n)$ vanishes if $n \neq hk$ and $i = \min(n,hk)$ except when n = 2 for which we get the pure braid group of the sphere.

The dual case. Let $Gr(k, n)^*$ be the Grassmannian manifold parametrizing k-dimensional subspaces in the dual space $(\mathbb{C}^n)^*$. Then we can define the *i*th dual ordered configuration space $\mathcal{F}_h^i(k, n)^*$ as

$$\mathcal{F}_{h}^{i}(k,n)^{*} = \{(H_{1},\ldots,H_{h}) \in \mathcal{F}_{h}(Gr(k,n)^{*}) \mid \dim(H_{1}\cap\cdots\cap H_{h}) = i\}.$$

The spaces $\mathcal{F}_h^i(k,n)^*$ stratify the ordered configuration space $\mathcal{F}_h(Gr(k,n)^*)$ of the Grassmannian manifold $Gr(k,n)^*$.

Taking annihilators, we get homeomorphisms Ann: $Gr(n-k,n) \to Gr(k,n)^*$ which induce homeomorphisms between the (n-i)th ordered configuration space $F_h^{n-i}(n-k,n)$ and the *i*th dual ordered configuration space $F_h^i(k,n)^*$. As a consequence we get that the spaces $F_h^{\max(0,n-hk)}(n-k,n)^*$ are simply connected manifolds except when n = 2. In this case the fundamental group is the pure braid group of the sphere.

*i*th ordered configuration spaces of hyperplane arrangements. Let us remark that when k = n - 1 we get an *h*-uple of hyperplanes in \mathbb{C}^n , i.e. an ordered arrangement of hyperplanes. In this case, if h = 1 then i = n - 1 and we get that the *i*th ordered configuration space is simply the Grassmannian manifold, i.e. $\mathcal{F}_1^{n-1}(n-1,n) = Gr(n-1,n)$. If h > 1, since the sum of two (different) hyperplanes is the whole space \mathbb{C}^n , we get that $h \ge 2$ forces i = nand the following equalities hold

$$\mathcal{F}_h^n(n-1,n) = \mathcal{F}_h(Gr(n-1,n)) = \mathcal{F}_h(\mathbb{C}\mathbb{P}^{n-1}).$$

Hence, the fundamental group of the *i*th ordered configuration space of hyperplane arrangements $\mathcal{F}_h^i(n-1,n)$ vanishes except when n=2. In this case we get the fundamental group of the sphere \mathbb{CP}^1 .

Taking duals, we have $\mathcal{F}_h^i(n-1,n)^* \cong \mathcal{F}_h^{n-i}(1,n)$ and the fundamental groups of the latter spaces have been computed in [BaS]. We get that the space of *h*-uples of distinct hyperplanes in \mathbb{C}^n whose intersection has dimension equal to *i* is simply connected except for i = n - 1.

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