

Manfredini, Sandro; Settepanella, Simona

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Institute of Economics
Scuola Superiore Sant'Anna

Piazza Martiri della Libertà, 33 - 56127 Pisa, Italy
ph. +39 050 88.33.43
institute.economics@sssup.it

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WORKING PAPER SERIES

On the Configuration Spaces of Grassmannian Manifolds

Sandro Manfredini[†]
Simona Settepanella[°]

[†]Department of Mathematics, University of Pisa, Italy

[°] Institute of Economics and LEM, Scuola Superiore Sant'Anna, Pisa, Italy

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On the Configuration Spaces of Grassmannian Manifolds

Sandro MANFREDINI* Simona SETTEPANELLA†

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Abstract

Let $\mathcal{F}_h^i(k, n)$ be the i th ordered configuration space of all distinct points $H_1, \dots, H_h$ in the Grassmannian $Gr(k, n)$ of k -dimensional subspaces of $\mathbb{C}^n$, whose sum is a subspace of dimension i . We prove that $\mathcal{F}_h^i(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ of dimension $i(n - i) + hk(i - k)$ and its fundamental group is trivial if $i = \min(n, hk)$, $hk \neq n$ and $n > 2$ and equal to the braid group of the sphere $\mathbb{C}P^1$ if $n = 2$. Eventually we compute the fundamental group in the special case of hyperplane arrangements, i.e. $k = n - 1$.

Keywords:

complex space, configuration spaces,
braid groups.

MSC (2010): 20F36, 52C35, 57M05, 51A20.

*Department of Mathematics, University of Pisa, manfredi@dm.unipi.it

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1 Introduction

Let M be a manifold. The *ordered configuration space* of h distinct points in M , $\mathcal{F}_h(M) = \{(x_1, \dots, x_h) \in M^h \mid x_i \neq x_j, i \neq j\}$, has been widely studied after it has been introduced by Fadell and Neuwirth [FaN] and Fadell [Fa] in the sixties. It is well known that for a simply connected manifold M of dimension ≥ 3 , the *pure braid group* on h strings of M , $\pi_1(\mathcal{F}_h(M))$, is trivial, while in low dimension there are non trivial pure braids. For example, the pure braid group of the sphere $S^2 \approx \mathbb{C}P^1$ is not trivial with the following presentation:

$$\pi_1(\mathcal{F}_h(\mathbb{C}P^1)) \cong \langle \alpha_{ij}, 1 \leq i < j \leq h-1 \mid (YB\,3)_{h-1}, (YB\,4)_{h-1}, D_{h-1}^2 = 1 \rangle$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{24}\alpha_{34}) \cdots (\alpha_{1k}\alpha_{2k} \cdots \alpha_{k-1\,k})$ (see [Bi] and [FaH] for the Yang-Baxter relations $(YB, 3)_{h-1}, (YB, 4)_{h-1}$).

In a recent paper ([BaS]) Berceanu and Parveen introduced new configuration spaces. They stratify the classical configuration spaces $\mathcal{F}_h(\mathbb{C}P^n)$ with complex submanifolds $\mathcal{F}_h^i(\mathbb{C}P^n)$ defined as the ordered configuration spaces of all h points in $\mathbb{C}P^n$ generating a projective subspace of dimension i . Then they compute the fundamental groups $\pi_1(\mathcal{F}_h^i(\mathbb{C}P^n))$ proving that they are trivial except when $i = 1$ providing, in this last case, a presentation for $\pi_1(\mathcal{F}_h^1(\mathbb{C}P^n))$ similar to those of the pure braid group of the sphere.

In a subsequent paper ([MPS]), the authors apply similar techniques to the affine case, i.e. to $\mathcal{F}_h(\mathbb{C}^n)$, showing that the situation is similar except in one case. More precisely they prove that, if $\mathcal{F}_h^{i,n} = \mathcal{F}_h^i(\mathbb{C}^n)$ denotes the ordered configuration space of all h points in $\mathbb{C}^n$ generating an affine subspace of dimension i , then the spaces $\mathcal{F}_h^{i,n}$ are simply connected except for $i = 1$ or $i = n = h-1$ and, in the last cases, they provide a presentation of the fundamental groups $\pi_1(\mathcal{F}_h^{i,n})$.

In this paper we generalize the result in [BaS] to the Grassmannian manifold $Gr(k, n)$ parametrizing k -dimensional subspaces of $\mathbb{C}^n$. We define the i th ordered configuration space $\mathcal{F}_h^i(k, n)$ as the ordered configuration space of all distinct points $H_1, \dots, H_h$ in the Grassmannian $Gr(k, n)$ such that the sum $(H_1 + \cdots + H_h)$ is an i -dimensional space.

We prove that the i th ordered configuration space $\mathcal{F}_h^i(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ and we compute its dimension. As a corollary, we prove that if $n \neq hk$ and $i = \min(n, hk)$ then the i th ordered configuration space $\mathcal{F}_h^i(k, n)$ has trivial fundamental group except

when $n = 2$ where we get the pure braid group of the sphere, that is

$$\begin{aligned}\pi_1(\mathcal{F}_h^{\min(n,hk)}(k,n)) &= 0 \quad \text{if } (k,n) \neq (1,2) \\ \pi_1(\mathcal{F}_1^1(1,2)) &= \pi_1(\mathcal{F}_2(\mathbb{CP}^1)).\end{aligned}\tag{1}$$

In the particular case of hyperplane arrangements, i.e. $k = n - 1$, we remark that the fundamental group of the i th ordered configuration space $\mathcal{F}_h^i(n-1, n)$ vanishes except when $n = 2$. Moreover using a dual argument we get similar results for the fundamental group of the ordered configuration space of all distinct k -dimensional subspaces $H_1, \dots, H_h$ in $\mathbb{C}^n$ such that the intersection $(H_1 \cap \dots \cap H_h)$ is an i -dimensional subspace.

We conjecture that a similar result to the one obtained in [BaS] for projective spaces holds also for Grassmannian manifolds and the fundamental group of the i th ordered configuration space $\mathcal{F}_h^i(k, n)$ vanishes except for low values of i . This will be the object of a forthcoming paper together with a generalization of the Pappus's construction in [BaS].

2 Main Section

For $0 < k < n$, let us consider the Grassmannian manifold $Gr(k, n)$ parametrizing k -dimensional subspaces of the n -dimensional complex space $\mathbb{C}^n$, and its ordered configuration spaces $\mathcal{F}_h(Gr(k, n))$.

The spaces $\mathcal{F}_h^i(k, n)$. Let's define the i th ordered configuration space $\mathcal{F}_h^i(k, n)$ as the space of all distinct points $H_1, \dots, H_h$ in the Grassmannian $Gr(k, n)$ whose sum is an i -dimensional subspace of $\mathbb{C}^n$, i.e.

$$\mathcal{F}_h^i(k, n) = \{(H_1, \dots, H_h) \in \mathcal{F}_h(Gr(k, n)) \mid \dim(H_1 + \dots + H_h) = i\}.$$

It is an easy remark that the following results hold:

1. in order to get a non empty set, $h = 1$ forces $i = k$ and we have $\mathcal{F}_1^k(k, n) = Gr(k, n)$;
2. in order to get a non empty set, $i = 1$ forces $k = h = 1$, and we have $\mathcal{F}_1^1(1, n) = Gr(1, n) = \mathbb{CP}^{n-1}$;
3. for $h \geq 2$, $\mathcal{F}_h^i(k, n) \neq \emptyset$ if and only if $i \geq k + 1$ and $i \leq \min(hk, n)$;

4. for $i = hk \leq n$, then the h subspaces giving a point of $\mathcal{F}_h^{hk}(k, n)$ are in direct sum;
5. for $h \geq 2$, $\mathcal{F}_h(Gr(k, n)) = \coprod_{i=2}^n \mathcal{F}_h^i(k, n)$;
6. for $h \geq 2$, the adjacency of the strata is given by

$$\overline{\mathcal{F}_h^i(k, n)} = \mathcal{F}_h^i(k, n) \coprod \mathcal{F}_h^{i-1}(k, n) \coprod \dots \coprod \mathcal{F}_h^2(k, n).$$

By the above remarks, it follows that the case $h = 1$ is trivial, so from now on we will consider $h > 1$ (and hence $i > k$).

We want to show that $\mathcal{F}_h^i(k, n)$ is (when non empty) a complex submanifold of $Gr(k, n)^h$ and compute its dimension. In order to do it we need to briefly recall a few easy facts and introduce some notation.

The determinantal variety. Let's recall that the determinantal variety $D_r(m, m')$ is the variety of $m \times m'$ matrices with complex entries of rank less than or equal to $r \leq \min(m, m')$. It is an analytic (algebraic, in fact) variety of dimension $r(m + m' - r)$ whose set of singular points is given by those matrices of rank less than r . From now on, $D_r(m, m')^*$ will denote the set of non-singular points of the determinantal variety $D_r(m, m')$, that is the set of $m \times m'$ matrices of rank equal to r .

A system of local coordinates for $Gr(k, n)^h$. Let $V_0 \subset \mathbb{C}^n$ be a subspace of dimension $\dim V_0 = n - k$, then the set

$$U_{V_0} = \{H \in Gr(k, n) \mid H \oplus V_0 = \mathbb{C}^n\}$$

is an open dense subset of $Gr(k, n)$.

Fix a basis $B = \{w_1, \dots, w_k, v_1, \dots, v_{n-k}\}$ of $\mathbb{C}^n$ such that $\{v_1, \dots, v_{n-k}\}$ is a basis of V_0 . Then we get a (complex) coordinate system on U_{V_0} as follows. Let H be an element in U_{V_0} , then the affine subspaces $V_0 + w_j$ intersect H in one point u_j for any $j = 1, \dots, k$ and $\{u_1, \dots, u_k\}$ will be a basis of H . Hence H is uniquely determined by a $n \times k$ matrix of the form $\begin{pmatrix} I \\ A \end{pmatrix}$, where I is the $k \times k$ identity matrix and A is the $(n - k) \times k$ matrix of the coordinates of $u_1 - w_1, \dots, u_k - w_k$ with respect to the vectors $\{v_1, \dots, v_{n-k}\}$. The coefficients of A give complex coordinates in $U_{V_0} \cong \mathbb{C}^{k(n-k)}$.

Let $(H_1, \dots, H_h)$ be a point in $Gr(k, n)^h$, the open sets $U_{H_1}, \dots, U_{H_h}$ in the Grassmannian manifold $Gr(n-k, n)$ have non empty intersection, that is there exists an element $V_0 \in Gr(n-k, n)$ such that $V_0 \oplus H_j = \mathbb{C}^n$ for all $j = 1, \dots, h$. Thus, $Gr(k, n)^h$ is covered by the open sets $U_{V_0}^h$ as V_0 varies in $Gr(n-k, n)$. Taking a basis as defined above, each element in $U_{V_0}^h$ is uniquely determined by a $n \times hk$ matrix of the form $\begin{pmatrix} I & I & \cdots & I \\ A_1 & A_2 & \cdots & A_h \end{pmatrix}$ and the coefficients of $(A_1 \ A_2 \ \cdots \ A_h)$ give complex coordinates in $U_{V_0}^h \cong \mathbb{C}^{hk(n-k)}$.

A system of local coordinates for $\mathcal{F}_h^i(k, n)$. In terms of the above coordinates, $(H_1, \dots, H_h) \in U_{V_0}^h$ belongs to $\mathcal{F}_h^i(k, n)$ if and only if $A_j \neq A_l$ if $j \neq l$ and $\text{rank} \begin{pmatrix} I & I & \cdots & I \\ A_1 & A_2 & \cdots & A_h \end{pmatrix} = i$. Let us remark that

$$\begin{aligned} \text{rank} \begin{pmatrix} I & I & \cdots & I \\ A_1 & A_2 & \cdots & A_h \end{pmatrix} &= \text{rank} \begin{pmatrix} I & I & \cdots & I \\ 0 & A_2 - A_1 & \cdots & A_h - A_1 \end{pmatrix} \\ &= k + \text{rank} (A_2 - A_1 \ \cdots \ A_h - A_1). \end{aligned}$$

We can then change coordinates taking the coefficients of $B_j = A_j - A_1$ instead of those of A_j for $j = 2, \dots, h$. In these new coordinates $U_{V_0} \cap \mathcal{F}_h^i(k, n)$ corresponds in $\mathbb{C}^{hk(n-k)}$ to $\mathbb{C}^{k(n-k)} \times D_{i-k}(n-k, hk-k)^*$ minus the closed sets given by $B_j = 0$ for $2 \leq j \leq h$ and by $B_j = B_l$ for $2 \leq j, l \leq h, j \neq l$. So far we have proved the following theorem.

Theorem 2.1. *The i th ordered configuration space $\mathcal{F}_h^i(k, n)$ is a complex submanifold of the Grassmannian manifold $Gr(k, n)$ of dimension*

$$d_h^i(k, n) = i(n-i) + hk(i-k).$$

The computation on dimension comes from the easy equality

$$k(n-k) + (i-k)(n-k+hk-k-(i-k)) = i(n-i) + hk(i-k).$$

Moreover, $d_h^i(k, n) = hk(n-k)$ if and only if $i = n$ or $i = hk$ and so, as a function of i , $d_h^i(k, n)$ is strictly increasing for $i \leq \min(n, hk)$.

The fundamental group of $\mathcal{F}_h^{\min(n, hk)}(k, n)$. The space $\mathcal{F}_h^{\min(n, hk)}(k, n)$ is an open subset of the ordered configuration space $\mathcal{F}_h(Gr(k, n))$ and all other (non void) $\mathcal{F}_h^j(k, n)$ have strictly lower dimension. Moreover, if $i = n$ the

difference of dimensions $d_h^i(k, n) - d_h^{i-1}(k, n)$ equals $1 + hk - n$ and if $i = hk$ it equals $1 + n - hk$. Then if $n \neq hk$, all (non void) $\mathcal{F}_h^j(k, n)$ with $j < \min(n, hk)$ have real codimension at least 4 in $\mathcal{F}_h(Gr(k, n))$. Then, if $n \neq hk$ and $i = \min(n, hk)$, the fundamental group of $\mathcal{F}_h^i(k, n) = \mathcal{F}_h(Gr(k, n)) \setminus \overline{\mathcal{F}_h^{i-1}(k, n)}$ is the same as the fundamental group of $\mathcal{F}_h(Gr(k, n))$ (since, by the adjacency of the strata, the closure $\overline{\mathcal{F}_h^{i-1}(k, n)}$ is the finite union of complex subvarieties of $\mathcal{F}_h(Gr(k, n))$ of real codimension at least 4).

Let us recall that the complex Grassmannian manifolds $Gr(k, n)$ are simply connected and have real dimension at least 4 except $Gr(1, 2) = \mathbb{CP}^1$ and that for a simply connected manifold of real dimension at least 3 the pure braid groups vanish, i.e. $\pi_1(\mathcal{F}_h(Gr(k, n))) = 0$ if $(k, n) \neq (1, 2)$.

Then we get the following corollary:

Corollary 2.2. *The fundamental group of the i th ordered configuration space $\mathcal{F}_h^i(k, n)$ vanishes if $n \neq hk$ and $i = \min(n, hk)$ except when $n = 2$ for which we get the pure braid group of the sphere.*

The dual case. Let $Gr(k, n)^*$ be the Grassmannian manifold parametrizing k -dimensional subspaces in the dual space $(\mathbb{C}^n)^*$. Then we can define the i th dual ordered configuration space $\mathcal{F}_h^i(k, n)^*$ as

$$\mathcal{F}_h^i(k, n)^* = \{(H_1, \dots, H_h) \in \mathcal{F}_h(Gr(k, n)^*) \mid \dim(H_1 \cap \dots \cap H_h) = i\}.$$

The spaces $\mathcal{F}_h^i(k, n)^*$ stratify the ordered configuration space $\mathcal{F}_h(Gr(k, n)^*)$ of the Grassmannian manifold $Gr(k, n)^*$.

Taking annihilators, we get homeomorphisms $\text{Ann}: Gr(n - k, n) \rightarrow Gr(k, n)^*$ which induce homeomorphisms between the $(n - i)$ th ordered configuration space $F_h^{n-i}(n - k, n)$ and the i th dual ordered configuration space $\mathcal{F}_h^i(k, n)^*$.

As a consequence we get that the spaces $F_h^{\max(0, n-hk)}(n - k, n)^*$ are simply connected manifolds except when $n = 2$. In this case the fundamental group is the pure braid group of the sphere.

i th ordered configuration spaces of hyperplane arrangements. Let us remark that when $k = n - 1$ we get an h -uple of hyperplanes in $\mathbb{C}^n$, i.e. an ordered arrangement of hyperplanes. In this case, if $h = 1$ then $i = n - 1$ and we get that the i th ordered configuration space is simply the Grassmannian manifold, i.e. $\mathcal{F}_1^{n-1}(n - 1, n) = Gr(n - 1, n)$. If $h > 1$, since the sum of two

(different) hyperplanes is the whole space $\mathbb{C}^n$, we get that $h \geq 2$ forces $i = n$ and the following equalities hold

$$\mathcal{F}_h^n(n-1, n) = \mathcal{F}_h(Gr(n-1, n)) = \mathcal{F}_h(\mathbb{CP}^{n-1}).$$

Hence, the fundamental group of the i th ordered configuration space of hyperplane arrangements $\mathcal{F}_h^i(n-1, n)$ vanishes except when $n = 2$. In this case we get the fundamental group of the sphere $\mathbb{CP}^1$.

Taking duals, we have $\mathcal{F}_h^i(n-1, n)^* \cong \mathcal{F}_h^{n-i}(1, n)$ and the fundamental groups of the latter spaces have been computed in [BaS]. We get that the space of h -uples of distinct hyperplanes in $\mathbb{C}^n$ whose intersection has dimension equal to i is simply connected except for $i = n - 1$.

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