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# Stochastic Differential Utility as the Continuous-Time Limit of Recursive Utility

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# STOCHASTIC DIFFERENTIAL UTILITY AS THE CONTINUOUS-TIME LIMIT OF RECURSIVE UTILITY

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ABSTRACT. We establish a convergence theorem that shows that discrete-time recursive utility, as developed by Kreps and Porteus (1978), converges to stochastic differential utility, as introduced by Duffie and Epstein (1992), in the continuous-time limit of vanishing grid size.

KEY WORDS: Stochastic differential utility, recursive utility, convergence, backward stochastic differential equation.

JEL CLASSIFICATION: D81, D91

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## 1. INTRODUCTION

Recursive preferences have become important in a large number of applications including macroeconomic and finance theory, and, in particular, equilibrium asset pricing. In the literature, the notion of stochastic differential utility has been accepted as the continuous-time analog of recursive utility. This correspondence implicitly underlies a large number of applications of stochastic differential utility. However, a rigorous proof of this connection has not been given yet. This paper closes that gap: We show that, in a general semimartingale framework and under standard assumptions on the aggregator, stochastic differential utility is the continuous-time limit of recursive utility.

It is well-known that the standard discounted expected utility paradigm restricts the relationship between preferences for smoothing across time and across states. To address this issue, the concept of discrete-time recursive utility was developed by Kreps and Porteus (1978), Epstein and Zin (1989), Weil (1990), and others, making it possible to disentangle risk aversion from the elasticity of intertemporal substitution. Stochastic differential utility was introduced by Epstein (1987) in a deterministic setting and by Duffie and Epstein (1992) in a stochastic setting as a continuous-time version of recursive utility. Epstein (1987), Duffie and Epstein (1992) and the subsequent literature define stochastic differential utility axiomatically in continuous time, but do not establish a rigorous connection to discrete-time recursive utility. Heuristic links to recursive utility are provided in Duffie and Epstein (1992), Svensson (1989) and Skiadas (2008).

The limitations of discounted expected utility are particularly apparent in an important area of research: the theory of equilibrium asset pricing. The implications of expected utility are known to be incompatible with various stylized facts in empirical findings; for instance, the excess return of stocks implied by expected utility is much too high for realistic risk aversion parameters (this is the “equity premium puzzle” of Mehra and Prescott (1985)). In the last 25 years, recursive preferences have therefore become a key ingredient in the asset pricing literature; see, e.g., Duffie and Epstein (1992b), Obstfeld (1994), Tallarini (2000), Bansal and Yaron (2004), Uhlig (2007), Hansen, Heaton, and Li (2008), Guvenen (2009), Kaltenbrunner and Lochstoer (2010), Gabaix (2012), Borovička, Hansen, Hendricks, and Scheinkman (2011) and Wachter (2011). These papers demonstrate that recursive preferences are a highly relevant modeling tool which allows researchers to address various open questions in economics and finance. Despite their empirical relevance, the literature lacks a rigorous result relating discrete-time recursive utility to stochastic differential utility. Thus, so far discrete-time and continuous-time models coexist (e.g., Bansal and Yaron (2004) is a recent discrete-time model, while Wachter (2011) is set in continuous time).<sup>1</sup>

Our main result (Theorem 4.1) shows that, under general conditions, the Kreps-Porteus recursive utilities associated to a sequence of approximating consumption plans converge to the stochastic differential utility of the limiting consumption plan as the

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<sup>1</sup>This is in stark contrast to, for instance, the theory of option pricing: There are several results, including, e.g., Cox, Ross, and Rubinstein (1979) and Duffie and Protter (1992), on the convergence of option prices in discrete-time models to their continuous-time counterparts.

grid size tends to zero. We also provide a rigorous justification of a classical formula, first derived by Epstein (1987) in a deterministic setting, that is commonly used to obtain a continuous-time aggregator from its discrete-time counterpart. Theorem 4.1 is based on assumptions that may be violated in some specific applications. We therefore also provide a convergence theorem that applies to bounded consumption plans under milder assumptions on the aggregator. In particular, we apply our convergence results to the empirically important Epstein-Zin-Weil parametrization of recursive utility. Finally, we provide a general sufficient condition for recursive utility with a nonstandard certainty equivalent to converge to stochastic differential utility.

The remainder of this paper is structured as follows: In Section 2 we fix the framework for the paper's analysis. Section 3 briefly reviews recursive and stochastic differential utility. In Section 4 we present our main convergence result. Section 5 contains the proof. Section 6 establishes a more general convergence result for bounded consumption plans. In Section 7 we provide a basis for further research on convergence with nonstandard certainty equivalents, and Section 8 concludes.

## 2. MATHEMATICAL SETTING

We work on a filtered probability space  $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$  whose filtration  $\mathfrak{F} = \{\mathfrak{F}_t\}_{t \in [0, T]}$  satisfies the “usual conditions” of right-continuity and  $\mathbb{P}$ -completeness. We assume that  $\mathfrak{F}_0$  is  $\mathbb{P}$ -trivial, and whenever  $M = \{M_t\}_{t \in [0, T]}$  is an  $(\mathfrak{F}, \mathbb{P})$ -martingale, we work with a càdlàg version of  $M$ .

**Consumption Plans.** Let  $\mathbf{C} \subseteq \mathbb{R}$  be an interval modeling the consumption space (typically,  $\mathbf{C} \subseteq [0, \infty)$ ). We call a consumption rate process  $c = \{c_t\}_{t \in [0, T]}$  **feasible** if  $c$  is  $\mathbf{C}$ -valued and  $\mathfrak{F}$ -progressively measurable with

$$(1) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} |c_t|^2 \right] < \infty.$$

We denote by  $\mathcal{A}$  the class of feasible consumption plans. Moreover we fix a terminal payoff  $X$ , given as a  $\mathbf{C}$ -valued  $\mathfrak{F}_T$ -measurable random variable with  $\mathbb{E}[|X|^2] < \infty$ .

**Approximating Sequences.** We study an arbitrary but fixed sequence  $\{\pi^N\}_{N \in \mathbb{N}}$  of partitions  $\pi^N = [t_0^N, \dots, t_N^N]$  of  $[0, T]$  where<sup>2</sup>  $0 = t_0^N \leq t_1^N \leq \dots \leq t_N^N = T$  and  $\pi^N \subseteq \pi^{N+1}$ . We set  $\Delta_k^N \triangleq t_{k+1}^N - t_k^N$ ,  $k = 0, \dots, N-1$  and assume that

$$\Delta^N \triangleq \max_{k=0, \dots, N-1} \Delta_k^N \rightarrow 0 \text{ as } N \rightarrow \infty.$$

A sequence  $\{c^N\}_{N \in \mathbb{N}}$  of  $\mathbf{C}$ -valued adapted processes is said to be an **approximating sequence** for the feasible plan  $c$  if  $c^N$  is right-continuous and piecewise constant on  $\pi^N$  for each  $N \in \mathbb{N}$  and<sup>3</sup>

$$(2) \quad c^N \rightarrow c \text{ a.e. on } [0, T] \times \Omega \text{ as } N \rightarrow \infty$$

subject to the integrability condition

$$(3) \quad \mathbb{E} \left[ \sup_{N \in \mathbb{N}, t \in [0, T]} |c_t^N|^2 \right] < \infty.$$

Note that (2) and (3) imply (1) for a suitable modification of  $c$ .

<sup>2</sup>To simplify notation, we assume that  $\pi^N$  has exactly  $N+1$  grid points.

<sup>3</sup> $[0, T] \times \Omega$  is endowed with the product measure  $dt \otimes \mathbb{P}$ .

*Examples of Approximating Sequences.* (a) If  $c \in \mathcal{A}$  has paths with left and right limits and  $c^N$  is the discretization of  $c$  along  $\pi^N$ , i.e.

$$c_t^N \triangleq c_{t_k^N} \text{ for } t \in [t_k^N, t_{k+1}^N), \quad k = 0, \dots, N-1$$

then  $\{c^N\}_{N \in \mathbb{N}}$  constitutes an approximating sequence.

(b) The approach in (a) applies in particular to piecewise-constant consumption plans. For instance,  $c$  may be represented by a continuous-time Markov chain. In this case,  $c^N$  represents the canonical tree-type approximation of  $c$ .

(c) If  $\{c^N\}_{N \in \mathbb{N}}$  is an approximating sequence for  $c$ , then  $\{\bar{c}^N\}_{N \in \mathbb{N}}$  is also one where  $\bar{c}^N$  is the space discretization of  $c^N$  along  $\zeta^N$ , i.e.

$$\bar{c}_t^N \triangleq z_k^N \text{ if } c_t^N \in [z_k^N, z_{k+1}^N) \text{ for } t \in [0, T]$$

with a sequence  $\{\zeta^N\}_{N \in \mathbb{N}}$  of partitions  $\zeta^N = [z_0^N, \dots, z_N^N]$  of the consumption space  $\mathbf{C}$  such that  $\max_{k=0, \dots, N-1} |z_{k+1}^N - z_k^N| \rightarrow 0$  as  $N \rightarrow \infty$ .

(d) In combination with (a), case (c) covers binomial tree-type approximations to Brownian models, given suitable integrability conditions.

Note that (b) and (d) correspond to the settings analyzed by Skiadas (2013).

### 3. RECURSIVE UTILITY AND STOCHASTIC DIFFERENTIAL UTILITY

To model preferences for intertemporal consumption in a stochastic setting, we are generally interested in mappings of the form

$$\mathbf{v} : \mathcal{A} \rightarrow \mathbb{R}, \quad c \mapsto \mathbf{v}(c)$$

such that  $c$  is preferred to  $c'$  if and only if  $\mathbf{v}(c) > \mathbf{v}(c')$ . If such a representation is available, the functional  $\mathbf{v}$  is referred to as a **utility index**. Clearly, if  $\mathbf{v}$  and  $\tilde{\mathbf{v}}$  are utility indices and there exists a strictly increasing function  $\varphi$  such that  $\tilde{\mathbf{v}} = \varphi \circ \mathbf{v}$ , then they describe the same preferences; in this case,  $\mathbf{v}$  and  $\tilde{\mathbf{v}}$  are said to be **ordinally equivalent**.

#### 3.1 RECURSIVE UTILITY

Recursive utility is a paradigm to construct utility indices in a discrete-time framework. Following Kreps and Porteus (1978), its two main components are

(i) a discrete-time **intertemporal aggregator**  $W$ , i.e. a mapping

$$W : [0, T] \times \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}, \quad (\Delta, c, v) \mapsto W(\Delta, c, v)$$

with  $W(0, c, v) = v$  for all  $(c, v) \in \mathbf{C} \times \mathbf{C}$ ; and

(ii) a **certainty equivalent**  $\mathbf{m}$ , i.e. a mapping

$$\mathbf{m} : \mathcal{M}(\mathbf{C}) \rightarrow \mathbf{C}, \quad \mu \mapsto \mathbf{m}(\mu)$$

such that  $\mathbf{m}(\delta_c) = c$  for all  $c \in \mathbf{C}$ , where  $\mathcal{M}(\mathbf{C})$  denotes the space of all probability distributions on  $\mathbf{C}$  such that  $\int_{\mathbf{C}} |v|^2 \mu(dv) < \infty$  and  $\delta_c$  is the Dirac measure at  $c$ .

The pair  $(W, \mathbf{m})$  is referred to as a (discrete-time) **aggregator**. The current utility of consumption  $c\Delta$  during a time interval of length  $\Delta$  and a random payoff with distribution  $\mu$  at the end of this interval is computed via the aggregation

$$(\Delta, c, \mu) \mapsto W(\Delta, c, \mathbf{m}(\mu)).$$

More precisely, given an aggregator  $(W, \mathbf{m})$  the recursive utility index of a feasible consumption stream  $c^N$  that is piecewise constant on  $\pi^N$  is given by

$$\mathbf{v}^N(c^N) \triangleq V_0^N$$

with the continuation value process  $V^N = \{V_{t_k}^N\}_{k=0, \dots, N-1}$  defined recursively via<sup>4</sup>

$$(4) \quad V_{t_k}^N = W\left(\Delta_k^N, c_{t_k}^N, \mathbf{m}(\mathcal{L}_{t_k}^N(V_{t_{k+1}}^N))\right), \quad k = N-1, \dots, 0, \quad V_{t_N}^N = X.$$

Here  $\mathcal{L}_t(Y)$  denotes the conditional distribution, given time- $t$  information  $\mathfrak{F}_t$ , of the random variable  $Y$ . Thus at each time  $t_k^N$  the agent aggregates current consumption  $c_{t_k}^N$  with the time- $t_k^N$  certainty equivalent  $\mathbf{m}(\mathcal{L}_{t_k}^N(V_{t_{k+1}}^N))$  of consumption starting in the next period.

The recursion (4) shows that  $W$  models intertemporal preferences, while  $\mathbf{m}$  captures risk preferences. In Sections 3 – 6 we assume that  $\mathbf{m}$  can be represented in the form characterized by Kreps and Porteus (1978):

$$(5) \quad \mathbf{m}(\mu) = u^{-1}\left(\int_{\mathcal{C}} u(v) \mu(dv)\right), \quad \mu \in \mathcal{M}(\mathcal{C})$$

with  $u : \mathcal{C} \rightarrow \mathbb{R}$  a strictly increasing continuous function of linear growth.<sup>5</sup> The interpretation is that preferences between (atemporal) random payoffs  $Y$  and  $Z$  are characterized by their expected utilities  $\mathbb{E}[u(Y)]$  and  $\mathbb{E}[u(Z)]$ .

**Normalization.** The transformation  $v \mapsto \tilde{v} \triangleq u(v)$  leads to the aggregation

$$(\Delta, c, \mu) \mapsto \tilde{W}(\Delta, c, \mathbf{m}_0(\mu))$$

corresponding to the aggregator  $(\tilde{W}, \mathbf{m}_0)$  where

$$\tilde{W} : [0, T] \times \mathcal{C} \times \mathbf{V} \rightarrow \mathbf{V}, \quad \tilde{W}(\Delta, c, v) \triangleq u(W(\Delta, c, u^{-1}(v)))$$

and  $\mathbf{m}_0$  denotes the risk-neutral certainty equivalent on  $\mathbf{V} \triangleq u(\mathcal{C})$ ,

$$\mathbf{m}_0(\mu) \triangleq \int_{\mathbf{V}} v \mu(dv), \quad \mu \in \mathcal{M}(\mathbf{V}).$$

The backward recursion (4) for  $\tilde{V}^N = \{\tilde{V}_{t_k}^N\}_{k=0, \dots, N}$ ,  $\tilde{V}_{t_k}^N = u(V_{t_k}^N)$  now reads

$$(6) \quad \tilde{V}_{t_k}^N = \tilde{W}\left(\Delta_k^N, c_{t_k}^N, \mathbb{E}_{t_k}^N[\tilde{V}_{t_{k+1}}^N]\right), \quad k = N-1, \dots, 0, \quad \tilde{V}_{t_N}^N = \xi$$

and we set  $\tilde{\mathbf{v}}^N(c^N, X) \triangleq \tilde{V}_0^N$ . Here  $\mathbb{E}_t$  denotes conditional expectation given  $\mathfrak{F}_t$  and  $\xi \triangleq u(X)$ . Importantly, the aggregator  $(W, \mathbf{m})$  describes (via (4)) the *same* preferences as  $(\tilde{W}, \mathbf{m}_0)$  (via (6)): The corresponding utility indices are linked via  $\tilde{\mathbf{v}}^N(c^N) = \tilde{V}_0^N = u(V_0^N) = u(\mathbf{v}^N(c^N))$  so they are ordinally equivalent.

<sup>4</sup>Existence of recursive utility is guaranteed in our setting by conditions (A1), (A2), (A3) and (A4) below. We refer to Marinacci and Montrucchio (2010) for existence and uniqueness results under more general conditions.

<sup>5</sup>The growth condition ensures that  $\mathbf{m}$  is well-defined. If this is guaranteed by other means, the condition can be relaxed; see also Section 6.

*Remark.* The terminal payoff  $X$  is transformed to the terminal utility  $\xi = u(X)$ . Since  $u$  is of linear growth, we also have  $\mathbb{E}[|\xi|^2] < \infty$ .

The above normalization procedure is feasible for every certainty equivalent of Kreps-Porteus type (5). Therefore, in the following we directly work with a normalized aggregator  $(W, \mathbf{m}_0)$  (or simply  $W$ , for short). Thus we study the discrete-time recursive utility index given by

$$\mathbf{v}^N(c^N) \triangleq V_0^N \text{ for } c^N \in \mathcal{A} \text{ piecewise constant on } \pi^N$$

where  $V^N = \{V_{t_k^N}^N\}_{k=0, \dots, N}$  satisfies the backward aggregation

$$(\text{Rec}^N) \quad V_{t_k^N}^N = W\left(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_{t_k^N}[V_{t_{k+1}^N}^N]\right), \quad k = 0, \dots, N-1, \quad V_{t_N^N}^N = \xi.$$

Here the terminal utility  $\xi$ , represented by an  $\mathfrak{F}_T$ -measurable random variable with  $\mathbb{E}[|\xi|^2] < \infty$ , and the temporal aggregator  $W : [0, T] \times \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{R}$  are given.<sup>6</sup>

*Remark.* If  $W$  is additive in the sense that  $W(\Delta, c, v) = \alpha \Delta u(c) + (1 - \delta \Delta)v$ , we recover the classical special case of discounted expected utility:

$$(7) \quad \mathbf{v}^N(c^N) = V_0^N = \mathbb{E}\left[\alpha \sum_{k=0}^{N-1} \beta_k^N u(c_{t_k^N}^N) \Delta_k^N + \beta_N^N u(X)\right]$$

where  $\xi = u(X)$  and  $\beta_k^N \triangleq \prod_{j=0}^{k-1} (1 - \delta \Delta_j^N)$  is the discrete time- $t_k^N$  discount factor.

### 3.2 STOCHASTIC DIFFERENTIAL UTILITY

Stochastic differential utility was introduced by Duffie and Epstein (1992) as a continuous-time analog of recursive utility. Although they provide a heuristic connection to recursive utility, Duffie and Epstein (1992) give a rigorous definition of stochastic differential utility in continuous time only: For a feasible consumption plan  $c$  and a given (continuous-time) aggregator  $f : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{R}$  they define

$$\mathbf{v}(c) \triangleq V_0$$

where  $\xi$  is an  $\mathfrak{F}_T$ -measurable random variable and the process  $V = \{V_t\}_{t \in [0, T]}$  is given by the backward stochastic differential equation (BSDE)

$$(\text{SDU}) \quad V_t = \mathbb{E}_t\left[\int_t^T f(c_s, V_s) ds + \xi\right], \quad t \in [0, T]$$

or alternatively, in equivalent differential notation,

$$dV_t = -f(c_t, V_t) dt + dM_t, \quad V_T = \xi$$

with  $M = \{M_t\}_{t \in [0, T]}$  an  $(\mathfrak{F}, \mathbb{P})$ -martingale.<sup>7</sup>

*Remark.* Similarly as in the discrete-time case, if  $f$  takes the additive form  $f(c, v) = \alpha u(c) - \delta v$  and  $\xi = u(X)$  we obtain standard expected utility with

$$V_t = \mathbb{E}_t\left[\alpha \int_t^T e^{-\delta s} u(c_s) ds + e^{-\delta T} u(X)\right], \quad t \in [0, T].$$

<sup>6</sup>We assume that  $W(\Delta, c, \cdot)$  is defined on  $\mathbb{R}$ . This is a technical requirement that can be relaxed, see Section 6.

<sup>7</sup>See Proposition A.2 in Appendix A for conditions on  $f$  that ensure existence and uniqueness.



## 4. THE CONVERGENCE RESULT

Throughout Sections 4 and 5 we suppose that a normalized discrete-time aggregator  $W$  and a terminal utility  $\xi$  with  $\mathbb{E}[|\xi|^2] < \infty$  are given. For a feasible consumption plan  $c$  and an approximating sequence  $\{c^N\}_{N \in \mathbb{N}}$  we study the sequence  $\{V^N\}_{N \in \mathbb{N}}$  of recursive utility processes constructed via  $(\text{Rec}^N)$  and the process  $V$  given by (SDU). When there is a risk of confusion, we denote  $V^N$  by  $V^N(c^N)$  and  $V$  by  $V(c)$  to highlight the dependencies on  $c^N$  and  $c$ , respectively.

**Conditions (A1), (A2), (A3), (A4).** We impose the following regularity conditions on the aggregator  $W$ : There is a function  $f : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{R}$  such that

- (A1)  $W(\Delta, c, v) = v + f(c, v)\Delta + \varepsilon(\Delta, c, v)\Delta$   
with  $|\varepsilon(\Delta, c, v)| \leq h(\Delta)(1 + |c| + |v|)$ ,  $(\Delta, c, v) \in [0, T] \times \mathbb{C} \times \mathbb{R}$
- (A2)  $|f(c, v) - f(c, w)| \leq L|v - w|$  for  $c \in \mathbb{C}$ ,  $v, w \in \mathbb{R}$
- (A3)  $|f(c, 0)| \leq K(1 + |c|)$  for  $c \in \mathbb{C}$
- (A4)  $f(\cdot, v)$  is continuous on  $\mathbb{C}$  for  $v \in \mathbb{R}$

where  $K, L > 0$  are constants and  $h : [0, T] \rightarrow [0, \infty)$  is a continuous increasing function with  $h(0) = 0$ . In Theorem 4.1 below we identify  $f$  as the continuous-time aggregator corresponding to  $W$ . Condition (A1) is a regularity condition for the discrete-time aggregator  $W$ ; it essentially requires that  $W$  grows linearly in  $\Delta$  at  $\Delta = 0$ . Note that (A1) is always satisfied for additive utility. (A1) implies in particular that  $W(0, c, v) = v$  and that  $f$  can be derived from  $W$  via

$$(8) \quad \frac{\partial W}{\partial \Delta}(0, c, v) = f(c, v) \text{ for all } (c, v) \in \mathbb{C} \times \mathbb{R}.$$

Conditions (A2) and (A3) are Lipschitz and linear growth conditions for the aggregator in (SDU) that are standard in the literature on BSDEs; see, e.g., Duffie and Epstein (1992) and El Karoui, Peng, and Quenez (1997). Condition (A4) is natural if a well-defined continuous-time limit is to be achieved.

*Remark.* Epstein (1987) and Duffie and Epstein (1992) specify the discrete-time aggregator as a smooth function  $W : [0, T] \times \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $W(0, c, v) = v$  and define the continuous-time generator  $\tilde{f}$  via  $\tilde{f}(c, v) \triangleq \frac{\partial W}{\partial \Delta}(0, c, v) / \frac{\partial W}{\partial v}(0, c, v)$ . In our setting this definition coincides with (8) since  $\frac{\partial W}{\partial v}(0, c, v) = 1$ .

**Recursive Utility in Continuous Time.** Let  $V^N = \{V_{t_k^N}^N\}_{k=0, \dots, N}$  denote the continuation value process associated to the consumption plan  $c^N \in \mathcal{A}$  on the time grid  $\pi^N$  via  $(\text{Rec}^N)$ . For the analysis of convergence we extend the definition of  $V^N$  to the whole time interval  $[0, T]$  such that  $V_t^N$  remains unaltered for  $t \in \pi^N$ :

$$V_t^N \triangleq W\left(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_t[V_{t_{k+1}^N}^N]\right) \text{ for } t \in [t_k^N, t_{k+1}^N) \text{ and } k = 0, \dots, N-1.$$

We now state the **main result** of this paper: Recursive utility converges to stochastic differential utility in the continuous-time limit.

**Theorem 4.1 (Convergence).** *Suppose that (A1), (A2), (A3) and (A4) hold and consider a feasible consumption plan  $c \in \mathcal{A}$ . Let  $\{c^N\}_{N \in \mathbb{N}}$  be an approximating sequence and let the recursive utility process  $V^N$  associated to  $c^N$  be given by*

$$(\text{Rec}^N) \quad V_t^N \triangleq W\left(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_t[V_{t_{k+1}^N}^N]\right), \quad t \in [t_k^N, t_{k+1}^N), \quad k = 0, \dots, N-1, \quad V_T^N = \xi.$$

*Moreover, let  $V$  denote the stochastic differential utility process of  $c$  given by*

$$(\text{SDU}) \quad V_t = \mathbb{E}_t\left[\int_t^T f(c_s, V_s)ds + \xi\right], \quad t \in [0, T]$$

*where  $W$  and  $f$  are related via (8), i.e.  $f(c, v) = \frac{\partial W}{\partial \Delta}(0, c, v)$ . Then it follows that*

$$\left\|\sup_{t \in [0, T]} |V_t^N - V_t|\right\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

*In particular, the associated utility indices satisfy*

$$\mathbf{v}^N(c^N) \rightarrow \mathbf{v}(c) \text{ as } N \rightarrow \infty.$$

## 5. PROOF OF THE CONVERGENCE THEOREM

We use the following extension of the definition of  $f$ :

$$(9) \quad f : [0, T] \times \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{R}, \quad f(\Delta, c, v) \triangleq \frac{W(\Delta, c, v) - v}{\Delta}.$$

With this notation, (A1) can be restated as

$$(A1') \quad |f(\Delta, c, v) - f(c, v)| \leq h(\Delta)(1 + |c| + |v|), \quad (\Delta, c, v) \in [0, T] \times \mathbb{C} \times \mathbb{R}.$$

Conditions (A1), (A2) and (A3) imply that there is a constant  $K_1 > 0$  with

$$(A3') \quad |f(\Delta, c, v)| \leq K_1(1 + |c| + |v|) \text{ for } (\Delta, c, v) \in [0, T] \times \mathbb{C} \times \mathbb{R}.$$

Using (9) the recursion  $(\text{Rec}^N)$  can be equivalently reformulated as

$$(10) \quad V_t^N \triangleq \mathbb{E}_t[V_{t_{k+1}^N}^N] + \Delta_k^N f(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_t[V_{t_{k+1}^N}^N]) \text{ for } t \in [t_k^N, t_{k+1}^N)$$

where  $k = 0, \dots, N-1$  and  $V_{t_N^N}^N = \xi$ .

**Outline of the Proof.** Iterating (10) yields the representation<sup>8</sup>

$$(\text{SDU}^N) \quad V_t^N = \mathbb{E}_t\left[\sum_{k=l}^{N-1} \Delta_k^N f(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]) + \xi\right], \quad t \in [t_l^N, t_{l+1}^N).$$

Our goal is to show that  $V^N \rightarrow V$  as  $N \rightarrow \infty$ , where

$$(\text{SDU}) \quad V_t = \mathbb{E}_t\left[\int_t^T f(c_s, V_s)ds + \xi\right].$$

Apparently  $(\text{SDU}^N)$  mimics a Riemann sum approximation of (SDU). This motivates the following strategy for the convergence proof: Starting from  $(\text{SDU}^N)$  with  $f(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N])$ , we replace

$$f(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]) \xrightarrow{\text{Step 1}} f(c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]) \xrightarrow{\text{Step 2}} f(c_{t_k^N}^N, V_{t \vee t_k^N}^N)$$

and, starting from (SDU) with  $f(c_s, V_s)$  we replace

$$\int_{t_l^N}^T f(c_{t_k^N}^N, V_{t_k^N}^N) \xleftarrow{\text{Step 4}} \int_t^T f(c_{t_k^N}^N, V_{t_k^N}^N) \xleftarrow{\text{Step 3}} \int_t^T f(c_s, V_s).$$

<sup>8</sup> $s \vee t \triangleq \max\{s, t\}$ . Note that  $t \vee t_k^N = t_k^N$  for every summand except possibly the first.

In the second part of the proof, we “meet halfway” and show with a Gronwall-Bellman argument that  $(\text{SDU}^N)$  with  $f(c_{t_k}^N, V_{t_k}^N)$ , i.e.

$$V_t^N = \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N f(c_{t_k}^N, V_{t_{k+1}}^N) + \xi \right] + \text{error}, \quad t \in [t_l^N, t_{l+1}^N]$$

and  $(\text{SDU})$  with  $f(c_{t_k}^N, V_{t_k}^N)$ , i.e.

$$V_t = \mathbb{E}_t \left[ \int_{t_l^N}^T \tilde{f}_s^N ds + \xi \right] + \text{error}, \quad t \in [t_l^N, t_{l+1}^N]$$

where  $\tilde{f}_s^N \triangleq f(c_{t_k}^N, V_{t_k}^N)$  for  $s \in [t_k^N, t_{k+1}^N)$ , are close and the error terms are small. Step 1 is crucial since in this step we carry out the transition from the discrete-time aggregator  $W$  to its continuous-time counterpart  $f$  via the linearization (A1). In Step 3 we switch from the continuous-time plan  $c$  to the approximating sequence  $\{c^N\}_{N \in \mathbb{N}}$ . Steps 2 and 4 are intermediate steps. Finally, the Gronwall-Bellman argument establishes the key link between the recursive utility processes  $V^N$  and the stochastic differential utility process  $V$ .

*Remark.* Our approach is related to stability results for BSDEs (such as, e.g., Antonelli (1996) and Coquet, Mackevičius, and Mémin (1998)) and to the recent literature on numerical approximations of BSDEs (see, e.g., Bouchard and Touzi (2004), Zhang (2004) and Bouchard and Elie (2008), among many others). We refer to Bouchard, Elie, and Touzi (2009) for an overview of this strand of research. This literature focuses on settings with finite martingale multiplicity (e.g., Brownian or jump-diffusion models). By contrast, in this paper we allow for a general semimartingale framework, so existing results are not directly applicable to our convergence problem. In addition, even with finite martingale multiplicity, recursive utility imposes a nonstandard discretization scheme (see, e.g., (10)): It involves nested conditional expectations and possibly path-dependent stochastic aggregators that depend explicitly on the time discretization.

**Proof of Theorem 4.1.** For reasons of readability, the first part of the proof is subdivided into lemmas that correspond to Steps 1, 2, 3 and 4 above. Necessary a priori estimates are deferred to Appendix A. Throughout this section, we consider a fixed consumption plan  $c \in \mathcal{A}$  and an approximating sequence  $\{c^N\}_{N \in \mathbb{N}}$  and assume that (A1), (A2), (A3) and (A4) are satisfied.

**Step 1.** By  $(\text{SDU}^N)$  we have

$$(11) \quad V_t^N = \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N f(c_{t_k}^N, \mathbb{E}_{t \vee t_k^N} [V_{t_{k+1}}^N]) + \xi \right] + R_t^{1,N} \quad \text{for } t \in [t_l^N, t_{l+1}^N]$$

where  $R^{1,N} = \{R_t^{1,N}\}_{t \in [0, T]}$  is given by

$$R_t^{1,N} \triangleq \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N \left\{ f(c_{t_k}^N, \mathbb{E}_{t \vee t_k^N} [V_{t_{k+1}}^N]) - f(c_{t_k}^N, \mathbb{E}_{t \vee t_k^N} [V_{t_{k+1}}^N]) \right\} \right]$$

for  $t \in [t_l^N, t_{l+1}^N)$ . This error term can be estimated as follows:

**Lemma 5.1.** *There is a constant  $K_3 > 0$  such that*

$$\left\| \sup_{t \in [0, T]} |R_t^{1,N}| \right\|_2 \leq K_3 h(\Delta^N) \quad \text{for all } N \geq N_0(c, \{c^N\}_{N \in \mathbb{N}}).$$

*Proof.* For  $t \in [t_l^N, t_{l+1}^N)$  we have from (A1') and the conditional triangle inequality

$$\begin{aligned} |R_t^{1,N}| &\leq \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N |f(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]) - f(c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N])| \right] \\ &\leq h(\Delta^N) \mathbb{E}_t \left[ \sum_{k=l}^{N-1} (1 + |c_{t_k^N}^N| + |\mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]|) \Delta_k^N \right] \\ &\leq h(\Delta^N) \mathbb{E}_t \left[ \sum_{k=0}^{N-1} (1 + |c_{t_k^N}^N| + |V_{t_{k+1}^N}^N|) \Delta_k^N \right]. \end{aligned}$$

Applying Doob's  $L^2$ -inequality to the  $(\mathfrak{F}, \mathbb{P})$ -martingale  $M^N = \{M_t^N\}_{t \in [0, T]}$  defined by  $M_t^N \triangleq \mathbb{E}_t[\sum_{k=0}^{N-1} (1 + |c_{t_k^N}^N| + |V_{t_{k+1}^N}^N|) \Delta_k^N]$  for  $t \in [0, T]$  it follows that

$$(12) \quad \|\sup_{t \in [0, T]} |R_t^{1,N}|\|_2 \leq h(\Delta^N) \|\sup_{t \in [0, T]} |M_t^N|\|_2 \leq 2h(\Delta^N) \|M_T^N\|_2.$$

On the other hand, by Proposition A.1 and (27) in Appendix A we have

$$\begin{aligned} \|M_T^N\|_2 &\leq \sum_{k=0}^{N-1} (1 + \|c_{t_k^N}^N\|_2 + \|V_{t_{k+1}^N}^N\|_2) \Delta_k^N \\ &\leq \sum_{k=0}^{N-1} (2 + \|c_{t_k^N}^N\|_2^2 + K_2[1 + \|\xi\|_2 + \|c\|_2^2]) \Delta_k^N \\ &\leq (2 + K_2[1 + \|\xi\|_2 + \|c\|_2^2])T + \sum_{k=0}^{N-1} \|c_{t_k^N}^N\|_2^2 \Delta_k^N \\ &\leq (2 + K_2[1 + \|\xi\|_2 + \|c\|_2^2])T + \|c\|_2^2 + 1, \quad N \geq N_0(c, \{c^N\}_{N \in \mathbb{N}}). \end{aligned}$$

In combination with (12) this completes the proof.  $\square$

**Step 2.** By (11) we have

$$(13) \quad V_t^N = \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N f(c_{t_k^N}^N, V_{t \vee t_k^N}^N) + \xi \right] + R_t^{1,N} + R_t^{2,N} \text{ for } t \in [t_l^N, t_{l+1}^N)$$

where  $R^{2,N} = \{R_t^{2,N}\}_{t \in [0, T]}$  is given by

$$R_t^{2,N} \triangleq \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N \left\{ f(c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]) - f(c_{t_k^N}^N, V_{t \vee t_k^N}^N) \right\} \right], \quad t \in [t_l^N, t_{l+1}^N).$$

**Lemma 5.2.** *There is a constant  $K_4 > 0$  such that*

$$\|\sup_{t \in [0, T]} |R_t^{2,N}|\|_2 \leq K_4 \Delta^N \text{ for all } N \geq N_0(c, \{c^N\}_{N \in \mathbb{N}}).$$

*Proof.* Condition (A2) implies

$$\begin{aligned} |R_t^{2,N}| &\leq \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N |f(c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]) - f(c_{t_k^N}^N, V_{t \vee t_k^N}^N)| \right] \\ &\leq L \Delta^N \mathbb{E}_t \left[ \sum_{k=l}^{N-1} |\mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N] - V_{t \vee t_k^N}^N| \right] \\ &= L \Delta^N \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N |f(\Delta_k^N, c_{t_k^N}^N, \mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N])| \right] \end{aligned}$$

where  $t \in [t_l^N, t_{l+1}^N)$  and the identity uses (10). Moreover, (A3') yields

$$\begin{aligned} |R_t^{2,N}| &\leq K_1 L \Delta^N \mathbb{E}_t \left[ \sum_{k=l}^{N-1} \Delta_k^N [1 + |c_{t_k^N}^N| + |\mathbb{E}_{t \vee t_k^N}[V_{t_{k+1}^N}^N]|] \right] \\ &\leq K_1 L \Delta^N \mathbb{E}_t \left[ \sum_{k=0}^{N-1} \Delta_k^N [1 + |c_{t_k^N}^N| + |V_{t_{k+1}^N}^N|] \right]. \end{aligned}$$

Now the desired estimate follows exactly as in the proof of Lemma 5.1.  $\square$

For future reference, we rewrite (13) in integral form as<sup>9</sup>

$$(14) \quad V_t^N = \mathbb{E}_t \left[ \int_{t_l^N}^T f_s^{N,t} ds + \xi \right] + R_t^{1,N} + R_t^{2,N} \text{ for } t \in [t_l^N, t_{l+1}^N)$$

where  $f^{N,t} = \{f_s^{N,t}\}_{s \in [0,T]}$  is given by

$$f_s^{N,t} \triangleq f(c_{t_k^N}^N, V_{t \vee t_k^N}^N) \text{ for } s \in [t_k^N, t_{k+1}^N), \quad k = 0, \dots, N-1.$$

**Step 3.** By definition of  $V$  we have

$$(15) \quad V_t = \mathbb{E}_t \left[ \int_t^T \tilde{f}_s^N ds + \xi \right] + R_t^{3,N} \text{ for } t \in [t_l^N, t_{l+1}^N)$$

where  $\tilde{f}^N = \{\tilde{f}_s^N\}_{s \in [0,T]}$  is defined by

$$\tilde{f}_s^N \triangleq f(c_{t_k^N}^N, V_{t_k^N}^N) \text{ for } s \in [t_k^N, t_{k+1}^N), \quad k = 0, \dots, N-1$$

and where  $R^{3,N} = \{R_t^{3,N}\}_{t \in [0,T]}$  is given by

$$R_t^{3,N} \triangleq \mathbb{E}_t \left[ \int_t^T \{f(c_s, V_s) - \tilde{f}_s^N\} ds \right], \quad t \in [0, T].$$

**Lemma 5.3.** *We have*

$$\left\| \sup_{t \in [0,T]} |R_t^{3,N}| \right\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

*Proof.* For every  $t \in [0, T]$  we have

$$|R_t^{3,N}| \leq M_t^N \triangleq \mathbb{E}_t \left[ \int_0^T |f(c_s, V_s) - \tilde{f}_s^N| ds \right]$$

with an  $(\mathfrak{F}, \mathbb{P})$ -martingale  $M^N = \{M_t^N\}_{t \in [0,T]}$  satisfying  $M_T^N = \int_0^T |f(c_s, V_s) - \tilde{f}_s^N| ds$ . Using Doob's  $L^2$ -inequality and Cauchy's inequality we obtain

$$(16) \quad \left\| \sup_{t \in [0,T]} |R_t^{3,N}| \right\|_2 \leq \left\| \sup_{t \in [0,T]} M_t^N \right\|_2 \leq 2 \|M_T^N\|_2 \\ = 2 \left\| \int_0^T |f(c_s, V_s) - \tilde{f}_s^N| ds \right\|_2 \leq 2\sqrt{T} \mathbb{E} \left[ \int_0^T |f(c_s, V_s) - \tilde{f}_s^N|^2 ds \right]^{\frac{1}{2}}.$$

By (2), the càdlàg property of  $V$ , (A3) and (A4) we have  $\tilde{f}^N \rightarrow f(c, V)$  a.e. on  $[0, T] \times \Omega$ , so by (3), Proposition A.2, (A3') and dominated convergence we obtain

$$\mathbb{E} \left[ \int_0^T |f(c_s, V_s) - \tilde{f}_s^N|^2 ds \right] \rightarrow 0 \text{ as } N \rightarrow \infty. \quad \square$$

*Remark.* Lemma 5.3 is the only one of the error estimates established in this section that does not specify an explicit convergence rate. This cannot be expected unless we impose stricter assumptions (e.g., a Lipschitz condition) on the dependence of  $W$  on  $c$  and on the regularity of  $c$  with respect to time. In this case, the literature on numerical solutions of BSDEs suggests that a convergence rate of  $\sqrt{\Delta^N}$  could be expected in a jump-diffusion setting; see Zhang (2004), Bouchard and Elie (2008) or Bouchard, Elie, and Touzi (2009) and the references therein.

**Step 4.** By (15) we have

$$(17) \quad V_t = \mathbb{E}_t \left[ \int_{t_l^N}^{t_{l+1}^N} f(c_{t_l^N}^N, V_t) ds + \int_{t_{l+1}^N}^T \tilde{f}_s^N ds + \xi \right] + R_t^{3,N} + R_t^{4,N} \\ = \mathbb{E}_t \left[ \int_{t_l^N}^T \hat{f}_s^{N,t} ds + \xi \right] + R_t^{3,N} + R_t^{4,N} \text{ for } t \in [t_l^N, t_{l+1}^N)$$

<sup>9</sup>The additional superindex  $t$  indicates that  $f^{N,t}$  depends explicitly on  $t$ .

where  $\hat{f}^{N,t} = \{\hat{f}_s^{N,t}\}_{s \in [0,T]}$  is given by

$$\hat{f}_s^{N,t} \triangleq f(c_{t_l^N}^N, V_t), \quad s \in [t_l^N, t_{l+1}^N), \quad \hat{f}_s^{N,t} \triangleq \tilde{f}_s^N, \quad s \in [t_{l+1}^N, T]$$

and  $R^{4,N} = \{R_t^{4,N}\}_{t \in [0,T]}$  is given by

$$R_t^{4,N} \triangleq -\int_{t_l^N}^t f(c_{t_l^N}^N, V_t) ds + \mathbb{E}_t \left[ \int_{t_l^N}^{t_{l+1}^N} \{\tilde{f}_s^N - f(c_{t_l^N}^N, V_t)\} ds \right], \quad t \in [0, T].$$

**Lemma 5.4.** *There is a constant  $K_5 > 0$  such that*

$$\left\| \sup_{t \in [0,T]} |R_t^{4,N}| \right\|_2 \leq K_5 \Delta^N \text{ for all } N \in \mathbb{N}.$$

*Proof.* By (A3') we have

$$\left| \int_{t_l^N}^t f(c_{t_l^N}^N, V_t) ds \right|, \quad \left| \int_{t_l^N}^{t_{l+1}^N} \{\tilde{f}_s^N - f(c_{t_l^N}^N, V_t)\} ds \right| \leq 2\Delta^N K_1 \left[ 1 + \sup_{N \in \mathbb{N}, s \in [0,T]} |c_s^N| + |V_s| \right]$$

so the desired conclusion follows immediately.  $\square$

**Gronwall-Bellman Argument.** We are now in a position to complete the proof of Theorem 4.1. For every  $t \in [0, T]$  we have

$$(14) \quad V_t^N = \mathbb{E}_t \left[ \int_{t_l^N}^T f_s^{N,t} ds + \xi \right] + R_t^{1,N} + R_t^{2,N}$$

$$(17) \quad V_t = \mathbb{E}_t \left[ \int_{t_l^N}^T \hat{f}_s^{N,t} ds + \xi \right] + R_t^{3,N} + R_t^{4,N}$$

where  $l$  chosen such that  $t \in [t_l^N, t_{l+1}^N)$  and by Lemmas 5.1, 5.2, 5.3 and 5.4

$$\left\| \sup_{t \in [0,T]} |R_t^{i,N}| \right\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty \text{ for } i = 1, 2, 3, 4.$$

Here  $f^{N,t} = \{f_s^{N,t}\}_{s \in [0,T]}$  and  $\hat{f}^{N,t} = \{\hat{f}_s^{N,t}\}_{s \in [0,T]}$  are given by

$$\begin{aligned} f_s^{N,t} &= f(c_{t_l^N}^N, V_t^N), \quad s \in [t_l^N, t_{l+1}^N), & f_s^{N,t} &= f(c_{t_k^N}^N, V_{t_k^N}^N), \quad s \in [t_k^N, t_{k+1}^N) \\ \hat{f}_s^{N,t} &= f(c_{t_l^N}^N, V_t), \quad s \in [t_l^N, t_{l+1}^N), & \hat{f}_s^{N,t} &= \tilde{f}_s^N = f(c_{t_k^N}^N, V_{t_k^N}^N), \quad s \in [t_k^N, t_{k+1}^N) \end{aligned}$$

for  $k = l + 1, \dots, N - 1$ . Combining these results it follows that the process  $D^N = \{D_t^N\}_{t \in [0,T]}$  given by  $D_t^N \triangleq V_t^N - V_t$  for  $t \in [0, T]$  satisfies

$$D_t^N = \mathbb{E}_t \left[ \int_{t_l^N}^T r_s^{N,t} ds \right] + R_t^N \text{ for } t \in [t_l^N, t_{l+1}^N)$$

where  $\left\| \sup_{t \in [0,T]} |R_t^N| \right\|_2 \rightarrow 0$  as  $N \rightarrow \infty$  and  $r^{N,t} = \{r_s^{N,t}\}_{s \in [0,T]}$  is defined by  $r_s^{N,t} \triangleq f_s^{N,t} - \hat{f}_s^{N,t}$ , i.e.

$$\begin{aligned} r_s^{N,t} &= f(c_{t_l^N}^N, V_t^N) - f(c_{t_l^N}^N, V_t), \quad s \in [t_l^N, t_{l+1}^N), \\ r_s^{N,t} &= f(c_{t_k^N}^N, V_{t_k^N}^N) - f(c_{t_k^N}^N, V_{t_k^N}), \quad s \in [t_k^N, t_{k+1}^N), \quad k = l + 1, \dots, N - 1. \end{aligned}$$

The Lipschitz condition (A2) implies that

$$|r_s^{N,t}| \leq L |D_t^N|, \quad s \in [t_l^N, t_{l+1}^N) \text{ and } |r_s^{N,t}| \leq L |D_{t_k^N}^N|, \quad s \in [t_k^N, t_{k+1}^N)$$

for  $k = l + 1, \dots, N - 1$ . Therefore,

$$|D_t^N| \leq L \mathbb{E}_t \left[ \Delta_l^N |D_t^N| + \sum_{k=l+1}^N \Delta_k^N |D_{t_k^N}^N| \right] + \mathbb{E}_t [R^N]$$

where  $R^N \triangleq \sup_{t \in [0,T]} |R_t^N|$ . If  $N$  is sufficiently large so that  $L\Delta^N \leq \frac{1}{2}$ , we get

$$(18) \quad |D_t^N| \leq 2L \sum_{j=l+1}^N \Delta_j^N \mathbb{E}_t [|D_{t_j^N}^N|] + 2\mathbb{E}_t [R^N].$$

Analogously it follows that  $|D_{t_k^N}^N| \leq 2L \sum_{j=k+1}^N \Delta_j^N \mathbb{E}_{t_k^N}[|D_{t_j^N}^N|] + 2\mathbb{E}_{t_k^N}[R^N]$  for  $k = l + 1, \dots, N$  and therefore

$$(19) \quad \mathbb{E}_t[|D_{t_k^N}^N|] \leq 2L \sum_{j=k+1}^N \Delta_j^N \mathbb{E}_t[|D_{t_j^N}^N|] + 2\mathbb{E}_t[R^N], \quad k = l + 1, \dots, N.$$

Applying the discrete Gronwall Lemma A.3 to (18) and (19) yields  $|D_t^N| \leq 2\mathbb{E}_t[R^N]e^{2LT}$ . Since  $t \in [0, T]$  is arbitrary and  $\|R^N\|_2 \rightarrow 0$  as  $N \rightarrow \infty$ , we get

$$\left\| \sup_{t \in [0, T]} |D_t^N| \right\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty$$

from Doob's L<sup>2</sup>-inequality. This completes the proof of Theorem 4.1.  $\square$

## 6. CONVERGENCE WITHOUT (A1), (A2), (A3), (A4)

In this section we provide a convergence result for bounded consumption plans. It applies to a wide class of aggregators that are relevant in applications but do not, in general, satisfy the conditions of Section 4.

**A General Class of Aggregators.** The discrete-time temporal aggregator is specified in terms of a strictly increasing function  $g : \mathbb{C} \subseteq [0, \infty) \rightarrow \mathbb{R}$  via

$$W(\Delta, c, v) \triangleq g^{-1}(\delta \Delta g(c) + (1 - \delta \Delta)g(v))$$

where  $\delta \in (0, 1]$  models a subjective discount rate. With  $\mathbf{V} \triangleq u(\mathbb{C})$  the corresponding normalized aggregator reads

$$(20) \quad W : [0, T] \times \mathbb{C} \times \mathbf{V} \rightarrow \mathbb{R}, \quad W(\Delta, c, v) \triangleq u \circ g^{-1}(\delta \Delta g(c) + (1 - \delta \Delta)g \circ u^{-1}(v))$$

where  $u$  is the function inducing the certainty equivalent via (5). The functions  $g : \mathbb{C} \rightarrow \mathbb{R}$  and  $u : \mathbb{C} \rightarrow \mathbb{R}$  are assumed to be of class  $C^2$  with

$$(21) \quad g'(c) \triangleq \frac{\partial g}{\partial c}(c) > 0 \text{ and } u'(c) \triangleq \frac{\partial u}{\partial c}(c) > 0, \quad c \in \mathbb{C}.$$

Note that (20) becomes additive for  $u = g$ , so discounted expected utility is a special case. By (8) the associated continuous-time aggregator is

$$(22) \quad f : \mathbb{C} \times \mathbf{V} \rightarrow \mathbb{R}, \quad f(c, v) = \delta \frac{u'(u^{-1}(v))}{g'(u^{-1}(v))} [g(c) - g(u^{-1}(v))].$$

A direct application of Theorem 4.1 may not be possible if (A1), (A2), (A3) and (A4) are not satisfied globally.<sup>10</sup> In addition,  $W(\Delta, c, \cdot)$  and  $f(c, \cdot)$  may not be defined on all of  $\mathbb{R}$ . Below we present a suitable extension of Theorem 4.1 that applies in this context.

*Example.* The empirically relevant specification proposed by Epstein and Zin (1989) and Weil (1989) is obtained by choosing  $\mathbb{C} = (0, \infty)$ ,  $g(c) = \frac{1}{1-\phi} c^{1-\phi}$  and  $u(c) = \frac{1}{1-\varrho} c^{1-\varrho}$ . The parameters  $\varrho > 0$  and  $\phi > 0$  model relative risk aversion and the

<sup>10</sup>In this case, the continuous-time limit is not necessarily well-defined. For instance, if  $u(c) = g(c) = -c^{-2}$  and the consumption plan  $(c, \xi)$  is given by  $c_t = T - t$ ,  $t \in [0, T]$  and  $\xi = 0$ , then the recursive utility indices satisfy (see (7))

$$\mathbf{v}^N(c, \xi) = -\delta \sum_{k=0}^{N-1} \beta_k^N \frac{1}{(T-t_k^N)^2} \Delta_k^N \leq -\delta \beta_{N-1}^N \frac{1}{\Delta_k^N} \rightarrow -\infty \text{ as } N \rightarrow \infty.$$

inverse of the elasticity of intertemporal substitution. The corresponding normalized aggregator is given by

$$W(\Delta, c, v) = \frac{1}{1-\varrho} \left[ \delta \Delta c^{1-\phi} + (1-\delta \Delta) [(1-\varrho)v]^{\frac{1-\phi}{1-\varrho}} \right]^{\frac{1-\varrho}{1-\phi}}$$

with  $V = (0, \infty)$  if  $\varrho < 1$  and  $V = (-\infty, 0)$  if  $\varrho > 1$ . The associated continuous-time aggregator

$$f(c, v) = \delta^{\frac{1-\varrho}{1-\phi}} v \left[ \left( \frac{c}{([1-\varrho]v)^{\frac{1}{1-\varrho}}} \right)^{1-\phi} - 1 \right]$$

coincides with the standard parametrization used by Duffie and Epstein (1992), Fisher and Gilles (1998) and Wachter (2011), among others.

**Convergence for Bounded Consumption Plans.** Consider a consumption plan  $c$  and an approximating sequence  $\{c^N\}_{N \in \mathbb{N}}$  that take values in a closed and bounded subinterval  $C_0 \subseteq C$  and suppose that the terminal utility  $\xi$  takes values in  $V_0 \triangleq u(C_0)$ .<sup>11</sup> By (21) the inverse functions  $g^{-1}$  and  $u^{-1}$  are of class  $C^2$  on  $g(C_0)$  and  $u(C_0)$ , respectively, and it follows from the inverse mapping theorem that  $W$  is of class  $C^2$  on  $[0, T] \times C_0 \times V_0$ . Applying the mean value theorem twice yields

$$f(\Delta, c, v) - f(c, v) = \frac{\partial W}{\partial \Delta}(\Delta', c, v) - \frac{\partial W}{\partial \Delta}(0, c, v) = \frac{\partial^2 W}{\partial \Delta^2}(\Delta'', c, v)\Delta'$$

where  $0 < \Delta'' < \Delta' < \Delta$ . Since  $W$  is of class  $C^2$  it follows that  $\frac{\partial^2 W}{\partial \Delta^2}$  is bounded on  $[0, T] \times C_0 \times V_0$ , so the restriction  $W_0$  of  $W$  to  $[0, T] \times C_0 \times V_0$  satisfies (A1). Similarly, (A2) holds for  $W_0$ . Finally, (A3) and (A4) are obvious.

Notice that for the aggregation  $(\text{Rec})^N$  which determines  $V^N = V^N(c^N)$  only the restriction  $W_0$  is relevant. Indeed, one easily checks that  $g^{-1}(\delta \Delta g(c) + (1-\delta \Delta)g(v)) \in C_0$  whenever  $c, v \in C_0$  and  $\delta \Delta \leq 1$ , so  $V^N$  takes values in  $V_0$  for all but finitely many  $N \in \mathbb{N}$ . Hence we obtain

**Theorem 6.1 (Convergence).** *Let  $W$  be given by (20) where  $g : C \rightarrow \mathbb{R}$  and  $u : V \rightarrow \mathbb{R}$  are  $C^2$  functions that satisfy (21) and suppose that  $\xi$  takes values in a bounded closed subset of  $V$ . If  $c \in \mathcal{A}$  is a consumption plan and  $\{c^N\}_{N \in \mathbb{N}}$  is an approximating sequence that take values in bounded closed subintervals of  $C$ , then*

$$\left\| \sup_{t \in [0, T]} |V_t^N - V_t| \right\|_2 \rightarrow 0 \text{ and in particular } \mathbf{v}^N(c^N) \rightarrow \mathbf{v}(c) \text{ as } N \rightarrow \infty.$$

Here  $\mathbf{v}^N$  and  $\mathbf{v}$  denote, respectively, the recursive utility index on  $\pi^N$  and the stochastic differential utility index with aggregator  $f$  given by (22), and  $V^N$  and  $V$  are the associated continuation value processes.

*Proof.* Choose a closed interval  $C_0 \subseteq C$  such that  $c$  and  $c^N$  take values in  $C_0$  for  $N \in \mathbb{N}$  and  $\xi$  takes values in  $V_0 \triangleq u(C_0)$ . Let  $W_0$  denote the restriction of  $W$  to  $[0, T] \times C_0 \times V_0$ ; by the preceding discussion  $W_0$  satisfies (A1), (A2), (A3) and (A4) on  $[0, T] \times C_0 \times V_0$ . Now select an extension  $\bar{W}_0$  of  $W_0$  to  $[0, T] \times C \times \mathbb{R}$  in such a way that (A1), (A2), (A3) and (A4) hold for  $\bar{W}_0$  on  $[0, T] \times C \times \mathbb{R}$ . Now we can apply Theorem 4.1 with the aggregator  $\bar{W}_0$ . As observed above, the process  $V^N = V^N(c^N)$  takes values in  $V_0$  for all but finitely many  $N \in \mathbb{N}$  and, in particular, does not depend on the choice of the extension  $\bar{W}_0$ ; since  $\|V_t^N - V_t\| \rightarrow 0$  as  $N \rightarrow \infty$  it follows that  $V = V(c)$  cannot depend on the choice of  $\bar{W}_0$  either. The claim now follows from Theorem 4.1.  $\square$

<sup>11</sup>This is consistent: The corresponding monetary payoff  $X = u^{-1}(\xi)$  is  $C_0$ -valued.



*Remark.* For Epstein-Zin-Weil utility, Theorem 6.1 requires consumption to be bounded from below by a positive constant. This may not be satisfied in specific applications as it interferes with the homotheticity of the Epstein-Zin aggregator.

## 7. BEYOND KREPS-PORTEUS CERTAINTY EQUIVALENTS

Recently certainty equivalents outside the Kreps-Porteus specification have become important in the modeling of ambiguity, source-dependent risk aversion, and further behavioral aspects of decisions under risk. In this section we provide a basis for research on recursive utility with nonstandard risk preferences: We establish a sufficient criterion for a general certainty equivalent to lead to stochastic differential utility in the continuous-time limit.

**Normalization.** Consider a consumption plan  $c \in \mathcal{A}$ , an approximating sequence  $\{c^N\}_{N \in \mathbb{N}}$  and the associated backward recursion (4) that defines recursive utility,

$$V_{t_k}^N = W\left(\Delta_k^N, c_{t_k}^N, \mathbf{m}_{t_k}^N(V_{t_{k+1}}^N)\right), \quad k = 0, \dots, N-1, \quad V_{t_N}^N = X.$$

Here  $\{\mathbf{m}_t\}_{t \in [0, T]}$  is a family of abstract time- $t$  certainty equivalents (not necessarily of Kreps-Porteus type). As in Section 3 the normalization  $v \mapsto \tilde{v} \triangleq u(v)$  with  $u$  strictly increasing leads to the ordinally equivalent representation

$$(23) \quad \tilde{V}_{t_k}^N = \tilde{W}\left(\Delta_k^N, c_{t_k}^N, \tilde{\mathbf{m}}_{t_k}^N(\tilde{V}_{t_{k+1}}^N)\right), \quad k = 0, \dots, N-1, \quad \tilde{V}_{t_N}^N = \xi$$

where  $\xi \triangleq u(X)$ , the normalized time- $t$  certainty equivalent  $\tilde{\mathbf{m}}_t$  is given by

$$(24) \quad \tilde{\mathbf{m}}_t(\tilde{V}) \triangleq u\left(\mathbf{m}_t(u^{-1}(\tilde{V}))\right)$$

and  $\tilde{W}(\Delta, c, \tilde{v}) \triangleq u \circ W(\Delta, c, u^{-1}(\tilde{v}))$ .

While, in general, it may not be possible to achieve an exact normalization as in Section 3 for the Kreps-Porteus case, motivated by the analysis in Skiadas (2013) we consider the case when an approximate normalization is available:

$$\tilde{\mathbf{m}}_t(\tilde{V}) = \mathbb{E}_t^{\mathbb{Q}}[\tilde{V}] + R_t$$

with a probability measure  $\mathbb{Q}$  and a “small” error  $R_t$ . In the following we establish a general convergence criterion based on an approximation of this form.

**Theorem 7.1 (Convergence with Approximate Normalization).** *Consider a normalized aggregator  $W = \tilde{W}$  as in (23) that satisfies (A1), (A2), (A3), (A4). Let  $\mathbb{Q}$  be an equivalent probability measure such that  $\{c^N\}_{N \in \mathbb{N}}$  is also an approximating sequence for  $c$  with respect to  $\mathbb{Q}$  and  $\mathbb{E}^{\mathbb{Q}}[|\xi|^2] < \infty$ . If the normalized certainty equivalents (24) satisfy*

$$(25) \quad \mathbf{m}_{t_k}^N(V_{t_{k+1}}^N) = \mathbb{E}_{t_k}^{\mathbb{Q}}[V_{t_{k+1}}^N] + R_{t_k}^N \text{ for } k = 0, \dots, N-1$$

where  $\sum_{k=1}^{N-1} \|R_{t_k}^N\|_{1, \mathbb{Q}} \rightarrow 0$  as  $N \rightarrow \infty$ , then we have  $\mathbf{v}^N(c^N) \rightarrow \mathbf{v}(c)$  as  $N \rightarrow \infty$  where  $\mathbf{v}(c) = V_0$  and  $V$  denotes the stochastic differential utility process

$$V_t = \mathbb{E}_t^{\mathbb{Q}} \left[ \int_t^T f(c_s, V_s) ds + \xi \right], \quad t \in [0, T] \text{ with } f(c, v) = \frac{\partial W}{\partial \Delta}(0, c, v).$$

*Proof.* By (25) we can rewrite the recursion (23) as

$$V_{t_k}^N = \mathbb{E}_{t_k}^{\mathbb{Q}} \left[ \Delta_k^N f(\Delta_k^N, c_{t_k}^N, \mathbf{m}_{t_k}^N(V_{t_{k+1}}^N)) + V_{t_{k+1}}^N + R_{t_k}^N \right], \quad k = 0, \dots, N-1$$

and similarly as in Section 5 iterating this leads to

$$V_{t_l}^N = \mathbb{E}_{t_l}^{\mathbb{Q}} \left[ \sum_{k=l}^{N-1} \Delta_k^N f(\Delta_k^N, c_{t_k}^N, \mathbf{m}_{t_k}^N(V_{t_{k+1}}^N)) + \xi + \sum_{k=l}^{N-1} R_{t_k}^N \right].$$

Now Theorem 4.1 implies  $\max_{k=0, \dots, N} |\tilde{V}_{t_k}^N - V_{t_k}^N| \rightarrow 0$  in  $L^2(\mathbb{Q})$  as  $N \rightarrow \infty$  where

$$\tilde{V}_{t_l}^N \triangleq \mathbb{E}_{t_l}^{\mathbb{Q}} \left[ \sum_{k=l}^{N-1} \Delta_k^N f(\Delta_k^N, c_{t_k}^N, \mathbb{E}_{t_k}^{\mathbb{Q}}[\tilde{V}_{t_{k+1}}^N]) + \xi \right], \quad l = 0, \dots, N.$$

Thus it suffices to show that

$$\max_{k=0, \dots, N} \|V_{t_k}^N - \tilde{V}_{t_k}^N\|_{1, \mathbb{Q}} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

To verify this note that, by (A1') and (A2),  $|f(\Delta, c, v) - f(\Delta, c, w)| \leq L|v - w| + 2h(\Delta)[1 + |c| + |v| + |w|]$ , so by (25) and the conditional triangle inequality

$$\begin{aligned} |V_{t_l}^N - \tilde{V}_{t_l}^N| &\leq \mathbb{E}_{t_l}^{\mathbb{Q}} \left[ \sum_{k=l}^{N-1} \Delta_k^N L |V_{t_{k+1}}^N - \tilde{V}_{t_{k+1}}^N| \right] + \mathbb{E}_{t_l}^{\mathbb{Q}} \left[ \sum_{k=l}^{N-1} (1 + \Delta_k^N L) |R_{t_k}^N| \right] \\ &\quad + \mathbb{E}_{t_l}^{\mathbb{Q}} \left[ 2h(\Delta_k^N) \sum_{k=l}^{N-1} \Delta_k^N (|c_{t_k}^N| + |V_{t_{k+1}}^N| + |\tilde{V}_{t_{k+1}}^N|) \right], \quad l = 0, \dots, N. \end{aligned}$$

We have  $\sup_{N \in \mathbb{N}} \max_{k=0, \dots, N} \|c_{t_k}^N\|_{2, \mathbb{Q}} < \infty$  since  $\{c^N\}_{N \in \mathbb{N}}$  is an approximating sequence under  $\mathbb{Q}$ . As in Proposition A.1 we can use (A3') to establish the a priori estimate  $\sup_{N \in \mathbb{N}} \max_{k=0, \dots, N} \|V_{t_k}^N\|_{1, \mathbb{Q}} < \infty$ . By the first part of the proof, we have  $\sup_{N \in \mathbb{N}} \max_{k=0, \dots, N} \|\tilde{V}_{t_k}^N\|_{2, \mathbb{Q}} < \infty$  and thus it follows that

$$\|V_{t_l}^N - \tilde{V}_{t_l}^N\|_{1, \mathbb{Q}} \leq \sum_{k=l}^{N-1} \Delta_k^N L \|V_{t_{k+1}}^N - \tilde{V}_{t_{k+1}}^N\|_{1, \mathbb{Q}} + (1 + \Delta^N L) \sum_{k=1}^{N-1} \|R_{t_k}^N\|_{1, \mathbb{Q}} + Kh(\Delta^N)$$

with a constant  $K > 0$ . Now the Gronwall Lemma A.3 yields the result.  $\square$

*Example: Drift Uncertainty.* Suppose  $W$  is a  $(\mathbb{P}, \mathfrak{F})$ -Wiener process and  $\{\mu_t^i(s)\}_{s \in [t, T]}$  is a family of progressively measurable processes for each  $t \in [0, T]$  and  $i = 1, \dots, d$ . For  $t, t + \Delta \in [0, T]$  define the drift-adjusted probability  $\mathbb{Q}_{t, t+\Delta}^i$  on  $\mathfrak{F}_{t+\Delta}$  by

$$\frac{d\mathbb{Q}_{t, t+\Delta}^i}{d\mathbb{P}} = \exp \left\{ \int_t^{t+\Delta} \mu_t^i(s) dW(s) - \frac{1}{2} \int_t^{t+\Delta} |\mu_t^i(s)|^2 ds \right\}$$

and assume the normalized time- $t$  certainty equivalent is given by

$$\mathbf{m}_t(V) = \Phi_t(\mathbb{E}_{t, t+\Delta}^i[V] : i = 1, \dots, d) \text{ for } V \in \mathfrak{F}_{t+\Delta}.$$

Here  $\mathbb{E}_{t, t+\Delta}^i$  denotes  $\mathbb{Q}_{t, t+\Delta}^i$ -conditional expectation given  $\mathfrak{F}_t$ , and for each  $t \in [0, T]$  the random function  $\Phi_t : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $\mathfrak{F}_t$ -measurable, satisfies  $\Phi_t(x, \dots, x) = x$  for all  $x \in \mathbb{R}$  and there is a constant  $L > 0$  such that

$$|\Phi_t(x) - \Phi_t(y)| \leq L|x - y| \text{ for } x, y \in \mathbb{R}^d.$$

Examples include the robust specification  $\Phi_t(x) = \min(x)$  and the second-order expectation  $\Phi_t(x) = \sum_{i=1}^d \alpha_t^i x^i$  with adapted probability weights  $\alpha_t^i = \{\alpha_t^i\}_{t \in [0, T]}$ .

In this setting, if there are a progressive process  $\mu^0 = \{\mu^0(t)\}_{t \in [0, T]}$  and a continuous increasing function  $h : [0, T] \rightarrow [0, \infty)$  with  $h(0) = 0$  such that

$$(26) \quad |\mu_t^i(s) - \mu^0(t)| \leq h(s - t)\sqrt{s - t} \text{ for } s \geq t \text{ and } i = 1, \dots, d$$

then condition (25) in Theorem 7.1 is satisfied with  $\mathbb{Q}$  defined via

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left\{ \int_0^T \mu^0(s) dW(s) - \frac{1}{2} \int_0^T |\mu^0(s)|^2 ds \right\}$$

provided the integrability condition  $\sup_{N \in \mathbb{N}} \max_{k=0, \dots, N} \|V_{t_k^N}^N\|_{2, \mathbb{Q}} < \infty$  holds. A proof is provided in Appendix B.

## 8. CONCLUSION

**Summary.** The convergence results of this paper rigorously establish stochastic differential utility as the continuous-time limit of Kreps-Porteus recursive utility. Thus they close an important gap in the literature and justify the use of stochastic differential utility in continuous-time models in macroeconomics and finance.

**Directions for Future Research.** The extension of our analysis to the case of an infinite time horizon is an interesting subject for further study. It would also be interesting to establish a more general convergence result for the special case of Epstein-Zin-Weil utility that also covers unbounded consumption rate processes, based on the analysis of Schroder and Skiadas (1999). In a Brownian setting, one can further extend the analysis of this paper to the notion of generalized stochastic differential utility analyzed by Lazrak (2004).

In Section 7 we have provided a basis for the analysis of preferences with nonstandard certainty equivalents. These results can be extended and generalized, in particular to specifications when an approximate normalization is not available. Finally, we expect our results to be relevant for the numerical evaluation of stochastic differential utility. A detailed analysis of these challenging aspects is left for future research.

## APPENDIX A. A PRIORI ESTIMATES AND GRONWALL INEQUALITY

Throughout this appendix, let  $c \in \mathcal{A}$  be a consumption plan with an associated approximating sequence  $\{c^N\}_{N \in \mathbb{N}}$ , let  $\mathbb{E}[\|\xi\|^2] < \infty$  and let the aggregator  $W$  satisfy (A1), (A2), (A3) and (A4). By (2), (3) and dominated convergence we have

$$\|c^N - c\|_2^2 \rightarrow 0 \text{ and in particular } \|c^N\|_2^2 = \sum_{k=0}^{N-1} \Delta_k^N \|c_{t_k^N}^N\|_2^2 \rightarrow \|c\|_2^2 \text{ as } N \rightarrow \infty$$

where  $\|c\|_2^2 \triangleq \mathbb{E}[\int_0^T c_t^2 dt]$ . Hence there exists  $N_0(c, \{c^N\}_{N \in \mathbb{N}}) \in \mathbb{N}$  such that

$$(27) \quad \sum_{k=0}^{N-1} \Delta_k^N \|c_{t_k^N}^N\|_2^2 \leq 1 + \|c\|_2^2 \text{ for all } N \geq N_0(c, \{c^N\}_{N \in \mathbb{N}}).$$

The following  $L^2$ -bounds are used in the proof of Theorem 4.1.

**Proposition A.1 (A Priori Estimate for Recursive Utility).** *For every approximating sequence  $\{c^N\}_{N \in \mathbb{N}}$  of  $c \in \mathcal{A}$  there is a constant  $K_2 > 0$  such that*

$$\left\| \sup_{t \in [0, T]} |V_t^N(c^N)| \right\|_2 \leq K_2 [1 + \|\xi\|_2 + \|c\|_2^2] \text{ for all } N \geq N_0(c, \{c^N\}_{N \in \mathbb{N}}).$$

*Proof.* Consider (10) and use the conditional triangle inequality and (A3') to get

$$\begin{aligned} |V_t^N| &\leq \mathbb{E}_t[|V_{t_{k+1}}^N|] + \Delta_k^N \mathbb{E}_t[|f(\Delta_k^N, c_{t_k}^N, \mathbb{E}_t[V_{t_{k+1}}^N])|] \\ &\leq \mathbb{E}_t[|V_{t_{k+1}}^N|] + \Delta_k^N K_1 \mathbb{E}_t[1 + |c_{t_k}^N| + |\mathbb{E}_t[V_{t_{k+1}}^N|]|] \\ &\leq \mathbb{E}_t[(1 + K_1 \Delta_k^N) |V_{t_{k+1}}^N| + K_1 \Delta_k^N (1 + |c_{t_k}^N|)] \text{ for } t \in [t_k^N, t_{k+1}^N]. \end{aligned}$$

Iterating this estimate and using the fact that  $1 + x \leq e^x$  for  $x \in \mathbb{R}$  we find

$$\begin{aligned} (28) \quad |V_t^N| &\leq \mathbb{E}_t \left[ \prod_{k=l}^{N-1} (1 + K_1 \Delta_k^N) |\xi| + \sum_{k=l}^{N-1} \prod_{j=l}^{k-1} (1 + K_1 \Delta_j^N) K_1 \Delta_k^N (1 + |c_{t_k}^N|) \right] \\ &\leq M_t^N \triangleq \mathbb{E}_t \left[ e^{K_1 T} |\xi| + K_1 e^{K_1 T} \sum_{k=0}^{N-1} \Delta_k^N (1 + |c_{t_k}^N|) \right], \quad t \in [t_l^N, t_{l+1}^N] \end{aligned}$$

where the process  $M^N = \{M_t^N\}_{t \in [0, T]}$  is an  $(\mathfrak{F}, \mathbb{P})$ -martingale with

$$\begin{aligned} \|M_T^N\|_2 &\leq e^{K_1 T} \|\xi\|_2 + K_1 e^{K_1 T} \sum_{k=0}^{N-1} \Delta_k^N (1 + \|c_{t_k}^N\|_2) \\ &\leq e^{K_1 T} \|\xi\|_2 + K_1 e^{K_1 T} \sum_{k=0}^{N-1} \Delta_k^N (2 + \|c_{t_k}^N\|_2^2) \\ &\leq e^{K_1 T} \|\xi\|_2 + K_1 e^{K_1 T} (2T + \|c\|_2^2 + 1), \quad N \geq N_0(c, \{c^N\}_{N \in \mathbb{N}}). \end{aligned}$$

Here we have used (27) and the fact that  $x \leq 1 + x^2$  for all  $x \in \mathbb{R}$ . Since

$$\left\| \sup_{t \in [0, T]} |V_t^N| \right\|_2 \leq \left\| \sup_{t \in [0, T]} M_t^N \right\|_2 \leq 2 \|M_T^N\|_2$$

by (28) and Doob's L<sup>2</sup>-inequality, this completes the proof.  $\square$

**Proposition A.2 (A Priori Estimate for SDU).** *For every  $c \in \mathcal{A}$  there exists a unique càdlàg process  $V = V(c) = \{V_t(c)\}_{t \in [0, T]}$  with  $\mathbb{E}[\sup_{t \in [0, T]} |V_t(c)|^2] < \infty$  such that (SDU) is satisfied. Moreover, there is a constant  $K'_2 > 0$  such that*

$$\left\| \sup_{t \in [0, T]} |V_t(c)| \right\|_2 \leq K'_2 [1 + \|\xi\|_2 + \|c\|_2^2] \text{ for every } c \in \mathcal{A}.$$

*Proof.* By (A2) and (A3) the function  $f$  satisfies a Lipschitz and linear growth condition. Hence the claim follows from well-known results on BSDEs; see, e.g., Duffie and Epstein (1992) or Antonelli (1993).  $\square$

**Lemma A.3 (Discrete Gronwall Inequality).** *Let  $\{d_k\}_{k=0, \dots, N}$  satisfy*

$$d_k \leq \varepsilon + \sum_{j=k+1}^N \alpha_j d_j \text{ for } k = 0, \dots, N$$

*where  $\{\alpha_k\}_{k=0, \dots, N}$  is a sequence of non-negative numbers and  $\varepsilon \geq 0$ . Then*

$$d_k \leq \varepsilon \prod_{j=k+1}^N (1 + \alpha_j) \leq \varepsilon e^{\sum_{j=k+1}^N \alpha_j} \text{ for all } k = 0, \dots, N.$$

*Proof.* Set  $x_k \triangleq \varepsilon + \sum_{j=k+1}^N \alpha_j x_j$ ,  $k = N, \dots, 0$ . By induction

$$(29) \quad x_k = \varepsilon \prod_{j=k+1}^N (1 + \alpha_j), \quad k = 0, \dots, N.$$

Indeed, this is clear for  $k = N$ ; if (29) holds for  $j = k+1, \dots, N$ , then

$$\begin{aligned} x_k &= \varepsilon + \sum_{j=k+1}^N \alpha_j x_j = \varepsilon + \sum_{j=k+1}^N \alpha_j \varepsilon \prod_{i=j+1}^N (1 + \alpha_i) \\ &= \varepsilon \left[ 1 + \sum_{j=k+1}^N \prod_{i=j}^N (1 + \alpha_i) - \prod_{i=j+1}^N (1 + \alpha_i) \right] = \varepsilon \prod_{i=k+1}^N (1 + \alpha_i). \end{aligned}$$

Another induction argument yields  $d_k \leq x_k$  for  $k = 0, \dots, N$ .  $\square$

## APPENDIX B. PROOF OF (25) FOR DRIFT AMBIGUITY

We assume without loss that  $\mu^0 = 0$ . By definition of  $\mathbb{Q}_{t,t+\Delta}^i$  we have  $\mathbb{E}_{t,t+\Delta}^i[V] = \mathbb{E}_t^{\mathbb{Q}}[V] + \mathbb{E}_t^{\mathbb{Q}}[Z_{t,t+\Delta}^i V]$  with  $Z_{t,t+\Delta}^i \triangleq \frac{d\mathbb{Q}_{t,t+\Delta}^i}{d\mathbb{Q}} - 1$ . The Lipschitz property of  $\Phi_t$  yields

$$\begin{aligned} |\mathbf{m}_t(V) - \mathbb{E}_t^{\mathbb{Q}}[V]| &= |\Phi_t(\mathbb{E}_t^{\mathbb{Q}}[V] + \mathbb{E}_t^{\mathbb{Q}}[Z_{t,t+\Delta}^i V] : i = 1, \dots, d) - \mathbb{E}_t^{\mathbb{Q}}[V]| \\ &\leq L \max_{i=1, \dots, d} \mathbb{E}_t^{\mathbb{Q}}[Z_{t,t+\Delta}^i |V|] \text{ for } V \in \mathfrak{F}_{t+\Delta} \end{aligned}$$

and therefore we obtain

$$\|R_{t_k}^N\|_{1,\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[|\mathbf{m}_{t_k}^N(V_{t_{k+1}}^N) - \mathbb{E}_{t_k}^{\mathbb{Q}}[V_{t_{k+1}}^N]|] \leq L \max_{i=1, \dots, d} \mathbb{E}^{\mathbb{Q}}[Z_{t_k, t_{k+1}}^i |V_{t_{k+1}}^N|].$$

By Cauchy's inequality it follows that

$$(30) \quad \sum_{k=1}^{N-1} \|R_{t_k}^N\|_{1,\mathbb{Q}} \leq L' \sum_{k=1}^{N-1} \max_{i=1, \dots, d} \|Z_{t_k, t_{k+1}}^i\|_{2,\mathbb{Q}}$$

where  $L' \triangleq L \sup_{N \in \mathbb{N}} \max_{k=0, \dots, N} \|V_{t_k}^N\|_{2,\mathbb{Q}}$ . With  $\bar{\mu}_t^i \triangleq 2\mu_t^i$  we have from (26)

$$\begin{aligned} \|Z_{t,t+\Delta}^i\|_{2,\mathbb{Q}}^2 &= \mathbb{E}^{\mathbb{Q}}\left[\left(\exp\left\{\int_t^{t+\Delta} \mu_t^i(s) dW(s) - \frac{1}{2} \int_t^{t+\Delta} |\mu_t^i(s)|^2 ds\right\} - 1\right)^2\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\int_t^{t+\Delta} \bar{\mu}_t^i(s) dW(s) - \frac{1}{2} \int_t^{t+\Delta} |\bar{\mu}_t^i(s)|^2 ds + \int_t^{t+\Delta} |\mu_t^i(s)|^2 ds\right\}\right] - 1 \\ &\leq \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\int_t^{t+\Delta} \bar{\mu}_t^i(s) dW(s) - \frac{1}{2} \int_t^{t+\Delta} |\bar{\mu}_t^i(s)|^2 ds + \frac{1}{2} h(\Delta)^2 \Delta^2\right\}\right] - 1 \\ &= e^{\frac{1}{2} h(\Delta)^2 \Delta^2} - 1 \leq h(\Delta)^2 \Delta^2 \text{ for sufficiently small } \Delta > 0. \end{aligned}$$

Hence the assertion follows from (30) and the estimate

$$\sum_{k=1}^{N-1} \max_{i=1, \dots, d} \|Z_{t_k, t_{k+1}}^i\|_{2,\mathbb{Q}} \leq \sum_{k=1}^{N-1} h(\Delta_k^N) \Delta_k^N \leq h(\Delta^N) T \rightarrow 0 \text{ as } N \rightarrow \infty. \quad \square$$

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