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Reconciling the Rawlsian and the utilitarian approaches to the maximization of social welfare

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to the Maximization of Social Welfare

by

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Abstract

Under a deadweight loss of tax and transfer, there is tension between the optimal policy choices of a Rawlsian social planner and a utilitarian social planner. However, when with a weight greater than a certain critical value the individuals' utility functions incorporate distaste for low relative income, a utilitarian will select exactly the same income distribution as a Rawlsian.

Keywords: Maximization of social welfare; Rawlsian social welfare function; Utilitarian social welfare function; Deadweight loss; Distaste for low relative income

JEL Classification: D31; D60; H21; I38

1. Introduction

The Rawlsian approach to social welfare, built on the foundation of the “veil of ignorance” (Rawls, 1999, p. 118), measures the welfare of a society by the wellbeing of the worst-off individual (the maximin criterion). A utilitarian measures the welfare of a society by the sum of the individuals’ utilities. Starting from such different perspectives, the optimal income distribution chosen by a Rawlsian social planner usually differs from the optimal income distribution chosen by a utilitarian social planner.

Rawls (1999, p. 182) acknowledges that utilitarianism is the single most important ethical theory with which he has to contend. In utilitarian ethics, the maximization of general welfare may require that one person’s good is sacrificed to serve the greater good of the group of people. Rawlsian ethics, however, would never allow this. As Rawls’ Difference Principle states, social and economic inequalities should be tolerated only when they are expected to benefit the disadvantaged. Rawls (1958) explicitly argues that his principles are more morally justified than the utilitarian principles because his will never condone institutions such as slavery, whereas this need not be the case with utilitarian ethics. In such a situation, a utilitarian would simply weigh all the benefits and all the losses, so a priori we cannot exclude a configuration in which slavery will turn out to confer higher aggregate welfare than non-slavery. Rawls argues that if individuals were to select the concept of justice by which the society is to be regulated without knowing their position in the society (the “veil of ignorance”¹), they would choose principles that allow the least undesirable condition for the worst-off member over the utilitarian principles. This hypothetical contract is the basis of the Rawlsian society, and of the Rawlsian social welfare function.

Is it possible to reconcile the Rawlsian and the utilitarian approaches? In this paper, we present a protocol of reconciliation by introducing into the individuals’ utility functions a distaste for low relative income.² We show that when the strength of the individuals’

¹ “[N]o one knows his place in society, his class position or social status; nor does he know his fortune in the distribution of natural assets and abilities, his intelligence and strength, and the like” (Rawls, 1999, p. 118).

² Evidence from econometric studies, experimental economics, social psychology, and neuroscience indicates that humans routinely engage in interpersonal comparisons, and that the outcome of that engagement impinges on their sense of wellbeing. People are discontented when their consumption, income or social standing fall below those of others with whom they naturally compare themselves (those who

distaste for low relative income is greater than a critical value, which depends on the shape of utility functions of the individuals and the initial distribution of incomes, then even under the utilitarian criterion, the maximization of social welfare aligns with the maximization of the utility of the worst-off individual. Intuitively, the more a society is concerned about “free and equal personality,” the greater is the distaste for low relative income. Indeed, in a contribution to the study of social welfare, Harsanyi (1955) assigns a prominent role to interpersonal comparisons in the social welfare function. Thus, this paper presents an explanation in the spirit of Harsanyi (1955) and Rawls (1974), reconciling the Rawlsian and the utilitarian criteria of social welfare maximization.

Ours is not the first attempt to align Rawlsianism with utilitarianism. Arrow (1973) argued that if individuals are extremely risk averse, the maximization of social welfare is equivalent to the maximization of the utility of the worst-off individual.³ However, as we show below, equivalence of the two approaches can be achieved when individuals exhibit little risk aversion in the sense that their degree of relative risk aversion is small; specifically, less or equal to one. Yaari (1981) provided a proof that there exists a specific set of weights of the individuals’ utilities in the utilitarian social welfare function under which the utilitarian and the Rawlsian social optima coincide. However, our reconciliation protocol does not require any specific weighting of the individuals’ utilities in the social welfare function; specifically, the individuals are given the same weight each, equal to one, in the utilitarian welfare function. In comparison with Arrow (1973) and Yaari (1981), we obtain reconciliation under less stringent conditions with respect to the preference structure of the individuals and/or the construction of the social welfare function; namely, we align the utilitarian and the Rawlsian perspectives by taking into consideration the well documented concern of individuals at having low relative income.

Nor are we the first to study the interaction between comparison utility and optimal taxation policy. Probably the closest to our work is a paper by Boskin and Sheshinski (1978) who investigate optimal tax rates for utilitarian and maximin criteria of social

constitute their “comparison group”). Examples of studies that recognize such discontent include Stark and Taylor (1991), Zizzo and Oswald (2001), Luttmer (2005), Fliessbach et al. (2007), Blanchflower and Oswald (2008), Takahashi et al. (2009), Stark and Fan (2011), Stark and Hyll (2011), Fan and Stark (2011), Stark et al. (2012), and Card et al. (2012). Stark (2013) presents corroborative evidence from physiology.

³ Rawls (1974) comments that Arrow’s (1973) argument is not sufficiently compelling, intimating that “the aspirations of free and equal personality point directly to the maximin criterion.”

welfare under varying intensities of the distaste for low relative income in the individuals' utility functions. They find that when a distaste for low relative income affects strongly the utilities of the individuals, maximization of both utilitarian and maximin measures of social welfare calls for highly progressive taxation - a result that reaffirms a natural intuition: comparison utility increases optimal redistribution. However, having admitted a distaste for low relative income, Boskin and Sheshinski (1978) are generally not interested in the convergence of the utilitarian and maximin approaches.⁴ Our paper takes a step further to show not only that redistribution becomes more intensive as the individuals' distaste for low relative income is taken into account, but also that it is likely that in such a situation, the goals of the utilitarian social planner and the Rawlsian social planner are exactly congruent.

Hammond (1977) links equality of utilities, namely equality of the individuals' levels of utility, with utilitarianism. His approach is to tailor the utilitarian social welfare function such as to render it "equity-regarding." In our model, however, we incorporate the distaste for low relative income in the utility functions of the individuals, not in the preference structure of the social planner as such. Namely, the preference for equity flows from the bottom-up rather than being "imposed" top-down. In addition, Hammond (1977) merely mentions that deadweight losses in the tax and transfer system may bear significantly on the optimality of a redistribution aimed at conferring equity on a population. In contrast, in our model, the deadweight loss is the reason why the Rawlsian and utilitarian optimal distributions can differ in the first place.

Using an example of a two-person population, in the next section we illustrate the tension between the goal of a Rawlsian social planner and the goal of a "standard" utilitarian social planner, that is, a utilitarian social planner who is not worried about individuals' distaste for low relative income. In Section 3 we conduct an analysis of the distributions of income chosen by, respectively, a Rawlsian social planner, a "standard" utilitarian social planner, and a low-relative-income-sensitive utilitarian social planner, for a population consisting of more than two individuals. We prove the existence of a critical value for the intensity of the individuals' distaste for low relative income under which the

⁴ Frank (1985) and Ireland (2001) show that progressive taxation can be Pareto improving if people care about relative income.

optimal income distribution chosen by a Rawlsian social planner is the same as that chosen by a low-relative-income-sensitive utilitarian social planner. In Section 4 we present this critical value for the case of two individuals. Section 5 concludes.

2. The tension between the optimal policy choices of a Rawlsian and a utilitarian – an example

The following example illustrates the tension between the two approaches. In a two-person population, one individual, the “rich,” has 14 units of income; the other individual, the “poor,” has 2 units of income. Let the preferences of an individual be given by a logarithmic utility function, $u(x) = \ln x$, where $x > 0$ is the individual’s income. A social planner can revise the income distribution by transferring income between the two individuals - specifically from the “rich” to the “poor.” Because of a deadweight loss of tax and transfer, only a fraction of the taxed income ends up being transferred; suppose that half of the amount t taken from the “rich” ends up in the hands (or in the pocket) of the “poor.” Then, the post-transfer utility levels are $\ln(14-t)$ of the “rich,” and $\ln(2+t/2)$ of the “poor.”

How will a Rawlsian choose the optimal t ? Following the maximin criterion, he maximizes the social welfare function

$$SWF_R(t) = \min \{ \ln(14-t), \ln(2+t/2) \}$$

over $t \in [0,14]$. Clearly, as long as $2+t/2 \leq 14-t$, we will have $\min \{ \ln(14-t), \ln(2+t/2) \} = \ln(2+t/2)$. Therefore, a Rawlsian social planner will find it optimal to raise the income of the “poor” by means of a transfer from the “rich.” When equality of incomes is reached, the Rawlsian social planner will not take away any additional income from the “rich” because if he were to do so, the “rich” would become the worst-off member of the population, and social welfare would register a decline. Thus, a Rawlsian social planner will choose to equalize incomes, that is, set the optimal amount to be taken from the “rich” at $t^{R*} = 8$, which results in a post-transfer income of 6 of each individual.

A utilitarian social planner, however, maximizes the social welfare function that is the sum of the individuals' utilities

$$SWF_U(t) = \ln(14-t) + \ln(2+t/2)$$

over $t \in [0,14]$. The first order condition,

$$\frac{1}{2(2+t/2)} - \frac{1}{14-t} = 0,$$

yields the optimal amount to be taken from the “rich” $t^{U^*} = 5$.⁵ Therefore, the post-transfer incomes are 9 of the “rich,” and 4.5 of the “poor.”

From this simple example we see vividly how the objectives of the Rawlsian and the utilitarian social planners come into conflict. In the case of a population that consists of two individuals, a Rawlsian social planner equalizes incomes even if the redistribution process wastes considerable income due to a deadweight loss of tax and transfer. The objective of a utilitarian social planner does not allow him to condone such a sacrifice; to secure higher aggregate utility, he will tax less than the Rawlsian social planner ($t^{U^*} = 5 < t^{R^*} = 8$). Nor will the utilitarian social planner equalize incomes.

3. Reconciling the optimal choices of the Rawlsian and the utilitarian social planners

Consider a population of individuals $1, \dots, n$ whose incomes are a_1, \dots, a_n , respectively, such that $0 \leq a_1 \leq \dots \leq a_n$. Our interest is in finding out the income transfer policies of a Rawlsian social planner, a “standard” utilitarian social planner, and a low-relative-income-sensitive utilitarian social planner; we refer to these three social planners as RSP, SUSP, and RIUSP, respectively.

Let $f : \mathbf{R}_{\geq 0} \mapsto \mathbf{R}$ be a twice differentiable, strictly increasing, and strictly concave function. Let the utility function of individual i be

$$u_i(x_1, \dots, x_n) = (1-\beta) f(x_i) - \beta \text{RI}(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \quad (1)$$

⁵ The second order condition for a maximum, $SWF_U''(t) < 0$, holds.

where $x_i \geq 0$ is the individual's income, and RI is a measure (index) of low relative income of an individual earning x_i , specifically,

$$\text{RI}(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \sum_{j=1}^n \max\{x_j - x_i, 0\}, \quad (2)$$

where x_j for $j \in \{1, \dots, n\} \setminus \{i\}$ are the incomes of the other individuals in the population.⁶

The individual's taste for absolute income is weighted by $1 - \beta$, $\beta \in [0, 1)$, his distaste for low relative income by β . The coefficient β is a measure of the intensity of the individual's distaste for low relative income. When an individual is not concerned at having low relative income, $\beta = 0$.

Let there be a social planner who can transfer income from one individual to another in order to obtain what he considers to be an optimal income distribution. Let $t \in \left[0, \sum_{i=1}^n a_i\right]$ denote the possible total income that is to be taken away from individuals (henceforth "tax"). Due to a deadweight loss of tax and transfer, only a fraction of the taxed income ends up being transferred. We denote this fraction by $0 < \lambda \leq 1$.⁷ Therefore, the set on which we will search the maximum of the considered social welfare functions is

$$\Omega(a_1, \dots, a_n; \lambda) = \left\{ (x_1, \dots, x_n) \in \mathbf{R}_{\geq 0}^n : \lambda \sum_{i=1}^n \max\{a_i - x_i, 0\} = \sum_{i=1}^n \max\{x_i - a_i, 0\} \right\}.$$

⁶ The relationship between the plans or policies of a RSP and a RIUSP is not contingent, however, on the index of low relative income of an individual being defined as in (2). A relationship similar to the one demonstrated below could be shown to hold if, for example, RI were to be defined as

$$\text{RI}(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \sqrt{\sum_j \max\{x_i - x_j, 0\}^2}.$$

⁷ Throughout the analysis, we make an implicit assumption that the lump-sum transfer does not alter the individuals' behavior (and, consequently, neither their pre-transfer income) with respect to their work/leisure optimization. This assumption holds if income is taken to be exogenous. However, if income includes labor income, and if individuals optimally choose how much time to allocate to work, then a lump sum transfer may change the individuals' optimal labor supply under a distaste for low relative income. The poor individual may then work less when his income is increased as his marginal disutility from low relative income decreases when his income is raised by the transfer. A modeling of such a repercussion is presented in Sorger and Stark (2013).

The constraint defining the set Ω simply states that the transfer has to be equal to the deadweight-loss adjusted tax. Because Ω is a compact subset of \mathbf{R}^n , any continuous function defined on this set attains a maximal value.⁸

In the following three subsections we delineate the optimal income distributions chosen by each of the three social planners.

3a. The maximization problem of a Rawlsian social planner

The maximization problem of a RSP is

$$\max_{\Omega(a_1, \dots, a_n; \lambda)} \text{SWF}_R(x_1, \dots, x_n) = \max \left\{ \min \{u_1(x_1, \dots, x_n), \dots, u_n(x_1, \dots, x_n)\} \right\}. \quad (3)$$

It is easy to see that for every $k \in \{1, \dots, n-1\}$ we have that

$$u_i(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = u_i(x_1, \dots, x_{k+1}, x_k, \dots, x_n)$$

for $i \in \{1, \dots, n\}$, $i \neq k, k+1$, and that

$$u_k(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = u_{k+1}(x_1, \dots, x_{k+1}, x_k, \dots, x_n).$$

Therefore, without loss of generality, we may assume that $x_1 \leq \dots \leq x_n$. Moreover if $x_1 \leq \dots \leq x_n$, then the monotonicity of f and the definition of the RI function imply that $u_1(x_1, \dots, x_n) \leq u_2(x_1, \dots, x_n) \leq \dots \leq u_n(x_1, \dots, x_n)$.

Thus, if we denote by $x_1^{R^*}, \dots, x_n^{R^*}$ the optimal post-transfer incomes partition of a RSP, we will have

$$\max_{\Omega(a_1, \dots, a_n; \lambda)} \text{SWF}_R(x_1, \dots, x_n) = u_1(x_1^{R^*}, \dots, x_n^{R^*}),$$

where $x_1^{R^*} = \dots = x_n^{R^*}$. To prove this by contradiction, we suppose that there is $j \in \{1, \dots, n-1\}$ such that $x_j^{R^*} < x_{j+1}^{R^*}$. Then, there exists $\delta > 0$ which we can take from the

⁸ We note that as a sum of concave functions of the form $\lambda \max\{a_i - x_i, 0\} - \max\{x_i - a_i, 0\}$, the constraint

function $g(x_1, \dots, x_n) = \lambda \sum_{i=1}^n \max\{a_i - x_i, 0\} - \sum_{i=1}^n \max\{x_i - a_i, 0\}$ is also a concave function.

$j+1$ -th individual and give that which remains after the deadweight loss to the j -th individual. The said existence is a consequence of the “Three-slope inequality” applied to function f ,⁹ and of the fact that a smaller difference between incomes implies a smaller value of the index of low relative income; namely, for any $0 < \delta < (x_{j+1}^{R^*} - x_j^{R^*}) / (1 + \lambda)$, in which case $x_j^{R^*} < x_j^{R^*} + \lambda\delta < x_{j+1}^{R^*} - \delta < x_{j+1}^{R^*}$, we have that

$$\begin{aligned} & u_1(x_1^{R^*}, x_2^{R^*}, \dots, x_j^{R^*} + \lambda\delta, x_{j+1}^{R^*} - \delta, \dots, x_n^{R^*}) - u_1(x_1^{R^*}, x_2^{R^*}, \dots, x_j^{R^*}, x_{j+1}^{R^*}, \dots, x_n^{R^*}) \\ &= (1 - \beta) \left[f(x_j^{R^*} + \lambda\delta) + f(x_{j+1}^{R^*} - \delta) - f(x_j^{R^*}) - f(x_{j+1}^{R^*}) \right] \\ & - \beta \left[\text{RI}(x_1^{R^*}; x_2^{R^*}, \dots, x_j^{R^*} + \lambda\delta, x_{j+1}^{R^*} - \delta, \dots, x_n^{R^*}) - \text{RI}(x_1^{R^*}; x_2^{R^*}, \dots, x_j^{R^*}, x_{j+1}^{R^*}, \dots, x_n^{R^*}) \right] > 0 \end{aligned}$$

for any $\beta \in [0, 1)$ and $0 < \lambda \leq 1$. Therefore, the value of SWF_R at $(x_1^{R^*}, x_2^{R^*}, \dots, x_j^{R^*} + \lambda\delta, x_{j+1}^{R^*} - \delta, \dots, x_n^{R^*}) \in \Omega(a_1, \dots, a_n; \lambda)$ will be higher than $u_1(x_1^{R^*}, \dots, x_n^{R^*})$, which contradicts the fact that SWF_R attains a global maximum at $(x_1^{R^*}, \dots, x_n^{R^*})$. Thus, the solution of the problem of a Rawlsian social planner, (3), is a transfer such that the post-transfer incomes are all equal.

Lastly, we note that the distribution chosen by a RSP entails equality of incomes even when $\beta = 0$, namely, even if a concern of the individuals’ at having low relative income is excluded from the RSP’s social welfare function.

3b. The maximization problem of a “standard utilitarian social planner”

The objective of a SUSP who does not factor in a distaste for low relative income is to maximize $\sum_{i=1}^n f(x_i)$, where $(x_1, \dots, x_n) \in \Omega(a_1, \dots, a_n; \lambda)$. Let

$$\text{SWF}_U(x_1, \dots, x_n) = \sum_{i=1}^n f(x_i).$$

⁹ The “Three-slope inequality.” Suppose that $f : \mathbf{R} \mapsto \mathbf{R}$ is concave and that $x < y < z$. Then

$$\frac{f(z) - f(y)}{z - y} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(y) - f(x)}{y - x}.$$

A proof is in Borwein and Vanderwerff (2010), p. 19.

We now state and prove a claim and two corollaries that, in combination, characterize the distribution chosen by a SUSP, depending on the initial incomes in the population and on the scale of the deadweight loss of tax and transfer.

Claim 1. Let $a_1 \leq \dots \leq a_n$ and $\frac{f'(a_n)}{f'(a_1)} < \lambda$. Then $\max \text{SWF}_U(x_1, \dots, x_n)$ is obtained for $(x_1^{U^*}, \dots, x_n^{U^*}) \in \Omega(a_1, \dots, a_n; \lambda)$ such that there exist $1 \leq i \leq j \leq n$ and $\underline{a}, \bar{a} \in [a_1, a_n]$, $\underline{a} < a_{i+1} \leq \dots \leq a_j < \bar{a}$ with $x_1^{U^*} = x_2^{U^*} = \dots = x_i^{U^*} = \underline{a}$, $x_{i+1}^{U^*} = a_{i+1}, \dots, x_j^{U^*} = a_j$ and $x_{j+1}^{U^*} = \dots = x_n^{U^*} = \bar{a}$, where $\lambda = \frac{f'(\bar{a})}{f'(\underline{a})}$. Moreover, such \underline{a} and \bar{a} are unique.

Proof. The proof is in the Appendix.

The $\frac{f'(a_n)}{f'(a_1)} < \lambda$ condition in Claim 1, which relates the ratio of marginal utilities from income of the richest and of the poorest individuals to the extent of the deadweight loss, delineates the required “efficiency” of the tax-and-transfer scheme. To see the intuition underlying the condition, we note that due to the concavity of the f function, the marginal utility of the poorest individual, $f'(a_1)$, is the highest in the population, and the marginal utility of the richest individual, $f'(a_n)$, is the lowest.¹⁰ Thus, taking a small tax t from the richest individual and giving λt to the poorest yields the highest possible marginal social gain from the tax-and-transfer, namely $\lambda f'(a_1 + \lambda t) - f'(a_n - t)$. To begin with, in order for the marginal gain to be positive for $t = 0$, we must have that $\frac{f'(a_n)}{f'(a_1)} < \lambda$.

As the SUSP proceeds to increase the tax, the ratio of the marginal utilities of the richest and of the poorest individuals rises, and when the ratio is equal to the level of the deadweight loss, the marginal gain from the tax-and-transfer procedure drops to zero.

Corollary 1. If $\frac{f'(a_n)}{f'(a_1)} \geq \lambda$, then a SUSP will not tax any income.

Proof. The proof is in the Appendix.

¹⁰ Here we ignore the possibility that there are two or more richest (poorest) individuals. A more detailed exposition is in the proof of Claim 1.

Corollary 2. A SUSP will choose the RSP's plan if and only if $\beta = 1$.

Proof. The proof is in the Appendix.

Claim 1 together with Corollary 1 state that, in general, the optimal distribution of income chosen by a SUSP will differ from the optimal income distribution chosen by a RSP (unequal as opposed to equal, respectively). The outcome of the optimizations of a RSP and a SUSP are congruent only when $\beta = 1$ (cf. Corollary 2). In the next subsection we show, however, that if a utilitarian social planner acknowledges that sensing low relative income impinges on the well-being of the individuals and that the intensity of this sensing is high enough, then he will choose the same income distribution as a RSP.

3c. The maximization problem of a relative-income-sensitive utilitarian social planner

The aim of a RIUSP is to maximize the function

$$SWF_{RU}(x_1, \dots, x_n) = \sum_{i=1}^n u_i(x_1, \dots, x_n)$$

for $(x_1, \dots, x_n) \in \Omega(a_1, \dots, a_n; \lambda)$. The problem of a RIUSP differs from the problem of a SUSP because the “behavior” (and thereby the maximum) of SWF_{RU} depends on the intensity of the individuals' distaste for low relative income, as exhibited by β . Our aim then is to show that there exists $\beta < 1$ under which a RIUSP will behave just like a RSP (and set $x_i^{RU*} = x_i^{R*}$, where x_i^{RU*} is an optimal post-transfer income of individual i in a RIUSP plan, for $i \in \{1, \dots, n\}$), even if implementing a tax and transfer program involves a deadweight loss ($\lambda < 1$). For a given $a_1 \leq \dots \leq a_n$ we will thus need to ensure that a RIUSP will prefer to tax and transfer incomes until the post-transfer incomes are equalized.

Our strategy is to proceed as follows. For sufficiently large β , a RIUSP will prefer to tax and transfer income. Similarly as in the proof of Claim 1, we show that, indeed, if he needs to tax and transfer income, he will always prefer to tax the richest (if there are two or more individuals with the highest income, he will tax them equally) and transfer to the poorest (supporting equally each of the individuals with the lowest income if there are two or more who are the poorest). The matter to watch is that a RIUSP will not be able to attain

the maximal value of social welfare as long as the post-transfer incomes are not equal. This reasoning invites looking for a β for which the protocol of taxing the richest and supporting the poorest will lead to the RSP's plan. The main outcome of this procedure is the following claim.

Claim 2. For a given f and λ , there exists $\beta(a_1, \dots, a_n) < 1$ such that for every $\beta > \beta(a_1, \dots, a_n)$, $x_i^{\text{RU}^*} = x_i^{\text{R}^*}$ for $i \in \{1, \dots, n\}$.

Proof. The proof is in the Appendix.

Claim 2 states that for the general utility function (1), the optimal income distributions of a RSP and a RIUSP align if the acknowledged individuals' distaste for low relative income is high enough. In the next section we provide an example of the condition on β for the case of a population of two individuals.

4. An example: a population of two individuals

Consider a population that consists of two individuals: "1" with income a_1 , and "2" with income a_2 , such that $0 < a_1 < a_2$. We are interested in finding a condition on β under which the income transfer policies of a RSP and a RIUSP align. To this end, we use a simplified notation compared to that of Section 3. Let the utility function of an individual with income x be

$$u(x) = (1 - \beta) f(x) - \beta \text{RI}(x; y), \quad (4)$$

where f is defined as in Section 3, and

$$\text{RI}(x; y) = \max\{y - x, 0\}, \quad (5)$$

where y is the income of the other individual.

In the case of only two individuals, it is convenient to formulate the optimization problem of a social planner as the maximization of social welfare with respect to the level of the tax. Below, we derive a condition under which the optimal level of the tax chosen by a RIUSP, namely t^{RU^*} , is the same as the tax chosen by a RSP, namely t^{R^*} .

In Section 3a we showed that a RSP will equalize the incomes for any $\beta \in [0,1)$ and $\lambda \in (0,1]$, that is, that

$$x_1^{R^*} = x_2^{R^*} = \frac{a_1 + \lambda a_2}{1 + \lambda},$$

yielding the tax

$$t^{R^*} = \frac{a_2 - a_1}{1 + \lambda}.$$

The maximization problem of a RIUSP is

$$\begin{aligned} \max_{0 \leq t \leq a_2} \text{SWF}_{\text{RU}}(t) &= \max_{0 \leq t \leq a_2} \{u(a_1 + \lambda t) + u(a_2 - t)\} \\ &= \max_{0 \leq t \leq a_2} \{(1 - \beta) f(a_1 + \lambda t) - \beta \text{RI}(a_1 + \lambda t; a_2 - t) \\ &\quad + (1 - \beta) f(a_2 - t) - \beta \text{RI}(a_2 - t; a_1 + \lambda t)\}. \end{aligned} \quad (6)$$

We now state the following corollary.

Corollary 3. If $\beta \geq \beta(a_1, a_2)$, where

$$\beta(a_1, a_2) = \frac{(1 - \lambda) f' \left(\frac{a_1 + \lambda a_2}{1 + \lambda} \right)}{(1 + \lambda) + (1 - \lambda) f' \left(\frac{a_1 + \lambda a_2}{1 + \lambda} \right)} \quad (7)$$

(such that $\beta(a_1, a_2)$ is a positive number strictly smaller than one), then $t^{\text{RU}^*} = t^{R^*}$, and $x_i^{\text{RU}^*} = x_i^{R^*}$ for $i = 1, 2$.

Proof: The proof is in the Appendix.

In Corollary 3 we derive a condition for the specific case of two individuals which is similar to that presented in Claim 2. We can now link our “reconciliation protocol” with two risk-related issues. First, by way of an example, we show that the alignment of the tax policies of the RIUSP and RSP does not hinge on the individuals being extremely risk

averse (the condition for reconciliation required by Arrow, 1973). To this end, let $f(x) = \ln x$, $x > 0$.¹¹ In this case, condition (7) becomes

$$\beta(a_1, a_2) = \frac{1 - \lambda}{1 - \lambda + a_1 + a_2 \lambda},$$

namely, $\beta(a_1, a_2)$ is positive and smaller than one, and Corollary 3 holds. Let $r_x \equiv -x \frac{u''(x)}{u'(x)}$ denote the Arrow-Pratt measure of relative risk aversion, where u is defined as in (4). Then, with $f(x) = \ln x$, for both cases where $x > y$ or $x < y$ (cf. (5)), we have that $r_x \leq 1$. Thus, our “reconciliation protocol” holds in spite of the individuals’ degree of relative risk aversion being small (less or equal to one).

Second, we also do not require the utility function to be of the “ruin-averse” type (cf. Cabrales et al., 2013), that is, of it to observe the property $\lim_{x \rightarrow 0^+} u(x) = -\infty$. Let

$f(x) = x^\gamma$, $x > 0$, $0 < \gamma < 1$. In this case, condition (7) becomes

$$\beta(a_1, a_2) = \frac{\gamma(1 - \lambda)(1 + \lambda)^{-\gamma}}{\gamma(1 - \lambda)(1 + \lambda)^{-\gamma} + (a_1 + a_2 \lambda)^{1-\gamma}},$$

so, once again, $\beta(a_1, a_2)$ is positive and smaller than one, and Corollary 3 holds. For this utility function, namely for $u(x) = (1 - \beta)x^\gamma - \beta \text{RI}(x; y)$, we have, however, that $\lim_{x \rightarrow 0^+} u(x) > -\infty$. (And here, still, $r_x < 1$.)

5. Conclusion

We have shown that the utilitarian optimal tax policy may align with the Rawlsian optimal tax policy if utility depends not only on an individual’s own income, but also on others’ income. In other words, when a utilitarian social planner incorporates the individuals’

¹¹ Strictly speaking, a logarithmic function does not satisfy the conditions on $f(x)$ assumed in Section 3 because it is not defined for $x = 0$. However, because $\lim_{x \rightarrow 0^+} (\ln x)' = \infty$, excluding zero from the domain of the optimization problems discussed in Sections 3 and 4 does not change the presented results.

distaste for low relative income, he can well end up seeing eye to eye with a Rawlsian social planner in the choices that they make concerning the optimal tax-and-transfer policy. The demonstration of this congruence offers a second avenue for reconciling these two interpretations of the policy that would flow from the objective perspective of an individual behind the Rawlsian veil of ignorance, the first being Arrow's point that Harsanyi's expected utility (utilitarian) social welfare function reduces to the Rawlsian (maximin) social welfare function if the concavity of the individual's utility from income is severe enough. We show that this congruence is attributable to a change in the stance of the utilitarian social planner, and is not contingent on the individuals being particularly risk averse.

We provide a new reason to think that incorporating such "comparison utility" into optimal tax models will have large effects on the results. While more and more economists recognize that such comparisons are undoubtedly an aspect of reality, up until the present, comparison utilities have had only a minor impact on how we think about benchmarking optimal tax policies. This paper demonstrates that neglecting comparisons may substantially bias those benchmarks.

Utilitarianism has its own well-defined ethical foundations. If a distaste for low relative income is to be introduced into utilitarianism, how would this be worked out in a consistent way? Research on this issue will enrich social welfare theory and strengthen the ethical foundation of income redistribution policies.

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Appendix

Proof of Claim 1

We first prove the existence of $\underline{a}, \bar{a} \in [a_1, a_n]$, $\underline{a} \leq \bar{a}$ such that $\frac{f'(\bar{a})}{f'(\underline{a})} = \lambda$, and we then prove that the maximum of SWF_U on $\Omega(a_1, \dots, a_n; \lambda)$ is attained at the point $(x_1^{U*}, \dots, x_n^{U*})$, where $x_1^{U*} = x_2^{U*} = \dots = x_1^{U*} = \underline{a}$, $x_{i+1}^{U*} = a_{i+1}, \dots, x_j^{U*} = a_j$, and $x_{j+1}^{U*} = \dots = x_n^{U*} = \bar{a}$.

Consider a tax-and-transfer policy in which we start by taxing the richest individual, transferring this amount (adjusted for the deadweight loss) to the poorest. In case we hit a point in which the income of the poorest (richest) reaches the income of the second poorest (richest), we start to support (tax) them evenly, and so on. We define functions

$$\underline{a}_k(t) = a_k + \frac{\lambda}{k}(t - t_{k-1}) \text{ for } t \in [t_{k-1}, t_k],$$

where $t_0 = 0, t_k = \left(ka_{k+1} - \sum_{m \leq k} a_m\right) / \lambda$ for $k \in \{1, \dots, n-1\}$, and

$$\bar{a}_1(t) = a_{n-1+1} - \frac{t - \bar{t}_{1-1}}{1} \text{ for } t \in [\bar{t}_{1-1}, \bar{t}_1],$$

where $\bar{t}_0 = 0, \bar{t}_1 = \sum_{m \geq n-1} a_m - (1+1)a_{n-1}$ for $1 \in \{1, \dots, n-1\}$. Then t_k (\bar{t}_1) is the level of tax after which the income of the k most poor (1 most rich) individuals reaches the level of income of the $k+1$ -th poorest (1+1-th richest). Consequently, as long as $\underline{a}_k(t) \leq \bar{a}_1(t)$, $\underline{a}_k(t)$ ($\bar{a}_1(t)$) for $t \in [t_{k-1}, t_k]$ ($t \in [\bar{t}_{1-1}, \bar{t}_1]$) is the “income path” of the k -th poorest (1-th richest) individual until he reaches the income of the $k+1$ -th poorest (1+1-th richest), after which he “joins” the income path of the $k+1$ -th poorest (1+1-th richest).

We define

$$\underline{a}(t) = \begin{cases} \underline{a}_1(t) & \text{for } t \in [t_0, t_1) \\ \dots & \\ \underline{a}_{n-1}(t) & \text{for } t \in [t_{n-2}, t_{n-1}] \end{cases}$$

and

$$\bar{a}(t) = \begin{cases} \bar{a}_1(t) & \text{for } t \in [\bar{t}_0, \bar{t}_1) \\ \dots \\ \bar{a}_{n-1}(t) & \text{for } t \in [\bar{t}_{n-2}, \bar{t}_{n-1}] \end{cases}$$

as the “income paths” of, respectively, the poorest individual and the richest individual for a given level of tax t .

Because $\underline{a}(0) = a_1 < a_n = \bar{a}(0)$,¹² $\underline{a}(t_{n-1}) = a_n > a_1 = \bar{a}(\bar{t}_{n-1})$, $\underline{a}(t)$ is continuous and strictly increasing, and $\bar{a}(t)$ is continuous and strictly decreasing, there exists $t_0^* \leq \min\{\bar{t}_{n-1}, \bar{t}_{n-1}\}$ such that $\underline{a}(t_0^*) = \bar{a}(t_0^*)$. We note that for $t \leq t_0^*$ the tax and transfer policy is self-contained; that is, the sum of the transfers is equal to the sum of the taxes (corrected for the deadweight loss).

From the strict monotonicity and concavity of f , and the strict monotonicity and continuity of $\underline{a}(t)$ and $\bar{a}(t)$, the fraction $\frac{f'(\bar{a}(t))}{f'(\underline{a}(t))}$ is a strictly increasing and continuous

function of t . Thus, because $\frac{f'(\bar{a}(0))}{f'(\underline{a}(0))} = \frac{f'(a_n)}{f'(a_1)} < \lambda$ and $\frac{f'(\bar{a}(t_0^*))}{f'(\underline{a}(t_0^*))} = 1 \geq \lambda$, there exists a

unique $t^* \in (0, t_0^*]$ such that $\underline{a}(t^*), \bar{a}(t^*) \in [a_1, a_n]$, $\underline{a}(t^*) \leq \bar{a}(t^*)$, and $\frac{f'(\bar{a}(t^*))}{f'(\underline{a}(t^*))} = \lambda$. We set

$\underline{a} = \underline{a}(t^*)$ and $\bar{a} = \bar{a}(t^*)$, and we obtain the point $(x_1^{U^*}, \dots, x_n^{U^*}) \in \Omega(a_1, \dots, a_n; \lambda)$ as

$x_1^{U^*} = \dots = x_i^{U^*} = \underline{a}$ for i individuals such that $a_1, \dots, a_i \leq \underline{a}(t^*)$, and $x_{j+1}^{U^*} = \dots = x_n^{U^*} = \bar{a}$ for j

individuals such that $a_{j+1}, \dots, a_n \geq \bar{a}(t^*)$, and with $x_{i+1}^{U^*} = a_{i+1}, \dots, x_j^{U^*} = a_j$.

We now proceed to the proof that $(x_1^{U^*}, \dots, x_n^{U^*})$ is the maximum of $\text{SWF}_U(x_1, \dots, x_n)$ on $\Omega(a_1, \dots, a_n; \lambda)$. Because $\text{SWF}_U(x_1, \dots, x_n)$ is a strictly concave function - it is a sum of strictly concave functions $f(x_i)$ - maximized on a closed subset $\Omega(a_1, \dots, a_n; \lambda)$ characterized by a concave constraint function (cf. footnote 8 in the main

¹² Because $\frac{f'(a_n)}{f'(a_1)} < \lambda \leq 1$, we must have that $a_1 < a_n$.

text), then, if a local maximum exists, it is also a global maximum. We show that $(x_1^{U^*}, \dots, x_n^{U^*}) \in \Omega(a_1, \dots, a_n; \lambda)$ defined above is a local maximum, and thus it is also a global maximum of $\text{SWF}_U(x_1, \dots, x_n)$ on $\Omega(a_1, \dots, a_n; \lambda)$.

Starting from the point $(x_1^{U^*}, \dots, x_n^{U^*})$, we consider a tax in the amount of $t > 0$, which is distributed among individuals from set $K \subset \{1, \dots, n\}$, $K \neq \emptyset$, by the weights $\tau_k \geq 0$ for $k \in K$ such that $\sum_{k \in K} \tau_k = 1$, and $\tau_k t < x_k^{U^*}$. The amount λt is then distributed among the remaining of individuals, namely among the set $L = \{1, \dots, n\} \setminus K$, according to weights $\nu_l \geq 0$, $l \in L$, such that $\sum_{l \in L} \nu_l = 1$. (Proceeding in this way, we obviously do not violate the conditions of the set $\Omega(a_1, \dots, a_n; \lambda)$). For clarity of notation, we set $\tau_l = 0$ for $l \in L$, and $\nu_k = 0$ for $k \in K$.

The change in the level of social welfare brought about by the above tax and transfer procedure is denoted as

$$\begin{aligned} \Delta \text{SWF}_U(t) &\equiv \text{SWF}_U(x_1^{U^*} + \nu_1 \lambda t - \tau_1 t, \dots, x_n^{U^*} + \nu_n \lambda t - \tau_n t) - \text{SWF}_U(x_1^{U^*}, \dots, x_n^{U^*}) \\ &= \sum_{l \in L} f(x_l^{U^*} + \nu_l \lambda t) + \sum_{k \in K} f(x_k^{U^*} - \tau_k t) - \sum_{i \in \{1, \dots, n\}} f(x_i^{U^*}). \end{aligned}$$

We have that

$$\Delta \text{SWF}'_{U+}(0) = \lambda \sum_{l \in L} [\nu_l f'(x_l^{U^*})] - \sum_{k \in K} [\tau_k f'(x_k^{U^*})].$$

Because $\min\{x_1^{U^*}, \dots, x_n^{U^*}\} = \underline{a}$, $\max\{x_1^{U^*}, \dots, x_n^{U^*}\} = \bar{a}$, and f is concave, we have that $f'(x_i^{U^*}) \leq f'(\underline{a})$ and $f'(x_i^{U^*}) \geq f'(\bar{a})$ for any $i \in \{1, \dots, n\}$. Thus, because $\frac{f'(\bar{a})}{f'(\underline{a})} = \lambda$,

$$\Delta \text{SWF}'_{U+}(0) \leq \lambda \sum_{l \in L} [\nu_l f'(\underline{a})] - \sum_{k \in K} [\tau_k f'(\bar{a})] = \lambda f'(\underline{a}) - f'(\bar{a}) = 0. \quad (\text{A1})$$

By means of the preceding tax and transfer procedure, we can characterize any permissible redistribution policy starting from $(x_1^{U^*}, \dots, x_n^{U^*})$. Therefore, (A1) and the uniqueness of the maximum render $(x_1^{U^*}, \dots, x_n^{U^*})$ a global maximum. Q.E.D.

Proof of Corollary 1

Under the assumption of Corollary 1, for $\Delta SWF_U(t)$ defined as

$$\Delta SWF_U(t) \equiv SWF_U(a_1 + \nu_1 \lambda t - \tau_1 t, \dots, a_n + \nu_n \lambda t - \tau_n t) - SWF_U(a_1, \dots, a_n)$$

and with ν_i and τ_i defined as in the proof of Claim 1, we have that $\Delta SWF_U'(0) \leq 0$, and thus, upon a reasoning akin to that in the proof of Claim 1, the proof of the corollary follows. Q.E.D.

Proof of Corollary 2

If $\lambda = 1$ then from the fact that f is concave we get immediately that the sum $\sum_{i=1}^n f(x_i)$ is maximized for $x_1 = x_2 = \dots = x_n$. On the other hand, the equal choices of a SUSP and a RSP entail that for \underline{a} and \bar{a} defined as in the proof of Claim 1, we will have that $\underline{a} = \bar{a}$ and thus, $\lambda = 1$. Q.E.D.

Proof of Claim 2

First, we show that if a RIUSP has to transfer income (whether or not he should, we will check in the second step), then he should tax the richest individual and support the poorest individual. This protocol follows directly from the concavity of f because if $a_1 \leq \dots \leq a_n$, we have that $f'(a_1) \geq \dots \geq f'(a_n)$. Therefore, for every $x, y \in [a_1, a_n]$ and a small t , we have $f(x + \lambda t) + f(y - t) \leq f(a_1 + \lambda t) + f(a_n - t)$.

Moreover, it is easy to see that the aggregate index of low relative income $\sum_{i=1}^n RI(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ decreases when the difference between the highest and the lowest incomes in the population shrinks. This implies that the best choice for a RIUSP (should he transfer any income) is to tax the richest individual n and give that which remains after the deadweight loss to the poorest individual 1. This reasoning continues to

hold as long as $a_n - t \geq a_{n-1}$, or as long as $a_1 + \lambda t \leq a_2$. Therefore, for

$$\begin{aligned}\eta(t) &= \sum_{i=1}^n u_i(a_1 + \lambda t, a_2, \dots, a_{n-1}, a_n - t) - \sum_{i=1}^n u_i(a_1, a_2, \dots, a_{n-1}, a_n) \\ &= (1 - \beta) [f(a_1 + \lambda t) + f(a_n - t)] + (n-1)\beta(1 + \lambda)t,\end{aligned}$$

if $\eta'_+(0) > 0$, then the social planner should transfer some income. In turn, this implies the following conditions on β :

1. If $\frac{f'(a_n) + n - 1}{f'(a_1) - (n - 1)} > \lambda$, then $\beta \geq \frac{f'(a_n) - \lambda f'(a_1)}{f'(a_n) - \lambda f'(a_1) + n - 1 + (n - 1)\lambda}$;
2. If $\frac{f'(a_n) + n - 1}{f'(a_1) - (n - 1)} \leq \lambda$, then $\beta > 0$.

We need to see to it that a RIUSP will tax the richest and support the poorest as long as the income of the poorest is smaller than a_2 , and the income of the richest is higher than a_{n-1} . Because the function $x \mapsto \frac{x}{c + x}$ (for $c > 0$) is increasing on its domain, it is sufficient to take $\beta \geq \beta_{11}$, where

$$\beta_{11} = \frac{f'(a_n - \min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\}) - \lambda f'(a_1 + \lambda \min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\})}{f'(a_n - \min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\}) - \lambda f'(a_1 + \lambda \min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\}) + (n-1)(1 + \lambda)}$$

if

$$f'(a_n - \min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\}) - \lambda f'(a_1 + \lambda \min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\}) < (n-1)(1 + \lambda),$$

and $\beta_{11} = 0$ otherwise. For such a β (which depends on $\lambda, a_1, a_2, a_{n-1}, a_n$ and f), taxing the richest (individual n) and supporting the poorest (individual 1) leads to one of two cases: $\min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\} = (a_2 - a_1)/\lambda$, or $\min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\} = a_n - a_{n-1}$.

We address these two cases in turn.

When $\min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\} = (a_2 - a_1)/\lambda$, then by means of tax and transfer we reach the point where the incomes are $a_2, a_2, a_3, \dots, a_{n-1}, a_n - (a_2 - a_1)/\lambda$. Let $t_1 = (a_2 - a_1)/\lambda$. We check what happens when two individuals have the same lowest

income. The concavity of f implies that the best choice is to tax the richest individual because still $\lambda f'(a_2) < f'(a_n - t_1)$. The question is how to divide what remains after the deadweight loss of the taxed income. Because f is concave, we get for every x and for every $\alpha \in [0, 1]$ that

$$\frac{f(a_2 + \alpha x) + f(a_2 + (1 - \alpha)x)}{2} \leq f\left(a_2 + \frac{1}{2}x\right),$$

and that the aggregate index of low relative income is also the smallest for an equal division of the taxed income. Therefore, the best choice is to give to the two poorest individuals (individual 1 and individual 2) the same amount. This means that in the next step, the aim of a RIUSP is to maximize

$$\begin{aligned} & \text{SWF}_{\text{RU}}\left(a_2 + \frac{\lambda}{2}t, a_2 + \frac{\lambda}{2}t, a_3, \dots, a_{n-1}, a_n - t_1 - t\right) \\ &= (1 - \beta) \left[2f\left(a_2 + \frac{\lambda}{2}t\right) + \sum_{i=3}^{n-1} f(a_i) + f(a_n - t_1 - t) \right] - \beta \sum_{i=1}^n \text{RI}(\dots) \end{aligned}$$

for $t \in [0, \min\{2(a_3 - a_2)/\lambda, a_n - a_{n-1} - t_1\}]$.¹³

When $\min\{(a_2 - a_1)/\lambda, a_n - a_{n-1}\} = a_n - a_{n-1}$, then the incomes are $a_1 + \lambda(a_n - a_{n-1}), a_2, \dots, a_{n-1}, a_{n-1}$. Let $t_1 = a_n - a_{n-1}$. If two individuals have the same highest income, then the concavity of f implies that the best choice is to tax both of them because still $\lambda f'(a_1 + \lambda t_1) < f'(a_{n-1})$. Moreover, because f is concave, we get for every x and for every $\alpha \in [0, 1]$ that

$$\frac{f(a_{n-1} - \alpha x) + f(a_{n-1} - (1 - \alpha)x)}{2} \leq f\left(a_{n-1} - \frac{1}{2}x\right)$$

and the aggregate index of low relative income is also the smallest for an equal division of the tax. Therefore, the best choice is to tax both the “richest” individuals (individual $n - 1$ and individual n) by the same amount. This means that the aim of a RIUSP is to maximize

¹³ For $t = \min\{2(a_3 - a_2)/\lambda, a_n - a_{n-1} - t_1\}$ we will have three individuals with the lowest income or two individuals with the highest income.

$$\begin{aligned} & \text{SWF}_{\text{RU}} \left(a_1 + \lambda t, a_2, \dots, a_{n-2}, a_n - t_1 - \frac{t}{2}, a_n - t_1 - \frac{t}{2} \right) \\ &= (1 - \beta) \left[f(a_1 + \lambda t) + \sum_{i=2}^{n-2} f(a_i) + 2f \left(a_n - t_1 - \frac{t}{2} \right) \right] - \beta \sum_{i=1}^n \text{RI}(\dots) \end{aligned}$$

for $t \in [0, \min \{(a_2 - a_1) / \lambda, 2(a_{n-1} - a_{n-2})\}]$.¹⁴

These considerations entail a procedure akin to that which is followed by a SUSP: we have $\underline{\alpha}, \bar{\alpha} \in [a_1, a_n]$ and $1 \leq i \leq j \leq n$, such that $x_1 = \dots = x_i = \underline{\alpha}$, $x_{i+1} = a_{i+1}, \dots, x_j = a_j$, and $x_{n-j} = \dots = x_n = \bar{\alpha}$. Then, we express the welfare function that a RIUSP seeks to maximize as a function of t ,

$$\begin{aligned} \text{SWF}_{\text{RU}}(t) &= (1 - \beta) \left[i \cdot f \left(\underline{\alpha} + \frac{\lambda}{i} t \right) + \sum_{i < k \leq j} f(a_k) + (n - j) \cdot f \left(\bar{\alpha} - \frac{t}{n - j} \right) \right] \\ &- \beta \left[(n - j) \sum_{i < k \leq j} \left(\bar{\alpha} - \frac{t}{n - j} - a_k \right) + i(n - j) \left(\bar{\alpha} - \frac{t}{n - j} - \underline{\alpha} - \frac{\lambda t}{i} \right) + i \sum_{i < k \leq j} \left(a_k - \underline{\alpha} - \frac{\lambda t}{i} \right) \right]. \end{aligned}$$

The right hand derivative of this function, evaluated at $t = 0$, is equal to

$$\text{SWF}'_{\text{RU}+}(0) = (1 - \beta) [\lambda f'(\underline{\alpha}) - f'(\bar{\alpha})] + \beta [(n - i)\lambda + j].$$

Because the function $x \mapsto \frac{x}{c + x}$ ($c > 0$) is increasing on its domain, the condition

under which a RIUSP will continue to tax the richest and support the poorest is $\beta > \beta_{ij}$, where

$$\beta_{ij} = \frac{f'(\bar{\alpha} - \min \{(a_{i+1} - \underline{\alpha}) / \lambda, \bar{\alpha} - a_j\}) - \lambda f'(\underline{\alpha} + \lambda \min \{(a_{i+1} - \underline{\alpha}) / \lambda, \bar{\alpha} - a_j\})}{f'(\bar{\alpha} - \min \{(a_{i+1} - \underline{\alpha}) / \lambda, \bar{\alpha} - a_j\}) - \lambda f'(\underline{\alpha} + \lambda \min \{(a_{i+1} - \underline{\alpha}) / \lambda, \bar{\alpha} - a_j\}) + (n - i)\lambda + j}$$

if

$$\lambda f'(\underline{\alpha} + \lambda \min \{(a_{i+1} - \underline{\alpha}) / \lambda, \bar{\alpha} - a_j\}) - f'(\bar{\alpha} - \min \{(a_{i+1} - \underline{\alpha}) / \lambda, \bar{\alpha} - a_j\}) \leq (n - i)\lambda + j,$$

and $\beta_{ij} = 0$ otherwise. Also, because $(n - i)\lambda + j > 0$ for any i, j such that $i + j < n$ and

¹⁴ For $t = \min \{(a_2 - a_1) / \lambda, 2(a_{n-1} - a_{n-2})\}$ we will have two individuals with the lowest income or three individuals with the highest income.

any $0 < \lambda \leq 1$, we get that $\beta_{ij} < 1$.

Our protocol yields the result that for $\beta > \beta_{\max} = \max_{(i,j)} \beta_{ij} < 1$ (namely, for all pairs (i, j) obtained in the protocol) there exist $\underline{\alpha}_0, \bar{\alpha}_0 \in [a_1, a_n]$ and $1 \leq i_0 \leq n$ such that $x_1 = \dots = x_{i_0} = \underline{\alpha}_0$ and $x_{n-i_0} = \dots = x_n = \bar{\alpha}_0$. At this point, a RIUSP maximizes

$$\begin{aligned} \text{SWF}_0(t) = & (1-\beta) \left[i_0 \cdot f \left(\underline{\alpha}_0 + \frac{\lambda}{i_0} t \right) + (n-i_0) f \left(\bar{\alpha}_0 - \frac{t}{n-i_0} \right) \right] \\ & - \beta \left[i_0(n-i_0) \left(\bar{\alpha}_0 - \frac{t}{n-i_0} - \underline{\alpha}_0 - \frac{\lambda t}{i_0} \right) \right]. \end{aligned}$$

Thus, because the choice of RSP is $x^{R^*} \in [\underline{\alpha}_0, \bar{\alpha}_0]$, we obtain the condition

$$\beta \geq \beta_{x^{R^*}} = \frac{(1-\lambda) f'(x^{R^*})}{(1-\lambda) f'(x^{R^*}) + i_0 + (n-i_0)\lambda}.$$

Let $\beta(a_1, \dots, a_n) = \max\{\beta_{\max}, \beta_{x^{R^*}}\} < 1$. For $\beta > \beta(a_1, \dots, a_n)$ we get that $x_1^{U^{**}} = x_2^{U^{**}} = \dots = x_n^{U^{**}} = x^{R^*}$, and the solution that a RIUSP will choose is the same as the solution that a RSP will choose. Q.E.D.

Proof of Corollary 3

For $t > \frac{a_2 - a_1}{1 + \lambda}$, individual 1 does not experience low relative income (because $a_1 + \lambda t > a_2 - t$), and we have that

$$\text{SWF}_{\text{RU}}(t) = (1-\beta) f(a_1 + \lambda t) + (1-\beta) f(a_2 - t) - \beta(a_1 + \lambda t - a_2 + t),$$

and

$$\begin{aligned} \text{SWF}'_{\text{RU}}(t) &= \lambda(1-\beta) f'(a_1 + \lambda t) - (1-\beta) f'(a_2 - t) - \beta(1 + \lambda) \\ &\leq -(1-\beta)(f'(a_2 - t) - f'(a_1 + \lambda t)) - \beta(1 + \lambda) \leq 0, \end{aligned}$$

because, from the concavity of f , we have that $f'(a_2 - t) > f'(a_1 + \lambda t)$ for

$a_2 - t < a_1 + \lambda t$. Therefore, a RIUSP will surely not choose a transfer such that 1 will end up having a higher post-transfer income than 2. In turn, for $0 \leq t \leq \frac{a_2 - a_1}{1 + \lambda}$, individual 2 does not experience low relative income, and the function maximized in (6) reduces to

$$\text{SWF}_{\text{RU}}(t) = (1 - \beta) f(a_1 + \lambda t) - \beta(a_2 - t - a_1 - \lambda t) + (1 - \beta) f(a_2 - t).$$

We have that

$$\text{SWF}'_{\text{RU}}(t) \Big|_{0 \leq t < \frac{a_2 - a_1}{1 + \lambda}} = \lambda(1 - \beta) f'(a_1 + \lambda t) + \beta(1 + \lambda) - (1 - \beta) f'(a_2 - t). \quad (\text{A2})$$

In addition, because f is concave,

$$\text{SWF}''_{\text{RU}}(t) \Big|_{0 \leq t < \frac{a_2 - a_1}{1 + \lambda}} = (1 - \beta) \left(\lambda^2 f''(a_1 + \lambda t) + f''(a_2 - t) \right) < 0. \quad (\text{A3})$$

From (A2) and (A3), we get that $t = \frac{a_2 - a_1}{1 + \lambda}$ is the solution to the maximization problem (6) if the left hand derivative of the social welfare function maintains

$$\text{SWF}'_{\text{RU}} \left(\frac{a_2 - a_1}{1 + \lambda} \right) \geq 0,$$

namely, if

$$\beta \left[(1 + \lambda) + (1 - \lambda) f' \left(\frac{a_1 + \lambda a_2}{1 + \lambda} \right) \right] - (1 - \lambda) f' \left(\frac{a_1 + \lambda a_2}{1 + \lambda} \right) \geq 0.$$

This condition translates directly into

$$\beta \geq \frac{(1 - \lambda) f' \left(\frac{a_1 + \lambda a_2}{1 + \lambda} \right)}{(1 + \lambda) + (1 - \lambda) f' \left(\frac{a_1 + \lambda a_2}{1 + \lambda} \right)} = \beta(a_1, a_2).$$

Obviously, we have that $\beta(a_1, a_2) < 1$. Q.E.D.