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Long memory and periodicity in intraday volatilities of stock index futures

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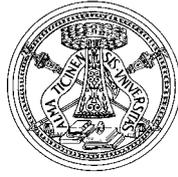
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Quaderni di Dipartimento

**Long memory and Periodicity in Intraday Volatilities
of Stock Index Futures**

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Long memory and Periodicity in Intraday Volatilities of Stock Index Futures *

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Abstract

This paper investigates the intraday volatility pattern of the E-mini SP500, quoted at the Chicago Mercantile Exchange, one of the most traded American Stock Index futures. The data set consists of round-the-clock hourly returns. The squared (and absolute) returns are characterized by long memory and periodicity. In order to jointly model the long memory and the periodic components in the returns volatility we introduce two new parameterizations. The Fractionally Integrated Periodic EGARCH (FI-PEGARCH) and the Seasonal Fractional Integrated Periodic EGARCH (SFI-PEGARCH). For both models we compute the population kurtosis and the autocorrelation function of power transformations of absolute returns. We find that during the Asian and European trading time the volatility is lower than during the American trading time when we observe a sharp increase. The results seem to confirm the fact that hourly returns sampled over the 24 hours across different markets are characterized by a strong seasonal pattern with a statistically significant persistence. Finally we present the in-sample and out-of-sample forecasts results of unrestricted and restricted long memory periodic volatility models.

JEL classification: C22, G13.

Keywords: PEGARCH models, Long memory, Asymmetric volatility, Stock Index Futures.

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1 Introduction

The increased availability of high frequency financial ¹ data has determined a growing number of research studies examining the complex intraday return dynamics. ²

The analysis of financial microdata, i.e. high frequency data, is complicated by both the presence of strong intraday seasonalities, especially in volatilities and volumes, and long memory. The latter is stronger when the sampling frequency increases. A number of papers have noted that the autocorrelation function of power transformations of absolute returns are best characterized by slowly mean-reverting hyperbolic rate of decay (see Ding, Granger, and Engle (1993)). Moreover, a well documented characteristic of intraday return volatility and traded volumes is the *U*-shaped pattern (or to be more precise, an inverted *J*): they reach their highest values at the market opening, then go down and reach their lower point around the lunch hours; finally they rise again at the market closing (Admati and Pfleiderer (1988), McNish and Wood (1990), Gerety and Mulherin (1992) Sheikh and Ronn (1994)).

To make matters more complex, numerous studies found weekend effects and other calendar effects at lower frequencies. In particular, the day-of-the-week effect has been studied in a number of papers: French (1980), Hamon and Jacquillat (1990), to name a few.

Andersen and Bollerslev (1997) observe that conclusions on return volatility and market microstructure variables at the intraday frequencies are likely subject to severe distortions due to the strong periodicity in returns.

This suggests that volatility modeling should be flexible enough to account for the presence of different persistent periodic components. In the literature we can identify two different approaches to this problem.

In the first one, pioneered by Andersen and Bollerslev (1997) the volatility process is the outcome of the simultaneous interaction of different components, among these an intraday long memory factor and an intraday periodic component. They allow the shape of the periodic pattern to also depend on the current overall level of return volatility. This approach has been frequently applied. Beltratti and Morana (1999) suppose that the cyclical component is stochastic and represented by a stochastic volatility model in state-space form. Baillie, Han, Myers, and Song (2007) study the volatility of high-frequency commodity futures returns and find that, after removing the intraday periodicity using a deterministic flexible Fourier form filter, an MA-FIGARCH model provides an excellent description of daily and high-frequency returns data. Moreover, Martens, Chang, and Taylor (2002) show that filtering out the seasonal pattern, that is deseasonalizing, improves the out-of-sample forecasting performance of volatility models. They also find that the volatility forecasts obtained from a GARCH model estimated with deseasonalized data, where a flexible Fourier Form is used to filter the intraday returns, are only marginally worse than those of Periodic-GARCH model, estimated with raw data.

¹We don't distinguish between intradaily and high-frequency data.

²We can identify at least three related but distinct literatures: the first one, analyses the transmission of information across international financial markets, that trade sequentially with little, if any, overlap in their trading hours. Early empirical papers include Engle, Ito, and Lin (1990) and Becker, Finnerty, and Friedman (1995), while more recent works are those of Connolly and Wang (2003), Wongswan (2003) and Ehrmann and Fratzscher (2003) who examine spillover effects between U.S. and other countries' financial markets. The second one examines the links between asset prices and macroeconomic fundamentals as embodied in news announcements effects: recent examples are Andersen and Bollerslev (1997), Balduzzi, Elton, and Green (2001), Hautsch and Hess (2002), Andersen, Bollerslev, Diebold, and Vega (2003, 2004). The third strand of the intraday time-series-oriented literature has been concerned with the role of information flow and other microstructure variables as determinants of intraday returns and volatilities: seminal works are those of Bollerslev and Domowitz (1993), and Goodhart, Hall, Henry, and Pesaran (1993). More recent works are Engle (2000), Bauwens and Giot (2002), Rydberg and Shephard (2002), and Liesenfeld and Pohlmeier (2003).

The second approach is based on seasonal fractional differencing. Arteche (2004) considers a long memory in stochastic volatility model and the Gauss semiparametric estimation of the memory parameter. A close approach is that of Bordignon, Caporin, and Lisi (2007a) with the Periodic Long-Memory GARCH models and Bordignon, M.Caporin, and F.Lisi (2007b) with a Gegenbauer-GARCH model.

Naturally, in order to establish to what extent the observed dynamics are due to long memory factors and to periodic components, we should dispose of models that can provide the most flexible representation of the interactions of these two features. A similar problem arises when we consider the interplay between non linearity and long memory, in the sense that non linear models can be considered, and then modeled, as long memory processes (Baillie and Kapetanios (2007)).

The purpose of the present paper is to introduce a new parameterization that allows for periodicity and long memory at different frequencies. We assume that the periodicity is known but allowed to vary, like in the Periodic GARCH models introduced by Bollerslev and Ghysels (1996). This class of models account for the periodic parameter variation observed in volatilities of high-frequency returns, namely they are a representation of nonrepetitive cycles. As Bollerslev and Ghysels (1996) underline the seasonal GARCH representation entails an informational loss in efficiency relative to the true periodic GARCH. In fact, seasonal adjustment, by definition, assumes that a time series can be split up into two independent components, a seasonal and a non-seasonal component, that is clearly impossible when the model is periodic.

Periodic models have found application in the stochastic volatility framework too. Tsiakas (2006) studies the day of the week, non-trading day, and month of the year seasonal effects in the daily returns and volatility of the S&P500 index adopting a periodic stochastic volatility model.

In the periodic GARCH models used so far with financial data (Bollerslev and Ghysels (1996), Franses and Paap (2000)), positive and negative shocks have the same impact on the volatility. A huge literature, starting from the seminal work of Black (1976) till the recent work of Dias and Embrechts (2004) has observed instead, the existence of a negative correlation between the current return and the future volatility, and this leverage effect becomes even stronger when we consider intra-daily data.

We present two new models to capture the long-memory features and the periodic patterns observed in high-frequency volatilities. In both models, the *Fractionally Integrated Periodic EGARCH* (FI-PEGARCH), which is an extension of the Fractionally Integrated EGARCH (Baillie, Bollerslev, and Mikkelsen (1996), Bollerslev and Mikkelsen (1996)), and the *Seasonally Fractionally Integrated Periodic EGARCH* (SFI-PEGARCH), the volatility process is periodic with long memory and leverage effect. The FI-PEGARCH and the SFI-PEGARCH have a different autocorrelation structure. The former is periodic with long memory in each period, the latter is periodic with seasonal long memory. A relevant characteristics of both models is that the population kurtosis and the autocorrelation functions of absolute observations vary with the season and with the asymmetric effects in the volatility process. Both models have enough flexibility, which is given by the long memory periodic structure, to account for the observed features of the high-frequency returns volatility process. The FI-PEGARCH and SFI-PEGARCH can be thought of as an extension of the Periodic Long-Memory GARCH models (Bordignon, Caporin, and Lisi (2007a)).

The models and the nested restrictions are fitted to the hourly returns of E-mini SP500 futures contracts quoted at the Chicago Mercantile Exchange. The empirical analysis shows that while the conditional mean can be modeled by a non-periodic process with no harm, the conditional volatility is characterized by a strong seasonal and persistent structure. Particularly,

we found that during the Asian and European trading time the volatility is much lower than during the American trading time when we observe a sharp increase.

The estimated fractional parameter d ranges, across the different models, between 0.1836 and 0.4112. The diagnostics tests and estimation statistics show that fractional models have a better fit than corresponding non fractional models, and, quite interestingly, that neglecting a complete modeling of the periodic pattern produces higher estimates of the persistence parameter, while a fully periodic model is far less persistent. This confirms that the long memory estimation is strictly connected to the periodicity modeling. Restricting the latter increases the former, and viceversa. This suggests that models which restricts the periodicity patterns such as those based on fractional seasonal filtering can lead to biased estimates, in particular of the long memory parameter. In the empirical analysis of the E-mini returns volatility, we find that the Fractionally Integrated PEGARCH is very close in terms of log-likelihood function and diagnostics to the Seasonally Fractionally Integrated PEGARCH.

In the examined case a restricted FI-PEGARCH(1,d,0) turns out to provide the best approximation to the underlying process. Result that is also confirmed by the analysis of the relative forecasting performances, based on the use of the hourly realized volatility as a proxy of the unobserved volatility process.

The sequel of the paper is organized as follows. In Section 2 we review the current state of the art for what concerns the modeling of long memory and periodicity. In section 3 we present a general expression for periodic EGARCH models. In section 4 we introduce the *Fractionally Integrated Periodic EGARCH* (FI-PEGARCH) model and the *Seasonally Fractionally Integrated Periodic EGARCH* (SFI-PEGARCH), along with a discussion of the properties. Section 5 presents the data and the preliminary analysis. Section 7 reports the empirical results while Section 9 concludes.

2 Periodicity and long memory in intraday return volatility

2.1 Long memory

Long memory is defined in terms of decay rates of long-lag autocorrelations, or in the frequency domain in terms of rates of explosion of low frequency spectra. A long-lag autocorrelation definition of long memory is

$$\gamma(\tau) = c\tau^{2d-1} \quad \tau \rightarrow \infty$$

the correlations of long memory process decay with a hyperbolic rate. They are not summable. An alternative, although not equivalent, definition of long range dependence can be given by using the spectral density $f(\lambda)$ of the process:

$$\lim_{\lambda \rightarrow 0^+} \frac{f(\lambda)}{c_f |\lambda|^{-2d}} = 1 \quad 0 < c_f < \infty.$$

The spectral density $f(\lambda)$ has a pole and behaves like a constant c_f times λ^{-2d} at the origin. A popular approach to the modeling of long memory is represented by the ARFIMA class introduced by Granger and Joyeux (1980) and Hosking (1981). They generalize the class of ARIMA models by allowing a fractional degree of differencing.

It is possible to have long memory at one or more other frequencies between 0 and π (Arteche and Robinson (2000)). In this case the spectral density

$$\lim_{\lambda \rightarrow 0^+} \frac{f(\omega + \lambda)}{c_f |\lambda|^{-2d}} = 1 \quad 0 < c_f < \infty.$$

where $\omega \in (0, \pi)$. There is a pole at ω if $d > 0$ and a zero if $d < 0$. ω represents either one of the seasonal frequencies or the cycle. For instance, the model

$$(1 - L^S)^d y_t = \epsilon_t \quad t = 1, 2, \dots$$

where L is the lag operator, and ϵ_t is a short memory process, has a spectral density for y_t which has poles at $\omega_j = 2\pi j/S$ with $j = 1, \dots, S/2$.

2.2 Seasonal patterns and long memory

A general problem encountered in modeling the intraday volatilities is to account for periodic or seasonal patterns and long memory. In the literature we have different possible solutions:

- Deterministic filtering (Andersen and Bollerslev (1997), Baillie, Han, Myers, and Song (2007)) (or stochastic filtering (Beltratti and Morana (1999)))
- Seasonal fractional differencing (Woodward, Cheng, and Gray (1998), Arteche (2004), Bordignon, Caporin, and Lisi (2007a), Bordignon, M.Caporin, and F.Lisi (2007b))

The deterministic filtering model by Andersen and Bollerslev (1997) is based on the following decomposition of the return for intraday period n and day t :

$$y_{t,n} = E(y_{t,n}) + (h_t^{1/2} s_{t,n} z_{t,n} N^{-1/2})$$

h_t is the conditional variance of daily returns, $s_{t,n}$ is a deterministic function to represent intraday seasonality, $z_{t,n} \sim i.i.d.(0, 1)$ independent of the daily volatility process h_t , and N number of return intervals per day. Taking logs

$$x_{t,n} \equiv 2 \log |y_{t,n} - E(y_{t,n})| - \log(h_t) + \log(N) = \log(s_{t,n}^2) + \log(z_{t,n}^2)$$

A generated $\hat{x}_{t,n}$ is obtained as

$$\hat{x}_{t,n} = 2 \log |y_{t,n} - \bar{y}_{t,n}| - \log(\hat{h}_t) + \log(N)$$

the observable variable $x_{t,n}$ is then a nonlinear regression on the time interval n , and daily volatility σ_t , or

$$\begin{aligned} x_{t,n} &= f(\theta; t, n) + u_{t,n} \\ u_{t,n} &= \log(z_{t,n}^2) - E[\log(z_{t,n}^2)] \sim i.i.d.(0, 1) \end{aligned}$$

where

$$f(\theta; t, n) = \sum_{j=0}^J \sigma_t^j \left\{ \mu_{0j} + \mu_{1j} \frac{n}{N_1} + \mu_{2j} \frac{n^2}{N_2} + \sum_{p=1}^P [\delta_{c,p} \cos(2\pi pn/n) + \delta_{s,p} \sin(2\pi pn/n)] \right\}$$

with

$$N_1 = \frac{1}{N} \sum_{i=1}^N i = \frac{N+1}{2} \quad N_2 = \frac{1}{N} \sum_{i=1}^N i^2 = \frac{(N+1)(2N+1)}{6}$$

If $J = 0$ reduces to the flexible Fourier Form (Gallant (1981)). We can estimate the parameters by OLS. The intraday periodicity for interval n , on day t is then estimated as

$$\hat{s}_{t,n} = \frac{T \left[\exp\left(\frac{\hat{f}_{t,n}}{2}\right) \right]}{\sum_{t=1}^{T/N} \sum_{n=1}^N \exp(\hat{f}_{t,n}/2)}$$

where

$$\widehat{f}_{t,n} \equiv f(\widehat{\theta}; t, n).$$

The high frequency returns are then filtered by the estimated intraday periodicity series $\widehat{s}_{t,n}$ to generate the filtered returns

$$\widetilde{y}_{t,n} = \frac{y_{t,n}}{\widehat{s}_{t,n}}$$

The filtered returns are eventually modeled allowing long memory in volatility. This approach has been generalized in a stochastic volatility framework by Beltratti and Morana (1999).

An alternative to the above procedure is to model the seasonal long memory patterns present in the volatility process. For instance by introducing periodic lags in conditional variance as in Bordignon, Caporin, and Lisi (2007a). They introduce the Periodic long memory GARCH (PLM-GARCH):

$$\begin{aligned} y_t &= E(y_t | \Phi_{t-1}) + \epsilon_t \\ \epsilon_t &= \sqrt{h_t} \eta_t \end{aligned}$$

Φ_{t-1} being the information up to time $t-1$. ϵ_t^2 has a Seasonal ARFIMA, or SARFIMA(p, d, q) $_S$, specification:

$$(1 - L^S)^d \phi(L) \epsilon_t^2 = \omega + [1 - \beta(L)] v_t$$

$\phi(L) = \sum_{i=1}^m \phi_i L^i$, $\beta(L) = \sum_{i=1}^p \beta_i L^i$, with L , such that $x_{t-1} = Lx_t$. The long-memory parameter $0 \leq d \leq 1$ and $v_t = \epsilon_t^2 - \sigma_t^2$. The conditional variance of ϵ_t is then given by

$$(1 - \beta(L)) h_t = \omega + [1 - \beta(L) - (1 - L^S)^d \phi(L)] \epsilon_t^2.$$

The PLM-EGARCH is readily obtained as:

$$(1 - L^S)^d \phi(L) [\ln h_t - \bar{\omega}] = \alpha(L) g_t(\eta_{t-1})$$

with $\bar{\omega} = E[\ln \sigma_t^2]$. A possible extension is represented by the k -factor representation of the Gegenbauer and GARMA models put forward by Woodward, Cheng, and Gray (1998) (Arteche (2004), Bordignon, M.Caporin, and F.Lisi (2007b)). Bordignon, M.Caporin, and F.Lisi (2007b) define a Gegenbauer-GARCH as:

$$\left[(1 - L)^{d_0} (1 + L)^{d_h I(E)} \prod_{j=1}^{h-1} (1 - 2 \cos(\lambda_j) L + L^2)^{d_j} \right] \phi(L) \epsilon_t^2 = \gamma + [1 - \beta(L)] v_t$$

where $I(E) = 1$ if S is even and zero otherwise and $h + 1 = [S/2] + 1 - I(E)$. $\lambda_j (j = 0, \dots, h)$ are frequencies at which the long-memory behavior occurs, $d_j (j = 0, \dots, h)$ are long-memory parameters indicating how slowly the autocorrelations are damped. The conditional variance:

$$\begin{aligned} h_t &= \gamma + \beta(L) h_t \\ &+ \left\{ 1 - \beta(L) - \left[(1 - L)^{d_0} (1 + L)^{d_h I(E)} \prod_{j=1}^{h-1} (1 - 2 \cos(\lambda_j) L + L^2)^{d_j} \right] \phi(L) \right\} \epsilon_t^2 \end{aligned}$$

This implies that in the G-GARCH framework each frequency is modeled by means of a specific long-memory parameter d_i .

3 The Periodic EGARCH process

Periodic models for the volatility process constitute an alternative representation for the seasonal patterns observed in volatility. Periodic GARCH have been introduced by Bollerslev and Ghysels (1996), and used for the analysis of periodicity in volatilities by Franses and Paap (2000), Taylor (2004), among others, and in stochastic volatility model by Tsiakas (2006). Representing the Periodic EGARCH as a vector process makes clear how we can extend this model to include the possibility of long memory.

We denote by P_t the asset price at time t , and by y_t the continuously compounded return $y_t = 100 \times [\ln(P_t) - \ln(P_{t-1})]$, where y_t is observed S times intradaily, for a total number of observations which is T .

The Periodic EGARCH(p, q) process (Bollerslev and Ghysels (1996)) $\{y_t\}$, defined on some probability space $(\Omega, \mathcal{A}, \mathcal{P})$, is a time varying coefficient model for the conditional variance of the returns:

$$\begin{aligned} y_t &= \eta_t \sqrt{h_t} & t = 1, \dots, T \\ h_t &= \text{Var}[y_t | \Phi_{t-1}^s] & t = 1, \dots, T \end{aligned}$$

where $\eta_t \sim i.i.d.(0, 1)$ is defined on the same probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Φ_{t-1}^s is a modified Borel σ -field filtration in which the Borel σ -field filtration based on the realization of the $\{y_t\}$ process up to time $t - 1$ is augmented by a process defining the stage of the periodic cycle at each point in time.

The logarithm of the conditional variance process is modeled as:

$$\ln h_t = \omega_s + \sum_{i=1}^p \delta_{is} \ln h_{t-i} + \sum_{j=1}^q \alpha_{js} g_s(\eta_{t-j}) \quad t = 1, \dots, T \quad s = 1, \dots, S$$

or using the lag operator:

$$(1 - \delta_s(L)) \ln h_t = \omega_s + \alpha_s(L) g_s(\eta_{t-1})$$

where

$$g_s(\eta_t) = \psi_s [|\eta_t| - E(|\eta_t|)] + \gamma_s \eta_t$$

$$\alpha_s(L) = 1 + \alpha_{1s} L + \dots + \alpha_{qs} L^q$$

$$\delta_s(L) = \delta_{1s} L + \dots + \delta_{ps} L^p$$

The Periodic EGARCH(p,q) can be rewritten as a vector process, with $S > p$

$$\begin{aligned}
& \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\delta_{12} & 1 & 0 & \dots & 0 \\ -\delta_{23} & -\delta_{13} & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & -\delta_{pS} & \dots & -\delta_{1S} & 1 \end{bmatrix} \begin{bmatrix} \ln h_{t+1} \\ \vdots \\ \ln h_{t+S} \end{bmatrix} = \\
& \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_S \end{bmatrix} + \begin{bmatrix} 0 & \dots & \delta_{p1} & \dots & \delta_{11} \\ 0 & \dots & 0 & \delta_{p2} & \dots & \delta_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \ln h_{t+1-S} \\ \vdots \\ \ln h_t \end{bmatrix} + \\
& + \begin{bmatrix} \alpha_{q1} & \dots & \alpha_{11} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} g_1(\eta_{t-q+1}) \\ \vdots \\ g_1(\eta_t) \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 \\ \alpha_{q2} & \dots & \alpha_{12} \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} g_2(\eta_{t-q+2}) \\ \vdots \\ g_2(\eta_{t+1}) \end{bmatrix} + \dots + \\
& + \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \alpha_{qS} & \dots & \alpha_{1S} \end{bmatrix} \begin{bmatrix} g_S(\eta_{t+S-q}) \\ \vdots \\ g_S(\eta_{t+S-1}) \end{bmatrix}
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{X}_{t+s} &= [\ln h_{t+1}, \dots, \ln h_{t+S}]' \\
\mathbf{G}_{t+j} &= [g_{j+1}(\eta_{t-q+1+j}), \dots, g_{j+1}(\eta_{t+j})]', \quad j = 0, \dots, S-1
\end{aligned}$$

$$\mathbf{\Psi}_i = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ \alpha_{qi} & \dots & \alpha_{1i} \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \quad i = 1, \dots, S$$

the system can be compactly expressed as

$$\mathbf{\Phi}_0 \mathbf{X}_{t+s} = \boldsymbol{\omega} + \mathbf{\Phi}_1 \mathbf{X}_t + \mathbf{\Psi}_1 \mathbf{G}_t + \dots + \mathbf{\Psi}_s \mathbf{G}_{t+s-1}$$

or

$$\mathbf{X}_{t+s} = \mathbf{\Phi}_0^{-1} \boldsymbol{\omega} + \mathbf{\Phi}_0^{-1} \mathbf{\Phi}_1 \mathbf{X}_t + \mathbf{\Phi}_0^{-1} \mathbf{\Psi}_1 \mathbf{G}_{t+s-1} + \dots + \mathbf{\Phi}_0^{-1} \mathbf{\Psi}_s \mathbf{G}_{t+s-1}$$

The matrix $\mathbf{\Phi}_0$ is a lower triangular matrix with no zeros on the diagonal, it is always invertible.

$$\mathbf{\Phi}(L) \mathbf{X}_{t+s} = \mathbf{\Psi}(L) \mathbf{G}_{t+s}$$

where

$$\begin{aligned}
\mathbf{\Phi}(L) &= \mathbf{\Phi}_0 - \mathbf{\Phi}_1 L^S \\
\mathbf{\Psi}(L) &= \mathbf{\Psi}_1 L + \dots + \mathbf{\Psi}_s L^S
\end{aligned}$$

The system is periodically stable if and only if the roots of

$$|\mathbf{\Phi}_0 - \mathbf{\Phi}_1 z^S| = 0$$

lie outside the unit circle, i.e. $|z| > 1$.

For instance, the Periodic EGARCH(1,0) process

$$\ln h_t = \omega_s + \delta_{1s} \ln h_{t-1} + g_s(\eta_{t-1})$$

is weakly stationary if and only if

$$\left| \prod_{s=1}^S \delta_{1s} \right| < 1. \quad (3.1)$$

We can identify three distinct types of integrated process.

- *Nonseasonally and Nonperiodically Integrated* Periodic EGARCH(p, q) if $(1 - \delta_s(L))$ contains the common factor $1 - L$, but the vector process $(1 - L)\mathbf{X}_{t+s}$ is stationary.
- *Seasonally integrated* Periodic EGARCH(p, q) if $(1 - \delta_s(L))$ contains the common factor $1 - L^S$, with the matrix representation for $(1 - L^S)\mathbf{X}_{t+s}$ being a stationary process.
- *Periodically integrated* if $|\Phi_0 - \Phi_1 z^S|$ contains the factor $(1 - L^S)$ but this is not common to each polynomial $(1 - \delta_s(L))$.

4 Long memory in periodic models

A model that captures the salient features of the volatility of high-frequency returns should account for both seasonally periodic patterns and long memory. We propose two models, the Fractionally Integrated Periodic EGARCH (FI-PEGARCH) and the Seasonally Fractionally Integrated Periodic EGARCH (SFI-PEGARCH).

In the FI-PEGARCH the log conditional variance has long memory with a periodic short memory while the SFI-PEGARCH is characterized by seasonal long memory with a periodic short memory structure.

4.1 Fractionally Integrated Periodic EGARCH

When we assume that long memory is common to all different seasons, that is $(1 - \delta_s(L)) = (1 - \beta_s(L))(1 - L)^d$ we obtain the *Fractionally Integrated* Periodic EGARCH(p, d, q) model:

$$(1 - \beta_s(L))(1 - L)^d (\ln h_t - \omega_s) = \alpha_s(L) g_s(\eta_{t-1}) \quad (4.1)$$

with the roots of $(1 - \beta_s(z)) = 0$ strictly outside the unit circle. This excludes the possibility of periodic integration. All long memory properties of the model are captured in $(1 - L)^d$. The process is, by analogy to the ARFIMA models, covariance stationary and invertible for $-1/2 \leq d \leq 1/2$, and strictly stationary and ergodic for $d < 1/2$ (from Theorem 2.1 in Nelson (1991))(Bollerslev and Mikkelsen (1996)).

When $d > 0$ the process is long memory. It is particularly useful, especially in the estimation process, to express the process in infinite ARCH form. The infinite ARCH representation of the FI-PEGARCH(p, d, q) process is:

$$\ln h_t = \omega_s + (1 - \beta_s(1))^{-1} (1 - L)^{-d} \alpha_s(L) g_s(\eta_{t-1}). \quad (4.2)$$

For instance, the *Fractionally Integrated* Periodic EGARCH(1,d,1) can be represented as an infinite ARCH process, where $-1/2 < d < 1/2$:

$$\ln h_t = \omega_s + (1 - \beta_{1s}L)^{-1} (1 - L)^{-d} (1 + \alpha_{1s}L) g_s(\eta_{t-1}) \quad (4.3)$$

with

$$\begin{aligned}
(1 - \beta_{1s}L)^{-1}(1 - L)^{-d} &= \left(1 + \sum_{i=1}^{\infty} \beta_{1s}^i L^i\right) \left(1 + \sum_{k=1}^{\infty} \left(\prod_{j=1}^k \frac{j-1+d}{j}\right) L^k\right) \\
&= 1 + \sum_{i=1}^{\infty} \lambda_{i,s} L^i = \lambda_s(L)
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\lambda_{0,s} &= 1 \\
\lambda_{i,s} &= \pi_i + \pi_{i-1}\beta_{1s} + \dots + \pi_1\beta_{1s}^{i-1} + \pi_0\beta_{1s}^i, \quad i > 1 \\
\pi_k &\equiv \prod_{j=1}^k \frac{j-1+d}{j}, \quad \pi_0 = 1
\end{aligned} \tag{4.5}$$

or

$$\lambda_{i,s} = \sum_{j=0}^{i-1} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} \beta_s^{i-j-1} \tag{4.6}$$

where $\Gamma(\cdot)$ is the Gamma function. Finally, the process can be compactly rewritten as

$$\ln h_t = \omega_s + \lambda_s(L)(1 + \alpha_{1s}L)g_s(\eta_{t-1}).$$

Given that $(1 - L)^0 = 1$ this model nests the covariance stationary Periodic EGARCH(p, q) model. Moreover, it can be extended to the case where the long-memory parameter d is periodic (Koopman, Ooms, and Carnero (2007)).

4.2 Seasonally Fractionally Integrated Periodic EGARCH

If we consider the possibility that each regime be seasonally persistent, that is $(1 - \delta_s(L)) = (1 - \beta_s(L))(1 - L^S)^d$, (Ooms and Franses (2001) propose a seasonal periodic long memory model for the conditional mean), we have the *Seasonally Fractionally Integrated* Periodic EGARCH(p, d, q):

$$(1 - \beta_s(L))(1 - L^S)^d(\ln h_t - \omega_s) = \alpha_s(L)g_s(\eta_{t-1}) \quad t = 1, 2, \dots, T \tag{4.7}$$

Just as in the case of FI-PEGARCH model the process is covariance stationary and invertible for $-1/2 \leq d \leq 1/2$, and strictly stationary and ergodic for $d < 1/2$, with $d > 0$ the process is characterized by long memory.

The infinite ARCH representation of SFI-PEGARCH($1, p, q$) is given by:

$$\ln h_t = \omega_s + (1 - \beta_s(L))^{-1}(1 - L^S)^{-d}\alpha_s(L)g_s(\eta_{t-1}) \quad t = 1, 2, \dots, T$$

For a SFI-PEGARCH($1, d, 1$) the infinite ARCH representation is:

$$\ln h_t = \omega_s + (1 - \beta_{1s}L)^{-1}(1 - L^S)^{-d}(1 + \alpha_{1s}L)g_s(\eta_{t-1})$$

The inverse of the fractional differential operator is:

$$\begin{aligned}
(1 - L^S)^{-d} &= 1 + \sum_{k=1}^{\infty} \left(\prod_{j=1}^k \frac{j-1+d}{j}\right) L^{kS} \\
&= 1 + \sum_{k=1}^{\infty} \pi_k L^{kS}
\end{aligned}$$

with $\pi_k \equiv \prod_{j=1}^k \frac{j-1+d}{j}$. A compact expression for $(1 - \beta_{1s}L)^{-1}(1 - L^S)^{-d}$ is given by

$$\begin{aligned} (1 - \beta_{1s}L)^{-1}(1 - L^S)^{-d} &= 1 + \beta_{1s}L + \dots + (\beta_{1s}^S + \pi_1)L^S + (\beta_{1s}\pi_1 + \beta_{1s}^{S+1})L^{S+1} + \dots \\ &= 1 + \sum_{k=1}^{\infty} \lambda_{k,s}L^k = \lambda_s(L). \end{aligned}$$

We can express the coefficients of the polynomial $\lambda_s(z)$, as follows

$$\lambda_{i \cdot S + j, s} = \begin{cases} \beta_{1s}^{iS+j} + \beta_{1s}^{(i-1)S+j}\pi_1 + \dots + \beta_{1s}^{S+j}\pi_{i-1} & i = 1, 2, \dots \quad j = \dots, -2, -1, 1, 2, \dots \\ \beta_{1s}^{iS} + \beta_{1s}^{(i-1)S}\pi_1 + \dots + \beta_{1s}^S\pi_{i-1} + \pi_i & i = 1, 2, \dots \quad j = 0 \end{cases} \quad (4.8)$$

where $k = i \cdot S + j$. From eq.(4.8) is evident that whenever $s = S$ the λ_k coefficient is increased by π_S , with $\pi_S > 0$, when $d > 0$. Therefore the SFI-PEGARCH(1,d,q) in ARCH(∞) form is written as:

$$\ln h_t = \omega_s + \lambda_s(L)\alpha_s(L)g_s(\eta_{t-1}) \quad (4.9)$$

The differences between these two models can be highlighted plotting the $\lambda_s(L)$ for both. For instance, setting $\beta_s = 0.8$ and $d = 0.3$ in FI-PEGARCH(1, d, 0) and in SFI-PEGARCH(1, d, 0) we have two different patterns as in fig.1. The polynomial of FI-PEGARCH(1, d, 0) shows a smoothing decaying, while, as expected, the SFI-PEGARCH is characterized by a jagging behavior.

4.3 The kurtosis

He, Tersvirta, and Malmsten (2002) have derived the kurtosis and the autocorrelations of absolute and squared observations for the short memory first order EGARCH process. From Theorem 2 (p.870) in He, Tersvirta, and Malmsten (2002), we can compute the kurtosis for season s .

Proposition 4.1 *Consider the FI-PEGARCH(1, d, 0) or the SFI-PEGARCH(1, d, 0) process and assume that the fourth moment $\nu_4 \equiv E(\eta_t^4) < \infty$, that $E[e^{g_s(\eta_t)}]$ and $E[e^{2g_s(\eta_t)}]$ exist and that $|\beta_{1s}| < 1, \forall s$ holds. Then the kurtosis of y_t for period s exists and is given by:*

$$k_{y,s} = \nu_4 \frac{\prod_{i=0}^{\infty} E[\exp\{2\lambda_{i,s}g_s(\eta_t)\}]}{\{\prod_{i=0}^{\infty} E[\exp\{\lambda_{i,s}g_s(\eta_t)\}]\}^2} \quad (4.10)$$

where $\lambda_{i,s}$ is defined in (4.5) and (4.8) for the FI-PEGARCH(1, d, 0) and the SFI-PEGARCH(1, d, 0) respectively.

Proof. See the Appendix A.

It is evident from the expression (4.10) that for both models the kurtosis results to be periodic. Moreover, the kurtosis increases as the magnitude of the leverage parameter (γ_s) increases (see for an analogous analysis Ruiz and Vega (2008)). In fig.2 the kurtosis of the FIPEGARCH(1, d, 0) is displayed. It is evident that the tail thickness of the unconditional distribution of the process increases with the short-memory parameter β_{1s} as well as with the parameter ψ_s which represents the effect of large shocks to the log-conditional variance. Moreover, larger values for the long memory parameter, d , are reflected in an increase of the kurtosis.

4.4 The autocorrelation function of absolute observations

It is possible to derive the autocorrelation function of $|y_t|^{2m}$ for season s by applying Proposition 4.1 and a recursion similar to that of Theorem 3 (p.872) in He, Tersvirta, and Malmsten (2002).

Proposition 4.2 *Consider the FI-PEGARCH(1,d,0) or the SFI-PEGARCH(1,d,0) and assume that $\nu_{4m} \equiv E(\eta_t^{4m}) < \infty$, $E[|\eta_t|^{2m} e^{mg_s(\eta_t)}] < \infty$ and $E[e^{2mg_s(\eta_t)}] < \infty$ for $0 < m < \infty$, and that $|\beta_{1s}| < 1, \forall s$ holds. Then the autocorrelation function of $|y_t|^{2m}$ for season s and lag n equals*

$$\rho_{m,s}(n) = \frac{E[y_t^{2m} y_{t-n}^{2m}] - [E(y_t^{2m})][E(y_{t-n}^{2m})]}{(E[y_t^{4m}] - [E(y_t^{2m})]^2)^{1/2} (E[y_{t-n}^{4m}] - [E(y_{t-n}^{2m})]^2)^{1/2}} \text{ for } n \geq 1. \quad (4.11)$$

where the numerator is

$$\begin{aligned} E[y_t^{2m} y_{t-n}^{2m}] - [E(y_t^{2m})][E(y_{t-n}^{2m})] &= \nu_{2m} \exp\{m[(\omega_r + \omega_s) + \beta_{1s}^n(\omega_r - \omega_s)]\} \times \\ & E \left[\eta_t^{2m} \exp\{m\chi_{n-1,s} g_s(\eta_t)\} \right] \prod_{i=0}^{n-2} E \left[\exp\{m\chi_{i,s} g_s(\eta_t)\} \right] \times \\ & \prod_{i=n}^{\infty} E \left[\exp\{m\chi_{i,s} g_s(\eta_t)\} \right] \prod_{i=1}^{\infty} E \left[\exp\{m(1 + \beta_{1s}^n)\lambda_{i,r} g_r(\eta_t)\} \right] - \\ & \nu_{2m}^2 \exp\{m(\omega_s + \omega_r)\} \left(\prod_{i=0}^{\infty} E \left[\exp\{m\lambda_{i,s} g_s(\eta_t)\} \right] \right) \times \\ & \left(\prod_{i=0}^{\infty} E \left[\exp\{m\lambda_{i,r} g_r(\eta_t)\} \right] \right) \end{aligned}$$

r denotes the season which occurs at $t - n$.³ The denominator is

$$\begin{aligned} & (E[y_t^{4m}] - [E(y_t^{2m})]^2)^{1/2} (E[y_{t-n}^{4m}] - [E(y_{t-n}^{2m})]^2)^{1/2} = \\ & \exp\{m(\omega_r + \omega_s)\} \left[\nu_{4m} \prod_{i=1}^{\infty} E \left[\exp\{2m\lambda_{i,s} g_s(\eta_t)\} \right] - \nu_{2m}^2 \left(\prod_{i=0}^{\infty} E \left[\exp\{m\lambda_{i,s} g_s(\eta_t)\} \right] \right)^2 \right]^{1/2} \times \\ & \left[\nu_{4m} \prod_{i=1}^{\infty} E \left[\exp\{2m\lambda_{i,r} g_r(\eta_t)\} \right] - \nu_{2m}^2 \left(\prod_{i=0}^{\infty} E \left[\exp\{m\lambda_{i,r} g_r(\eta_t)\} \right] \right)^2 \right]^{1/2} \quad (4.12) \end{aligned}$$

with $\lambda_s(L) = (1-L)^{-d}(1-\beta_{1s}L)^{-1}$ and $\chi_s(L) = (1-L)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1} L^{i-1}$ for the FI-PEGARCH(1,d,0)

and $\lambda_s(L) = (1-L^S)^{-d}(1-\beta_{1s}L)^{-1}$, and $\chi_s(L) = (1-L^S)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1} L^{i-1}$ for the SFI-PEGARCH(1,d,0).

Proof. See the Appendix.

From Proposition 4.2 follows that the autocorrelation function of the seasonal fractionally integrated EGARCH(1,d,0):

$$(1 - \beta_1 L)(1 - L^s)^d (\ln h_t - \omega) = g(\eta_{t-1}) \quad t = 1, 2, \dots, T$$

³If $n = 1$ then $\prod_{i=0}^{n-2} E \left[\exp\{m\chi_{i,s} g_s(\eta_t)\} \right] = 1$.

is given by

Using the theorem A1.1 in Nelson (1991) we can further simplify the expressions for the kurtosis and the periodic autocorrelation function, since under the hypothesis of $\eta_t \sim i.i.d.N(0, 1)$, we have that:

$$E [\exp \{m\lambda_{i,s}g_s(\eta_t)\}] = \left\{ \Phi[m(\psi_s + \gamma_s)\lambda_{i,s}] \exp [m^2\lambda_{i,s}^2(\gamma_s + \psi_s)^2/2] + \Phi[m(\psi_s - \gamma_s)\lambda_{i,s}] \exp [m^2\lambda_{i,s}^2(\psi_s - \gamma_s)^2/2] \right\} \times \exp [-m\lambda_{i,s}\psi_s\sqrt{(2/\pi)}] < \infty.$$

for finite real scalars m and $\lambda_{i,s}$. For $m > 0$ and any finite real $\chi_{n-1,s}$ (see Lemma 1 in He, Tersvirta, and Malmsten (2002)) we have that

$$E [|\eta_t|^{2m} \exp \{m\chi_{n-1,s}g_s(\eta_t)\}] = \frac{1}{\sqrt{2\pi}} \Gamma(2m + 1) \exp \left\{ -\sqrt{\frac{2}{\pi}} m\psi_s\chi_{n-1,s} \right\} \times \exp \left\{ \frac{1}{4} m^2(\gamma_s + \psi_s)^2 \chi_{n-1,s}^2 \right\} \times \left\{ D_{-(2m+1)}[-m\chi_{n-1,s}(\gamma_s + \psi_s)] + \exp \{ -m^2 \chi_{n-1,s}^2 (\gamma_s \psi_s) \} D_{-(2m+1)}[-m\chi_{n-1,s}(\psi_s - \gamma_s)] \right\}$$

where $D_{-(2m+1)}(\cdot)$ is the parabolic cylinder function (see Gradshteyn and Ryzhik (1980)), defined as

$$D_{(-p)}[q] = \frac{\exp\{-q^2/4\}}{\Gamma(p)} \int_0^\infty x^{p-1} \exp\{-qx - x^2/2\} dx \quad p > 0$$

and $\Gamma(\cdot)$ is the gamma function.

When $p = 2$, He, Tersvirta, and Malmsten (2002) show that

$$\int_0^\infty x^2 \exp(-qx - \frac{x^2}{2}) dx = -q + \sqrt{2\pi}(1 + q^2)\Phi(-q) \exp(q^2/2)$$

5 The data

Our data set consists of transaction prices of the E-mini stock index futures SP500, recorded every 60 minutes on the electronic market Chicago Mercantile Exchange, from March 14, 2004 through March 10, 2006, comprising 12,312 observations. The prices of the futures contracts used have been built with the futures contracts closest to expiration, while missing data were substituted with the last available price.⁴ The hourly returns are shown in fig.(3). The contract, besides being the most important stock index futures, has the characteristics of being quoted almost 24 hours a day: to be more precise, there is normal "pit" trading from 9.30 to 16.15 - New York time - (named *Regular Trading Hour, RTH*), and an electronic session from 16.45 until 9.15 the day after (*Globex*). The Monday morning session in Asian markets starts at 17.00 NY time, that correponds to sunday afternoon in the US. The index contains the most liquid stocks from the corresponding market and hence the problem of spurious autocorrelation induced by non-synchronous trading should not arise. The main descriptive statistics are reported in Table 1. The sample kurtosis per hour varies across the day, and the night time returns are characterized by extremely leptokurtic distributions. There is also some evidence of asymmetry. This

⁴Data from Computer Information Systems 101 Holly Ridge Monroe, LA 71203 U.S.A.

confirms that the E-mini stock index futures SP500 hourly returns are not normally distributed.

Seasonality. The sample autocorrelation functions (see fig. 3) don't suggest a seasonal pattern in the hourly returns. In the following analysis we do not make any attempts to correct for the lower frequency interdaily patterns. Moreover, in the series of Emini-SP500 returns we have an increase in the percentage of zeros returns in the night time (fig.8). This is common feature of assets priced over the 24 hours, and is related to the traded volume. It is evident that the Asian and European trading times are mostly affected by this.

For what concerns volatility, we compute an ex-post volatility measure, namely the aggregated squared returns over the hour (i.e. *hourly realized volatility*), from the 5-minutes returns. Even though this estimate can be biased, in this context it represents a first attempt to describe the intraday pattern of the volatility process. In particular, to evaluate the intraday periodicity of the returns volatility we plot in figure 6 the average realized volatility over the hour interval for the 24 hours. This exhibits the classical *U-shaped* pattern. It reveals a pronounced difference in the volatility over the day. This pattern is closely linked to the cycle of market activity in the various trading times around the globe. It increases sharply during the overlap of afternoon trading in Europe and the opening of the U.S. market.⁵

Long memory. According to Granger and Ding (1996) a series is said to have long-memory if it displays a slowly declining autocorrelation and an infinite spectrum at zero frequency. The autocorrelograms of squared and absolute returns in figure 4 clearly show the presence of a strong persistent periodic behavior. In particular, they seem to be characterized by a decay at a very slow mean-reverting hyperbolic rate.

The spectrum estimates (using Bartlett's window) of absolute and squared returns have a peak in correspondence of a cycle with a period of 24 hours (see fig. 5). The profile of the spectrum estimate of the hourly realized volatility is very close to that of the absolute and squared hourly returns, and like those has a peak at a frequency corresponding to a cyclical component of period 24 (see fig. 7). The estimates of the fractional difference parameter obtained with the Geweke, Porter-Hudak estimator (*GPH* estimator) (Geweke and Porter-Hudak (1983)), and the local Whittle local estimator see table 2) and the associated test statistics suggest the presence of persistence or long memory behavior in both series. The GPH estimator of persistence in volatility is based on an ordinary linear regression of the log-periodogram of a series that serves as a proxy of a financial time series, i.e. the squared returns or the absolute returns.

6 Estimation

In this paper we estimate the model parameters θ by maximization of the conditional likelihood function. Let $f(\eta_t, \psi)$ be the density of η_t , the log-likelihood for the t -th observation is given by:

$$\ln L(\eta_t; \theta) = \ln f(\eta_t; \psi) - \frac{1}{2} \ln h_t.$$

We assume that the standardized innovations are normal. When the assumption of conditional normality is violated, asymptotically valid inference can be carried out on the basis of the results of the theory of quasi maximum likelihood estimation (Bollerslev and Wooldridge (1992)). However adopting alternative conditional distributions can lead to significant differences in parameter estimates in finite samples.

⁵Andersen and Bollerslev (1997) note that "return volatility varies systematically over the trading day and that this pattern is highly correlated with the intraday variation of trading volume".

The computation of the conditional log-likelihood necessitates the truncation of the infinite lag polynomial. In order to retain the long-run dependencies in the conditional variance process we set the truncation lag at 1,008. One well known problem in the estimation of long-memory ARCH models is the treatment of initial conditions required to start up the recursions for the conditional variance function. Given the sample size, we use as starting values in the recursion for the calculation of $\log h_t$ the first 1008 standardized residuals, $\hat{\eta}_t$, where the conditional variances are replaced with the sample variance. Thus the sample size used in the estimation is reduced to 11311 observations.

7 Empirical Results

In this section we present the estimation results for a battery of models fitted to the Emini-SP500 data. The starting point is that a seasonal periodic heteroskedastic long memory model may be adequate to capture the dynamics in the conditional variance of the series.

We also consider a Periodic Long Memory EGARCH(1,0) (PLM-EGARCH, hereafter) as a benchmark. This model is particularly interesting because is based on fractional seasonal filter of the log-conditional variance, which is a simple alternative to the FI-PEGARCH and SFI-PEGARCH. First, we present models based on 24 seasons, then we illustrate the results for restricted models.

7.1 Models with twenty-four seasons

From the preliminary analysis the conditional mean equation is specified as

$$y_t = c + \eta_t \sqrt{h_t}$$

while for the log-conditional variance we consider the following models:

- PLM-EGARCH(1,0):

$$(1 - \beta L)(1 - L^S)^d (\ln h_t - \omega) = \psi(|\eta_{t-1}| - E|\eta_{t-1}|) + \gamma \eta_{t-1}, \quad s = 1, \dots, 24$$

- PEGARCH(1,0) with periodic ω , ψ and β and constant γ :

$$(1 - \beta_s L) \ln h_t = \omega_s + \psi_s(|\eta_{t-1}| - E|\eta_{t-1}|) + \gamma \eta_{t-1}, \quad s = 1, \dots, 24$$

- FI-PEGARCH(1,d,0) with periodic ω , ψ and β and constant γ :

$$(1 - \beta_s L)(1 - L)^d (\ln h_t - \omega_s) = \psi_s(|\eta_{t-1}| - E|\eta_{t-1}|) + \gamma \eta_{t-1}, \quad s = 1, \dots, 24$$

- SFI-PEGARCH(1,d,0) with periodic ω , ψ and β and constant γ :

$$(1 - \beta_s L)(1 - L^S)^d (\ln h_t - \omega_s) = \psi_s(|\eta_{t-1}| - E|\eta_{t-1}|) + \gamma \eta_{t-1}, \quad s = 1, \dots, 24$$

The estimates are presented in table 3.⁶ The main results can be summarized as follows:

1. *Conditional Mean:* The Ljung-Box tests on the standardized residuals in levels and squared, computed for the normal case, reported in table 4, clearly show that the conditional mean can be modeled by a non-periodic process with no harm.

⁶To economize on the estimation output we omit standard errors.

2. *PLM-EGARCH*: The diagnostic tests in table 4 show that the PLM-EGARCH (Bordignon, Caporin, and Lisi (2007a)) provides an adequate fit of the long memory component of the log-volatility while is clearly unable to catch the periodic patterns in volatility, as it is well shown by the rejection of the null hypothesis of the seasonality test, based on a simple regression of the standardized squared returns on a set of seasonal dummies.
3. *PEGARCH*: The estimated β 's satisfy the condition of stationarity for a periodic EGARCH model, in fact $\prod_s \hat{\beta}_s < 1$. The leverage effect, given by γ , is strongly statistically significant and the sign is negative which confirms the presence of an asymmetric effect of returns shocks on volatility. The diagnostic tests in table (4) suggest that the estimated PEGARCH is able to fit the periodic features of the volatility process but is unable to fit the long memory components as the rejection of the null of the Ljung-Box test statistic of long autocorrelation in the squared standardized residuals testifies.
4. *FI-PEGARCH*: The estimates of the coefficients of the size effect, ψ_s , are markedly different, across the the 24 hours, nonetheless they are similar in signs. An analogous consideration holds for the estimated β 's. As in the PEGARCH γ is strongly statistically significant in all cases considered. The long-memory coefficient is within the stationary bounds. The estimate of fractional integration parameter d of the squared standardized residuals, using the GPH estimator, indicates that there is no persistence that survives after the filtering. The models are estimated considering different truncations of the infinite autoregressive polynomial in (4.4), and the presented results are based on a truncation equal to 1008. In our experience, the estimates of the long memory parameter turn out to be sensible to this choice.

For what concerns the possible nested restrictions, the Wald test statistic, table 4, rejects the null of FI-PEGARCH(1,d,0). The squared standardized residuals have no seasonal components as the F statistics of the seasonality test confirms (table 4).

5. *SFI-PEGARCH*: The estimates of the ω 's are fairly close to those of FI-PEGARCH. Differences arise in the p-values of the t of the coefficients of the size effect, ψ_s . Furthermore the estimated β 's follow a different pattern. Again γ is strongly statistically significant. Like in the FI-PEGARCH case, the estimate of the long memory parameter of the squared standardized residuals is not statistically different from zero. Just as in the case of FI-PEGARCH the estimated long-memory parameter is within stationary bounds. The residuals have no seasonal components as the F statistics of seasonality test confirms. Moreover, we find that during the Asian and European trading time the volatility is much lower than during the American trading time when we observe a sharp increase. These results seem to confirm the fact that hourly returns sampled over the 24 hours across different markets are characterized by different seasonal patterns with a statistically significant persistence.⁷ The Wald test statistics for the null of $d = 0$ and constant β 's are, like the FI-PEGARCH, strongly rejected.
6. *FI-PEGARCH vs SFI-PEGARCH*: The diagnostic tests highlight no major differences among the two models and the Schwarz criteria as well as the log-likelihood are very close.

⁷We do not report the graphs of the estimated conditional volatilities for sake of space. However, they are available from the authors upon request.

7.2 Models with constrained patterns of periodicity

When S is large, like the case we are examining, the number of periodic parameters can be limited assuming that their seasonal variation is described by periodic functions like combinations of cosine and sinus. Following Jones and Brelsford (1967) we assume that the periodic parameters in the FI-PEGARCH(1,d,0) and SFI-PEGARCH(1,d,0) are modeled in the following way:

$$\begin{aligned}\omega_s &= \tilde{\omega}_0 + \tilde{\omega}_1 \cos\left(\frac{2\pi s}{S} - \tilde{\omega}_2\pi\right) \\ \beta_s &= \tilde{\beta}_0 + \tilde{\beta}_1 \cos\left(\frac{2\pi s}{S} - \tilde{\beta}_2\pi\right) \\ \psi_s &= \tilde{\psi}_0 + \tilde{\psi}_1 \cos\left(\frac{2\pi s}{S} - \tilde{\psi}_2\pi\right) \\ \gamma_s &= \tilde{\gamma}_0 + \tilde{\gamma}_1 \cos\left(\frac{2\pi s}{S} - \tilde{\gamma}_2\pi\right)\end{aligned}$$

For parameter identification we restrict $\tilde{\omega}_2, \tilde{\beta}_2, \tilde{\psi}_2, \tilde{\gamma}_2 \in [0, 1)$ as $\cos(x + k\pi) = (-1)^k \cos(x)$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. This specification is very close to the Flexible Fourier Form GARCH adopted by Taylor (2004). Estimation results are reported in table 5. The estimated parameters of the FI-PEGARCH are all significant, while the parameters in the function of γ_s for the SFI-PEGARCH result to be statically not different from zero. The \hat{d} for both models are sensibly higher than those estimated in the unconstrained models. This finding confirms that restricting the periodicity pattern leads to an increase in the estimate of the fractional integration parameter.

7.3 Models with three seasons

It is evident that models with twenty-four seasons can be heavily overparameterized, and this can lead to some loss of efficiency.

We can hypothesize a more parsimonious model based on three periods. The first corresponds to the ASIA-EUROPE trading period (17.00 – 09.00), the second to the morning in the US (10.00 – 13.00) and the third to the afternoon (14.00 – 16.00).

We estimate the FI-PEGARCH(1, d , 0) and SFI-PEGARCH(1, d , 0), where all parameters are periodic.

The FI-PEGARCH(1, d , 0):

$$(1 - \beta_s L)(1 - L)^d (\ln h_t - \omega_s) = \psi_s (|\eta_{t-1}| - E|\eta_{t-1}|) + \gamma_s \eta_{t-1}, \quad s = 1, \dots, 24$$

The SFI-PEGARCH(1, d , 0):

$$(1 - \beta_s L)(1 - L^S)^d (\ln h_t - \omega_s) = \psi_s (|\eta_{t-1}| - E|\eta_{t-1}|) + \gamma_s \eta_{t-1}, \quad s = 1, \dots, 24$$

Let

$$\theta_s = \omega_s, \psi_s, \beta_s, \gamma_s$$

the parameters in the models follow the following pattern:

$$\theta_s = \begin{cases} \theta_1 & s = 1, \dots, 17 \\ \theta_2 & s = 18, \dots, 21 \\ \theta_3 & s = 22, 23, 24 \end{cases}$$

The results are reported in table 7. In this case, like the models with restricted periodic parameters, the estimated d 's are larger than those estimated for the unrestricted models. The diagnostic tests in table 8 suggest that the SFI-PEGARCH provides a slight better fit. However both models fail to account for the periodicity as the seasonality test clearly highlights. The Ljung-Box statistic for the autocorrelation in the squared standardized disturbances at 500 lags rejects the null hypothesis for both models.

8 Forecasts

8.1 In-sample forecasting performance

In this section we present some measures of the in-sample forecasting performances of the models presented in the previous sections.

The evaluation and comparison of univariate (and multivariate) volatility forecasts, is made difficult by the fact that the object of interest is unobservable, even ex post. Thus the evaluation and comparison of volatility forecasts must rely on direct or indirect methods of overcoming this difficulty. Direct methods use a volatility proxy, i.e. some observable variable that is related to the latent variable of interest.

The estimated models considered in the previous sections are used to generate time-consistent 1-step-ahead forecasts of conditional return volatility. These forecasts are obtained by first estimating the parameters of the model on the full sample and performing a series of static one-step ahead forecasts. The forecasting performances are evaluated on the basis of the bias, the Mean Absolute Error (MAE), the Mean Square Error (MSE). The MAFE is a commonly used measure for forecast evaluation but it imposes the same penalty on over- and under-predictions of volatility and is not invariant to scale transformations. The mean absolute percentage error (MAPE) accommodates possible heteroskedasticity in forecast errors but it can be unstable if volatility is very low. Following Andersen and Bollerslev (1998) we report the statistics of the Mincer-Zarnowitz regressions of the hourly realized volatility on a constant and the various model forecasts based on time $t - 1$ information:

$$rv_t = a + b\hat{h}_t + e_t \quad t = 1, \dots, T$$

The OLS parameter estimates will be less accurately estimated the larger the variance of $(rv_t - \hat{h}_t)$, where rv_t is the hourly realized volatility and \hat{h}_t is the volatility forecast. The results are shown in table 9. For the purpose of benchmark provision we also consider the forecasts generated by a Periodic EGARCH(1, 0) and a PLM-EGARCH(1, d , 0). First of all it is important to note that omitting the long memory component, like in the Periodic EGARCH(1, 0), sensibly affects the quality of the forecasts. In the case of Periodic EGARCH(1, 0) we find the lowest R^2 and the largest MSE.

Second, the SFI-PEGARCH models behave worse than the corresponding FI-PEGARCH models in terms of MAE, MAPE and MSE, while for the bias the results are mixed. Moreover the PLM-EGARCH has a better performance than the restricted SFI-PEGARCH in terms of R^2 while the situation is reversed for the other statistics. Finally, The FI-PEGARCH(1, d , 0) with 3 periods is the best model in terms of MSE and R^2 of the Mincer-Zarnowitz regression. In fig. 9 and 10 the average of the in-sample forecasts for the FI-PEGARCH and SFI-PEGARCH are reported along with the average hourly realized volatility. It is interesting to note that the unrestricted models fit much better than the restricted ones, which means that they are able to catch the change in volatility observed in the round-the-clock returns. Instead, the latter seem to be penalized by the imposed restricted periodicity patterns. Moreover, this seems to

be independent of the kind of long memory structure adopted. Indeed, the fit provided by FI-PEGARCH model with three periods (mid panel fig. 9) and the one with constrained periodicity (bottom panel fig. 9) are very similar to the ones obtained with the restricted SFI-PEGARCH, in fig.10 mid and bottom panel, respectively.

8.2 Out-of-sample forecasting performance

While in-sample forecasting provides a general indication of the predictive ability, it is only the evaluation of the out-of-sample forecasts that can assess the models capability of providing sensible predictions for financial decision making. The forecasts are based on parameter estimates from rolling samples with fixed sample sizes of 11312 hours. For every date $t > 11312$ in the sample, we estimate the parameters of each specification over the 11312 data points up to including date t . We calculate 120 out-of-sample forecasts. We compute the Mincer-Zarnowitz regressions and the F test statistic for the null hypothesis that $a = 0 \cap b = 1$, which corresponds to the unbiasedness of the forecast. Table 9 reports the results. It turns out that the best model in terms of bias, MAE, and MSE is the SFI-PEGARCH with three periods. Moreover is important to note that for this model the null hypothesis is accepted.

9 Conclusions

This paper presents a new approach to the modeling of volatility of financial returns that takes into account persistence, periodicity and asymmetry.

The proposed models are used to investigate the features of the returns volatility of the E-mini SP500 futures contracts. The empirical application shows that the proposed models, and in particular the FI-PEGARCH, provide an adequate description of high-frequency volatility returns. Furthermore the results confirm that the volatility is characterized by long memory, periodicity and asymmetric responses to return shocks. They confirm that restricting the periodicity patterns lead to a larger estimate of the long memory parameter, which comes out to be biased. This seems to suggest that a model of the periodicity pattern, which cannot fully account for the time-varying periodicity overestimates the long memory component. This seems to be characteristic of models based on seasonal fractional differencing. It is worth emphasizing that predictability and seasonality of stock returns and volatility found in this work need not imply market inefficiency. Although our results can be useful in the real-world investment process, they do not imply that profitable trading strategies yielding superior returns when adjusted for transaction cost exist. A further investigation into the economic significance of futures returns volatility seasonality is therefore called for. Besides, there are other lines for future research: for example, how to achieve estimation parsimony when dealing with a high number of seasons and/or high autoregressive order is surely one of the most interesting and compelling topic of periodic modeling. In a similar fashion, a multivariate extension can imply a wealth of parameters to estimate and a dramatic loss of efficiency.

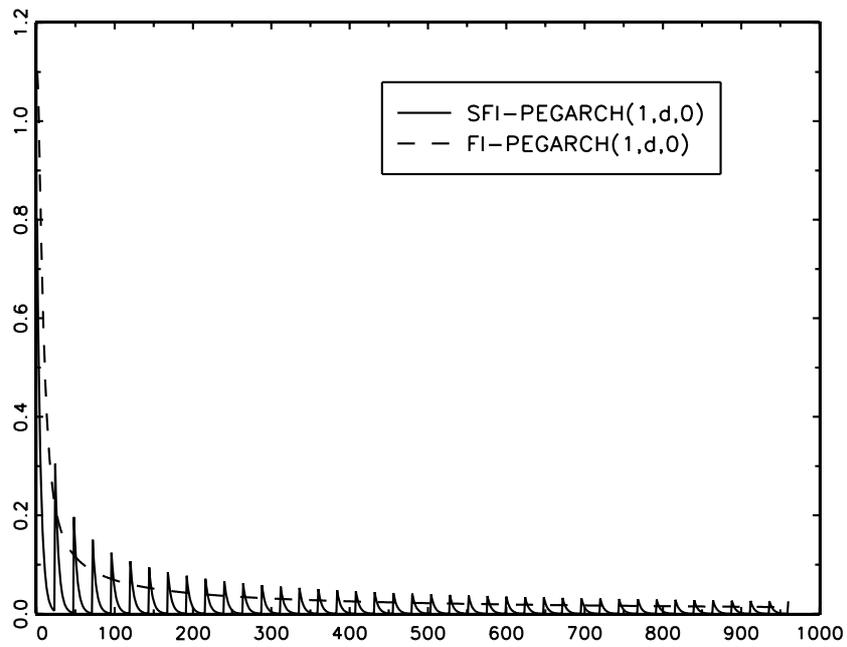


Figure 1: Series expansion of $(1 - L)^{-d}(1 - \beta_1 L)^{-1}$ and $(1 - L^S)^{-d}(1 - \beta_1 L)^{-1}$ with $d = 0.3$, $\beta_1 = 0.8$

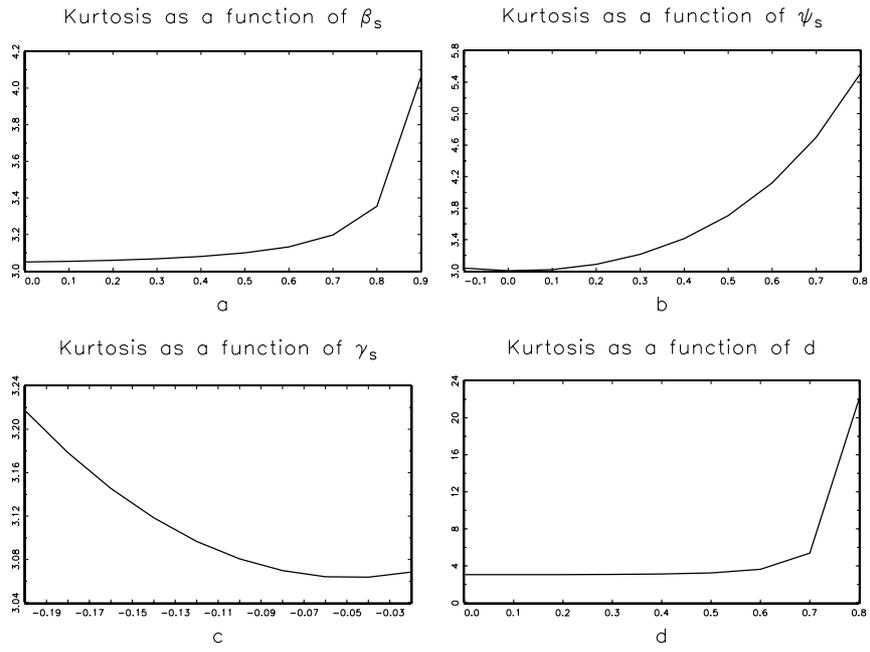


Figure 2: FI-PEGARCH(1,d,0) - Kurtosis as a function of model parameters

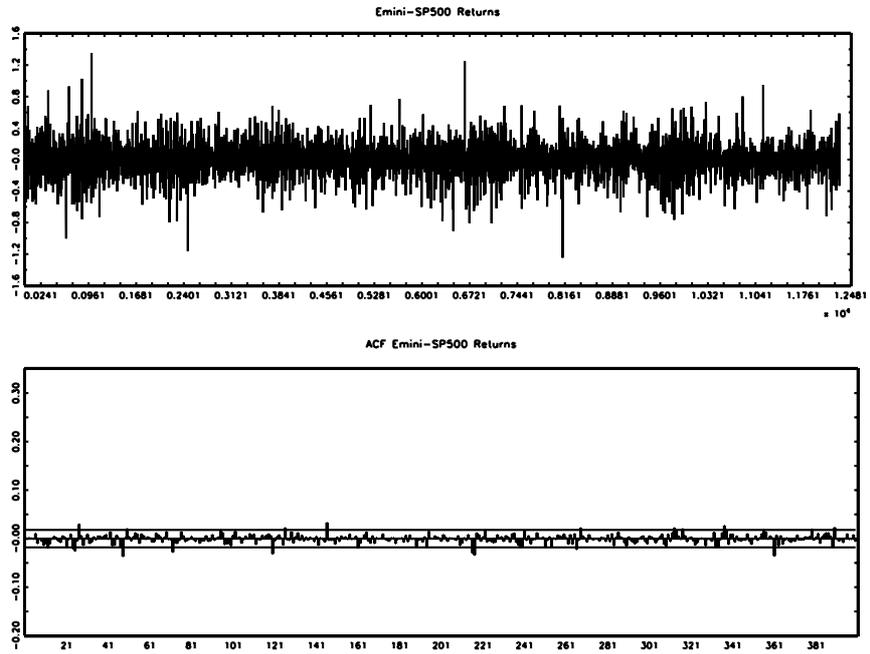


Figure 3: Hourly returns and the autocorrelation function

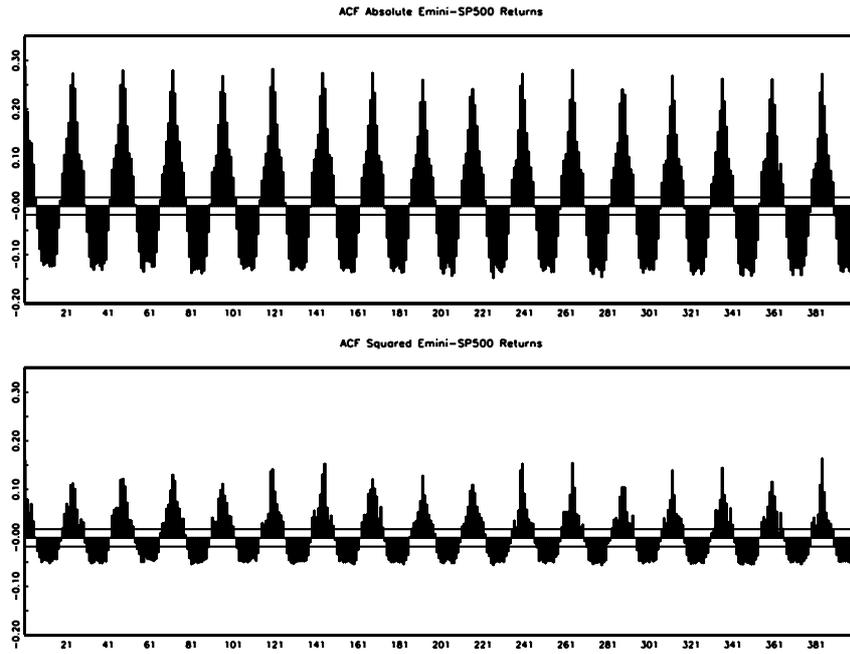


Figure 4: Autocorrelation functions of squared and absolute Emini-SP500 returns

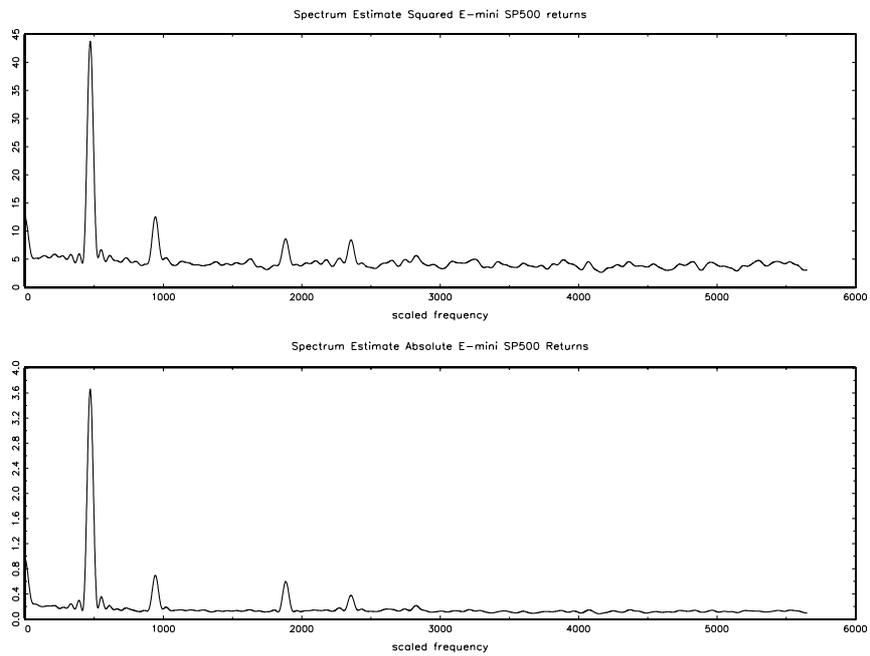


Figure 5: Spectrum estimates of squared and absolute Emini-SP500 returns

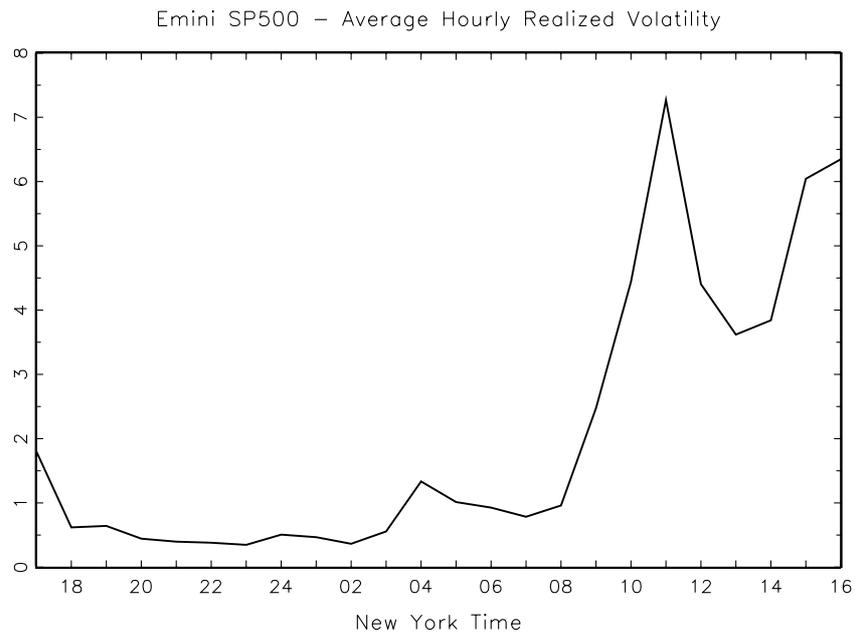


Figure 6: Average RV per hour

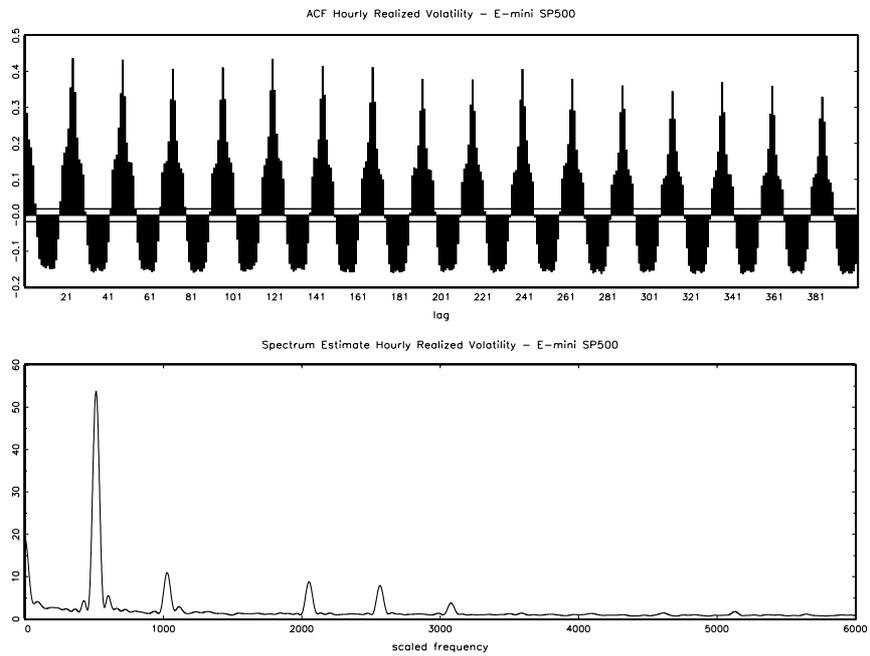


Figure 7: Autocorrelogram and Spectrum Estimate of Emini SP 500 Realized Volatility

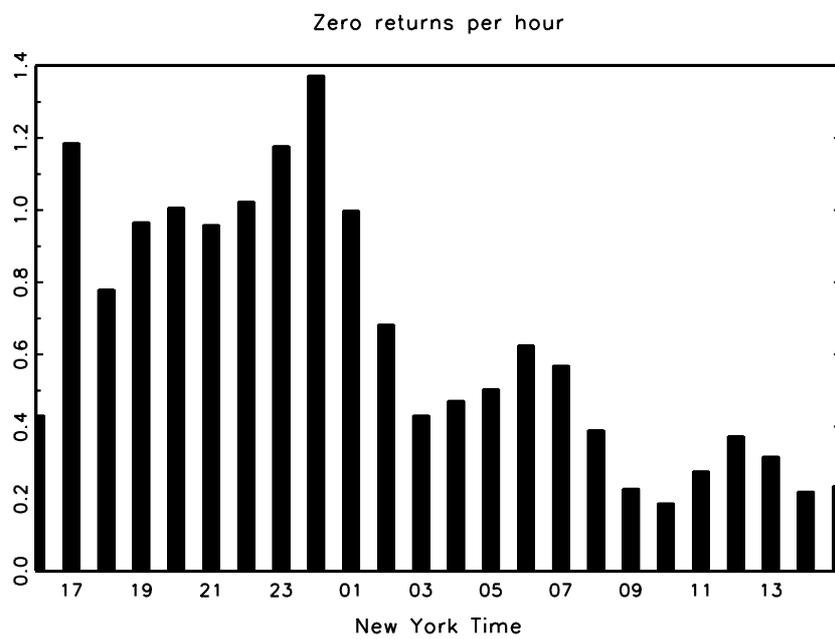


Figure 8: Percentages of zero returns per hour

	Mean	Std. Dev.	Skewness	Kurtosis	1st quartile	Median	3rd quartile
16-17	-0.0514	1.3256	-0.6030	5.0005	-0.8042	0.0000	0.6849
17-18	-0.0519	0.5732	-1.7946	19.8279	-0.2217	0.0000	0.2075
18-19	0.0749	0.6773	-1.1281	25.7040	-0.2143	0.0000	0.4153
19-20	0.0158	0.5019	1.7944	16.5357	-0.2206	0.0000	0.2204
20-21	0.0334	0.5564	0.6275	9.5416	-0.2111	0.0000	0.2227
21-22	-0.0104	0.5024	0.1781	14.4292	-0.2211	0.0000	0.2115
22-23	0.0123	0.4021	-0.3018	4.9239	-0.2103	0.0000	0.2193
23-24	-0.0051	0.6260	1.0740	32.5273	-0.2112	0.0000	0.2108
24-01	0.0584	0.5647	3.4963	35.5868	-0.2062	0.0000	0.2119
01-02	0.0151	0.4791	0.1724	5.2854	-0.2158	0.0000	0.2180
02-03	0.1200	0.5794	0.0328	3.8995	-0.2125	0.0000	0.4312
03-04	0.0859	1.0149	0.0847	3.9003	-0.5851	0.0000	0.6541
04-05	-0.0245	0.9196	-0.3762	6.3901	-0.5506	0.0000	0.4379
05-06	-0.0233	0.9411	-4.6837	62.6113	-0.4211	0.0000	0.4171
06-07	0.0003	0.6807	0.0648	4.6483	-0.4086	0.0000	0.4141
07-08	0.0603	0.8662	0.3258	7.3379	-0.4152	0.0000	0.6017
08-09	0.0690	1.4594	-0.5409	14.1448	-0.6189	0.0000	0.7897
09-10	0.0145	1.7878	0.0233	3.7637	-1.1869	0.0000	1.0931
10-11	-0.1205	2.4923	-0.0762	3.4942	-1.6165	-0.1945	1.2283
11-12	0.0121	2.0090	0.0886	5.1103	-1.0207	0.0000	1.0530
12-13	0.1134	1.7749	-0.0477	5.2063	-0.8020	0.0000	0.9757
13-14	-0.0639	1.8822	-0.0662	3.6934	-1.2435	0.0000	1.0336
14-15	0.0384	2.4907	0.3397	5.2149	-1.4386	0.0000	1.3461
15-16	-0.1051	2.5121	-0.0479	5.1651	-1.5684	0.0000	1.3574

Table 1: Descriptive statistics for E-mini SP500 hourly returns

y_t^2				
	GPH	$d = 0$	Whittle	$d = 0$
$m = 283$	0.132 [0.035]	3.716 (0.000)	0.147 [0.030]	4.951 (0.000)
$m = 110$	0.296 [0.061]	4.857 (0.000)	0.272 [0.048]	5.699 (0.000)
$ y_t $				
$m = 283$	0.253 [0.043]	5.912 (0.000)	0.245 [0.029]	8.245 (0.000)
$m = 110$	0.366 [0.063]	5.805 (0.000)	0.356 [0.048]	7.465 (0.000)

Table 2: Fractional integration estimation (d) of E-mini squared returns. Bandwidth parameter set to $m = T^{0.6} = 283$ and $m = \sqrt{T} = 110$, and testing (p-values in parenthesis, standard error in square brackets)

	PEGARCH(1,0)		FI-PEGARCH(1,d,0)		SFI-PEGARCH(1,d,0)	
	estimate	p-value	estimate	p-value	estimate	p-value
c	0.0283	0.000	0.0269	0.000	0.0264	0.000
ω_1	-1.0186	0.284	0.5303	0.000	0.5830	0.000
ω_2	-2.0680	0.000	-0.9670	0.000	-1.1316	0.000
ω_3	0.2031	0.262	-0.6454	0.000	-0.7657	0.000
ω_4	-1.3637	0.000	-1.0083	0.000	-0.9964	0.000
ω_5	8.7465	0.216	-1.1049	0.000	-1.0658	0.000
ω_6	0.9668	0.504	-1.1152	0.000	-1.0890	0.000
ω_7	-0.9025	0.001	-1.5939	0.000	-1.5289	0.000
ω_8	4.1819	0.001	-0.8860	0.000	-0.7688	0.000
ω_9	-0.7229	0.000	-1.0549	0.000	-1.1099	0.000
ω_{10}	-0.9889	0.000	-1.3726	0.000	-1.2582	0.000
ω_{11}	0.7381	0.174	-0.6718	0.000	-0.6413	0.000
ω_{12}	0.7860	0.001	0.2285	0.029	0.2507	0.020
ω_{13}	-0.1787	0.030	0.0346	0.742	0.0395	0.717
ω_{14}	-0.2780	0.030	-0.4452	0.000	-0.5530	0.000
ω_{15}	-0.4204	0.004	-0.5606	0.000	-0.5386	0.000
ω_{16}	0.2405	0.232	-0.2536	0.034	-0.2861	0.008
ω_{17}	0.9354	0.000	0.8662	0.000	0.6742	0.000
ω_{18}	0.5013	0.044	1.1961	0.000	1.1689	0.000
ω_{19}	0.4202	0.434	1.9086	0.000	1.9245	0.000
ω_{20}	0.0323	0.931	1.4371	0.000	1.3717	0.000
ω_{21}	-2.6784	0.009	1.5162	0.000	1.4513	0.000
ω_{22}	0.8347	0.000	1.3647	0.000	1.2512	0.000
ω_{23}	0.0819	0.874	1.9722	0.000	1.9457	0.000
ω_{24}	0.4080	0.567	1.8983	0.000	1.7702	0.000
ψ_1	0.2145	0.002	0.1748	0.041	0.0368	0.090
ψ_2	0.7379	0.000	0.6557	0.000	0.5263	0.000
ψ_3	0.9108	0.000	0.7693	0.000	0.8436	0.000
ψ_4	0.0305	0.001	0.0982	0.002	0.1033	0.000
ψ_5	0.0880	0.173	0.0245	0.026	0.0532	0.004
ψ_6	0.3648	0.000	0.3127	0.000	0.3266	0.000
ψ_7	0.1704	0.000	0.2543	0.001	0.2431	0.000
ψ_8	0.7533	0.000	0.4860	0.000	0.1765	0.000
ψ_9	0.2645	0.000	0.2625	0.000	0.3648	0.000
ψ_{10}	0.1853	0.005	0.4121	0.005	0.1262	0.000
ψ_{11}	0.2790	0.000	0.1272	0.000	0.1661	0.000
ψ_{12}	0.1583	0.023	0.1207	0.023	0.1262	0.018
ψ_{13}	0.1629	0.004	0.0533	0.004	0.0884	0.000
ψ_{14}	0.3931	0.000	0.4055	0.000	0.3653	0.000
ψ_{15}	0.0735	0.284	0.0472	0.022	0.0674	0.003
ψ_{16}	0.1782	0.005	0.0250	0.206	0.0370	0.091
ψ_{17}	0.0793	0.034	0.0216	0.007	-0.0350	0.703
ψ_{18}	0.1442	0.005	0.1316	0.143	0.1482	0.034
ψ_{19}	0.1077	0.029	0.2813	0.017	0.2020	0.003
ψ_{20}	0.0712	0.043	0.0603	0.086	0.0635	0.191
ψ_{21}	0.1462	0.076	0.1667	0.000	0.2183	0.000
ψ_{22}	0.0664	0.157	0.0305	0.145	0.0663	0.396
ψ_{23}	-0.0209	0.738	0.0429	0.098	0.0636	0.036
ψ_{24}	0.0524	0.600	0.2630	0.000	0.2583	0.000
β_1	0.8929	0.109	0.2710	0.513	0.9661	0.000
β_2	1.5351	0.000	0.3800	0.002	0.7395	0.000
β_3	0.8549	0.000	0.2229	0.006	0.5393	0.000
β_4	0.0088	0.022	0.9054	0.000	0.9653	0.000
β_5	7.2564	0.156	0.9746	0.000	0.9819	0.000
β_6	1.9289	0.084	0.7214	0.000	0.8974	0.000
β_7	0.6906	0.000	0.7723	0.000	0.9357	0.000
β_8	2.8731	0.000	0.2235	0.037	0.9272	0.000
β_9	0.5283	0.000	0.6298	0.000	0.7678	0.000
β_{10}	0.4180	0.000	0.3046	0.288	0.9542	0.000
β_{11}	1.1302	0.000	0.9025	0.000	0.9537	0.000
β_{12}	0.7048	0.000	0.8037	0.000	0.9222	0.000
β_{13}	0.7627	0.003	0.9266	0.000	0.9540	0.000
β_{14}	1.1841	0.000	-0.1568	0.379	0.2102	0.341
β_{15}	0.8692	0.000	0.9533	0.000	0.9775	0.000
β_{16}	0.7133	0.000	0.9411	0.000	0.9648	0.000
β_{17}	0.5857	0.004	0.9672	0.000	-0.2977	0.658
β_{18}	0.8644	0.015	0.5591	0.113	0.7527	0.000
β_{19}	1.2737	0.010	0.1593	0.718	0.8088	0.000
β_{20}	0.7225	0.001	0.7854	0.000	0.8038	0.000
β_{21}	2.8092	0.000	0.8653	0.000	0.9285	0.000
β_{22}	0.3837	0.001	0.9058	0.000	-0.5625	0.036
β_{23}	1.3539	0.001	0.9257	0.000	0.9573	0.000
β_{24}	0.7395	0.068	0.5940	0.000	0.7673	0.000
γ	-0.0290	0.000	-0.0337	0.000	-0.0626	0.000
d			0.2441	0.000	0.1836	0.000

Table 3: Model estimates with obtained Gaussian QML. The t statistics are computed using robust standard errors Bollerslev and Wooldridge (1992). P-values are reported.

	PLM-EGARCH(1,0)	PEGARCH(1,0)	FI-PEGARCH(1,d,0)	SFI-PEGARCH(1,d,0)
$Q_{\hat{\eta}_t}(5)$	4.335	9.641	7.421	9.257
<i>p-value</i>	0.502	0.086	0.191	0.099
$Q_{\hat{\eta}_t}(25)$	39.202	43.789	42.291	43.208
<i>p-value</i>	0.035	0.011	0.017	0.013
$Q_{\hat{\eta}_t}(500)$	536.447	565.318	549.934	555.603
<i>p-value</i>	0.126	0.023	0.061	0.043
$Q_{\hat{\eta}_t^2}(5)$	6.523	11.907	4.620	10.278
<i>p-value</i>	0.259	0.036	0.464	0.068
$Q_{\hat{\eta}_t^2}(25)$	21.242	43.505	18.761	28.219
<i>p-value</i>	0.679	0.012	0.808	0.298
$Q_{\hat{\eta}_t^2}(500)$	504.583	621.148	509.791	489.516
<i>p-value</i>	0.434	0.000	0.371	0.623
Seasonality test (<i>p-value</i>):	0.000	1.000	1.000	1.000
$\hat{d}(\hat{\eta}_t^2) m = 110$	0.0322	0.222	-0.0154	-0.00627
	(0.072)	(0.064)	(0.063)	(0.070)
Log-likelihood function	-16080.87	-14648.00	-14566.05	-14565.20
Schwarz criterion (SC)	2.617	2.653	2.424	2.424
$H_0 : d = 0$			36.009	59.046
<i>p-value</i>			0.000	0.000
$H_0 : \beta_s = \beta$			161.259	193.965
<i>p-value</i>			0.000	0.000

Table 4: Diagnostic tests. The test statistics are computed for models estimated with the normal density. Ljung-Box test statistics for standardized residuals ($Q_{\hat{\eta}_t}$) and square standardized residuals ($Q_{\hat{\eta}_t^2}$) are computed for the number of lags reported in parentheses. The Schwarz Information Criterion is computed as $SC = -L(\hat{\theta})/T + p \frac{\ln T}{T}$, where p is the number of parameters and T is the number of observations. The seasonality test is the F test of the regression of the standardized squared residual ($\hat{\eta}_t^2$) on a set of 23 dummies, one for each hour. \hat{d} is the estimated fractional integration parameter of the standardized squared residuals, standard errors are reported in parentheses. Wald test statistics for the null hypotheses in the table are reported.

	FI-PEGARCH(1,d,0)		SFI-PEGARCH(1,d,0)	
	estimate	p-value	estimate	p-value
c	0.0262	0.000	0.0358	0.000
$\tilde{\omega}_0$	0.9756	0.000	0.4009	0.000
$\tilde{\omega}_1$	-1.0642	0.000	-1.2548	0.000
$\tilde{\omega}_2$	0.6680	0.000	0.7074	0.000
$\tilde{\psi}_0$	0.1797	0.000	0.2482	0.000
$\tilde{\psi}_1$	0.1816	0.000	0.2671	0.000
$\tilde{\psi}_2$	0.9833	0.000	0.9362	0.000
$\tilde{\beta}_0$	0.3449	0.000	0.6797	0.000
$\tilde{\beta}_1$	0.6232	0.000	0.3105	0.000
$\tilde{\beta}_2$	0.0671	0.000	0.0959	0.000
$\tilde{\gamma}_0$	-0.0797	0.000	-0.0258	0.003
$\tilde{\gamma}_1$	0.0709	0.000	-0.0134	0.253
$\tilde{\gamma}_2$	0.0144	0.003	0.3102	0.528
d	0.4952	0.000	0.3526	0.000

Table 5: Quasi-maximum likelihood estimates of the models with 24 periods but with constrained parameters periodicity (see section 7.2).

	FI-PEGARCH(1,d,0)	SFI-PEGARCH(1,d,0)
$Q_{\hat{\eta}_t}(5)$	4.479	4.420
<i>p-value</i>	0.483	0.491
$Q_{\hat{\eta}_t}(25)$	34.377	38.738
<i>p-value</i>	0.100	0.039
$Q_{\hat{\eta}_t}(500)$	530.819	534.264
<i>p-value</i>	0.164	0.140
$Q_{\hat{\eta}_t^2}(5)$	7.610	5.073
<i>p-value</i>	0.179	0.407
$Q_{\hat{\eta}_t^2}(25)$	15.059	11.503
<i>p-value</i>	0.940	0.990
$Q_{\hat{\eta}_t^2}(500)$	732.945	571.422
<i>p-value</i>	0.000	0.015
F-test seasonality (<i>p-value</i>):	0.000	0.000
$\hat{d}(\hat{\eta}_t^2) m = 110$	-0.012 (0.081)	-0.0023 (0.073)
Log-likelihood function	-15307.579	-15386.09
Schwarz criterion (SC)	2.498	2.510

Table 6: Diagnostic tests for models with constrained periodic parameters. The test statistics are computed for models estimated with the normal density. Ljung-Box test statistics for standardized residuals ($Q_{\hat{\eta}_t}$) and square standardized residuals ($Q_{\hat{\eta}_t^2}$) are computed for the number of lags reported in parentheses. The Schwarz Information Criterion is computed as $SC = -L(\hat{\theta})/T + p \frac{\ln T}{T}$, where p is the number of parameters and T is the number of observations. The seasonality test is the F test of the regression of the standardized squared residual ($\hat{\eta}_t^2$) on a set of 23 dummies, one for each hour. \hat{d} is the estimated fractional integration parameter of the standardized squared residuals, standard error is in parenthesis.

	FI-PEGARCH(1,d,0)		SFI-PEGARCH(1,d,0)	
	estimate	p-value	estimate	p-value
c	0.0201	0.008	0.0126	0.072
ω_1	-0.0937	0.103	0.1808	0.002
ω_2	1.6477	0.000	1.6084	0.000
ω_3	1.8104	0.000	1.7112	0.000
ψ_1	0.3990	0.000	0.3835	0.000
ψ_2	0.0902	0.000	0.0014	0.300
ψ_3	0.1048	0.001	0.0907	0.000
β_1	0.1134	0.077	0.5557	0.000
β_2	0.7320	0.000	0.9916	0.000
β_3	0.6960	0.000	0.8793	0.000
γ_1	-0.0982	0.000	-0.0367	0.002
γ_2	-0.0222	0.093	-0.0140	0.000
γ_3	-0.0721	0.002	-0.0816	0.000
d	0.3010	0.000	0.4112	0.000

Table 7: Quasi-maximum likelihood estimates of models with three periods (see section 7.3).

	FI-PEGARCH(1,d,0)	SFI-PEGARCH(1,d,0)
$Q_{\hat{\eta}_t}(5)$	6.315	4.413
<i>p-value</i>	0.277	0.492
$Q_{\hat{\eta}_t}(25)$	36.723	39.271
<i>p-value</i>	0.061	0.035
$Q_{\hat{\eta}_t}(500)$	565.355	537.689
<i>p-value</i>	0.023	0.118
$Q_{\hat{\eta}_t^2}(5)$	1.017	6.500
<i>p-value</i>	0.961	0.261
$Q_{\hat{\eta}_t^2}(25)$ (25 lags)	30.437	28.170
<i>p-value</i>	0.208	0.300
$Q_{\hat{\eta}_t^2}(500)$	1175.941	605.941
<i>p-value</i>	0.000	0.001
F-test seasonality (<i>p-value</i>):	0.000	0.000
$\hat{d}(\hat{\eta}_t^2) m = 110$	-0.016 (0.071)	-0.0071 (0.072)
Log-likelihood function	-15956.20	-15712.17
Schwarz criterion (SC)	2.603	2.563

Table 8: Diagnostic tests of models estimated with three periods. The test statistics are computed for models estimated with the normal density. Ljung-Box test statistics for standardized residuals ($Q_{\hat{\eta}_t}$) and square standardized residuals ($Q_{\hat{\eta}_t^2}$) are computed for the number of lags reported in parentheses. The Schwarz Information Criterion is computed as $SC = -L(\hat{\theta})/T + p \frac{\ln T}{T}$, where p is the number of parameters and T is the number of observations. The seasonality test is the F test of the regression of the standardized squared residual ($\hat{\eta}_t^2$) on a set of 23 dummies, one for each hour. \hat{d} is the estimated fractional integration parameter of the standardized squared residuals, standard error is in parenthesis.

	\hat{a}	\hat{b}	R^2	bias	MAE	MAPE	MSE
PLM-EGARCH(1,d,0)	0.472	0.886	0.304	0.2850	1.1741	9.5550	8.0168
	<i>0.089</i>	<i>0.068</i>					
PEGARCH(1,0)	1.365	0.320	0.122	0.1646	1.1349	9.1552	16.1988
	<i>0.341</i>	<i>0.197</i>					
FI-PEGARCH(1,d,0) 24 periods	0.502	0.837	0.326	0.2235	1.0727	9.0179	7.8204
	<i>0.273</i>	<i>0.165</i>					
FI-PEGARCH(1,d,0) 24 periods (restricted)	0.064	1.069	0.348	0.1857	1.1805	9.4526	7.4280
	<i>0.039</i>	<i>0.027</i>					
FI-PEGARCH(1,d,0) 3 periods	0.116	1.081	0.370	0.2522	1.1133	9.2636	7.2146
	<i>0.042</i>	<i>0.031</i>					
SFI-PEGARCH(1,d,0) 24 periods	0.908	0.592	0.233	0.2043	1.0924	9.0345	9.9809
	<i>0.613</i>	<i>0.364</i>					
SFI-PEGARCH(1,d,0) 24 periods (restricted)	0.748	0.622	0.188	0.0316	1.2997	9.7735	9.9760
	<i>0.221</i>	<i>0.121</i>					
SFI-PEGARCH(1,d,0) 3 periods	0.053	1.149	0.296	0.2970	1.1474	9.6935	7.6781
	<i>0.033</i>	<i>0.027</i>					

Table 9: In-Sample forecasts of volatility. Properties of alternative models using the hourly realized volatility as a volatility proxy. The table reports the estimated intercept and slope coefficients from the Mincer-Zarnowitz regression, \hat{a} and \hat{b} , respectively, along with heteroskedastic robust standard error (in italics), and the R^2 , for all models. The table also reports the bias (sample mean of the forecast minus the ex post realized value), the Mean Absolute Error, the Mean Absolute Percentage Error, the Mean Square Error.

	\hat{a}	\hat{b}	R^2	$a = 0 \cap b = 1$	bias	MAE	MAPE	MSE
FI-PEGARCH(1,d,0) 24 periods	1.130	0.677	0.012	7.261	1.0230	1.1400	7.7039	7.3741
	<i>0.342</i>	<i>0.629</i>		<i>0.001</i>				
FI-PEGARCH(1,d,0) 24 periods (restricted)	0.064	1.069	0.282	0.944	-0.2605	1.1194	4.9725	4.9439
	<i>0.231</i>	<i>0.233</i>		<i>0.392</i>				
FI-PEGARCH(1,d,0) 3 periods	0.500	0.603	0.134	4.557	-0.0633	1.1602	5.8978	5.9068
	<i>0.166</i>	<i>0.188</i>		<i>0.012</i>				
SFI-PEGARCH(1,d,0) 24 periods	0.466	0.566	0.157	13.667	-0.2132	1.2246	5.9943	6.0179
	<i>0.113</i>	<i>0.119</i>		<i>0.000</i>				
SFI-PEGARCH(1,d,0) 24 periods (restricted)	0.443	0.522	0.215	2.847	-0.3882	1.2541	5.1985	6.3117
	<i>0.235</i>	<i>0.200</i>		<i>0.062</i>				
SFI-PEGARCH(1,d,0) 3 periods	0.053	1.149	0.228	0.067	0.0240	0.9790	5.6286	4.9308
	<i>0.037</i>	<i>0.032</i>		<i>0.935</i>				

Table 10: Out-of-Sample forecasts of volatility. Properties of alternative models using the hourly realized volatility. The table reports the estimated intercept and slope coefficients from the Mincer-Zarnowitz regression, \hat{a} and \hat{b} , respectively, along with heteroskedastic robust standard error (in italics), and the R^2 , for all models. The table also reports the bias (sample mean of the forecast minus the ex post realized value), the Mean Absolute Error, the Mean Absolute Percentage Error, the Mean Square Error.

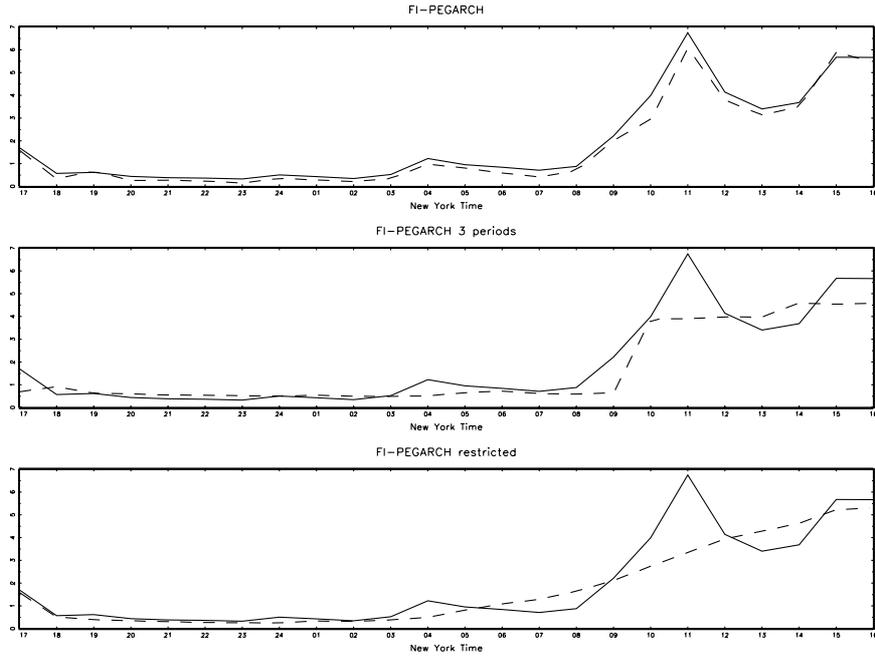


Figure 9: Average of in-sample forecasts of FI-PEGARCH(1,d,0) (dashed line) and average hourly realized volatility (solid line). Top panel FI-PEGARCH(1,d,0) with 24 periods, mid panel FI-PEGARCH(1,d,0) with three periods, FI-PEGARCH(1,d,0) with restricted pattern in the bottom panel

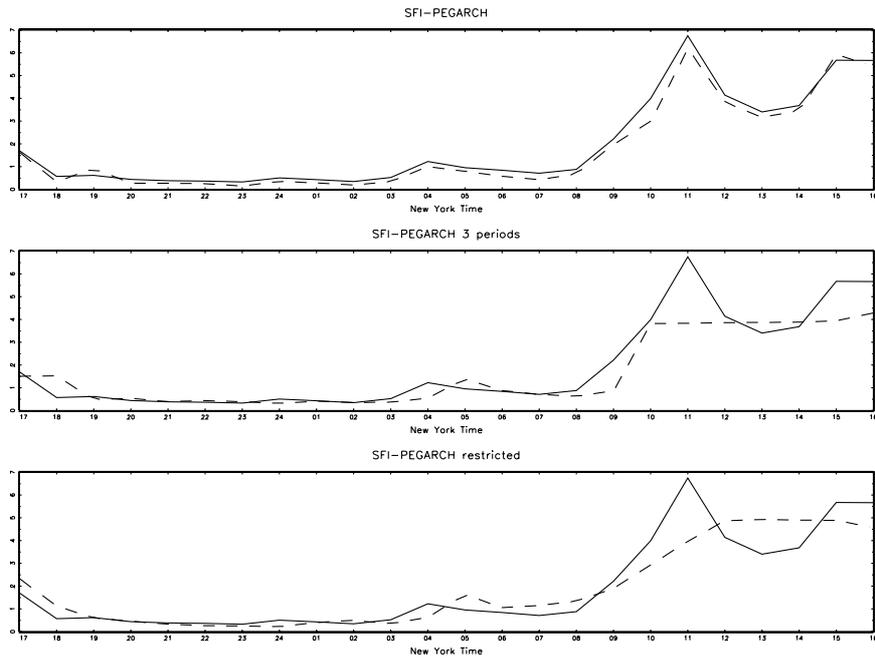


Figure 10: Average of in-sample forecasts of SFI-PEGARCH(1,d,0) (dashed line) and average hourly realized volatility (solid line). Top panel SFI-PEGARCH(1,d,0) with 24 periods, mid panel SFI-PEGARCH(1,d,0) with three periods, SFI-PEGARCH(1,d,0) with restricted pattern with restricted pattern in the bottom panel

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A Proofs

Proof of Proposition 4.1.

The FI-PEGARCH(1,d,0) can be written as:

$$(1 - \beta_{1s}L) \log h_t = (1 - \beta_{1s})\omega_s + (1 - L)^{-d}g_s(\eta_{t-1})$$

Take the exponential and raise both sides of the equation to the power m

$$h_t^m = \exp\{m(1 - \beta_{1s})\omega_s\} \exp\{m(1 - L)^{-d}g_s(\eta_{t-1})\} h_{t-1}^{m\beta_{1s}} \quad (\text{A.1})$$

If we solve (A.1) recursively we get

$$h_t^m = \exp\left\{m(1 - \beta_{1s})\omega_s \sum_{i=1}^n \beta_{1s}^{i-1}\right\} \exp\left\{m(1 - L)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1}g_s(\eta_{t-i})\right\} h_{t-n}^{m\beta_{1s}^n} \quad (\text{A.2})$$

Assuming that $|\beta_{1s}| < 1$, $\forall s$ and letting $n \rightarrow \infty$ then

$$h_t^m = \exp\{m\omega_s\} \exp\left\{m(1 - L)^{-d} \sum_{i=1}^{\infty} \beta_{1s}^{i-1}g_s(\eta_{t-i})\right\} \quad (\text{A.3})$$

using (4.4)

$$h_t^m = \exp\{m\omega_s\} \exp\{m\lambda_s(L)g_s(\eta_{t-1})\} \quad (\text{A.4})$$

Now, given that h_t and η_t are uncorrelated, the even moments of y_t are equal to

$$\begin{aligned} E[|y_t|^{2m}] &= E\left[|\eta_t\sqrt{h_t}|^{2m}\right] \\ &= E[|\eta_t|^{2m}] E[h_t^m] \\ &= \nu_{2m} \exp\{m\omega_s\} E[\exp\{m\lambda_s(L)g_s(\eta_{t-1})\}] \\ &= \nu_{2m} \exp\{m\omega_s\} \prod_{i=0}^{\infty} E[\exp\{m\lambda_{i,s}g_s(\eta_{t-1-i})\}]. \end{aligned}$$

where $\nu_{2m} = E[|\eta_t|^{2m}]$. Since the kurtosis of y_t is defined as $k_{y,s} = \frac{E[(y_t)^4]}{E[(y_t)^2]^2}$, setting $m = 1$ for the numerator and $m = 2$ for the denominator, respectively, we obtain

$$k_{y,s} = \nu_4 \frac{\prod_{i=0}^{\infty} E[\exp\{2\lambda_{i,s}g_s(\eta_{t-1-i})\}]}{\left\{\prod_{i=0}^{\infty} E[\exp\{\lambda_{i,s}g_s(\eta_{t-1-i})\}]\right\}^2}. \quad (\text{A.5})$$

The same procedure is followed for the SFI-PEGARCH(1,d,0), where $\lambda_s(L) = (1 - \beta_{1s}L)^{-1}(1 - L^S)^{-d}$.

Proof of Proposition 4.2

The autocorrelation function is defined as

$$\rho_{m,s}(n) = \frac{E[y_t^{2m} y_{t-n}^{2m}] - [E(y_t^{2m})][E(y_{t-n}^{2m})]}{(E[y_t^{4m}] - [E(y_t^{2m})]^2)^{1/2} (E[y_{t-n}^{4m}] - [E(y_{t-n}^{2m})]^2)^{1/2}}. \quad (\text{A.6})$$

Multiply both sides of (A.2) by h_{t-n}^m and $\eta_t^{2m} \eta_{t-n}^{2m}$, knowing that $\sum_{i=0}^{n-1} \beta_{1s}^i = (1 - \beta_{1s}^n)(1 - \beta_{1s})^{-1}$,

$$y_t^{2m} y_{t-n}^{2m} = \eta_t^{2m} \eta_{t-n}^{2m} \exp\{m(1 - \beta_{1s}^n)\omega_s\} \exp\left\{m(1 - L)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1} g_s(\eta_{t-i})\right\} h_{t-n}^{m(1+\beta_{1s}^n)}. \quad (\text{A.7})$$

and taking expectations yields

$$E[y_t^{2m} y_{t-n}^{2m}] = \nu_{2m} \exp\{m(1 - \beta_{1s}^n)\omega_s\} E\left[\eta_{t-n}^{2m} \exp\left\{m(1 - L)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1} g_s(\eta_{t-i})\right\}\right] \times E\left[h_{t-n}^{m(1+\beta_{1s}^n)}\right]. \quad (\text{A.8})$$

By eq.(A.4), we know that

$$E\left[h_{t-n}^{m(1+\beta_{1s}^n)}\right] = \exp\{m\omega_r(1 + \beta_{1s}^n)\} \prod_{i=1}^{\infty} E[\exp\{m(1 + \beta_{1s}^n)\lambda_{i,r} g_r(\eta_{t-n-i})\}] \quad (\text{A.9})$$

where r denotes the period corresponding to time $t - n$. Then, replacing $E\left[h_{t-n}^{m(1+\beta_{1s}^n)}\right]$ in (A.8) with the expression in (A.9) we obtain

$$E[y_t^{2m} y_{t-n}^{2m}] = \nu_{2m} \exp\{m[(\omega_r + \omega_s) + \beta_{1s}^n(\omega_r - \omega_s)]\} E\left[\eta_{t-n}^{2m} \times \exp\left\{m(1 - L)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1} g_s(\eta_{t-i})\right\}\right] \times \prod_{i=1}^{\infty} E[\exp\{m(1 + \beta_{1s}^n)\lambda_{i,r} g_r(\eta_{t-n-i})\}]$$

denoting $\chi_s(L) = (1 - L)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1} L^{i-1} = 1 + \chi_{1,s}L + \dots$, then

$$\begin{aligned} E[y_t^{2m} y_{t-n}^{2m}] &= \nu_{2m} \exp\{m[(\omega_r + \omega_s) + \beta_{1s}^n(\omega_r - \omega_s)]\} E\left[\eta_{t-n}^{2m} \prod_{i=0}^{\infty} \exp\{m\chi_{i,s} g_s(\eta_{t-1-i})\}\right] \times \\ &\quad \prod_{i=1}^{\infty} E[\exp\{m(1 + \beta_{1s}^n)\lambda_{i,r} g_r(\eta_{t-n-i})\}] \\ &= \nu_{2m} \exp\{m[(\omega_r + \omega_s) + \beta_{1s}^n(\omega_r - \omega_s)]\} E\left[\eta_{t-n}^{2m} \exp\{m\chi_{n-1,s} g_s(\eta_t)\}\right] \\ &\quad \prod_{i=0}^{n-2} E[\exp\{m\chi_{i,s} g_s(\eta_t)\}] \prod_{i=n}^{\infty} E[\exp\{m\chi_{i,s} g_s(\eta_t)\}] \prod_{i=1}^{\infty} E[\exp\{m(1 + \beta_{1s}^n)\lambda_{i,r} g_r(\eta_t)\}] \end{aligned}$$

Furthermore, we know that

$$E[y_t^{2m}] = \nu_{2m} \exp\{m\omega_s\} \prod_{i=0}^{\infty} E[\exp\{m\lambda_{i,s}g_s(\eta_t)\}]$$

The numerator in (4.11) is equal to

$$\begin{aligned} E[y_t^{2m} y_{t-n}^{2m}] - [E(y_t^{2m})][E(y_{t-n}^{2m})] &= \nu_{2m} \exp\{m[(\omega_r + \omega_s) + \beta_{1s}^n(\omega_r - \omega_s)]\} \\ & E[\eta_t^{2m} \exp\{m\chi_{n-1,s}g_s(\eta_t)\}] \prod_{i=0}^{n-2} E[\exp\{m\chi_{i,s}g_s(\eta_t)\}] \times \\ & \prod_{i=n}^{\infty} E[\exp\{m\chi_{i,s}g_s(\eta_t)\}] \prod_{i=1}^{\infty} E[\exp\{m(1 + \beta_{1s}^n)\lambda_{i,r}g_r(\eta_t)\}] - \\ & \nu_{2m}^2 \exp\{m(\omega_s + \omega_r)\} \left(\prod_{i=0}^{\infty} E[\exp\{m\lambda_{i,s}g_s(\eta_t)\}] \right) \times \\ & \left(\prod_{i=0}^{\infty} E[\exp\{m\lambda_{i,r}g_r(\eta_t)\}] \right). \end{aligned}$$

The variance of y_t^{2m} is given by

$$\begin{aligned} E[y_t^{4m}] - (E[y_t^{2m}])^2 &= \nu_{4m} \exp\{2m\omega_s\} \prod_{i=1}^{\infty} E[\exp\{2m\lambda_{i,s}g_s(\eta_t)\}] - \\ & \nu_{2m}^2 \exp\{2m\omega_s\} \left(\prod_{i=0}^{\infty} E[\exp\{m\lambda_{i,s}g_s(\eta_t)\}] \right)^2 \end{aligned}$$

then the denominator is

$$\begin{aligned} \exp\{m(\omega_r + \omega_s)\} & \left[\nu_{4m} \prod_{i=1}^{\infty} E[\exp\{2m\lambda_{i,s}g_s(\eta_t)\}] - \nu_{2m}^2 \left(\prod_{i=0}^{\infty} E[\exp\{m\lambda_{i,s}g_s(\eta_t)\}] \right)^2 \right]^{1/2} \times \\ & \left[\nu_{4m} \prod_{i=1}^{\infty} E[\exp\{2m\lambda_{i,r}g_r(\eta_t)\}] - \nu_{2m}^2 \left(\prod_{i=0}^{\infty} E[\exp\{m\lambda_{i,r}g_r(\eta_t)\}] \right)^2 \right]^{1/2} \end{aligned}$$

The same procedure can be followed for the SFI-PEGARCH(1,d,0) simply setting $\lambda_s(L) \equiv (1 - L^S)^{-d}(1 - \beta_{1s}L)^{-1}$, and $\chi_s(L) = (1 - L^S)^{-d} \sum_{i=1}^n \beta_{1s}^{i-1} L^{i-1}$.