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Loss aversion and perceptual risk aversion

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Abstract

This paper analyzes, for S-shaped value functions, the relations between loss aversion and perceptual risk aversion (i.e., computed with the perceived probability weights) in Cumulative Prospect Theory. We show that perceptual risk aversion for mixed sign lotteries is equivalent to weak loss aversion, so this is the right assumption to get a sensible behavior towards risk. No assumption on the probability distortion is needed, beside the basic ones. Next we show a case (the widely used power S-shaped value function) where the lack of loss aversion can lead to a puzzling behavior with respect to risk.

Keywords: Cumulative Prospect Theory, power S-shaped value function, reference dependence utility, loss aversion.

JEL Classification: D81

1 Introduction

Cumulative Prospect Theory (CPT) affords a framework where both rational and psychological factors explicitly enter the decision problem. Various works have proved that decisions are taken by evaluating gains and losses separately, rather than considering the final wealth as a whole: this happens because people differently perceive improvements or deterioration of their reference wealth (e.g., see Kahneman and Tversky (1979), Tversky and Kahneman (1992), Kahneman et al. (1991) and Benartzi and Thaler (1995)). The observation that losses matter more than gains is defined as loss aversion. Furthermore, the fact that the perception of a change in wealth decreases with its distance from the reference point is termed diminishing sensitivity and can be modelled by value (or utility) functions that are concave for gains and convex for losses: precisely S-shaped value functions. Moreover, the subjective perception and the evaluation of uncertainty cause a distortion of the real probabilities, a phenomenon to be taken into account when making a decision, by properly weighting the probabilities.

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Consider an individual facing the choice among various lotteries. In CPT (see Tversky and Kahneman (1992)), a preliminary ordering of a lottery \( X \), whose outcomes are \( x_1, x_2, \ldots, x_n \), with probabilities \( p_1, p_2, \ldots, p_n \), is necessary. That is, the possible outcomes are ranked in order to have

\[
x_1 > x_2 > \cdots > x_k \geq 0 > x_{k+1} > \cdots > x_n.
\]

Remark that the strong inequalities mean that events with the same outcome are put together and their probabilities are summed up (here is the combination operation described in Kahneman and Tversky (1979, p. 274)). Every lottery is evaluated through

\[
V(X) = \sum_{i=1}^{n} \pi_i v(x_i),
\]

where \( v : \mathbb{R} \to \mathbb{R} \) is the value function, and \( \pi_i \) are the weights assigned to the probabilities. Given two lotteries \( X \) and \( Y \), \( X \) is preferred to \( Y \) if and only if \( V(X) > V(Y) \) and \( X \) is preferred to the null lottery if and only if \( V(X) > 0 \).

The value function \( v \) is defined on the difference between the final wealth and a reference wealth level, say \( m \), rather than on the overall final wealth. For simplicity’s sake, we assume, as usual, that \( m = 0 \) (the status quo or the null lottery) and \( v(0) = 0 \). The value function is continuous, strictly increasing and its first derivative is piecewise continuous except, at most, at the origin, where it can display an infinite jump. Moreover, \( v(x) \) is convex for \( x < 0 \) and concave for \( x > 0 \), i.e. it is S-shaped.

The probability weights \( \pi_i \) are

\[
\pi_i = \begin{cases} 
  w^+(P[X \geq x_i]) - w^+(P[X > x_i]) & \text{if } i \leq k, \\
  w^-(P[X \leq x_i]) - w^-(P[X < x_i]) & \text{if } i > k,
\end{cases}
\]

being the weighting functions \( w^+ \) and \( w^- \) differentiable, strictly increasing and mapping a probability \( P \in [0, 1] \) into \([0, 1]\), with

\[
\begin{align*}
  w^+(0) &= w^-(0) = 0, & w^+(1) &= w^-(1) = 1, \\
  w^+, w^- &\in C^1(0, 1), & \frac{dw^+}{dp} &> 0, & \frac{dw^-}{dp} &> 0, \forall p \in (0, 1).
\end{align*}
\]

They distort the probabilities in order to reflect their subjective perception. In fact, as Kahneman and Tversky (1979) and Fennema and van Assen (1999) have shown, small probabilities are overevaluated, whereas large ones are underevaluated. Moreover, the perception of probabilities can be different for gains and for losses.

In this paper we consider the attitude with respect to risk in perceptual sense, that is all evaluations are made replacing the probabilities \( p_i \) with their corresponding weights \( \pi_i \). In order to emphasize this, we use the prefix “\( \pi \)-". Therefore, an individual who prefers the \( \pi \)-expected value \( E^\pi[X] = \sum_{i=1}^{n} x_i \pi_i \) of any given risky lottery \( X \) for sure to the lottery \( X \) will be said to display a \( \pi \)-risk _averse attitude_ (\( \pi \)-RA). In the same way, indifference or preference of the lotteries to its \( \pi \)-expected value for sure are defined as \( \pi \)-risk neutral and \( \pi \)-risk seeking (\( \pi \)-RS) attitude, respectively. A lottery \( X \) is said to be \( \pi \)-unfavorable, \( \pi \)-fair, \( \pi \)-favorable when \( E^\pi[X] \leq 0 \), respectively.

Starting from \( X \), we define two lotteries \( X^+ \) and \( X^- \): \( X^+ (X^-) \) contains the non-negative (non-positive) outcomes of \( X \), whereas all others are replaced by 0 and their probabilities are summed up. This allows to write (1) as

\[
V(X) = V(X^+) + V(X^-).
\]
The concept of loss aversion means, loosely speaking, that losses “matter” more than gains, (e.g., see Kahneman and Tversky (1979), Bowman et al. (1999), Köbberling and Wakker (2003) and Neilson (2002)). As pointed out in Köbberling and Wakker (2003), Maggi (2004) and Neilson (2002), there is not a fixed notion of loss aversion. Different proposals are presented and discussed in Maggi (2004) and Neilson (2002). Among them, we recall the weak and the strong loss aversion (see Neilson (2002)), just owing to their relevant role for π-RA with mixed sign lotteries.

Definition 1 An increasing value function \( v \) displays weak loss aversion if \( v(0) = 0 \) and
\[
\frac{v(y)}{y} \leq \frac{v(z)}{z}, \quad \forall y > 0, \forall z < 0. \tag{5}
\]

Definition 2 An increasing value function \( v \), differentiable except possibly at 0, displays strong loss aversion if \( v(0) = 0 \) and
\[
v'(y) \leq v'(z), \quad \forall y > 0, \forall z < 0.
\]

Neilson (2002) shows that the two definitions above are equivalent for S-shaped value functions. Moreover a simple rule to check weak and strong loss aversion for S-shaped value functions can be found (see Maggi (2004)). In fact, if \( v \) is S-shaped and differentiable except possibly at 0, the two loss aversion definitions are equivalent to
\[
\lim_{x \to 0^+} v'(x) \leq \lim_{x \to -\infty} v'(x).
\]

In section 2 the role of a precise definition of loss aversion (i.e. the weak one) in ensuring a π-RA behavior with respect to mixed sign lotteries is clarified, without requiring any assumption on the probability distortions, besides the basic ones. In section 3 we point out that, owing to the lack of loss aversion, the well-known S-shaped power value function (see (7) below) may lead, even in a simplified framework (with \( n = 2 \)), to a puzzling behavior facing to mixed sign lotteries. Namely, if the exponent \( \alpha \) for gains is smaller than \( \beta \), the one for losses, then a lottery – no matter whether π-favorable or π-unfavorable – is preferred to the status quo as soon as its magnitude is sufficiently small. In other words, an individual who is π-RA over the gains can display π-RS behavior with respect to some (not to all) mixed sign lotteries. These results arouse some questions on the reliability of the widely used power value function in CPT. They are briefly discussed in section 4. In general, this paper emphasizes the need for a deeper analysis of the various value functions used in CPT. This paper completes the analysis of the properties of the S-shaped power value function already pointed out in Maggi (2004) in the case of no probability distortions.

2 Loss aversion and π-risk aversion

An S-shaped value function describes an individual who is risk averse over gains and risk seeking over losses when the computation is made with the probabilities \( p_i \). Moreover, from the perceptional point of view resulting from (2), the S-shapedness of the value function also ensures that the individual is π-RA with respect to any lottery \( X^+ \) without losses, and π-RS with respect to any lottery \( X^- \) without gains. Indeed, given \( X^+ \) (for \( X^- \) the argument and the conclusion are symmetrical), the probability distortion defined by (2) yields a new set of weights \( \pi_i \) that can be treated
as probabilities, as they are positive and sum up to unity (as remarked by Tversky and Kahneman (1992) also). So, in perceptional terms, $\pi$-RA for gains and $\pi$-RS for losses correspond, respectively, to the concavity of $v$ for $x > 0$ and its convexity for $x < 0$:

$$\pi$-RA : \quad V(X^+) \leq V(E^+[X^+]), \text{ i.e. } \sum_{i=1}^{n} \pi_i v(x^+_i) \leq v\left(\sum_{i=1}^{n} \pi_i x^+_i\right),$$

$$\pi$-RS : \quad V(X^-) \geq V(E^-[X^-]), \text{ i.e. } \sum_{i=1}^{n} \pi_i v(x^-_i) \geq v\left(\sum_{i=1}^{n} \pi_i x^-_i\right),$$

both with $\pi_i > 0$, $\forall i$ and $\sum_{i=1}^{n} \pi_i = 1$.

However, the S-shapedness by itself does not guarantees a $\pi$-RA behavior with respect to all mixed sign risky lotteries. In fact, without any further assumption, some S-shaped value functions can lead to a $\pi$-RS attitude for mixed sign lotteries.

The following theorem directly links the definition (5) of loss aversion to a consistent behavior with respect to mixed sign lotteries.

**Theorem 3** Given a CPT model, with the probability weights fulfilling (2) and (3), assume $\pi$-risk aversion for gains and $\pi$-risk seeking for losses. Then weak loss aversion

$$v\left(\frac{y}{z}\right) \leq v\left(\frac{z}{x}\right), \quad \forall y > 0, \forall z < 0,$$

is necessary and sufficient to get $\pi$-risk aversion for mixed sign lotteries in the sense that the status quo is preferred to every non-null $\pi$-fair lottery.

**Proof.** Thanks to (4), the mixed sign lottery $X$ is refused, if compared to the status quo, if and only if $V(X^+) < -V(X^-)$. Being the individual $\pi$-averse for gains and $\pi$-risk seeking for losses, we get $V(X^+) \leq V(E^+[X^+]) = v(E^+[X^+])$ and $V(X^-) \geq V(E^-[X^-]) = v(E^-[X^-])$. For a non-null $\pi$-fair lottery $0 < E^+[X^+] = -E^-[X^-]$ by definition. Let $E^+[X^+] = c$. Let us draw on the non-negative orthant $R^+_2$ of a Cartesian plane $(x, w)$ the two graphs $A = \{(x, w) \in R^+_2 : w = v(x)\}$ and $B = \{(x, w) \in R^+_2 : w = -v(-x)\}$. Thanks to the concavity of the functions $v(x)$ and $(-v(-x))$, the convex closures of $A$ and $B$ are, respectively, the sets

$$\bar{A} = \left\{(x, w) \in R^+_2 : x \left(\lim_{x \to +\infty} \frac{v(x)}{x}\right) \leq w \leq v(x)\right\},$$

$$\bar{B} = \left\{(x, w) \in R^+_2 : x \left(\lim_{x \to -\infty} \frac{v(x)}{x}\right) \leq w \leq -v(-x)\right\}.$$  

Being $V(X^+)$ and $(-V(X^-))$ convex linear combinations of points lying in $A$ and $B$ respectively, then $(c, V(X^+)) \in \bar{A}$ and $(c, -V(X^-)) \in \bar{B}$. Therefore $\bar{A} \cap \bar{B} = (0, 0)$ if and only if (6) holds. Moreover, (6) implies that $v(x) \leq -v(-x), \forall x > 0$, therefore $V(X^+) < -V(X^-)$, so every fair lottery is less valuable than the status quo.

Theorem 3 deserves some remarks:

- Theorem 3 exactly defines the role of loss aversion and recognizes which formal definition of loss aversion, i.e. the weak one (6), leads to a reasonable behavior facing risk for mixed sign lotteries. In fact, if (6) holds, every mean preserving spread added to the status quo makes it less valuable. Although in CPT the individuals are risk seeking over losses, loss aversion (6) ensures that they are always risk averse over mixed sign $\pi$-fair lotteries.
• Remark that condition (6) restricts the value function, i.e. the function describing the perception of the value, independently of the subjective perception of probabilities. This means that, should a condition weaker than (6) be desired, then restricting the probability distortion is useless. Instead, some weaker conditions can be found by restricting the set of admissible lotteries. For instance, in the set of binary symmetric lotteries (see (8) below) the condition \( v(x) < -v(-x), \forall x > 0 \), is equivalent to \( \pi \)-RA for mixed sign lotteries.

• Needless to say, theorem 3 works even without probability distortion, i.e. with \( \pi_i = p_i, \forall i \). Hence, the result applies, in general, on reference dependent decision models.

In the next section we discuss some anomalies stemming from the lack of loss aversion for the well-known power value function.

3 The power S-shaped value function

A widely used S-shaped value function is the power S-shaped one (Benartzi and Thaler (1995), Thaler et al. (1997), Prelec (1998), Fennema and van Assen (1999), Tversky and Kahneman (1992) and Wakker and Zank (2002)):

\[
v(x) = \begin{cases} 
  x^\alpha, & x \geq 0, \\
  -\lambda (-x)^\beta, & x < 0,
\end{cases} \quad \text{with } 0 < \alpha \leq \beta \leq 1, \lambda \geq 1.
\] (7)

It is defined on all \( \mathbb{R} \), continuous, strictly increasing and S-shaped. We exclude the case \( \alpha = \beta = 1 \), i.e. a kinked linear value function, to be discussed below. This way the assumptions \( \beta \geq \alpha \) and \( \lambda \geq 1 \) mean that losses matter more than gains. It is easy to check that the power value function does not display weak loss aversion. Therefore, given the result of theorem 3, some anomalies can emerge. The assumption \( \alpha < \beta \) is commonly based on the argument that losses matters more than gains, e.g., see Wakker and Zank (2002) and Fennema and van Assen (1999). Theorem 3 states that this argument is not precise. In fact, when loss aversion (6) is not verified, it may well happen that some losses matter less than some gains.

In order to show this anomaly, proposition 4 below adopts a simplified framework, namely it deals with binary symmetric lotteries \( X \), i.e. of the kind

\[
X = (x,p; -x,1-p), \quad x > 0,
\] (8)

and with their rescalings \( \varepsilon X = (\varepsilon x, p; -\varepsilon x, 1-p) \), with \( \varepsilon > 0 \).

**Proposition 4** Assume CPT with a power value function (7) and \( 0 < \alpha < \beta \leq 1 \). Let \( X \) be any binary symmetric lottery, i.e. of the kind (8), with a given probability of gain \( p \in (0,1) \). Then there exists a critical magnitude \( \bar{x} \) of the stack for which, no matter how \( X \) is \( \pi \)-unfavorable or \( \pi \)-favorable (i.e. how much \( \frac{\pi_1}{\pi_2} \) is small or large), or how the risk is perceived (i.e. how the values \( \alpha, \beta \) and \( \lambda \geq 1 \) are fixed), the lottery \( \varepsilon X \) is indifferent to the status quo. Moreover, if the rescaling factor \( \varepsilon \) is smaller (larger) than \( \bar{x} = \frac{1}{x} \left( \frac{\alpha}{\lambda} \frac{w^+(p)}{w^-(1-p)} \right)^{\frac{1}{\beta-\alpha}} > 0 \), then the lottery \( \varepsilon X \) is preferred (refused) with respect to the status quo.
Proof. For a binary symmetric lottery, according to (2), the weights are \( \pi_1 = w^+(p) \) and \( \pi_2 = w^-(1-p) \), so \( V(\varepsilon X) = (\varepsilon x)\alpha w^+(p) - \lambda (\varepsilon x)\beta w^-(1-p) \). The lottery \( \varepsilon X \) is indifferent to the status quo if and only if \( V(\varepsilon X) = 0 \), that is

\[
e^{\beta-\alpha} = x^{\alpha-\beta} \frac{w^+(p)}{\lambda w^-(1-p)}.
\]  

(9)

No matter how \( X \), \( \lambda \geq 1 \) and the weighting functions \( w^+ \) and \( w^- \) fulfilling (3) are chosen, the right hand side of (9) is a positive constant \( \forall \varepsilon > 0 \). Instead, being \( \alpha < \beta \), the left hand side of (9) is a continuous, strictly increasing and unbounded function on \((0, +\infty)\) vanishing for \( \varepsilon \to 0^+ \). Hence the unique \( \varepsilon > 0 \) satisfying (9), say \( \varepsilon = \bar{\varepsilon} \), separates \((0, +\infty)\) into two subsets \((0, \bar{\varepsilon})\) and \((\bar{\varepsilon}, +\infty)\) where \( \varepsilon^{\beta-\alpha} \geq x^{\alpha-\beta} \frac{w^+(p)}{\lambda w^-(1-p)} \) respectively hold. That is, given \( X \), we get \( V(\varepsilon X) \geq 0 \Leftrightarrow \varepsilon \geq \bar{\varepsilon} = \frac{1}{\lambda} \left( \frac{w^+(p)}{w^-(1-p)} \right)^{\frac{1}{\beta-\alpha}} \).

Obviously, proposition 4 tells that, if \( E^\pi [X] = 0 \), then \( \pi \)-fair binary symmetric lotteries are more (less) valuable than the status quo if the stake \( \varepsilon \) is small (large) enough. In this case \( \varepsilon = \frac{1}{\lambda} \lambda = \pi \). Here are some remarks on proposition 4. They may complete what Köbberling and Wakker (2003) have pointed out for small lotteries.

- Proposition 4 confirms in an easy way that CPT with power S-shaped value function does not respect second order stochastic dominance neither in objective nor in perceptional terms. In fact, consider the binary \( \pi \)-fair symmetric lottery \( Z \) whose stake is \( \varepsilon \in (0, \pi) \), with \( \varepsilon = \lambda \). Proposition 4 implies that \( Z \) is preferred to the null lottery. Here the surprising anomaly is that a \( \pi \)-RS attitude can be displayed by an individual who is \( \pi \)-RA over both gains and fairly large mixed sign lotteries. This confirms that, as Levy and Wiener (1998) remarked, CPT should be deeply analyzed with mixed signs and complex lotteries.

- Another effect is that the preference order of two lotteries \( X \) and \( Y \) can differ from the one of \( \varepsilon X \) and \( \varepsilon Y \), with \( \varepsilon > 0 \). For example, consider the power value function (7) with \( \alpha = 0.7 \), \( \beta = 0.9 \), \( \lambda = 2 \), and the weighting functions

\[
w^+(p) = \frac{p^{\gamma^+}}{(p^{\gamma^+} + (1-p)^{\gamma^+})^{1-\gamma^+}}, \quad w^-(p) = \frac{p^{\gamma^-}}{(p^{\gamma^-} + (1-p)^{\gamma^-})^{1-\gamma^-}},
\]

with \( \gamma^+ = 0.6 \), \( \gamma^- = 0.7 \). The lottery \( X = (0.005, \frac{1}{2}; -0.005, \frac{1}{2}) \) is preferred to \( Y = (0.0025, \frac{1}{2}; -0.0025, \frac{1}{2}) \) as \( V(X) > V(Y) \), but the lottery \( 2Y \) is preferred to \( 2X \), so the preference order is reversed.

In the case \( \alpha = \beta < 1 \), as in Tversky and Kahneman (1992), loss aversion does not hold, either. In fact, the lottery \( X = (3, \frac{2}{5}; 2, \frac{1}{5}; -14, \frac{1}{5}) \) is fair, but with \( \alpha = \beta = 0.5 \), \( \lambda = 2 \), and \( \gamma^+ = \gamma^- = 1 \) we get \( V(X) > 0 \).

Forcing \( \alpha = \beta \to 1 \) in the power value function (7), we get the well-known \textit{kinked linear value function}

\[
v(x) = \begin{cases} x, & x \geq 0, \\ \lambda x, & x < 0, \end{cases} \quad \text{with } \lambda > 1,
\]

which fulfills weak loss aversion. In this case the individual is \( \pi \)-risk neutral over gains and over losses separately, but she/he is \( \pi \)-RA over mixed sign lotteries.
4 Concluding remarks

Theorem 3 shows that, in a CPT model, weak loss aversion is a condition equivalent to perceptual risk aversion for mixed sign lotteries. This result exactly characterizes the role of loss aversion from the behavioral point of view. Therefore, a consistent behavior facing risk is obtained by restricting the function describing the perception of the value and no further assumption on the probability perception is needed, beside the basic ones (2). Consequently, when no probability distortion is in place, the result keeps valid.

It is worth to note that this result gives a final answer to the question of how to define loss aversion. In fact, weak loss aversion (6) is the right notion of loss aversion in the sense that it is necessary and sufficient in order to get a sensible behavior with respect to mixed sign lotteries. In fact, if (6) does not hold, then there exist lotteries that are perceived as fair, and yet are preferred than the status quo. This means that the losses of these lotteries matter less than their gains, although the lotteries are $\pi$-fair.

The results contained in proposition 4 may arouse some questions on the reliability of the power CPT value function. Their relevance comes from the wide use of the power function for both theoretical and empirical analysis. However, they may lead to different, or even opposite, interpretations, which even appear in simple symmetric lotteries.

Beginning from the negative one, the irrational-like behavior emerging when $\alpha < \beta$ may lead to the conclusion that the power value function is not a suitable tool to analyze the choices of an individual facing risk. A serious anomaly is that the same individual can display different attitudes toward risk, depending on the magnitude (and not only on the structure) of the lotteries to be evaluated. All this may be ascribed to the lack of weak loss aversion.

Another interpretation of the consequences of proposition 4 starts from a positive attitude towards them, even though it may appear more speculative. Considering that Prospect Theory is grounded on the perception of gains and losses (measured with respect to a reference point), the power value can be a tool consistent with a widely observed behavior. Moreover, the psychological perception theory, developed by Cognitive Psychology and Psychophysics, has shown that power laws generally describe the sensorial stimuli perception: brightness, loudness, sweetness, duration, etc., and – why not? – value. Following this approach, the puzzling features pointed out in section 3 can fit some irrational-like common behavior. In fact, when $\alpha < \beta$ the fair or even unfavorable lotteries that are preferred to the status quo have a small stack. Apart from pathological gamblers with a compulsive behavior, many fairly rational people accept even manifestly unfair lotteries, provided that the magnitude of the stack is small enough. This can reflect the difficulty in perception of small changes in wealth. A point that deserves a deeper analysis is whether the weighting function exhausts the distortion effect of perception on real uncertainty, or – on the contrary – some room is still available for other types of misevaluation to be captured by, say, the power value function.
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