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## Investment under Risk with Discrete and Continuous Assets

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Tinbergen Institute Discussion Paper

# Investment under Risk with Discrete and Continuous Assets: Solution and Estimation

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# Investment under risk with discrete and continuous assets: Solution and estimation

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## Abstract

This paper considers a general class of stochastic dynamic choice models with discrete and continuous decision variables. This class contains a variety of models that are useful for modeling intertemporal household decisions under risk. Our examples are drawn from the field of development economics. We formalize this class as a dynamic programming problem, then propose a solution method that relies on value function iteration. Finally, in an example we show how our algorithm can be applied to solve and estimate a dynamic model with discrete and continuous controls.

Keywords: value function iteration, mixed continuous/discrete controls, stochastic dynamic choice model

JEL classifications: C61, C63, C51, E12, G11, Q12

## 1 Introduction

When a German manufacturing company decides to set up a new factory to increase its capacity; when an American consumer is thinking about buying a new car or a home; when an Ethiopian farmer decides to invest in a pair of oxen to increase his productivity in cultivating crops, they are all facing the same kind of decision: they want to buy an object that has a high cost compared to their income, and only whole units of it can be purchased. In order to afford this investment, they need to save or take a loan in continuous assets, like a savings or credit account. This paper analyzes a class of models that deal with these kinds of decision problems under risk, assuming that the agents are rational and fully informed about the probability distribution of future outcomes.

It is not hard to see that in an economy where it is easy to borrow, discrete investment decisions can easily be handled by taking a loan in a continuous asset. In this case, it is not that important to explicitly model the discrete choice, unless the investment causes a shift in the profit function. This happens if the discrete and the continuous assets are not perfect substitutes, i.e. they affect the income of the agent in a different way. On the other hand, consider a credit constrained agent, who needs to accumulate savings before he can invest in the discrete asset. He needs to delay the investment until he can finance it from his savings. In this case, the optimal savings and investment decisions of the agent will be quite different from the case where it is possible to buy increments of all assets.

Farming households in developing countries usually face both credit constraints and a shift in productivity when they invest in a discrete asset, like cattle or land. Both credit constraints and the discrete nature of investments hinder poor households in engaging in more productive activities. If the prospect of higher earnings is too far away, poor households will not be motivated to save enough to invest in the technology enhancing asset,

thereby inducing themselves to remain poor. When the poor do not have the possibility to escape poverty we talk about a poverty trap. It is a central issue in development economics to find out whether poverty traps keep households from accumulating wealth. For example, Zimmerman and Carter (2003) find the existence of poverty traps in a stochastic dynamic programming model with a savings and an investment asset (both continuous) under risk and subsistence constraints. In the application part of this paper, we take our turn at investigating the presence of poverty traps using a simplistic model including indivisibility and credit constraints.

However, the main focus of this paper is to describe a solution method for a class of stochastic dynamic choice models with both discrete and continuous controls. We encountered this class of problems when investigating the vulnerability of African farming households to risk. In general, this paper can be useful for researchers analyzing intertemporal consumption and wealth accumulation decisions of households under risk. For example, Elbers *et al.* (2007) and Pan (2008) both estimate an infinite-horizon structural dynamic choice model with one continuous asset, livestock, to assess the effect of risk on livestock accumulation of Zimbabwean and Ethiopian households, respectively. Using counterfactual scenarios they also decompose the total effect of risk into ex ante behavioral responses, and ex post adjustment effects. In order to measure these effects, it is necessary to estimate all structural parameters of the dynamic choice model, which requires the solution of the dynamic programming problem. The results of the estimations imply that in both regions the anticipation of shocks significantly changes the livestock accumulation decisions of households, with the effect that households hold less livestock.

Elbers *et al.* (2007) and Pan (2008) use only one continuous asset. On the other hand, Rosenzweig and Wolpin (1993) (referred to as R&W in the following) estimate a finite-horizon discrete dynamic choice model under credit constraints using data from India. In their model, households can accumulate discrete values of bullock holdings (0, 1 or 2) to buffer income risk, but they assume that bullocks also play a role in crop income generation.<sup>1</sup> Additionally, they assume that installing a pump set, an irreversible discrete asset (0 or 1), can further increase farming productivity. In the estimation of the structural parameters, a numerical identification problem<sup>2</sup> might arise due to the limited number of asset combinations. Using simulation exercises, Elbers *et al.* (2008) indeed argue that R&W needed to postulate the value of the discount factor because it is numerically not well-identified in their estimation.

This identification problem can be overcome by introducing a continuous asset in the dynamic choice model additional to the discrete asset(s). In a model of irreversible investment in well construction and continuous asset holdings, Fafchamps and Pender (1997) investigate how the discrete and irreversible nature of the investment discourages poor farming households from undertaking a highly profitable investment under risk and credit constraints. They estimate the preference parameters of this infinite-horizon dynamic choice model using a Full Information Maximum Likelihood estimator that consists of a set of nested algorithms that iterate on the likelihood function and on the Bellman equation using value function iteration.<sup>3</sup> Fafchamps and Pender utilize a subsample of the

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<sup>1</sup>In their model, obtaining a bullock can take two forms: it is possible to purchase it, but households can also decide to engage in breeding, which will produce a bullock in 3 periods.

<sup>2</sup>Numerical identification refers to the ability of numerical methods to recover the parameters of the model in practice, given the available data. In theory, the maximum number of parameters that can be identified in the estimation procedure is limited by the number of observed realizations of the dependent variable.

<sup>3</sup>The income distribution of households, hence the magnitude of risk, is estimated separately, prior to the estimation of the preference parameters. Fafchamps and Pender (1997) do not consider ex ante risk

dataset used in R&W, and find that the discount rate of the households is around 18 percent, which is much higher than the 5 percent assumed in R&W.<sup>4</sup>

Dercon (1998) performs simulation experiments on a finite-horizon model that incorporates discrete cattle and continuous sheep holdings to shed light on the relevance of lumpiness in livestock accumulation decisions. The model incorporates income risk, but assets are assumed safe. Calibrating the model to features of rural Tanzania, he finds that missing credit markets and the indivisibility of cattle appear to be important driving mechanisms in the households' asset allocation decisions.

Drawing on the findings on Dercon (1998), Vigh (2008) estimates the structural parameters of an infinite-horizon dynamic choice model with discrete oxen and continuous other livestock holdings. Using these estimates Vigh determines the size of ex ante and ex post effects of risk on livestock accumulation, following Elbers *et al.* (2007) and Pan (2008). Her results suggest that when asset indivisibility is introduced in the model, risk can have a positive ex ante effect on livestock holdings. Risk in the model is realized through oxen-dependent income shocks and asset shocks. To estimate the model, Vigh uses a Simulated Pseudo Maximum Likelihood estimator that updates the asset accumulation rules of households using policy function iteration. However, due to the discrete nature of the oxen holdings, her accumulation rules are only first order approximations to the true policy functions.

In all of the above mentioned examples, the estimation of the structural parameters involves the solving of a dynamic programming problem. Introducing discrete controls in the model complicates the analysis because in this case it is not possible to rely on first order conditions to find the optimal asset accumulation rules. Instead, in finite horizon problems it is possible to use backward solution as done in Rosenzweig and Wolpin (1993) and Dercon (1998). Fafchamps and Pender (1997) simplify the model structure by using an absorbing state (owning a well), and use value function iteration to estimate the policy function of the continuous asset and the timing of the irreversible investment. For the case of the stochastic Ramsey model with a continuous asset, Pan (2008) notes that value function iteration produces less accurate results than policy function iteration. However, the policy functions of a dynamic programming model with mixed continuous/discrete controls cannot be properly estimated using solely policy function iteration.

The purpose of this paper is twofold. First, we introduce a general class of mixed discrete-continuous dynamic choice models, and propose a solution method using value function iteration. Second, the use and accuracy of this method is demonstrated on a simple example with a continuous saving and a binary investment asset under income risk. Reference is made to the existence of poverty traps and estimation of the structural parameters of such a model.

In the remaining sections of the paper we introduce the class of dynamic choice models that we are interested in, and formulate it as a dynamic programming problem. Next, the solution algorithm is described using value function iteration. We demonstrate the use of this solution method on a simple application. Finally, we draw the conclusions on the proposed solution method.

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coping strategies.

<sup>4</sup>The difference in the estimated preference parameters can be caused by the different specification of the utility function, i.e. the Fafchamps and Pender (1997) model does not include a minimum consumption parameter.

## 2 The model

We start with defining a class of economic models to investigate the agent's decision to invest in a lumpy asset under risk. This class can be written using a stochastic dynamic choice model with the following structural assumptions: (1) in each period the agent chooses  $d$ , a feasible number of lumpy asset he wants to own, and  $x$ , a feasible value of his other continuous savings; (2) making these decisions the agent aims to maximize the expectation of his discounted life-time utility from consumption,  $c$ ; (3) the agent anticipates that his choices this period affect the future realizations of his income and also his future decisions on all assets; (4) the agent knows and rationally anticipates the probability distribution of shock realizations that affect his income stream; (5) the agent rationally chooses the optimal level of investment considering its effect on his future income flows; (6) the agent decides on his asset holdings knowing the realization of  $s_t$ , the shocks in the given period, but he is uncertain about the shock outcomes next period,  $s_{t+1}$ ; (7) the agent faces the same decision problem over an infinite time horizon. Using these assumptions, the maximization problem of the agent in period  $\tau$  can be formulated as

$$\max E_\tau \sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t). \quad (1)$$

with

$$c_t = F(x_{t-1}, d_{t-1}, s_t) - g(x_t, d_t) \quad (2)$$

and satisfying

$$c_t \geq 0 \quad (3)$$

$$d_t \in \mathbb{D} = \{0, 1, \dots, D\} \quad (4)$$

$$x_t \in \mathbb{X} \subseteq \mathbb{R} \quad (5)$$

$$d_t, x_t \text{ are measurable w.r.t. the event space (sigma-field)} \quad (6)$$

generated by  $\{F(x_{t-s}, d_{t-s}, s_{t-s+1}) | i = 1, \dots, t - \tau + 1\}$

for  $t = \tau, \tau + 1, \dots, T \leq \infty$ , given  $d_{\tau-1}, x_{\tau-1}$ .

In the maximization problem  $d_t$  denotes the discrete asset,  $x_t$  the continuous asset and  $s_t$  the shock variables.  $\mathbb{X}$  specifies all possible values of the continuous asset.<sup>5</sup>  $F(\cdot)$  is the function that determines the wealth at hand.  $g(\cdot)$  specifies the cost of investment in  $d$  and  $x$  in terms of the consumption good. We assume that the shocks to the wealth function are distributed i.i.d. Note that (3) represents the budget constraint stating that the difference between income and non-consumption expenditure has to be non-negative.

## 3 Dynamic programming formulation

The utility maximization problem outlined in (1)-(6) can be reformulated as a dynamic programming problem

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<sup>5</sup> $\mathbb{X}$  can be defined flexibly. For example, it can be applied to an entry decision model with continuous investment levels in  $x$ . In this model  $d$  is the binary entry decision variable such that  $d = 1$  if entry and  $d = 0$  otherwise. Then,  $x = 0$  if  $d = 0$  and  $x \geq X^{min}$  if  $d = 1$ , which implies  $\mathbb{X} = \{0\} \cup \{x \in \mathbb{R} | x \geq X^{min}\}$  with  $X^{min} > 0$ .

$$V(F(x_{t-1}, d_{t-1}, s_t)) = \max_{x_t, d_t} u(F(x_{t-1}, d_{t-1}, s_t) - g(x_t, d_t)) + \beta E_t V(F(x_t, d_t, s_{t+1}))$$

$$s.t. \tag{7}$$

$$F(x_{t-1}, d_{t-1}, s_t) \geq g(x_t, d_t) \tag{8}$$

$$d_t \in \{0, 1, \dots, D\} \tag{9}$$

$$x_t \in \mathbb{X} \subseteq \mathbb{R} \tag{10}$$

where  $V(\cdot)$  denotes the value function in the Bellman equation (7). The state variable of the problem is wealth at hand, denoted by  $w_t \equiv F(x_{t-1}, d_{t-1}, s_t)$ . Note, that the dimension of the state space is one, irrespective of the number of controls ( $x_t, d_t$ ) and shock ( $s_t$ ) variables. Therefore, the curse of dimensionality does not apply. This formulation of the problem assumes that there exists a market for all assets such that it is possible to buy and sell each asset for a constant relative price in any period without transaction costs. In this case, it is possible to describe the wealth of the household with only one state variable,  $w_t$ . Hence, knowing  $w_t$  and the distribution of shocks is sufficient to make the optimal investment ( $d(w_t)$ ) and savings ( $x(w_t)$ ) decisions.

Before proceeding to the description of the solution algorithm, we summarize all the assumptions of the model together with the ones implied by the dynamic programming formulation of model (1)-(6):

**Assumption 1 (A1):** Properties of the utility function:  $u'(\cdot) > 0$  and  $u''(\cdot) < 0$ .

**Assumption 2 (A2):** Properties of the profit function:  $F_x(\cdot) > 0$  and  $F_s(\cdot) > 0$ , where  $F_i(\cdot)$  denotes the derivative of  $F(\cdot)$  w.r.t. variable  $i$ . Further,  $F(x, d_1, s) - F(x, d_0, s) > 0$  for all  $d_1 > d_0 \geq 0$ .

**Assumption 3 (A3):** There is a functioning market for all assets in  $x$  and  $d$ , and there are no transaction costs when buying and selling the assets.

**Assumption 4 (A4):** The cost of investment is a deterministic function of  $x$  and  $d$ , i.e.  $s$  does not appear in  $g(\cdot)$ .

**Assumption 5 (A5):** Shocks are i.i.d., hence  $\text{cov}(s_t, s_\tau) = 0$  for all  $\tau \neq t$ .<sup>6</sup>

**Assumption 6 (A6):** There is a finite number of possible values for the discrete asset,  $d$ , with  $D \ll \infty$ .

**Assumption 7 (A7):** The continuous asset is bounded from below.

If a dynamic choice problem can be written in the form of (1)-(6) and Assumptions 1-7 are satisfied then the solution method of section 4 can be used to solve the problem. This algorithm will be efficient if there is at least one discrete control in the problem at hand. In case of problems with only continuous controls, Pan (2008) shows for the stochastic Ramsey model that a solution method using policy function iteration outperforms the method with value function iteration.

Note that formulation (1)-(6) is more general than that of Fafchamps and Pender (1997) because it allows a decision on the discrete asset holdings in every period, irrespective of the choices before. It is straightforward to extend the number of assets in  $x$  and  $d$ .

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<sup>6</sup>In case of serially correlated shocks, the state space should be expanded by at least  $s_t$ , therefore we do not consider this case.



However, an expansion of the decision space will increase the computational burden of approximating the policy function.

## 4 Solution algorithm using value function iteration

In this section, we describe a solution algorithm for problem (7)-(10). The algorithm is based on value function iteration and it approximates the solution of the dynamic programming problem in a non-parametric way. Since the problem is recursive, in the description of the algorithm we leave away the time dimension, and use shorthand notation  $x = x_t$ ,  $d = d_t$ ,  $s^+ = s_{t+1}$ ,  $w = F(x_{t-1}, d_{t-1}, s_t)$  and  $w^+ = F(x, d, s^+)$  to rewrite (7) as

$$V(w) = \max_{d,x} u(w - g(x, d)) + \beta EV(w^+). \quad (11)$$

### 4.1 Discretization of the state space and integration

To approximate the expectation in equation (11) we discretize the next period's state variable,  $w^+ \in \mathbb{R}$ , as  $w^+ \in \{w_0^+, w_1^+, \dots, w_N^+\}$  for a sufficiently large  $N$ . We denote by  $p(w_n^+|x, d)$  the probability that next period's wealth at hand will be  $w_n^+$  given  $x$  and  $d$ . Then,

$$V(w) = \max_{d,x} u(w - g(x, d)) + \beta \sum_{n=0}^N p(w_n^+|x, d)V(w_n^+). \quad (12)$$

The evaluation of  $V(w)$  involves the calculation of the conditional probabilities  $p(w_n^+|x, d)$  for all  $w_n^+$  during the maximization procedure w.r.t.  $d$  and  $x$ . The expectation of next period's value function can also be approximated using a discretization of the shock variable,  $s$ . Take  $M$  discrete realizations of the income shock,  $s_m$ , and the corresponding probability weights,  $\theta_m^s$ , for  $m = 1, \dots, M$ . Then,

$$V(w) = \max_{d,x} u(w - g(x, d)) + \beta \sum_{m=1}^M \theta_m^s V(F(x, d, s_m)). \quad (13)$$

### 4.2 Numerical and Monte Carlo integration methods

The shocks and corresponding weights in (13) can be drawn in a number of ways. The most popular of these are Monte Carlo integration and Gaussian quadrature methods. Gauss-Hermite quadrature can be used if the underlying distribution has a factor of  $\exp(-z^2)$ . Hence, Gauss-Hermite quadrature can be used in case of normally or log-normally distributed disturbances. However, the approximation will only be accurate if the function in the integral is close to a polynomial in the random variable, because the algorithm chooses nodes and probability weights in such a way that with  $M$  number of nodes the integral of a polynomial up to order  $2M - 1$  can be solved exactly.<sup>7</sup>

In case of Monte Carlo integration, a random sample from the true distribution is drawn with probability weights  $1/M$ . This algorithm performs well with multi-dimensional integrals (with dimension 3 or larger), however it is not as efficient as numerical integration in case of a single dimension.<sup>8</sup>

<sup>7</sup>For more information on the quadrature methods consult chapter 7 in Judd (1998) or Chapter 4 in Press *et al.* (1992).

<sup>8</sup>For more information on Monte Carlo integration see chapter 8 in Judd (1998).

As an alternative, we propose a third method that we use fruitfully in the application of Section 5. We take equidistant nodes on interval  $(-4, 4)$  for the standard normal disturbances,  $(e_1, e_2, \dots, e_M)$ , and then use these to construct the log-normal shocks as  $s_m = \exp(\alpha_0 + \alpha_1 e_m)$ , where  $\alpha_0$  and  $\alpha_1$  are chosen such that  $\sum_{m=1}^M \theta_m^s s_m = 1$  and  $\sum_{m=1}^M \theta_m^s (\log s_m - \sum_m \theta_m^s \log s_m)^2 = \sigma^2$ . Probability weights,  $\theta_m^s$ , are based on the standard normal density of  $e_m$  normalized such that  $\sum \theta_m^s = 1$ .

Any of the three methods outlined above can be applied in the solution algorithm with the appropriate choices for  $\theta_m^s$  and  $s_m$ . However, depending on the nature of the problem usually one integration method will prove to be more efficient and/or accurate than the others for the specific case. Therefore, it is a good idea to experiment with different methods to find the most suitable one for the problem at hand.

### 4.3 Value function iteration algorithm

Before we can start the value function iteration algorithm, we need to first specify the state space,  $w_i$ , the shock realizations  $s_m$  and their probability weights  $\theta_m^s$ , and initial guesses for the value and policy functions at each  $w_i$ . We initially choose equidistant wealth at hand realizations for  $w_i$  on a relevant interval for the problem. Shocks and their probability weights are chosen as discussed in section 4.2. An initial approximation of the value function is necessary so that we can calculate the RHS values of  $V(\cdot)$  in (13). The initial guesses of the policy functions are used as starting values in the value function maximization step. Choosing good starting values for the value and policy functions are important in achieving a fast and accurate convergence.

In the iteration procedure we use the approximated value and policy functions from the previous iteration,  $V^{r-1}(\cdot)$  and  $\psi_x^{r-1}(\cdot), \psi_d^{r-1}(\cdot)$ , and their linear interpolation or extrapolation<sup>9</sup> to obtain the value of the value and policy functions at a given wealth level. Therefore, in each iteration we evaluate the policy function at wealth levels  $\{w_1, w_2, \dots, w_N\}$ , which can serve as the value of next period's value function at states  $\{w_1, w_2, \dots, w_N\}$  in the next iteration. Thus, in each iteration we solve

$$V^r(w_i) = \max_{d_i, x_i} u(w_i - g(x_i, d_i)) + \beta \sum_{m=1}^M \theta_m^s \hat{V}^{r-1}(w_m) \quad (14)$$

with  $w_m = F(x_i, d_i, s_m)$  and  $\hat{V}^{r-1}(\cdot)$  the interpolated or extrapolated values of the value function for  $i = 0, 1, \dots, N$ .<sup>10</sup>

The iteration procedure can be summarized in the following steps:

**Step 1. Initialization:** take  $N$  values of  $w_i$ , such that  $w_1 < w_2 < \dots < w_N < \infty$ . These are the nodes where the value function will be evaluated. Specify initial values  $V(w_i)$  and  $\psi_x^0(w_i), \psi_d^0(w_i)$  for each  $w_i$ , which give the starting approximation for the value and policy functions. Using the equidistant integration method in section 4.2, draw  $M$  realizations of shocks,  $s_m$ , and the corresponding probability weights,  $\theta_m^s$ .

<sup>9</sup>Linear interpolation or extrapolation uses known points of a function to approximate function values at any points by calculating a weighted average of the neighboring known points. The weights are based on the distance between the point to be evaluated and the neighboring known points. A simpler formulation is the following:  $y = y_0 + (x - x_0)(y_1 - y_0)/(x_1 - x_0)$ .

<sup>10</sup>A hat over a function denotes that we use interpolation or extrapolation to evaluate the function at the given value.

**Step 2.** Iteration: at each iteration,  $r$ , a new approximation of  $V^r(w_i)$  and  $\psi_x^r(w_i), \psi_d^r(w_i)$  is calculated for each value of  $w_i$ . In the evaluation, the approximations at the previous iteration are linearly interpolated or extrapolated to account for the approximation of the policy functions at wealth levels not included in  $w_i$ . At each iteration the following steps are implemented to find a new value function and optimal investment rule for each  $w_i$ :

**Step 2.1.** For all feasible values of the discrete asset,  $\bar{d}_k \in \{d_i \in \mathbb{D} | g(0, d_i) \leq w_i\}$ , maximize

$$V_k^r(w_i) = \max_{x_k} u(w_i - g(x_k, \bar{d}_k)) + \beta \sum_{m=1}^M \theta_m^s \hat{V}^{r-1}(F(x_k, \bar{d}_k, s_m)) \quad (15)$$

$$s.t. \quad g(x_k, \bar{d}_k) \leq w_i$$

and store the optimal values  $(V_k^r(w_i), x_k, \bar{d}_k)$ .

**Step 2.2.** Update the value function

$$V^r(w_i) = \max_k V_k^r(w_i)$$

and the policy functions

$$\{\psi_d^r(w_i), \psi_x^r(w_i)\} = \{\bar{d}_{k^*}, x_{k^*}\}$$

with  $k^* = \arg \max_k V_k^r(w_i)$ .

**Step 3.** Convergence: the approximation of the value and policy functions converges to their true values, because  $V(w)$  satisfies Blackwell's sufficient conditions.<sup>11</sup> Hence, the contraction mapping theorem applies to the value function iteration. In practice, we stop the iteration procedure when the difference between the values of  $(V^{r-1}(w_i), \psi_x^{r-1}(w_i), \psi_d^{r-1}(w_i))$  and  $(V^r(w_i), \psi_x^r(w_i), \psi_d^r(w_i))$  becomes very small for all  $i$ .

To demonstrate the use of the algorithm, in the next section we take a simple application and show how our solution method can be used to approximate the policy function.

## 5 Application

In this section, we discuss a simple example to demonstrate the solution of a dynamic choice model with both discrete and continuous controls. Additionally, we describe how the structural parameters of such a model can be estimated, but leave the application of the estimation procedure to a later stage.

### 5.1 The model

Imagine a farmer, who earns his income from cultivating land. He has an expected yield of  $y_0 = 0.5$  units, however its value is affected by weather shocks summarized in  $s$ . He has a possibility to save from his income through storing grains,  $x$ . For simplicity, we assume that grain is a safe asset. In each period he also has the option to rent a pair of oxen,  $d$ ,

<sup>11</sup>Blackwell's sufficient conditions require  $V(w)$  to be monotonic and satisfy discounting. Further,  $V(w)$  is bounded since  $0 < w_i < w_N < \infty$  for all  $i$ .

which he can use in ploughing. Oxen rental costs him 1 unit in each period, however, it increases his expected income to  $y_1 = 2$  in the next period. This farmer realizes that it is beneficial for him to rent the oxen if he has enough money, therefore he wants to make a savings plan that would tell him how much he should save in each period, given his wealth at hand.

This problem can be formalized in the dynamic programming framework with the following Bellman equation:

$$V(w) = \max_{x,d} u(w-x) + \beta EV(x-d + s(y_0 + d(y_1 - y_0))) \quad (16)$$

*s.t.*

$$w \geq x \quad (17)$$

$$x \geq d \quad (18)$$

$$d \in \{0, 1\} \quad (19)$$

with  $y_0 = 0.5$  and  $y_1 = 2$ . Let  $\beta = 0.9$  and assume CRRA utility  $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$  with  $\gamma = 0.95$ . The distribution of the shock is  $\log s \sim N(-\sigma^2/2, \sigma^2)$ . Note that  $x$  represents the total holdings of assets, while  $d$  denotes the part of assets that is invested in the more productive technology.

For this problem we approximate the optimal savings and investment decision using numerical optimization. However, for the deterministic case ( $s = 1$  with probability 1), we can derive the exact solution analytically. This is done in the next section, so that we can compare the results of the approximated value and policy functions with  $\sigma = 0$  to the analytical solution.

## 5.2 Benchmark: analytical solution of the deterministic case

First, we solve the deterministic case of (16)-(19) analytically. Notice that in this setting it is optimal for the agent to reach and stay at the steady state with  $d = 1$  and  $x = 1$ . We can simplify the problem by noting that once the farmer saves 1 in  $x$ , he will invest this money in  $d$ . Thus, we can rewrite the problem without  $d$  as

$$V(w) = \max_x u(w-x) + \beta V(y_0 + I_{x \geq 1}(y_1 - y_0 - 1) + x) \quad (20)$$

*s.t.*

$$w \geq x \quad (21)$$

$$x \geq 0. \quad (22)$$

We have to find the wealth level,  $\bar{w}_1$ , at which the agent will decide to invest 1 unit in  $d$ . At this wealth level the agent will be indifferent between investing 1 unit in  $d$  today and making the same investment tomorrow. Hence, we can write that

$$V^0(\bar{w}_1) \equiv u(\bar{w}_1 - 1) + \beta V(y_1) = \max_x u(\bar{w}_1 - x) + \beta u(y_0 + x - 1) + \beta^2 V(y_1) \equiv V^1(\bar{w}_1). \quad (23)$$

Using the FOC for  $x$  in the RHS and the equality in (23) we can find the switching wealth level,  $\bar{w}_1$  and the savings level just below it,  $x_1^l$ . Equation (23) has two solutions but we are only interested in the lower solution that gives us  $\bar{w}_1 = 1.093$ , which is the wealth level below which the agent waits one more period to reach  $d = 1$  and saves  $x = 0.780$  unit

$n$	$\bar{w}_n$	$\bar{x}_n^l$	$\bar{x}_n^h$	$V(\bar{w}_n)$
0	2.000	1.000	1.000	0.000
1	1.093	0.780	1.000	-2.240
2	0.792	0.498	0.638	-3.483
3	0.532	0.230	0.335	-4.481
4	0.305	-0.017	0.069	-5.275

Table 1: Switch-points in the deterministic case using analytical solution,  $\beta = 0.9$ ,  $\gamma = 0.95$ ,  $y_0 = 0.5$  and  $y_1 = 2$

this period, and above which the agent invests in  $d = 1$  period. There is a discontinuity in the policy function of  $x$  at  $\bar{w}_1$ .

Next we want to find the switch point,  $\bar{w}_2$ , where the agent is indifferent between investing in  $d = 1$  next period and in two periods, and so on. A detailed description of the analytical solution can be found in Appendix A Table 1 reports all the switching wealth levels, savings decisions and value function values at these points. Note that an agent with 0.5 unit of income needs to save for three periods before he can invest in  $d$ . Therefore, in a risk-free world there is no poverty trap for this parametrization of the model, because each agent receives at least 0.5 unit of income, which is sufficient to build up savings to invest in the higher technology.

The policy function of total savings,  $x$ , is shown in Figure 1. The graph indicates that the policy function is linear between the switching points because on these intervals the Euler equation of the problem is satisfied.<sup>12</sup> Thus, the optimal savings decision is

$$x = \frac{\beta^{1/\gamma}}{1 + \beta^{1/\gamma}}w + \frac{1}{1 + \beta^{1/\gamma}}(x_{i+1}^* - y_0) \quad (24)$$

for wealth levels  $w_{i+1} < w < w_i$ , where  $i$  stands for the number of periods that are needed to reach  $d = 1$  and  $x_{i+1}^*$  denotes the optimal savings decision in the next period.

### 5.3 Solution method using value function iteration

Next, we solve (16)-(19) using value function iteration. For this model, Step 2.1 of the solution algorithm involves maximizing

$$V_0^r(w_i) = \max_{x_i^0 \geq 0} u(w_i - x_i^0) + \beta \sum_{m=1}^M \theta_m^s \bar{V}^{r-1}(x_i^0 + s_m y_0) \quad (25)$$

and

$$V_1^r(w_i) = \max_{x_i^1 \geq 1} u(w_i - x_i^1) + \beta \sum_{m=1}^M \theta_m^s \bar{V}^{r-1}(x_i^1 - 1 + s_m y_1) \quad (26)$$

for every wealth level, where it is feasible.

In (25) and (26) the values of  $s_m$  and  $\theta_m^s$  are chosen using the equidistant nodes approach described in section 4.2. This approach to numerical integration does better than the Gauss-Hermite quadrature or the Monte Carlo integration in case of this problem. The bad performance of Gauss-Hermite quadrature can have two reasons: (1) the value function cannot be well approximated with a polynomial, and (2) the algorithm chooses too many nodes at the tails of the distribution, while we are more interested about what

<sup>12</sup>See Appendix A for further details.

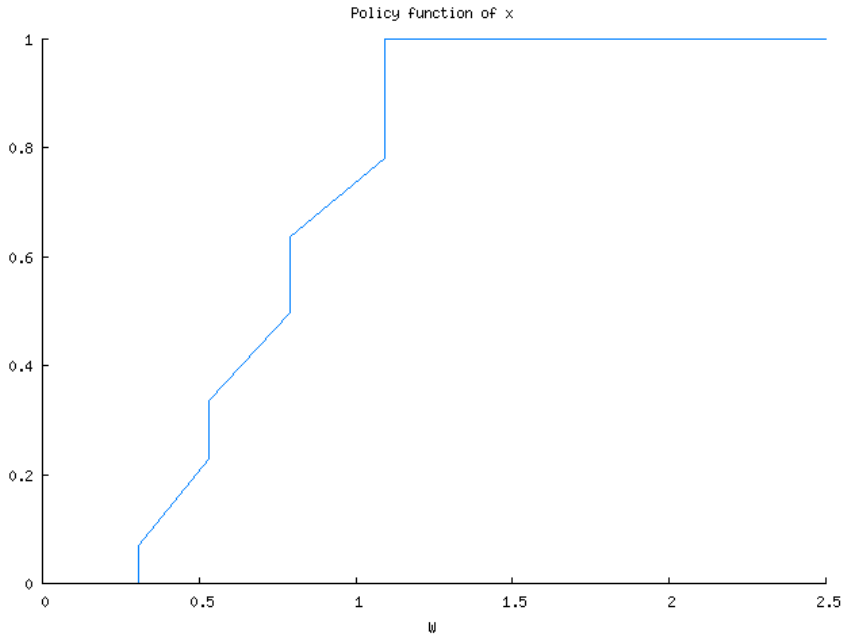


Figure 1: Policy function of the problem

happens within  $3\sigma$  distance around the mean. As a result, the policy functions obtained using Gauss-Hermite quadrature were coarse compared to the Monte Carlo integration or the equidistant method. The handicap of Monte Carlo integration is that we need more nodes to derive a precise estimate of the policy function compared to the equidistant nodes.

Further details on the implementation of the solution algorithm can be found in Appendix B.

#### 5.4 Simulation results

First, we report the results of the value function iteration for the deterministic case ( $\sigma = 0$ ). The results of the iteration procedure are plotted in Figure 2 and the estimates of the switch points are reported in Table 2. In the table  $w^l$  and  $w^h$  denote the grid-points where the discontinuities occur. The distance between  $w^l$  and  $w^h$  can be reduced by evaluating the policy function at more nodes, however we did not do a finer grid. For wealth level  $w^l$ ,  $x^l$  is the estimated stock of assets held by the household, while it is  $x^h$  at  $w^h$ .  $V(w^l)$  stands for the level of the value function at  $w^l$ . The shape of the obtained policy functions are very similar to the analytical solution. The values of  $(w, x^l, x^h, V(w))$  at all switch-points are estimated with a high precision. The true switch-points,  $\bar{w}$ , always fall in the range found by the value function iteration algorithm. This result is dependent on the use of the initial-value-search routine, which is described in Appendix B. Without this routine, the switch-points are estimated several grid-points away the true value except at the wealth level where the investment in  $d$  is made.<sup>13</sup> The values of  $x$  also have the right magnitudes. The values of  $x_n^l$  are slightly lower than the analytical solution  $\bar{x}_n^l$  because the estimated values  $w_n^l$  are also a little bit smaller than  $\bar{w}_n$ . The same holds with reverse sign for  $x_n^h$  and  $\bar{x}_n^h$ .

<sup>13</sup>In this case the discontinuity occurs due to a change in  $d$ , hence the flatness of the value function w.r.t.  $x$  is not an issue here.

$w^l$	$w^h$	$x^l$	$x^h$	$V(w^l)$
1.999	2.006	1.000	1.000	0.029
1.089	1.096	0.777	1.000	-2.219
0.789	0.795	0.496	0.639	-3.463
0.527	0.534	0.227	0.335	-4.465
0.300	0.307	0.000	0.071	-5.262

Table 2: Switch-points in the deterministic case using value function iteration

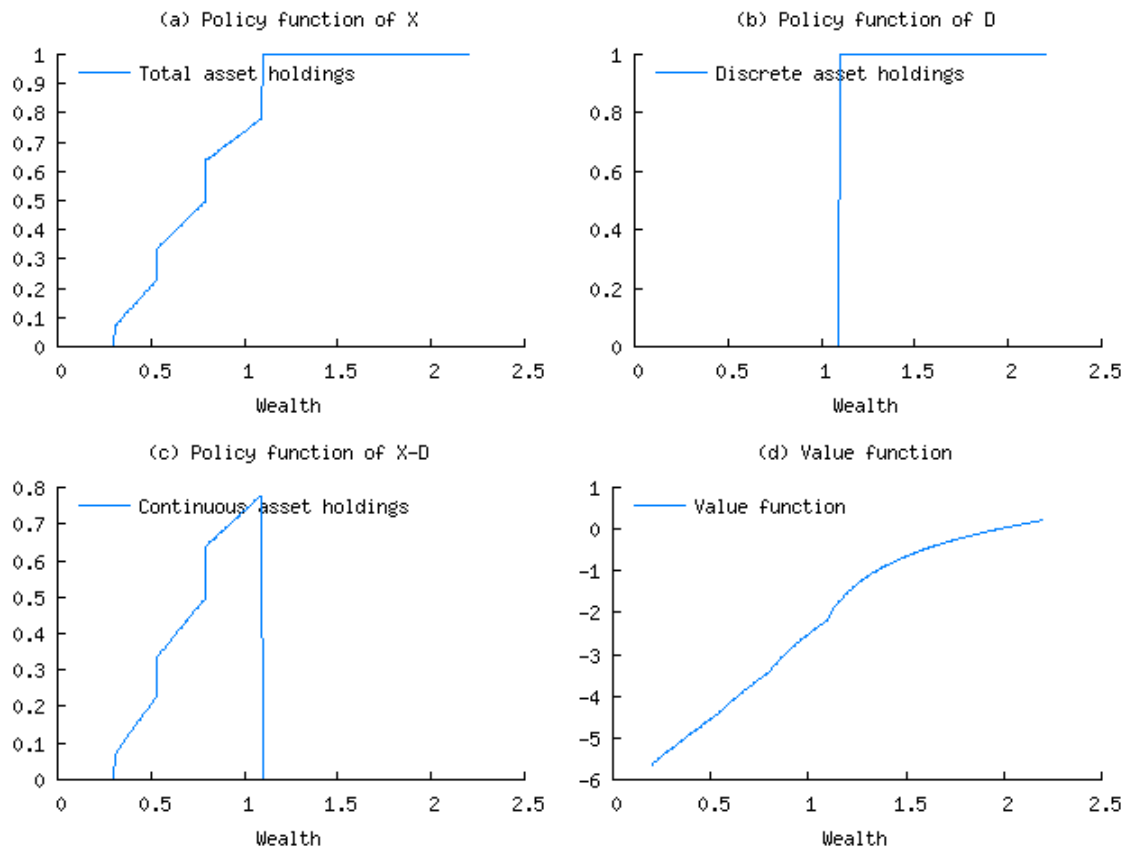


Figure 2: Value function iteration results

Next, we look at how the policy function of total asset holdings and the value function change shape as one of the parameter is altered from the set  $(\beta, \gamma, \sigma, y_0, y_1) = (0.9, 0.95, 0, 0.5, 2)$ . In Figures 3-6 we can observe the patterns in a risk-free environment. Figure 3 plots how the policy and value function depends on the time-preference parameter,  $\beta$ . We observe that the more patient households are, the more they are going to save from their income. Also, we see that only patient households (with  $\beta$  close to 0.9 and higher) can avoid the poverty trap by saving enough to invest in the better technology with return  $y_1$ . For households with a low  $\beta$  it is not optimal to accumulate assets in a risk-free environment with income below 1.

Looking at Figure 4, we can see how the policy and value functions behave as the rate of relative risk aversion,  $\gamma$ , changes. Note that the higher  $\gamma$  is the lower is the household's elasticity of intertemporal substitution. The graph shows that for wealth levels below  $y_1$  the households are less willing to give up consumption this period for a higher consumption in the next period. If the elasticity becomes too low ( $\gamma$  too high), the households do not have sufficient incentive to save for the better technology. This poverty trap can be observed in Figure 4 for  $\gamma = 1.6$ . On the other hand, if the household's wealth level becomes higher than the steady state level ( $w = y_1 = 2$  and  $x = 1$ ), the households with a higher  $\gamma$  will save more.

Figure 5 plots the policy and value function for different values of  $y_0$ . We observe that if the basic income of the households decreases then they will save more at low wealth levels so that they build up enough wealth to invest in the higher return activity. However, if  $y_0$  is too low such that the utility of consuming now becomes larger than the future discounted benefits of the higher income, then the households cannot grow out of poverty. As the payoff difference between the two activities,  $y_1 - y_0$  approaches the cost of switching between the activities (1), the switching becomes less attractive and the investment into the better technology occurs at higher wealth levels. If the difference in the returns becomes less than 1, the households will only save if their income is above  $y_0$  and will never invest in the high return activity. We observe the opposite pattern when we look at Figure 6, which plots the value and policy functions as  $y_1$  changes.

Next, we investigate the policy function under risk. We set  $\sigma = 0.25$ , and look at Figures 7-11 to discuss the shape of the policy and value function around the parameter set  $(\beta, \gamma, \sigma, y_0, y_1) = (0.9, 0.95, 0.25, 0.5, 2)$ . The change in the policy functions is very similar to the case under no risk, however, we observe a smoother policy function with less discontinuities in all cases.

Looking at Figure 9, which plots the policy function as the magnitude of risk increases, we observe that for wealth levels below 1 the policy function smoothes out as risk increases. At the end, we only have one discontinuity in the policy function when investing in the high return technology. The investment happens at higher wealth levels as risk increases. Also, the household builds up precautionary savings when his wealth level increases above  $y_1$  due to a positive shock. In this case, savings are the only way to cope with income risk and prevent the household from dropping back to the basic technology. Finally, we observe that the household's utility level decreases in the presence of risk.



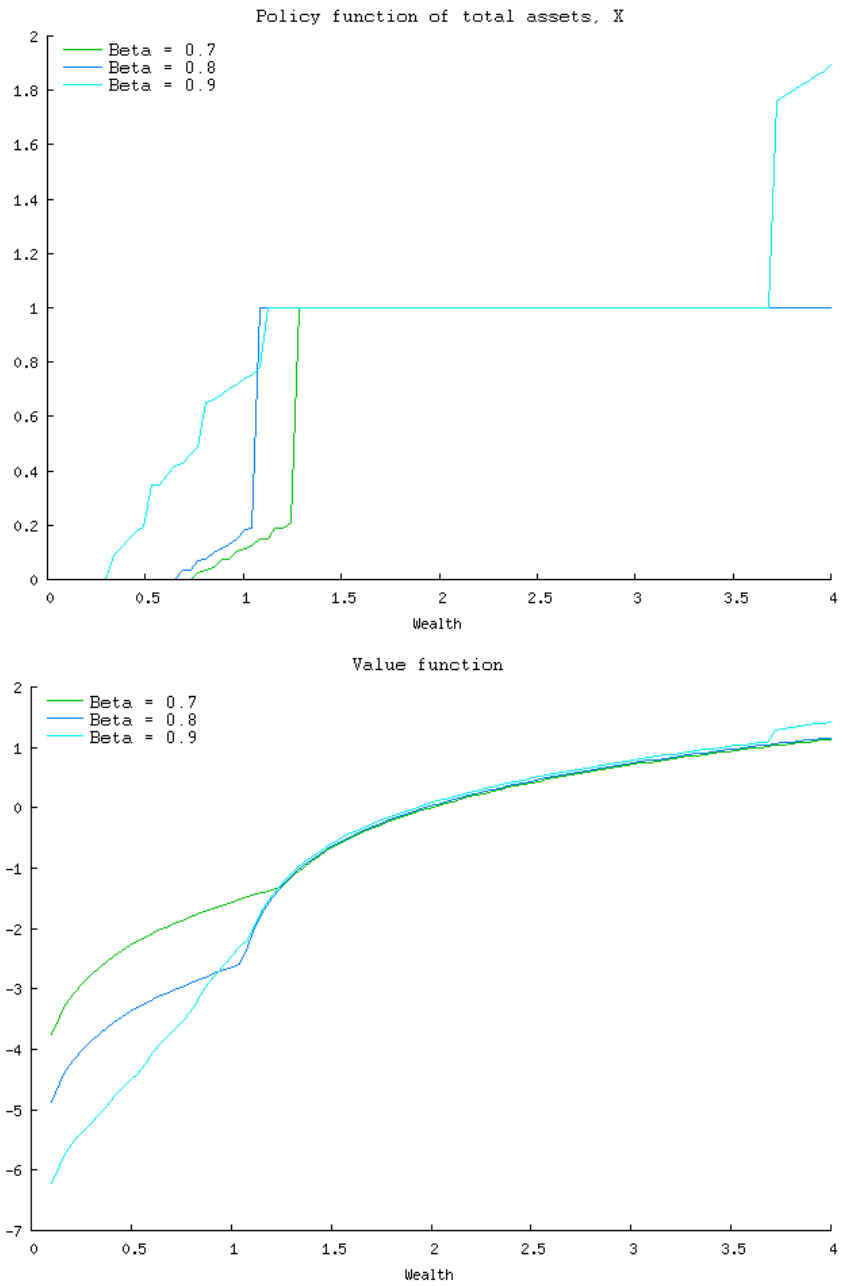


Figure 3: Policy and value function realizations for different values of  $\beta$  with  $\sigma = 0$

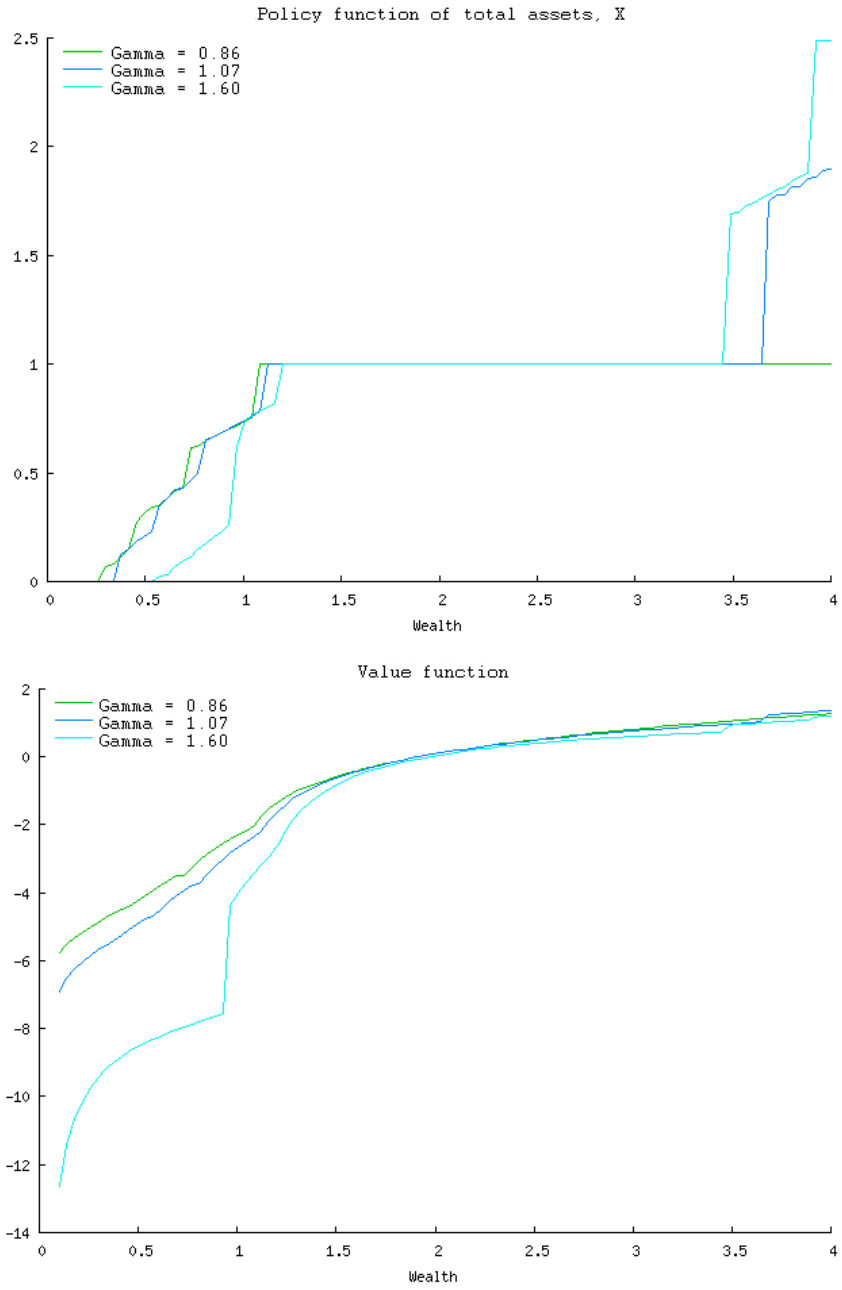


Figure 4: Policy and function realizations for different values of  $\gamma$  with  $\sigma = 0$

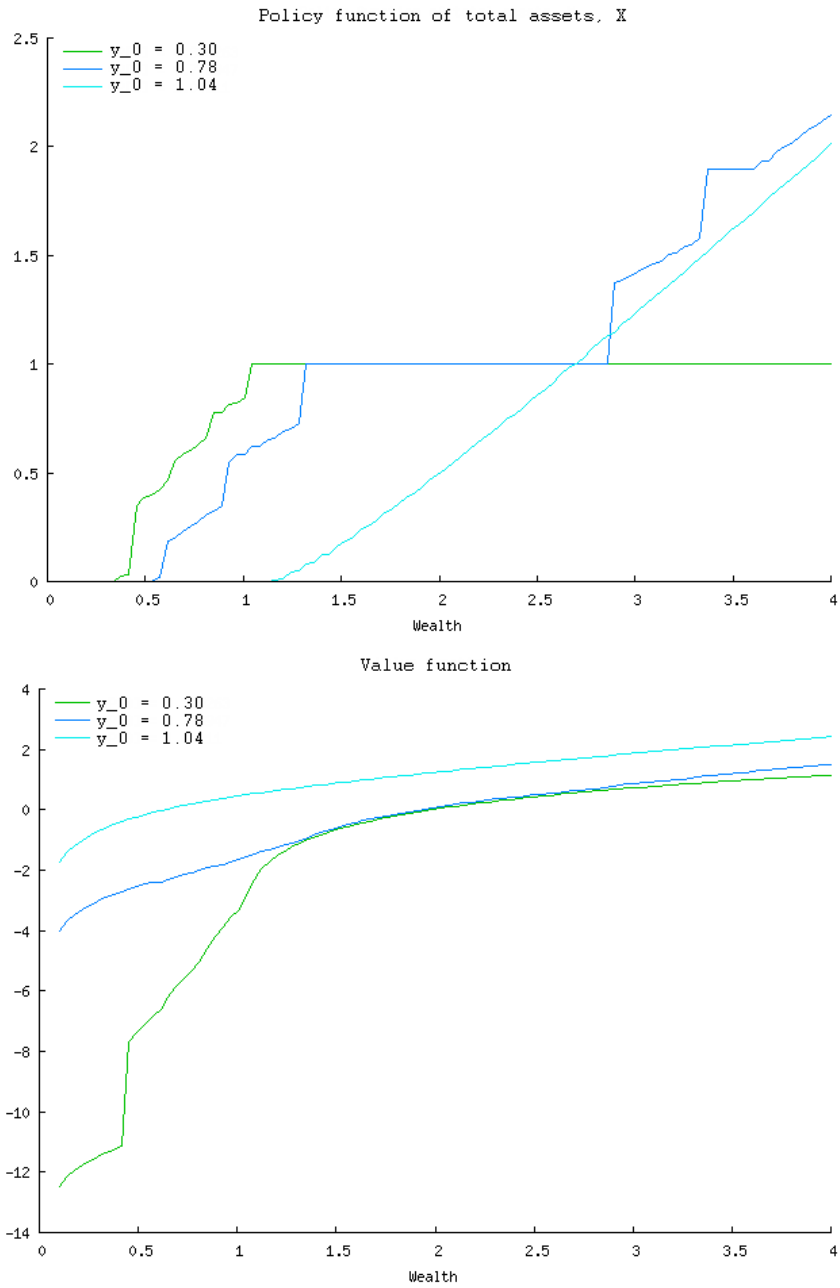


Figure 5: Policy and value function realizations for different values of  $y_0$  with  $\sigma = 0$

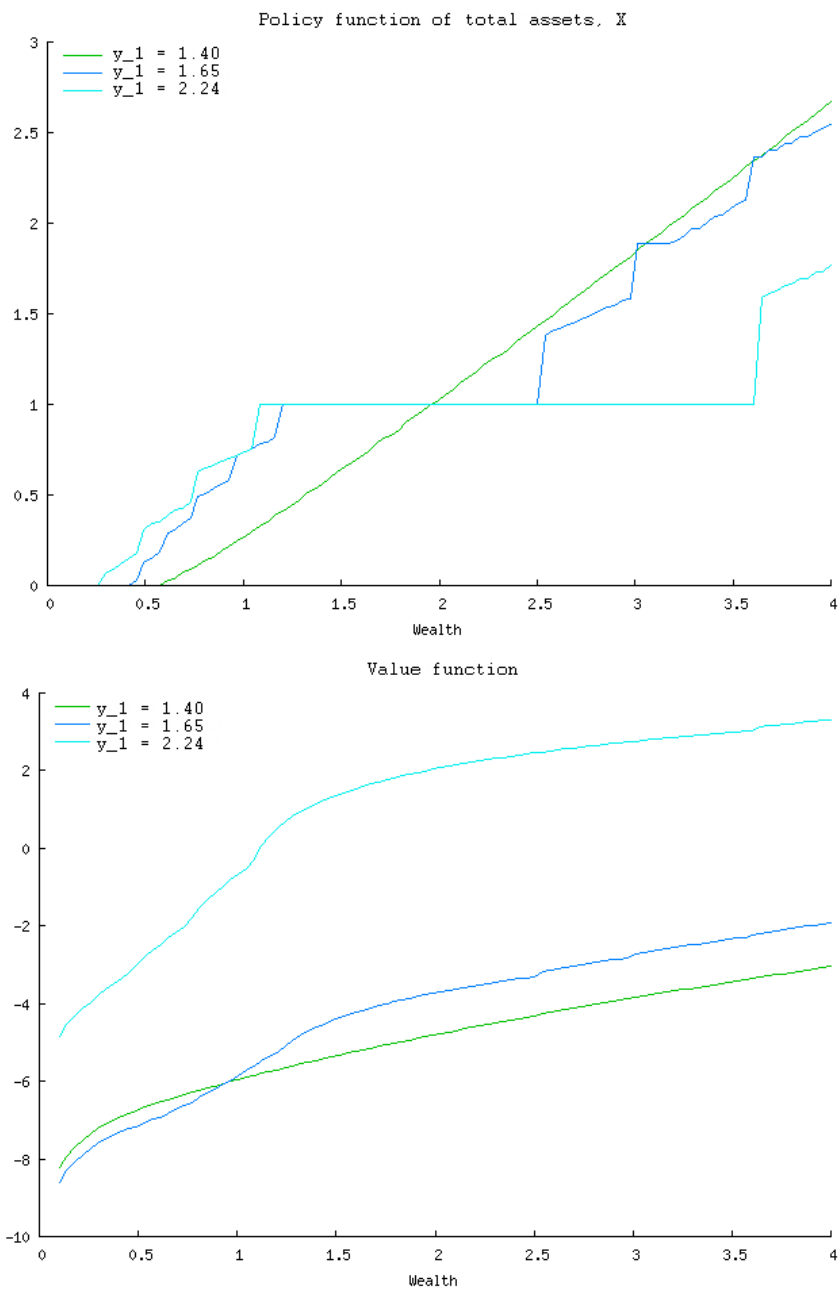


Figure 6: Policy and value function realizations for different values of  $y_1$  with  $\sigma = 0$

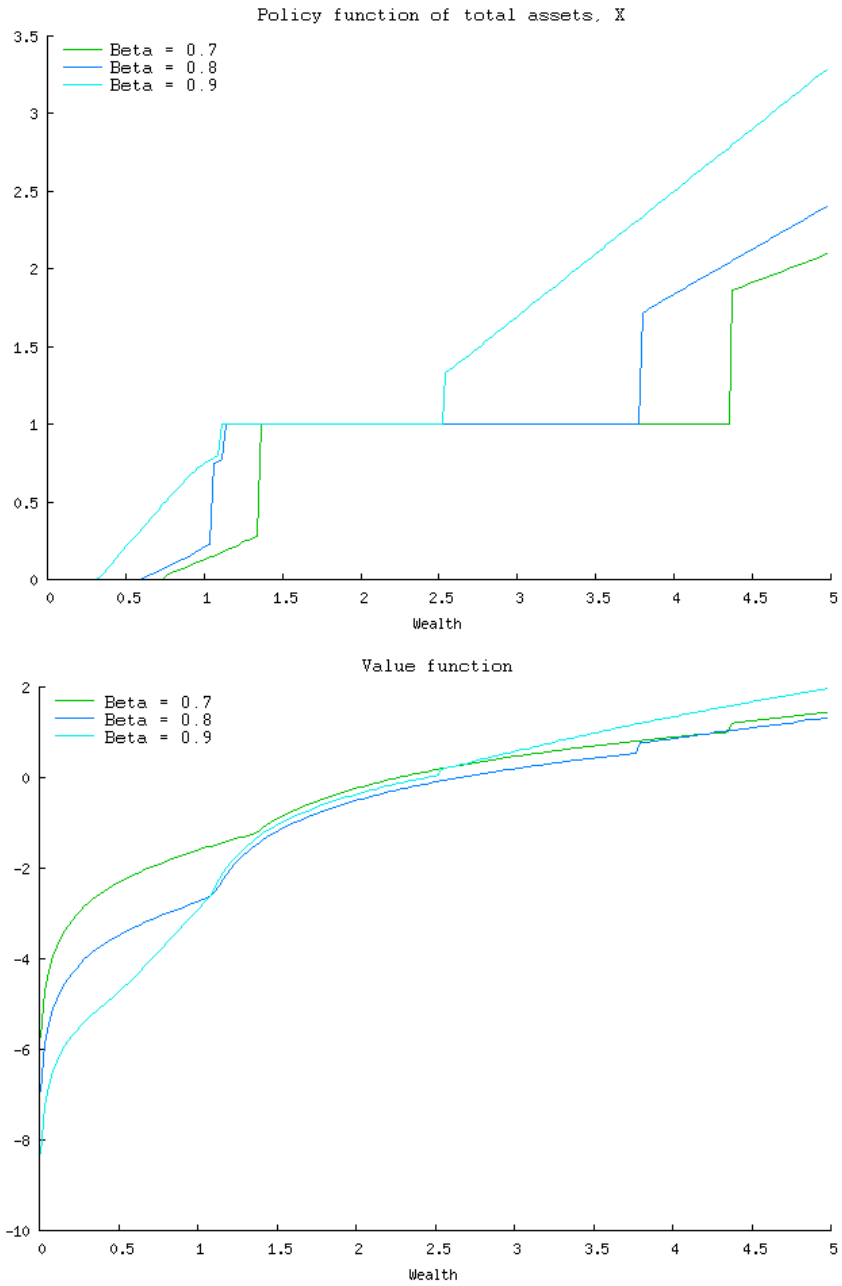


Figure 7: Policy and value function realizations for different values of  $\beta$  with  $\sigma = 0.25$

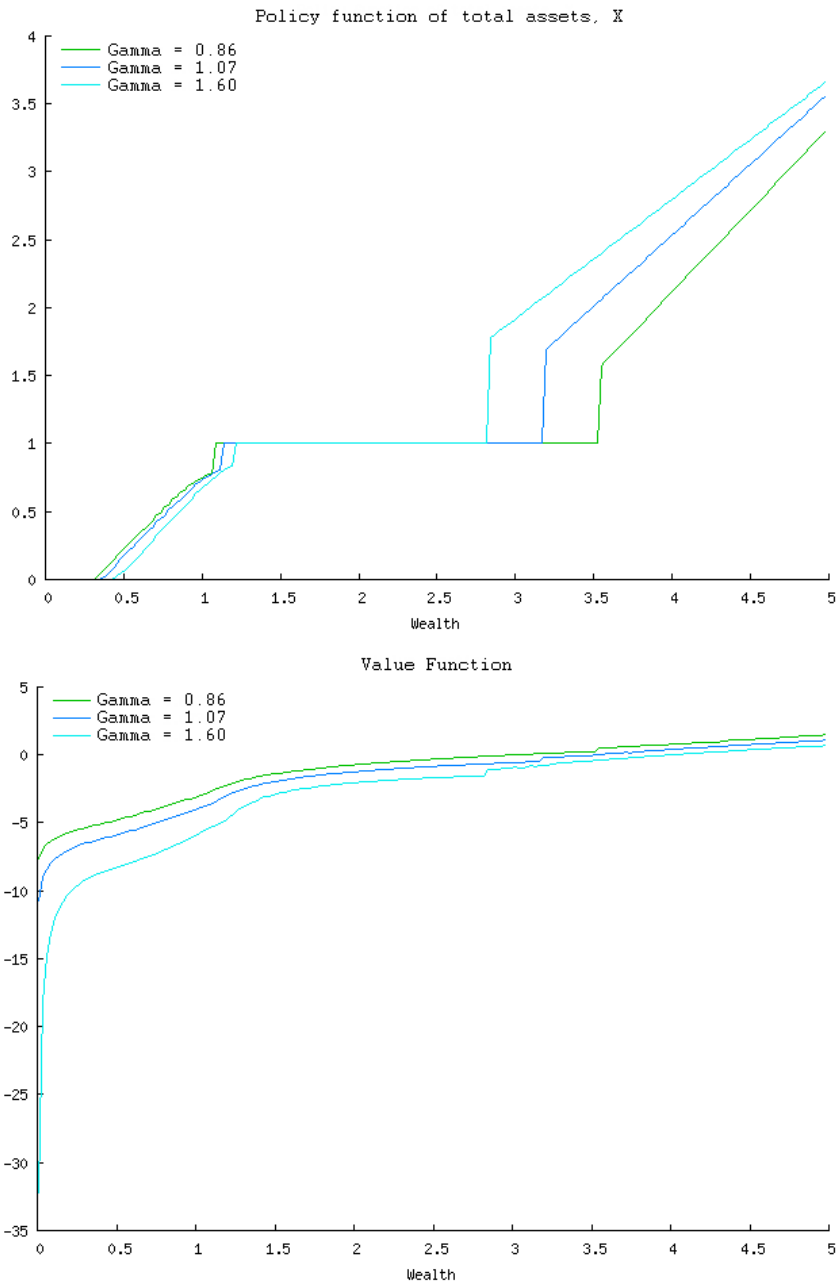


Figure 8: Policy and function realizations for different values of  $\gamma$  with  $\sigma = 0.25$

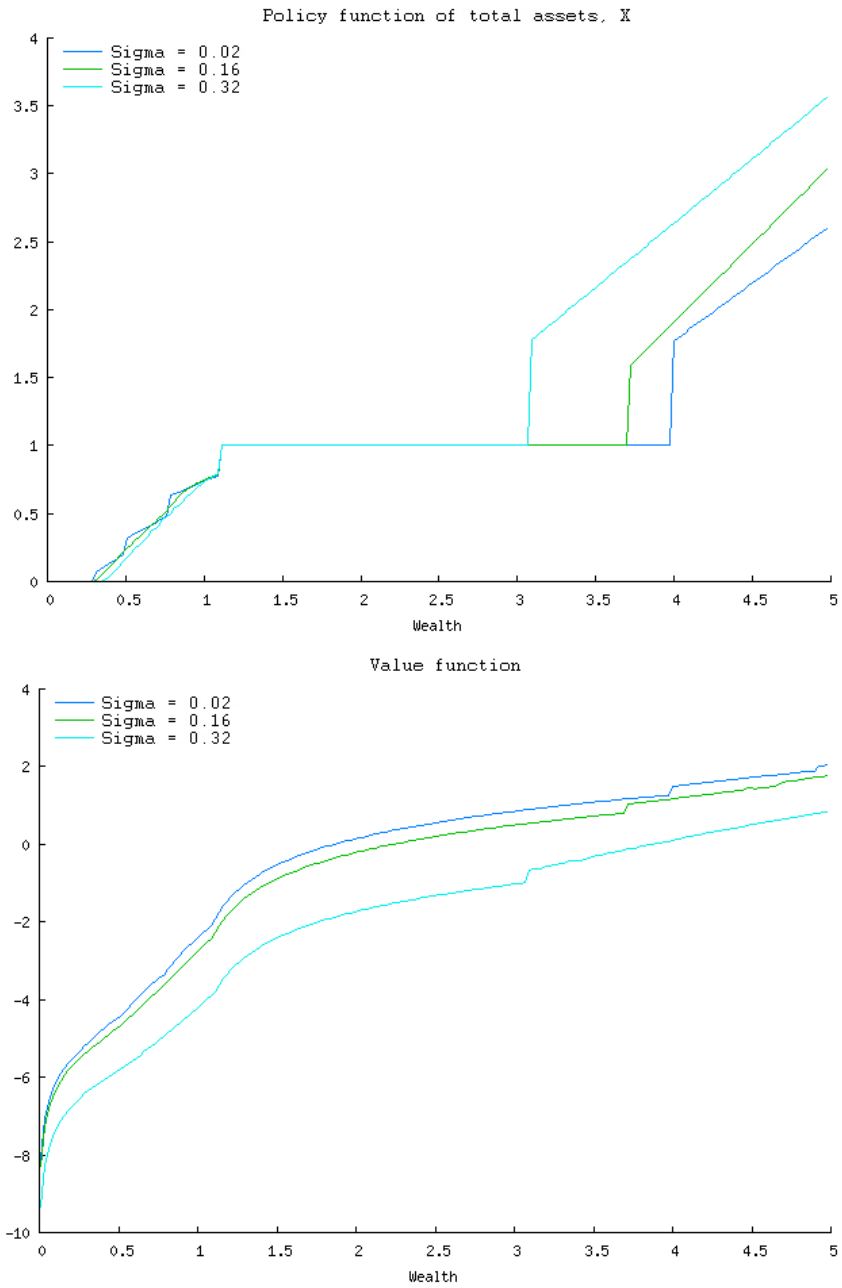


Figure 9: Policy and function realizations for different values of  $\sigma$

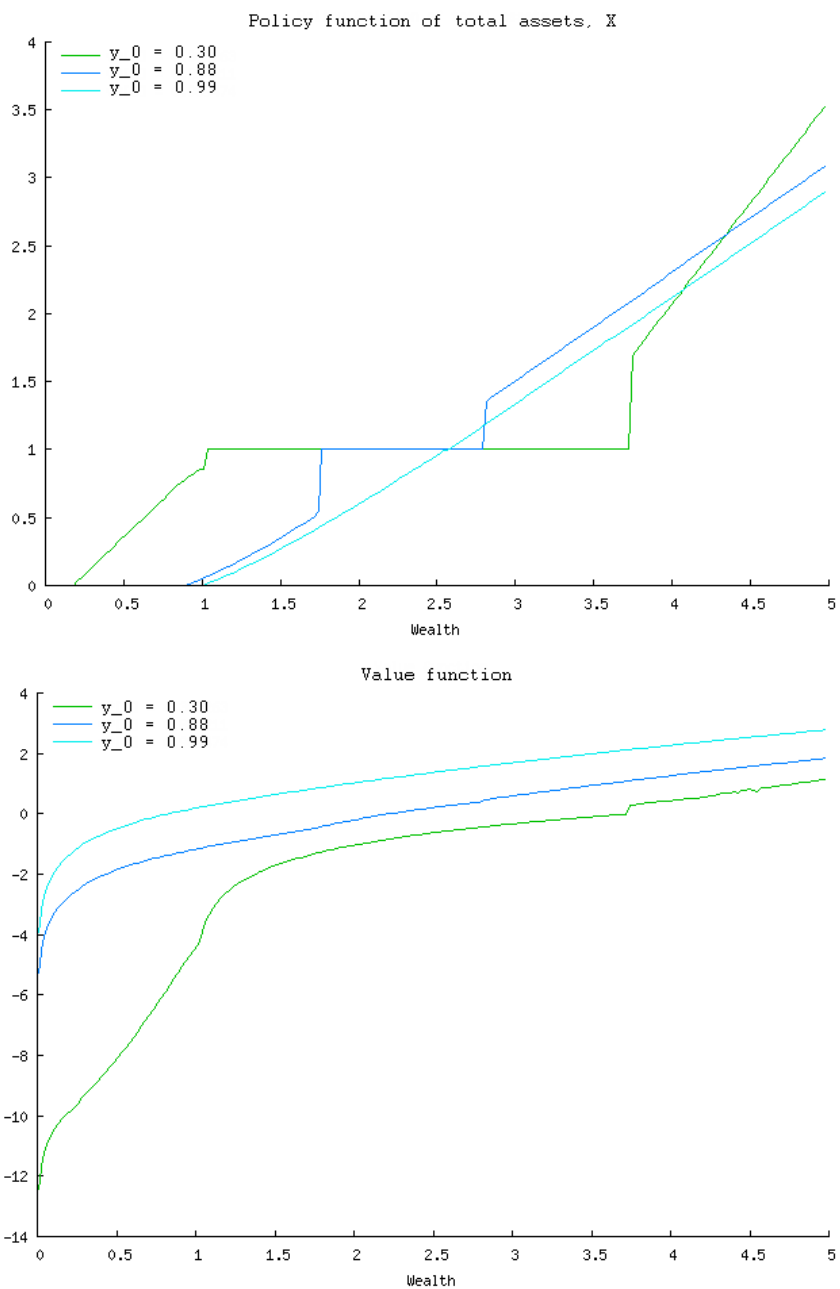


Figure 10: Policy and value function realizations for different values of  $y_0$  with  $\sigma = 0.25$



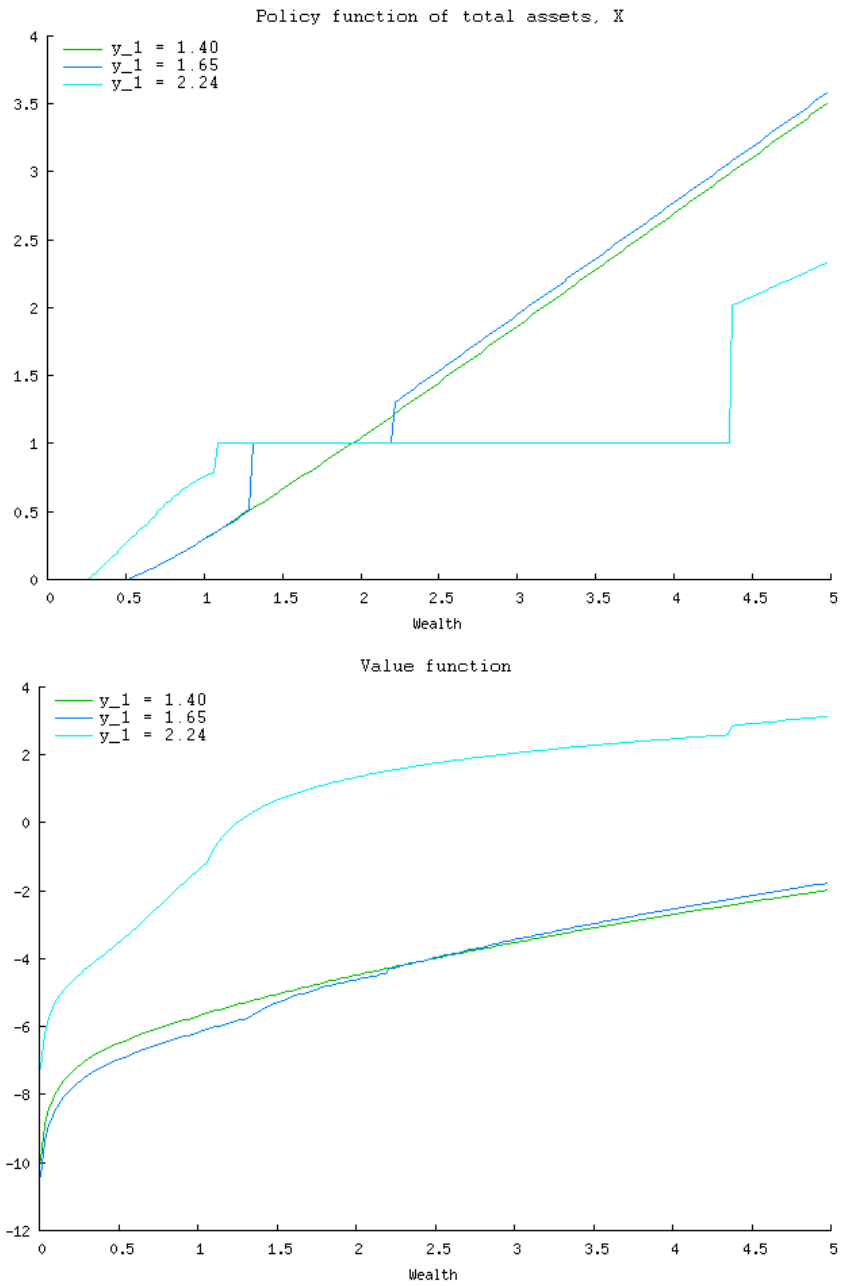


Figure 11: Policy and value function realizations for different values of  $y_1$  with  $\sigma = 0.25$

## 5.5 Estimation

For a sample of  $N$  observations, assuming that only the asset holdings for  $x$  and  $d$  are observed for two consecutive periods ( $\bar{x}$  and  $\bar{d}$  for the earlier period and  $x$  and  $d$  for the following period) but the structure of the income function is known, the log-likelihood function can be written as

$$\begin{aligned}\ell(\mathbf{x}, \mathbf{d}|\theta) &= \sum_{i=1}^N \log p_{\theta}(x = x_i, d = d_i | \bar{x}_i, \bar{d}_i) \\ &= \sum_{i=1}^N \sum_{d \in \{0,1\}} \log p_{\theta}(x = x_i | d_i, \bar{x}_i, \bar{d}_i) + \log P_{\theta}(d = d_i | \bar{x}_i, \bar{d}_i)\end{aligned}\quad (27)$$

where  $p_{\theta}(x = x_i, d = d_i | \cdot)$  is the joint probability of observing  $x = x_i$  and  $d = d_i$ , given previous observations  $\bar{x}_i$  and  $\bar{d}_i$  and parameters  $\theta = (\beta, \gamma, \sigma, y_0, y_1)$ , while  $p_{\theta}(x = x_i | \cdot)$  is the conditional probability of  $x = x_i$  given  $d_i$  and the past observations. Similarly,  $P_{\theta}(d = 0 | \cdot)$  denotes the probability that  $d_i = 0$  given the previous observations and parameter values. Note that  $d_i$  can only take values 0 and 1, therefore,  $P_{\theta}(d = 1 | \cdot) = 1 - P_{\theta}(d = 0 | \cdot)$ .

First, we derive  $P_{\theta}(d = 0 | \cdot)$ . We denote the policy function for the discrete asset by  $\psi_d(w)$ , while  $\underline{\psi}_d^{-1}(1)$  denotes the lowest wealth level for which  $d_i = 1$ . Risk in the model is represented by  $s$  with  $\log s \sim N(-\sigma^2/2, \sigma^2)$ .  $\Phi(\cdot)$  denotes the standard normal c.d.f.

$$P_{\theta}(d = 0 | \bar{x}_i, \bar{d}_i) = P_{\theta}(\psi_d(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0))) < 1) \quad (28)$$

$$= P_{\theta}(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0)) < \underline{\psi}_d^{-1}(1)) \quad (29)$$

$$= P_{\theta}\left(\log s < \log\left(\frac{\underline{\psi}_d^{-1}(1) - \bar{x}_i + \bar{d}_i}{y_0 + \bar{d}_i(y_1 - y_0)}\right)\right) \quad (30)$$

$$= \Phi\left(\frac{\sigma}{2} + \frac{1}{\sigma} \log(\underline{\psi}_d^{-1}(1) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0))\right) \quad (31)$$

Now, we turn to  $p_{\theta}(x = x_i | d_i)$ . We first derive  $P_{\theta}(x < x_i) = \int_{-\infty}^{x_i} p_{\theta}(x) dx$ . This is very similar to (28)-(31). We denote the policy function for the continuous asset by  $\psi_x(w)$ . Note that  $\phi(\cdot)$  denotes the standard normal p.d.f.

$$P_{\theta}(x < x_i | \bar{x}_i, \bar{d}_i) = P_{\theta}(\psi_x(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0))) < x_i) \quad (32)$$

$$= P_{\theta}(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0)) < \psi_x^{-1}(x_i)) \quad (33)$$

$$= P_{\theta}\left(\log s < \log\left(\frac{\psi_x^{-1}(x_i) - \bar{x}_i + \bar{d}_i}{y_0 + \bar{d}_i(y_1 - y_0)}\right)\right) \quad (34)$$

$$= \Phi\left(\frac{\sigma}{2} + \frac{1}{\sigma} \log(\psi_x^{-1}(x_i) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0))\right) \quad (35)$$

In  $P_{\theta}(x < x_i | d_i)$ , the conditioning on  $d_i$  restricts the range of  $x$ 's that have a positive probability. If  $d_i = 0$  it has to be that  $x \in [0, \psi_x(\underline{\psi}_d^{-1}(1))]$ , and  $P_{\theta}(x < x_i | d_i = 0) = 1$  for  $x_i \geq \psi_x(\underline{\psi}_d^{-1}(1))$ . Similarly,  $P_{\theta}(x < x_i | d_i = 1) > 0$  only for values  $x_i \geq \psi_x(\underline{\psi}_d^{-1}(1))$ . Let  $I(\cdot)$  denote the indicator function. Then,

$$P_\theta(x < x_i | d_i = 0) = I\left(0 \leq x_i < \psi_x(\underline{\psi}_d^{-1}(1))\right) P_\theta(x < x_i) + I\left(x_i \geq \psi_x(\underline{\psi}_d^{-1}(1))\right) \quad (36)$$

$$P_\theta(x < x_i | d_i = 1) = I\left(x_i \geq \psi_x(\underline{\psi}_d^{-1}(1))\right) P_\theta(x < x_i) \quad (37)$$

Then, the conditional density of  $x = x_i$  can be calculated for  $x_i \neq \{0, \psi_x(\underline{\psi}_d^{-1}(1))\}$  as

$$\begin{aligned} p_\theta(x = x_i | d_i) &= \frac{d}{dx} P_\theta(x < x_i | d_i) \\ &= I(\psi_d(\psi_x^{-1}(x_i)) = d_i) \frac{(\psi_x^{-1})'(x_i)}{\sigma(\psi_x^{-1}(x_i) - \bar{x}_i + \bar{d}_i)} \cdot \\ &\quad \phi\left(\frac{\sigma}{2} + \frac{1}{\sigma} \log(\psi_x^{-1}(x_i) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0))\right) \quad (38) \end{aligned}$$

The inverse of the policy function is not well-defined at  $x_i = 0$  and  $x_i = 1$ . For these observations we have to use the appropriate probabilities. Note that  $x_i = 0$  can only occur when  $d_i = 0$  and if  $x_i = 1$  then  $d_i = 1$ . Therefore,

$$P_\theta(x = 0 | \bar{x}_i, \bar{d}_i) = P_\theta(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0)) < \bar{\psi}_x^{-1}(0)) \quad (39)$$

$$= P_\theta(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0)) < \bar{\psi}_x^{-1}(0)) \quad (40)$$

$$= \Phi\left(\frac{\sigma}{2} + \frac{1}{\sigma} \log(\bar{\psi}_x^{-1}(0) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0))\right) \quad (41)$$

and

$$P_\theta(x = 1 | \bar{x}_i, \bar{d}_i) = P_\theta\left(\underline{\psi}_x^{-1}(1) \leq \bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0)) < \bar{\psi}_x^{-1}(1)\right) \quad (42)$$

$$\begin{aligned} &= P_\theta(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0)) < \bar{\psi}_x^{-1}(1)) \\ &\quad - P_\theta(\bar{x}_i - \bar{d}_i + s(y_0 + \bar{d}_i(y_1 - y_0)) < \underline{\psi}_x^{-1}(1)) \quad (43) \end{aligned}$$

$$\begin{aligned} &= \Phi\left(\frac{\sigma}{2} + \frac{1}{\sigma} \log(\bar{\psi}_x^{-1}(1) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0))\right) \\ &\quad - \Phi\left(\frac{\sigma}{2} + \frac{1}{\sigma} \log(\underline{\psi}_x^{-1}(1) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0))\right) \quad (44) \end{aligned}$$

where  $\underline{\psi}_x^{-1}(\chi)$  stands for the lowest wealth level for which  $x_i = \chi$  and  $\bar{\psi}_x^{-1}(\chi)$  for the highest wealth level for which  $x_i = \chi$ . Note that if  $\bar{\psi}_x^{-1}(0) < \bar{x}_i - \bar{d}_i$  then  $P_\theta(x_i = 0) = 0$  and if  $\underline{\psi}_x^{-1}(1) < \bar{x}_i - \bar{d}_i$  then the second term in (44) is 0.

Finally, we can write the log-likelihood function as

(45)

$$\begin{aligned}
\ell(\mathbf{x}, \mathbf{d}|\theta) &= \sum_{i=1}^N \log \left( d_i + (1 - 2d_i) \Phi \left( \frac{\sigma}{2} + \frac{1}{\sigma} \log(\underline{\psi}_d^{-1}(1) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0)) \right) \right) \\
&+ \sum_{i=1}^N I(x_i \neq \psi_x(\underline{\psi}_d^{-1}(d_i))) \left[ \log \left( (\psi_x^{-1})'(x) \right) - \log(\sigma(\psi_x^{-1}(x) - \bar{x}_i + \bar{d}_i)) \right. \\
&\quad \left. + \log \phi \left( \frac{\sigma}{2} + \frac{1}{\sigma} \log(\psi_x^{-1}(x_i) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0)) \right) \right] \\
&+ \sum_{i=1}^N I(x_i = 0) \log \Phi \left( \frac{\sigma}{2} + \frac{1}{\sigma} \log(\bar{\psi}_x^{-1}(0) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0)) \right) \\
&+ \sum_{i=1}^N I(x_i = 1) \cdot \\
&\quad \log \left[ \Phi \left( \frac{\sigma}{2} + \frac{1}{\sigma} \log(\bar{\psi}_x^{-1}(1) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0)) \right) \right. \\
&\quad \left. - \Phi \left( \frac{\sigma}{2} + \frac{1}{\sigma} \log(\underline{\psi}_x^{-1}(1) - \bar{x}_i + \bar{d}_i) - \frac{1}{\sigma} \log(y_0 + \bar{d}_i(y_1 - y_0)) \right) \right]
\end{aligned}$$

where  $I(\cdot)$  is the indicator function and  $\phi(\cdot)$  and  $\Phi(\cdot)$  denote the p.d.f. and c.d.f. of the standard normal distribution, respectively.

We use Simulated Annealing<sup>14</sup> in the estimation of the model because it does not require differentiability of the objective function and it is a global optimization method. For the estimation it is important that the policy function changes monotonically for changes in the parameter values, otherwise the likelihood function will not be smooth. Using the tools outlined in section 5.3 and Appendix B, our value function iteration algorithm satisfies this monotonicity condition.

The estimation procedure is rather time consuming and we are still experimenting with this Simulated Maximum Likelihood estimator, therefore we decided not to report our results at this stage.

## 6 Conclusion

In this paper we have defined a general class of stochastic dynamic choice models with discrete and continuous decision variables. We argue that this class contains a variety of models that are useful in analyzing intertemporal consumption and wealth accumulation decisions of households under risk. An essential feature of the models discussed is that all the necessary information for making the decisions on consumption and asset holdings is summarized in one variable, the wealth at hand of the agent, given the distribution of the stochastic variables. For these models, we propose a solution method using value function iteration.

This solution method is primarily developed for models that incorporate both discrete and continuous decision variables. Hence, it can be fruitfully applied in settings where agents face discontinuities (i.e. shifts) in their income function with respect to the investment asset(s). We are particularly interested in settings where credit constrained rural

<sup>14</sup>For details see Goffe *et al.* (1994).

households make savings decisions in continuous assets (small livestock) and investment decisions in discrete assets (cattle) under risk as in Vigh (2008). Therefore, we take a simplified version of Vigh (2008) to demonstrate the working of the algorithm. The example model can be solved analytically in the deterministic case, which allows us to study the accuracy of our solution method. Finally, we show for the example model how a Simulated Maximum Likelihood estimator of the model parameters can be based on the solution method.

## References

- [1] Dercon, S. (1998): Wealth, risk and activity choice: cattle in Western Tanzania. *Journal of Development Economics* 55(1), pp. 1-42.
- [2] Elbers, C.; Gunning, J.W. and Kinsey B. (2007): Growth and Risk: Methodology and Micro Evidence. *The World Bank Economic Review* 21(1), pp. 1-20.
- [3] Elbers, C.; Gunning, J.W. and Pan, L. (2008): Insurance and Rural Welfare: What Can Panel Data Tell Us? *Applied Economics*, January 2008.
- [4] Fafchamps, M. and Pender, J. (1997): Precautionary Saving, Credit Constraints, and Irreversible Investment: Theory and Evidence From Semiarid India. *Journal of Business and Economic Statistics* 15, pp. 180-194.
- [5] Goffe, W. L.; Ferrier, G. D. and Rogers, J. (1994): Global optimization of statistical functions with simulated annealing. *Journal of Econometrics* 60(1-2), pp. 65-99.
- [6] Judd, K. (1998): Numerical Methods in Economics. The MIT Press. Cambridge, Massachusetts.
- [7] Pakes, A. (1991): Dynamic structural models: problems and prospects. Mixed continuous discrete controls and market interactions. *Cowles Foundation Discussion Paper No. 984*.
- [8] Pan, L. (2008): Growth and Risk in Rural Ethiopia. *PhD Thesis*, Tinbergen Institute, Amsterdam.
- [9] Press, W.H.; Teukolsky S.A.; Vetterling W.T.; Flannery B.P. (1992): Numerical Recipes in C: The art of scientific Computing. Second Edition, Cambridge University Press, World Wide Web sample. [www.nr.com](http://www.nr.com)
- [10] Rosenzweig, M.R. and Wolpin, K.I. (1993): Credit Market Constraints, Consumption Smoothing, and Accumulation of Durable Production Assets in Low-Income Countries: Investments in Bullocks in India. *Journal of Political Economy* 101(21), pp. 223-244.
- [11] Vigh, M. (2008): Risk coping in rural Ethiopia: the liquidity and productivity of livestock. *MPhil Thesis*, Tinbergen Institute, Amsterdam.
- [12] Zimmerman, F.J. and Carter, M.R. (2003): Asset smoothing, consumption smoothing and the reproduction of inequality under risk and subsistence constraints. *Journal of Development Economics* 71, pp. 233-260.

## Appendix A. Analytical solution of the deterministic model

This section describes the analytical solution of the model with Bellman equation (20)-(22), which is presented again for convenience

$$V(w) = \max_x u(w - x) + \beta V(y_0 + I_{x>=1}(y_1 - y_0 - 1) + x) \quad (46)$$

*s.t.*

$$w \geq x \quad (47)$$

$$x \geq 0 \quad (48)$$

In order to solve the problem, first we have to find the wealth level,  $\bar{w}_1$ , at which the agent will decide to invest 1 in  $d$ . At this wealth level the agent will be indifferent between investing 1 in  $d$  today ( $V^0(\bar{w}_1)$ ) and making the same investment tomorrow ( $V^1(\bar{w}_1)$ ). Hence, we can write that

$$V^0(\bar{w}_1) \equiv u(\bar{w}_1 - 1) + \beta V(y_1) = \max_x u(\bar{w}_1 - x) + \beta u(y_0 + x - 1) + \beta^2 V(y_1) \equiv V^1(\bar{w}_1) \quad (49)$$

First we need to find  $x$  as a function of  $\bar{w}_1$  to be able to solve for  $\bar{w}_1$ . The FOC for the RHS of equation (49) yield the Euler equation

$$u'(\bar{w}_1 - x) = \beta u'(y_0 + x - 1). \quad (50)$$

Now, using the Euler equation we can solve for  $x$  and get

$$x = \frac{\beta^{1/\gamma}}{1 + \beta^{1/\gamma}} \bar{w}_1 + \frac{1}{1 + \beta^{1/\gamma}} (1 - y_0) \quad (51)$$

Substituting for  $x$  we are now able to solve for  $\bar{w}_1$  in (49). The equation has two solutions as can be seen on Figure 12. We are interested in the lower solution that gives us  $\bar{w}_1 = 1.080$ , which is the wealth level below which the agent waits one more period to reach  $d = 1$  and saves  $x = 0.720$  today, and above which the agent invests in  $d = 1$  today. Hence, we observe that there is a discontinuity in the policy function for  $x$  at  $\bar{w}_1$ .

Next we want to find the switch point,  $\bar{w}_2$ , where the agent is indifferent between investing in  $d = 1$  tomorrow and in two days. Hence, we want to solve

$$\begin{aligned} V^1(\bar{w}_2) &\equiv \max_x u(\bar{w}_2 - x) + \beta u(y_0 + x - 1) + \beta^2 V(y_1) \quad (52) \\ &= \max_{x_0, x_1} u(\bar{w}_2 - x_0) + \beta u(y_0 + x_0 - x_1) + \beta^2 u(y_0 + x_1 - 1) + \beta^3 V(y_1) \equiv V^2(\bar{w}_2) \end{aligned}$$

Again, we need to solve for the  $x$ 's first through the FOC's, which yield

$$x = \frac{\beta^{1/\gamma}}{1 + \beta^{1/\gamma}} \bar{w}_2 + \frac{1}{1 + \beta^{1/\gamma}} (1 - y_0) \quad (53)$$

$$x_0 = \frac{\beta^{1/\gamma}}{1 + \beta^{1/\gamma}} \bar{w}_2 + \frac{1}{1 + \beta^{1/\gamma}} (x_1 - y_0) \quad (54)$$

$$x_1 = \frac{\beta^{1/\gamma}}{1 + \beta^{1/\gamma}} (x_0 + y_0) + \frac{1}{1 + \beta^{1/\gamma}} (1 - y_0) \quad (55)$$

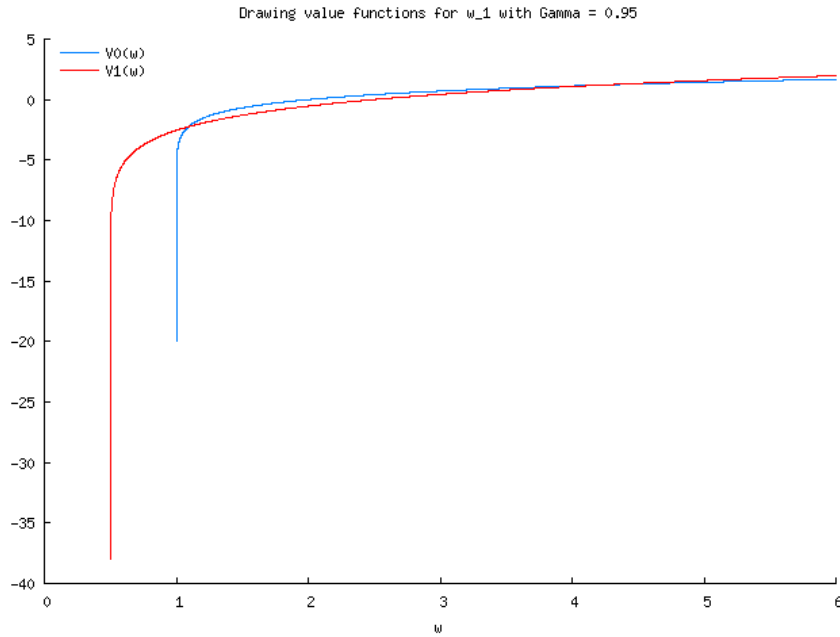


Figure 12: Solving for  $\bar{w}_1$ : value functions  $V^0(\bar{w}_1)$  and  $V^1(\bar{w}_1)$

Equation (52) has two solutions again, from which the lower one is of interest to us:  $\bar{w}_2 = 0.677$ . At this wealth level the savings of the agent who invests in  $d$  tomorrow is  $x = 0.530$ , and the savings of the agent who invests in  $d$  in 2 days is  $x_0 = 0.350$  today and  $x_1 = 0.659$  tomorrow. Between  $\bar{w}_1$  and  $\bar{w}_2$  agents reach  $d = 1$  in one period, and their savings decision follows the FOC of (51).

The next step is to find the switch point,  $\bar{w}_3$ , where the agent is indifferent between investing in  $d = 1$  in two days and in three days. However, we leave it to the reader to derive the remaining switch points.

## Appendix B. Notes on programming

This section contains our comments on the implementation of the solution algorithm for the model of section 5. The algorithm is programmed in Ox.<sup>15</sup>

We set  $N = 300$  with the values of  $w$  evenly spaced on  $[0.2, 2.2]$ . The iteration procedure terminates if the largest relative change in the value function and the policy functions become very small. Attention should be paid to the initial values of the policy and value functions and the starting values of the maximization algorithm. Choosing good starting values is important in achieving convergence. When setting up the program, it is a good idea to plot the function approximations after each iteration. For wealth level  $w_i$  we use  $\psi_x^0(w_i) = \min\{0.7w_i, 1\}$ ,  $\psi_d^0(w_i) = 1$  if  $\psi_x^0(w_i) \geq 1$  and 0 otherwise.  $V^0(w_i) = u(\psi_x^0(w_i) + y_0 + \psi_d^0(w_i)(y_1 - y_0 - 1))/(1 - \beta)$  for the initial function values. Before applying the initial-value-search routine, we observed that when initial values of  $x$  contain zeros, those grids will not move away from zero anymore. This can occur as a result of the flatness of the policy function at the switch points and the log transformation in the optimization, which make the output of the optimization algorithm sensitive to starting values. Therefore, to be on the safe side, it is a good idea to assume some savings for every wealth level in the initial values.

In the optimization problem of Step 2.1 we apply the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method<sup>16</sup> on the log-transformed variable  $\tilde{x} = \log(x - d)$ , such that the unconstrained maximization algorithm returns  $x > d$ . The starting value in the maximization is chosen as  $\tilde{x}_k^r = \log(\max\{\psi_x^{r-1}(w_i) - \bar{d}_k, \varepsilon\})$  with  $\varepsilon$  a small positive increment. This value is, however, modified in the neighborhood of the discontinuities in the policy function according to the initial-value-search routine.

The *initial-value-search algorithm* does the following: if  $|\psi_x^{r-1}(w_{i+1}) - \psi_x^{r-1}(w_i)| > w_{i+1} - w_i$  or  $|\psi_x^{r-1}(w_i) - \psi_x^{r-1}(w_{i-1})| > w_i - w_{i-1}$ , then in iteration  $r$  at  $w_i$  before executing the maximization routine, we evaluate value function  $V(\tilde{x}_b|w_i, d_i)$  at  $B$  equidistant values of  $\tilde{x}_b$  on range  $\exp(\tilde{x}_b) \in (\min\{\exp(\tilde{x}) - 0.05, \varepsilon\}, \min\{\psi_x^{r-1}(w_{i+1}), w_i - \varepsilon\})$ . From this we use  $\tilde{x}_b$  with maximal value of  $V(\tilde{x}_b|w_i, d_i)$  as the starting value in the maximization procedure. The larger we choose  $B$ , the closer the initial value of  $\tilde{x}$  is going to be to the optimum.

The initial-value-search routine is an important part of the solution algorithm because at the switch points the value function has the same value for two different saving strategies (invest in the advanced technology after  $k$  or  $k + 1$  periods) that contain different optimal savings in  $x$  today. Figure 12 in Appendix A illustrates the value function for different values of wealth level assuming optimal decisions for  $x$  and  $d$ . From a different angle, Figure 13 shows the shape of  $V(\tilde{x}_b|w_i, d_i)$  close to a switch point. This plot also highlights the importance of good starting values. Starting at around  $x = 0.5$  the BFGS algorithm does not find a value of  $x$  that yield higher optimum, however thorough evaluation of the value function using the initial-value-search routine shows that the value function takes its optimum close to  $x = 0.63$  instead of 0.5. Hence, without the initial-value-search routine we were not able to locate the discontinuity points in the policy function accurately.

We observe that many times the maximization algorithm reports weak or no convergence, however the value function iteration algorithm converges nonetheless. The problematic areas are (a) close to the discontinuities in the policy function, where the slope is

<sup>15</sup>The Ox code is available on request from Melinda Vigh (mvigh@feweb.vu.nl).

<sup>16</sup>See chapter 5 of Judd (1998) for more information on the BFGS method.



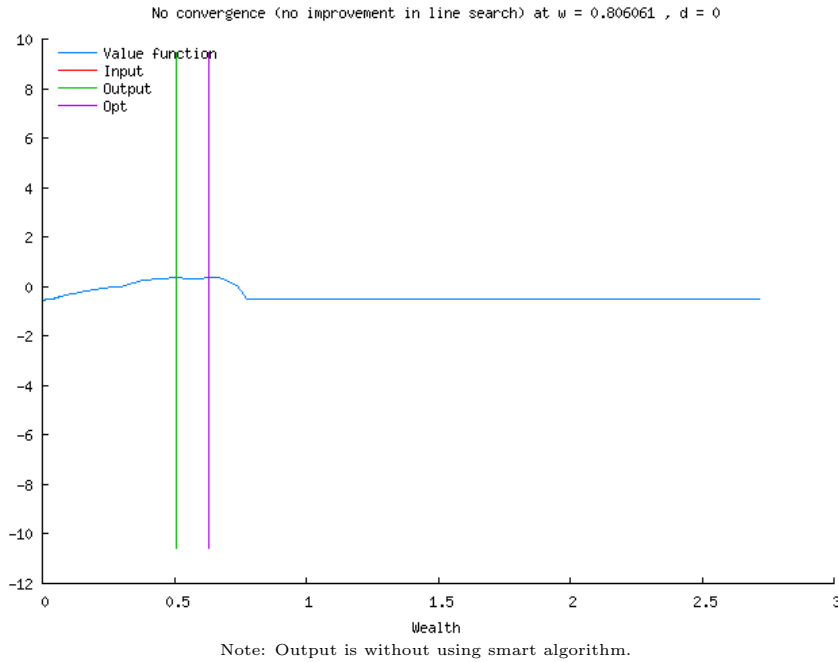


Figure 13: Optimization routine at a switch point

very flat (see Figure 13); (b) the areas where  $x - d = 0$ , because there the optimum of the log transformed problem is  $\log(x - d) = -\infty$ ; and (c) sometimes it also occurs for the optimum with  $d = 0$  when the optimal chose is  $d = 1$ . Applying the initial-value-search routine is able to reduce type (a) non-convergence messages.

In order to get a precise approximation of the critical points of the policy function (i.e. where the discontinuities occur), we change the grid of wealth levels after the policy function is close to convergence. More grid-points are added around the wealth levels where the slope of the policy function is changing, and less nodes are used at the intervals where the policy function is (close to) linear. Additional grid points are also added around the location where  $x$  becomes positive and where it becomes 1.

With a convergence criterion of 0.001 for the largest relative change in function values compared to the previous iteration, the policy functions converges in 10 iterations, while it takes 40 iterations for the value function to achieve convergence with the same tolerance level. In some cases it might occur that the algorithm diverges for a specific starting value. To avoid the breakdown of the program, we restart the value function iteration with new random starting values if the convergence condition becomes too large. This is useful when estimating the model parameters.