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Tinbergen Institute Discussion Paper

Retirement with Perfect Insurance

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Tinbergen Institute

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Retirement with perfect insurance

Grzegorz Kula*

November 2003

Abstract

This paper focuses on the relation between worker's productivity and retirement decision. Assuming that productivity follows geometric Brownian motion with drift, there exists such a level of productivity for which it is optimal to retire. The worker buys an insurance, which gives a constant income and retirement benefits in exchange for the total output. The level of income and benefits is set to maximize lifetime utility. In such framework we find the retirement threshold of productivity and the probability of retirement.

1 Introduction

In the second half of the last century the labor force participation of elderly decreased rapidly and substantially around the world. The prime suspect responsible for this process was the social security system, with increased benefits, guaranteed income, implicit earnings tax and the possibility of early retirement. Many papers¹ tried to show what elements of existing retirement systems have the strongest influence on the decision to retire and its timing, and what would be the results of a reform of this systems. The model described in this paper approaches the problem of retirement from different perspective than that of expected wages and rules of social security system. We ask two questions: first, can the drop in the participation rate of elderly be explained by the introduction of better, more efficient insurance schemes? With the increased insurance coverage and the huge variety of insurance schemes it seems quite possible that workers, or their employers choose a scheme that gives higher individual utilities and offers more flexibility than older retirement systems (or, from the historical point of view, the lack of retirement system). In this paper we consider the perfect insurance, i.e. an insurance, which gives to an individual a constant income while working and benefits while retired in exchange for the total lifetime output. The level of income and benefits is chosen to maximize the lifetime utility.

The lifetime insurance scheme proposed above may seem unrealistic. The two most common questions it raises are: why insurer bothers to maximize worker's utility and why the worker is allowed to retire, if, even with low productivity now, there is some possibility that it will increase in the future. Both

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¹For the recent literature review see Lumsdaine and Mitchell (1999).

these questions are answered when we assume that the insurer operates on perfectly competitive market. If insurance firms are forced to compete for clients, they are forced to offer an insurance policy that maximizes worker's utility subject to the insurance company budget constraint. The worker is allowed to retire when her/his expected productivity is lower than her/his cost of continued work. The potential gain for the insurer from keeping her/him working is small in such a case and possibility of retirement attracts clients. The same argument refers to the utility maximization - worker buys the insurance from the firm that offers the best conditions, and therefore insurance firms try to maximize utilities of their clients. The insurer has high potential gains anyway, since the insurance scheme is constructed in such a way that most productive individuals will stay working till the end of their lives.

One may ask why such an insurance scheme is not used in practice, if it has so many advantages? The reason for this is that this scheme can work only in the absence of principal-agent problem. The insurance does not include any control mechanism, which would give the insurer guarantee that the insured does not shirk. On the other hand the motivation for shirking is very big: if the worker lowers her/his output on purpose, the insurer sees only a fall in productivity and may allow the worker to retire. In the model we assume that principal-agent problem does not exist. This insurance scheme is obviously not used. It could be argued that it has some features of social security, like certain benefits not connected with the earlier contributions. However social security is more like the saving scheme since both are last income related.

The second question of this paper is: what is the relation between worker productivity and retirement decision? In many papers earnings are treated as certain, or having a white noise component (like in Gustman and Steinmeier (1986) or Rust and Phelan (1997)). Worker's productivity is usually assumed to rise with age or experience, at least for a part of labor career. However we cannot neglect neither an occurrence of sudden falls in the productivity nor its possible decline for older agents. We cannot also neglect the presence of transitory shocks in earnings. Many of these elements may be included into the model with assumption of stochastic productivity. If productivity is stochastic there will be such a level of productivity for which there would be no point in working any longer. Assuming the perfect insurance case in which worker's lifetime income is equal to the expected value of her/his output from work, we will calculate the threshold value of productivity, i.e. the work-quitting level of productivity.

We use a dynamic programming model. Such models are used quite often to study retirement behavior, since they provide "... a framework that is rich enough to accurately model the dynamic structure of Social Security rules and the uncertainties and sequential nature of individuals' decision making processes..." (Rust and Phelan (1997), p. 791). In our case it allows us to combine retirement decision with stochastic productivity. The individuals' decision in dynamic programming studies is usually to work or to retire. There are also more developed versions, like decision to work full-time, part-time or retire, and how much to consume in Rust (1989). The basis for a decision is the comparison of the expected discounted values of the utility over the remaining lifetime, resulting from the possible decisions. This is usually done through a value function, which is the solution of Bellman equation combining current and expected utilities. According to Lumsdaine et al. (1992), since the dynamic programming evaluates the maximum of future

disturbance terms, the implementation of the model depends on the assumed error structure. In this paper we do not include an error term with the assumed distribution. Instead, as was mentioned above, we treat worker's productivity as stochastic variable and assume that it follows geometric Brownian motion with drift.

This paper is organized as follows: Section 2 studies individual's utility and income when no retirement is allowed. Section 3 analyzes retirement in the infinite lifetime setup. In this case the model gives a critical, work-quitting threshold value of productivity as a function of starting productivity. In Section 4 we allow for an increasing probability of death, i.e. lifetime can still be infinite but the probability of death is rising with time. With such a possibility included the retirement threshold is a function of both the starting probability and time. Section 5 concludes.

2 No retirement

2.1 Infinite lifetime

Before turning to the problems of the retirement decision we study utility and income of an individual living for the infinite time without a possibility of retirement. Her/his productivity, defined as the value of output, is given by the geometric Brownian motion with drift²:

$$dP = \bar{\mu}Pdt + \sigma Pdz \quad (1)$$

where $\bar{\mu} \equiv \mu + \frac{\sigma^2}{2}$ is a drift parameter, σ - a variance parameter, and dz is the increment of a Wiener process, with $E(dz) = 0$ and $V[dz] = E[(dz)^2] = dt$. The discount rate ρ is exogenous, and $\bar{\mu} < \rho$: otherwise, with the expected rise in productivity bigger than the discount rate, expected future values of productivity would always be higher than its current value, rising even to infinity. The total lifetime income of an individual is equal to the expected discounted value of lifetime productivity calculated at the beginning of the labor career, i.e. at $t = 0$. The level of income is chosen to maximize worker's lifetime utility and it is spread equally over the entire lifetime. Thus it is equivalent to the case in which worker buys an insurance giving constant benefits in exchange for her/his life-time output.

Assume that an individual lives from $t = 0$ to infinity. The lifetime utility can be written as:

$$U(Y) = \int_0^\infty \frac{1}{1-\alpha} Y^{1-\alpha} e^{-\rho t} dt \Leftrightarrow U(Y) = \frac{1}{1-\alpha} \frac{Y^{1-\alpha}}{\rho} \quad (2)$$

and for $\alpha = 1$ it is $U(Y) = \frac{1}{\rho} \ln Y$, where α is a coefficient of relative risk aversion and Y is income/insurance benefit. The budget constraint is:

$$E \int_0^\infty P_t e^{-\rho t} dt = \int_0^\infty Y e^{-\rho t} dt \quad (3)$$

where the left hand side represents the expected total lifetime productivity and the right hand side the insurance wealth, i.e. lifetime income. Since the insurance wealth is equal to the expected discounted

²The productivity is defined as the value of output. For the empirical justification of the assumption that the productivity follows Brownian motion with drift see for example Topel and Ward (1992), where the evolution of earnings is shown to be approximately a random walk with drift.

value of lifetime productivity calculated at $t = 0$, it must depend only on the starting productivity P_0 and the stochastic process defining productivity. We cannot directly calculate the value of $\int_0^\infty P_t e^{-\rho t} dt$, we can however find its expected value. The expected value of productivity at any period t is equal to $E(P_t) = P_0 e^{\bar{\mu}t}$. Discounting this value with ρ yields:

$$\int_0^\infty P_0 e^{\bar{\mu}t} e^{-\rho t} dt = \frac{P_0}{\rho - \bar{\mu}} \quad (4)$$

Without loss of generality we normalize the starting productivity P_0 to one.

Since the income level over the lifetime is constant we can rewrite both the lifetime utility function and the budget constraint:

$$\rho U(Y) = \frac{1}{1 - \alpha} Y^{1 - \alpha} \quad (5)$$

$$Y - \frac{\rho}{\rho - \bar{\mu}} = 0 \quad (6)$$

Each individual wants to maximize her/his utility, subject to the budget constraint. Thus we must maximize (5) subject to (6). Therefore we can write the Lagrangian:

$$\rho U(Y, \lambda) = Y \left(\frac{Y^{-\alpha}}{1 - \alpha} - \lambda \right) + \lambda \frac{\rho}{\rho - \bar{\mu}} \quad (7)$$

The first order conditions are:

$$\begin{aligned} \rho \frac{\partial U^*(P, \lambda)}{\partial Y} &= Y^{-\alpha} - \lambda = 0 \\ \rho \frac{\partial U^*(P, \lambda)}{\partial \lambda} &= -Y + \frac{\rho}{\rho - \bar{\mu}} = 0 \end{aligned}$$

This gives the optimal income based on the initial productivity and expectations of its future values for all the values of the coefficient of relative risk aversion:

$$Y = \frac{\rho}{\rho - \bar{\mu}} \quad (8)$$

This optimal value of income, or consumption, is a natural consequence of the model: since insurance wealth is equal to the expected discounted total productivity, the benefits through the infinite lifetime cannot be higher than the interests from the insurance wealth - otherwise the stock of wealth could be depleted.

In optimum the Lagrange multiplier is equal

$$\lambda = \left(\frac{\rho}{\rho - \bar{\mu}} \right)^{-\alpha}$$

and the expected value of lifetime utility, when starting productivity is normalized to one, is equal:

$$\begin{aligned} U &= \text{for } \alpha \neq 1 : \frac{1}{(1 - \alpha)\rho} \left(\frac{\rho}{\rho - \bar{\mu}} \right)^{1 - \alpha} \\ &= \text{for } \alpha = 1 : \frac{1}{\rho} \ln \left(\frac{\rho}{\rho - \bar{\mu}} \right) \end{aligned} \quad (9)$$

Both income and lifetime utility are proportional to starting productivity. This is the expected result, because the higher the starting productivity, the higher is the expected total productivity. And, since

total income is equal to the expected discounted value of worker's output, it is straightforward that level of income and hence lifetime utility grows with starting productivity. On the other hand the Lagrange multiplier is a decreasing function of starting productivity. If we interpret λ as the shadow price of income, the interpretation follows directly: if the starting value of productivity is low, a worker expects that her/his productivity over the whole lifetime will be relatively low, and thus income obtained for this productivity will be low. Therefore when the level of income is low, its price is high.

Special case for $\alpha = 0$ When an individual is risk neutral, i.e. when $\alpha = 0$ the lifetime utility is:

$$U(Y) = \int_0^\infty Y e^{-\rho t} dt \Leftrightarrow U(Y) = \frac{Y}{\rho} \quad (10)$$

The budget constraint is the same as (3) and the Lagrangian is now given by:

$$\rho U(Y, \lambda) = Y(1 - \lambda) + \lambda \frac{\rho}{\rho - \bar{\mu}} \quad (11)$$

The first order give the optimal income based on the initial productivity and expectations of its future values:

$$Y = \frac{\rho}{\rho - \bar{\mu}}$$

The Lagrange multiplier is equal

$$\lambda = 1$$

and the expected value of lifetime utility is:

$$U = \frac{1}{\rho - \bar{\mu}} \quad (12)$$

Since people are risk neutral, there is no risk aversion to influence their utility. Thus utility is multiplicative in starting productivity, and the shadow price of income is equal 1.

As we can see from these results, in perfect insurance case with the infinite lifetime and without the possibility of retirement, independent from the value of risk aversion, the optimal consumption = benefit level is equal to the interests gained from the insurance wealth, which is equal to the expected total productivity.

2.2 Increasing probability of death

The notion of the constant probability of death, as in the infinite lifetime case, although tempting because of the analytical simplicity, is not very plausible. Therefore we introduce an increasing probability of death: an individual has a potential for the infinite life but the probability of death is rising with age. Let us assume that probability of death is a linear function of time δt , where $0 < \delta < 1$ is a trend in the dying rate. In such a case the lifetime utility function without the possibility of retirement is equal:

$$U(Y) = \int_0^\infty \frac{1}{1 - \alpha} Y^{1-\alpha} e^{-(\rho + \frac{\delta}{2}t)t} dt \quad (13)$$

with an equivalent of $U(Y) = \int_0^\infty \ln Y e^{-(\rho + \frac{\delta}{2}t)t} dt$ for $\alpha = 1$, where $\frac{\delta}{2}$ is necessary to make the solution of the differential equation in the case when retirement is allowed possible. The budget constraint is almost the same as in (3):

$$E \int_0^\infty P_t e^{-(\rho + \frac{\delta}{2}t)t} dt = \int_0^\infty Y e^{-(\rho + \frac{\delta}{2}t)t - \rho t} dt \quad (14)$$

The expected present value of lifetime productivity at the beginning of the labor career is now more complicated than in (4):

$$E \int_0^\infty P_t e^{-(\rho + \frac{\delta}{2}t)t} dt = \int_0^\infty e^{-\frac{\delta}{2}t^2 - (\rho - \bar{\mu})t} dt \quad (15)$$

with starting productivity normalized to one. Repeating the analysis above we optimize the Lagrangian. The first order conditions yield the optimal level of benefits given the starting value of productivity and its expectations in future:

$$Y = \frac{\int_0^\infty e^{-\frac{\delta}{2}t^2 - (\rho - \bar{\mu})t} dt}{\int_0^\infty e^{-(\rho + \frac{\delta}{2}t)t} dt} \quad (16)$$

This value is the same for all values of the coefficient of relative risk aversion. If we compare it with the result in (8) it is basically the same thing corrected for the increasing probability of death. Optimal income is equal to the interests from the insurance wealth, where the interest rate is increased by the probability of death - this follows from the denominator. The insurance wealth is equal to the expected discounted total productivity as showed in the budget constraint. The expected discounted total productivity is lower than in the infinite time case, however whether consumption is lower depends on the trend in productivity $\bar{\mu}$. If the expected trend is high, then benefits level is higher in the increasing probability of death case - life is shorter and the long spells of low productivity are less probable, thus it is possible to consume more. If the expected trend is low, or even negative, then benefits level is lower in the increasing probability of death case than in the infinite lifetime case - once the productivity starts falling the possibility of its future rise is much lower than with the infinite lifetime to wait. Therefore consumption must be lower.

The optimal utility is equal:

$$\begin{aligned} U(P_0) &= \text{for } \alpha \neq 1 : \frac{1}{1 - \alpha} \left(\frac{\int_0^\infty e^{-\frac{\delta}{2}t^2 - (\rho - \bar{\mu})t} dt}{\int_0^\infty e^{-(\rho + \frac{\delta}{2}t)t} dt} \right)^{1 - \alpha} \int_0^\infty e^{-(\rho + \delta t)t} dt \\ &= \text{for } \alpha = 1 : \ln \left(\frac{\int_0^\infty e^{-\frac{\delta}{2}t^2 - (\rho - \bar{\mu})t} dt}{\int_0^\infty e^{-(\rho + \frac{\delta}{2}t)t} dt} \right) \int_0^\infty e^{-(\rho + \delta t)t} dt \\ &= \text{for } \alpha = 0 : \int_0^\infty e^{-\frac{\delta}{2}t^2 - (\rho - \bar{\mu})t} dt \end{aligned} \quad (17)$$

As with consumption the optimal utility with increasing probability of death can be either higher or lower than the utility in the infinite lifetime case. This depends on the trend in productivity and on risk aversion. Only for risk neutrality we may say that utility is always lower in the increasing probability of death case, since it is equal to the expected discounted value of productivity, and this value is lower when there is a probability of death than when an individual lives and produces forever.

3 Retirement with the infinite lifetime

In this section we allow the worker to retire. If productivity falls below a certain level P^* at some moment T , an individual quits work forever. An individual buys an insurance giving, in exchange for the lifetime productivity, benefits ensuring constant utility. As a result there are two levels of benefits: one for work period, which compensates an individual for the disutility of working, and the other for retirement. The insurance wealth is equal to the expected discounted value of the lifetime productivity calculated at the beginning of the labor career, i.e. at $t = 0$.

Let the insurance benefit while working be given by Y_W and the benefit while retired by Y_R . The lifetime utility is equal:

$$\begin{aligned} U(Y_W, Y_R) &= \text{for } \alpha \neq 1 : E \int_0^T \frac{1}{1-\alpha} Y_W^{1-\alpha} e^{-\rho t} dt + E \int_T^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-\rho(T+t)} dt \quad (18) \\ &= \text{for } \alpha = 1 : E \int_0^T \ln Y_W e^{-\rho t} dt + E \int_T^\infty [\ln Y_R + \ln \varepsilon] e^{-\rho(T+t)} dt \end{aligned}$$

where ε is a cost of effort or the disutility from working - since people generally prefer leisure to work we assume that $\varepsilon > 1$. If people dislike working, then it is possible that their utility is actually higher than while working, even with lower benefits while retired. Thus the multiplier of the utility while retired must be bigger than one. The first element in (18) shows the discounted expected utility while working and the second the discounted expected utility while retired. The presence of the cost of effort in the utility while retired shows the gain of the utility due to not working.

The utility while working is just the utility while retired increased by the compensation for the disutility from working. Hence we can rewrite the lifetime utility³:

$$U(Y_W, Y_R) = \int_0^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-\rho t} dt + E \int_0^T \frac{1}{1-\alpha} (Y_W^{1-\alpha} - \varepsilon^{1-\alpha} Y_R^{1-\alpha}) e^{-\rho t} dt \quad (19)$$

The budget constraint is:

$$E \int_0^T P_t e^{-\rho t} dt = E \int_0^T Y_W e^{-\rho t} dt + E \int_T^\infty Y_R e^{-\rho(T+t)} dt \quad (20)$$

The right hand side of the budget constraint represents the insurance wealth, composed of the sum of benefits paid while working and the sum of benefits paid while retired. Since the insurance wealth is equal to the expected discounted value of life-time work earnings calculated at $t = 0$, it must depend only on starting productivity P_0 and the time spent working T . However, due to the randomness of P , which makes T also random, we can calculate the value of neither side of (20) directly. Transforming (20) we get:

$$E \int_0^T (P_t - Y_W + Y_R) e^{-\rho t} dt = \int_0^\infty Y_R e^{-\rho t} dt \quad (21)$$

Each individual maximizes her/his utility subject to the budget constraint. Thus we must maximize (19) subject to (21). This gives a Lagrangian:

$$\rho U(Y_W, Y_R, \lambda) = \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} - \lambda Y_R + \rho G(P) \quad (22)$$

³All the subsequent formulas and results are for $\alpha \neq 1$. Their equivalents for $\alpha = 1$ are presented in Appendix 2

where

$$G(P) = E \int_0^T \left[\frac{1}{1-\alpha} (Y_W^{1-\alpha} - \varepsilon^{1-\alpha} Y_R^{1-\alpha}) + \lambda (P - Y_W + Y_R) \right] e^{-\rho t} dt$$

and where we ignore the time subscript of the productivity. The first element of the Lagrangian is known, the other is stochastic, and we must find its expected value depending on productivity. To simplify the problem we first maximize the Lagrangian to find the values of income while working (Y_W) and Lagrange multiplier (λ) in terms of income while retired (Y_R). From the first order conditions we have that:

$$\lambda = Y_W^{-\alpha} \quad (23)$$

$$\lambda = \varepsilon^{1-\alpha} Y_R^{-\alpha} \quad (24)$$

Together (23) and (24) give the relation between Y_R and Y_W :

$$Y_W = \varepsilon^{\frac{\alpha-1}{\alpha}} Y_R \quad (25)$$

As we can see income while working does not have to be bigger than income while retired - it depends on risk aversion being bigger or smaller than one. Optimality requires equal marginal utilities of consumption while working and while retired. Actual levels of consumption, and hence income and benefits, depend on two opposite effects: the substitution and income effects. Income effect causes marginal utility while retired to fall, reducing consumption after retirement decision is taken. Substitution effect makes marginal utility of consumption to rise, increasing consumption while retired. The substitution effect dominates when $\alpha < 1$, hence for risk aversion smaller than one income while working is smaller than income while retired. Income effect is stronger when $\alpha > 1$ and then opposite situation takes place. For $\alpha = 1$ the effects cancel each other, and income while working is equal to income while retired.

By substitution of (23), (24) and (25) into the Lagrangian we can rewrite it as a function only of income while retired:

$$\rho U(Y_R) = \frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} + \rho G(P) \quad (26)$$

where

$$G(P) = E \int_0^T \left[\frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P \right] e^{-\rho t} dt$$

Now we can look for the expected value of $G(P)$. First define the value of $G(P)$ at any moment t :

$$G(P) = \left(\frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P \right) dt + E [G(P + dP) e^{-\rho dt}] \quad (27)$$

Using Ito's lemma and the definition of the stochastic process (1) in (27) yields the Bellman equation⁴

⁴ Applying Ito's Lemma and (1) gives:

$$\begin{aligned} G(P) &= \left(\frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P \right) dt + (1 - \rho dt) \left[G(P) + G_P(P) dP + \frac{1}{2} G_{PP}(P) (dP)^2 \right] \\ &= \left(\frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P \right) dt + (1 - \rho dt) \left[G(P) + \bar{\mu} P G_P(P) dt + \frac{\sigma^2}{2} P^2 G_{PP}(P) dt \right] \\ &= \left(\frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P \right) dt + (1 - \rho dt) G(P) + \bar{\mu} P G_P(P) dt + \frac{\sigma^2}{2} P^2 G_{PP}(P) dt \end{aligned}$$

where the last transformation results from the fact that $(dt)^2$ goes to zero faster than other terms. Now we have:

$$0 = \left(\frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P \right) dt - \rho dt G(P) + \bar{\mu} P G_P(P) dt + \frac{\sigma^2}{2} P^2 G_{PP}(P) dt$$

Dividing both sides by dt we obtain the differential equation (28).

defining $G(P)$:

$$0 = \frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P - \rho G(P) + \bar{\mu} P G_P(P) + \frac{\sigma^2}{2} P^2 G_{PP}(P) \quad (28)$$

The general solution of (28) is of the form:

$$G(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2} + \frac{\varepsilon^{1-\alpha} Y_R^{-\alpha} P}{\rho - \bar{\mu}} + \frac{1}{\rho} \frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) \quad (29)$$

where β_1 and β_2 are the roots of the equation:

$$\frac{1}{2} \sigma^2 \beta(\beta - 1) + \bar{\mu} \beta - \rho = 0$$

which arise from substitution of (29) into (28) and then equalization of the sum of all the terms which include P to zero. The roots are:

$$\begin{aligned} \beta_1 &= \frac{1}{2} - \frac{\bar{\mu}}{\sigma^2} + \sqrt{\left(\frac{\bar{\mu}}{\sigma^2} - \frac{1}{2} \right)^2 + 2 \frac{\rho}{\sigma^2}} > 1 \\ \beta_2 &= \frac{1}{2} - \frac{\bar{\mu}}{\sigma^2} - \sqrt{\left(\frac{\bar{\mu}}{\sigma^2} - \frac{1}{2} \right)^2 + 2 \frac{\rho}{\sigma^2}} < 0 \end{aligned} \quad (30)$$

Since we are interested only in stopping work, we can simplify the solution. When the productivity P rises to ∞ , the probability that an individual will quit work becomes very small. Therefore the value of the quitting option should go to zero as productivity increases. Hence, the coefficient A_1 corresponding to positive root β_1 should be zero. Thus the value of $G(P)$ is:

$$G(P) = A_2 P^{\beta_2} + \frac{\varepsilon^{1-\alpha} Y_R^{-\alpha} P}{\rho - \bar{\mu}} + \frac{1}{\rho} \frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) \quad (31)$$

where the first expression on the right hand side is the value of the option to stop working, while two other expressions constitute the value of continued work.

With $G(P)$ found we may start looking for P^* , i.e. the critical, work-quitting level of productivity. At such a critical threshold two conditions, value matching and smooth pasting, must be fulfilled:

$$G(P^*) = 0 \quad (32)$$

$$\frac{\partial G(P^*)}{\partial P^*} = 0 \quad (33)$$

Condition (32) is a value matching condition showing net gain/loss from stopping work - an individual quits work and does not have to undertake effort but has to live from the fixed level of insurance. Since a worker cannot change the levels of benefits, she/he cannot gain more wealth by retiring earlier or working longer. Thus at the moment of retirement the gains and losses from stopping work should equalize. Condition (33) is a smooth pasting condition, ensuring continuity and smoothness at the threshold P^* .

Now we have two equations with two unknowns. Solving together (32) and (33) we are able to define P^* and A_2 . Some transformations yield the appropriate formulas:

$$P^* = \frac{\beta_2}{1 - \beta_2} \frac{\rho - \bar{\mu}}{\rho} \frac{\alpha}{1 - \alpha} Y_R \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) \quad (34)$$

$$A_2 = \frac{\varepsilon^{1-\alpha} Y_R^{-\alpha} \left[\frac{\alpha}{1-\alpha} Y_R \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) \right]^{1-\beta_2}}{\rho (\beta_2 - 1) \left[\frac{\beta_2}{1-\beta_2} \frac{\rho - \bar{\mu}}{\rho} \right]^{\beta_2}} \quad (35)$$

(34) gives the optimal work-quitting threshold: whenever in the lifetime productivity level, while falling, hits P^* , an individual stops working forever. (35) yields a constant coefficient of the solution of the differential equation. Using (35) in (31) yields the full solution for $G(P)$. This solution may be substituted into the Lagrangian (22) but, since the Lagrangian maximizes the utility for the whole life and insurance income depend only on starting value of productivity and the expected time of work, productivity P in $G(P)$ is equal to starting productivity P_0 normalized to one. Thus the final Lagrangian is given by:

$$\rho U(Y_R) = \varepsilon^{1-\alpha} Y_R^{-\alpha} \left[\frac{\left[\frac{\alpha}{1-\alpha} Y_R \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) \right]^{1-\beta_2}}{(\beta_2 - 1) \left[\frac{\beta_2}{1-\beta_2} \frac{\rho - \bar{\mu}}{\rho} \right]^{\beta_2}} + \frac{\rho}{\rho - \bar{\mu}} + \frac{\alpha}{1-\alpha} Y_R \varepsilon^{\frac{\alpha-1}{\alpha}} \right] \quad (36)$$

The first order condition of (36) yields an equation defining benefits while retired:

$$\frac{\left[\frac{\alpha}{1-\alpha} Y_R \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) \right]^{1-\beta_2}}{(\beta_2 - 1) \left[\frac{\beta_2}{1-\beta_2} \frac{\rho - \bar{\mu}}{\rho} \right]^{\beta_2}} (1 - \beta_2 - \alpha) + \alpha Y_R \varepsilon^{\frac{\alpha-1}{\alpha}} - \alpha \frac{\rho}{\rho - \bar{\mu}} = 0 \quad (37)$$

This equation cannot be solved analytically. We can however solve it numerically assigning the values of the parameters in the model. This yields the value of Y_R and with it the values of Y_W and λ , which are then used to find the retirement threshold according to formula (34).

Some simulation results are presented in a Table 1 for different values of the coefficient of relative risk aversion, and for the following values of parameters⁵: $\rho = 0.05$, $\sigma = 0.1$, $\bar{\mu} = -0.01$ and 0.03 . Since cost of effort must be bigger than one for people to want to retire, we consider two values: $\varepsilon = 1.5$ and $\varepsilon = 2$. The results show that work-quitting threshold of productivity is always positive and growing with the disutility from work - the more an individual dislikes work, the more she/he is likely to retire early. P^* is smaller than the starting productivity, i.e. smaller than one, because as long as productivity is higher than one the insurance company is gaining and it is not likely to let the worker retire. We can also see that retirement threshold is decreasing with an increase in risk aversion. It is possible that more risk averse individuals are afraid to retire and thus retire later, but it does not explain this result, since in our model individual income and benefits are certain. Therefore risk aversion should not influence the retirement decision directly. The better answer follows from another result - income while working is increasing with risk aversion, while retirement benefits are decreasing. Thus a risk averse worker enjoys high working income, but knows that her/his retirement benefits will be low. Therefore she/he retires as late as possible. In other words, income while working and retirement benefits are given by marginal utilities of income, but at the moment of retirement an individual is comparing average utilities.

The question is why incomes from working are an increasing function of α , while retirement benefits are a decreasing one? As we said above when talking about substitution and income effects, the answer

⁵The trend in wages is equal 0.03, according Carroll (1992,2001). -0.01 allows us to consider negative trend without moving too far from the realistic values. The standard deviation of 0.1 is within range of Topel and Ward (1992) results and it is equal to Carroll (1992) result for the standard deviation of the permanent and transitory shocks to income. The interest rate of 0.05 is a compromise between the real interest rates, which are currently very low, and the individuals' time preference rates which are much higher, since people have usually quite a short time horizon.

is in the equation (25), where $\varepsilon^{\frac{\alpha-1}{\alpha}}$ is increasing with risk aversion. In order to keep marginal utilities of work and retirement equal at the moment of retirement, Y_W must increase and Y_R must decrease with rise of α . It shows that with the rise of risk aversion income effect becomes stronger. Another result concerning incomes is that they are falling when cost of effort is rising. One explanation is that since an individual who dislikes work is more likely to retire early, her/his salary must be lower in order to finance the retirement. Second argument is that it also arises from the relation between income and benefits in equation (25).

Comparing the results in Table 1 we may see that for risk aversion smaller than one income while working is lower than the retirement benefits, while for $\alpha > 1$ the situation is opposite. This also results from the interaction of substitution and income effects, since in the first case $\varepsilon^{\frac{\alpha-1}{\alpha}} < 1$, and in the second it is bigger than one. For $\alpha < 1$ it is also true that $\varepsilon^{\frac{\alpha-1}{\alpha}}$ is decreasing in ε (and increasing in ε for $\alpha > 1$), and that is why for such risk aversion the retirement benefits are increasing with disutility from work, and why for $\alpha > 1$ they are decreasing with ε . This last result is more intuitive - the more an individual dislikes work the less she/he needs to be paid to stay retired.

| Table 1: $\bar{\mu} = -0.01$ | | | | | | |
|---|---------------------|-------|-------|-------------------|-------|-------|
| Parameters' values: $\rho = 0.05, \sigma = 0.1$ | | | | | | |
| | $\varepsilon = 1.5$ | | | $\varepsilon = 2$ | | |
| α | Y_W | Y_R | P^* | Y_W | Y_R | P^* |
| 0.5 | 0.772 | 1.158 | 0.309 | 0.601 | 1.203 | 0.481 |
| 1 | 0.818 | 0.818 | 0.265 | 0.769 | 0.769 | 0.426 |
| 1.5 | 0.827 | 0.722 | 0.251 | 0.807 | 0.64 | 0.399 |
| 2 | 0.83 | 0.678 | 0.244 | 0.821 | 0.581 | 0.385 |

With the negative trend in productivity ($\bar{\mu} = -0.01$) each individual expects her/his productivity to fall, even if it is currently rising. That is why incomes are low relative to the starting productivity, and why the separation threshold grows relatively slow. In order to see what is happening when people do not expect any fall in productivity, in Table 2 we present the results of our model calculated for positive trend ($\bar{\mu} = 0.03$). In such a case individuals know that their productivity is likely to rise sooner or latter and that makes the insurer to behave less precautionary. Comparison of the results in Tables 1 and 2 shows that there are some important differences between two cases. First, the retirement threshold of productivity grows faster with the starting productivity than in the negative trend case. This is a straightforward consequence of the positive trend: if productivity is on average rising the retirement threshold is higher than when it is on average falling. Second, incomes are higher - since an employer/insurer expects higher lifetime productivity. Because the starting productivity is the basis for calculating income levels -and on average productivity is rising from the initial level - incomes are higher than its value. Since incomes are high, the Lagrange multiplier, the shadow price of income, is much lower than in the negative trend case.

The last result in table 2, which we need to consider is $Y_W \simeq 2.5$ in all but one case. This is not some kind of magical number. It is simply the interest earned from the discounted expected lifetime

productivity. With our values of parameters $\frac{\rho}{\rho-\bar{\mu}} = 2.5$. Hence, in almost all the cases income while working is equal to the "safe amount": faced with prospects of paying the benefits infinitely, the insurer does not want to pay more than the interests gained from the wealth she/he is expecting to earn, in order to not deplete this wealth when the productivity starts falling or when the worker retires.

| Table 2: $\bar{\mu} = 0.03$ | | | | | | |
|---|---------------------|-------|-------|-------------------|-------|-------|
| Parameters' values: $\rho = 0.05, \sigma = 0.1$ | | | | | | |
| | $\varepsilon = 1.5$ | | | $\varepsilon = 2$ | | |
| α | Y_W | Y_R | P^* | Y_W | Y_R | P^* |
| 0.5 | 2.49 | 3.736 | 0.432 | 2.064 | 4.129 | 0.716 |
| 1 | 2.499 | 2.499 | 0.351 | 2.453 | 2.453 | 0.59 |
| 1.5 | 2.5 | 2.184 | 0.329 | 2.486 | 1.973 | 0.534 |
| 2 | 2.5 | 2.041 | 0.318 | 2.494 | 1.763 | 0.507 |

Special case for $\alpha = 0$ As in the section 2 we examine the case of risk neutrality. The lifetime utility for $\alpha = 0$ is given by:

$$U(Y_W, Y_R) = E \int_0^T Y_W e^{-\rho t} dt + E \int_T^\infty \varepsilon Y_R e^{-\rho(T+t)} dt \quad (38)$$

Following the steps described above the results maximizing the Lagrangian are given by

$$Y_R = 0 \quad (39)$$

$$Y_W = \frac{\rho}{\rho - \bar{\mu}} \quad (40)$$

If we use them to compute the retirement threshold it appears that a risk-neutral individual never retires, since:

$$P^* = 0 \quad (41)$$

what follows also from formula (34) if we take limit $\lim_{\alpha \rightarrow 0^+} P^* = 0$.

The question is: why a risk-neutral individual never retires? It is inconsistent with the results presented above for risk-averse individuals: with the fall in risk aversion the retirement threshold is rising, thus the less risk-averse individuals are more likely to retire earlier. However, since levels of benefits are certain, the risk aversion should not influence the retirement decision directly, only through benefits. More risk averse individuals retire later because they have high income while working, but low income while retired, i.e. in their case the income effect dominates. Here the mechanism is opposite - only substitution effect matters, thus an individual never retires. As can be seen from the definition of the utility (38), with disutility from working $\varepsilon > 1$, consumption after the retirement gives higher utility than consumption while working per unit of income. A risk-neutral individual is by definition indifferent to risk and thus she/he will consume everything when consumption is more efficient, i.e. after the retirement. Therefore, with the infinite lifetime, she/he is willing to postpone retirement waiting for the possibility of higher consumption. However in the conditions just described, with the infinite lifetime and risk neutrality, such decision is never taken, and thus the result (41) is true.

3.1 Probability of retirement

Knowing the retirement thresholds we would like to find out what is the probability of reaching them in a given time. From Harrison (1990) we know that the probability that X will remain below the certain level y for the time t , when X follows standard Brownian motion, is given by⁶:

$$P\{M_t < y\} = \Phi\left(\frac{y - \mu t}{\sigma t^{\frac{1}{2}}}\right) - e^{\frac{2\mu y}{\sigma^2}} \Phi\left(\frac{-y - \mu t}{\sigma t^{\frac{1}{2}}}\right) \quad (42)$$

where $M_t \equiv \max\{X_s, 0 \leq s \leq t\}$ and $\Phi(\cdot)$ is the $N(0, 1)$ distribution function. However we are interested not in the probability of remaining below the certain level, but in the probability of falling below the certain level. This can be done if we transform (42) into the formula defining the probability that X will remain above the certain level y , as in Corollary B.3.4 in Musiela, Rutkowski (1998):

$$P\{m_t > y\} = \Phi\left(\frac{-y + \mu t}{\sigma t^{\frac{1}{2}}}\right) - e^{\frac{2\mu y}{\sigma^2}} \Phi\left(\frac{y + \mu t}{\sigma t^{\frac{1}{2}}}\right) \quad (43)$$

where $m_t \equiv \min\{X_s, 0 \leq s \leq t\}$. In order to find out the probability of falling below y , we need simply to subtract (43) from one:

$$P\{m_t \leq y\} = 1 - P\{m_t > y\} = \Phi\left(\frac{y - \mu t}{\sigma t^{\frac{1}{2}}}\right) + e^{\frac{2\mu y}{\sigma^2}} \Phi\left(\frac{y + \mu t}{\sigma t^{\frac{1}{2}}}\right) \quad (44)$$

To switch into our case we must change from standard Brownian motion starting at zero into geometric Brownian starting at $P_0 = 1$. The substitution of:

$$\begin{aligned} y &= \ln \frac{P^*}{P_0} = \ln P^* \\ \mu &= \bar{\mu} - \frac{\sigma^2}{2} \end{aligned}$$

yields the final formula:

$$P\{m_t \leq P^*\} = \Phi\left(\frac{\ln P^* - \left(\bar{\mu} - \frac{\sigma^2}{2}\right)t}{\sigma t^{\frac{1}{2}}}\right) + P^{*\frac{2\bar{\mu}}{\sigma^2}-1} \Phi\left(\frac{\ln P^* + \left(\bar{\mu} - \frac{\sigma^2}{2}\right)t}{\sigma t^{\frac{1}{2}}}\right)$$

where $P\{m_t \leq P^*\}$ is the probability that geometric Brownian motion with a starting value 1, drift $\bar{\mu}$ and standard deviation σ , will hit the lower boundary P^* before time t . The plots of this probability for the critical thresholds from the previous section are presented in Figures 1 and 2.

The plots in Figures 1 and 2 show probability that the worker retires before time t . Most of these results are intuitive: probability is an increasing function of time and cost of effort ε . The older people are, i.e. the longer their tenure, the more likely they are to retire. Also if individuals dislike to work, the possibility of retirement is increasing. It grows much faster with time for high ε and it seems to stabilize for high t - showing that for higher ages probability of retirement does not change much with time. It can be explained when we notice that it appears only for positive drift, suggesting that if an individual has not retired so far, the chance that productivity falls to the retirement threshold is low, or rather dominated by its expected rise. The probability of retirement is falling with the increase of risk aversion,

⁶This is the formula (8.11) in Harrison (1990). For examples of its use check Sarkar (2000) or Pawlina and Kort (2002).

Figure 1: Probability of retirement for $\bar{\mu} = -0.01$, parameters' values: $\rho = 0.05$, $\sigma = 0.1$

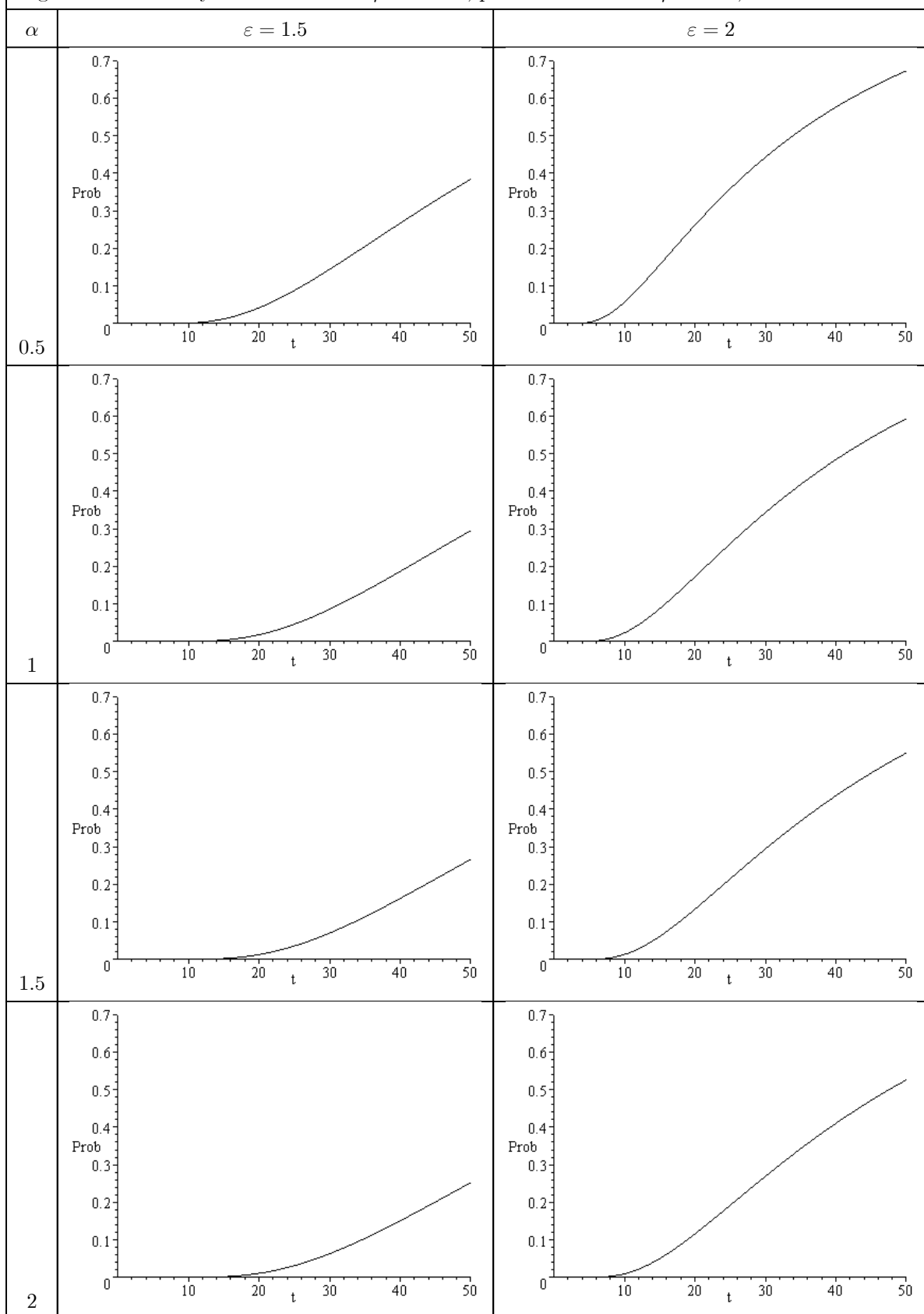
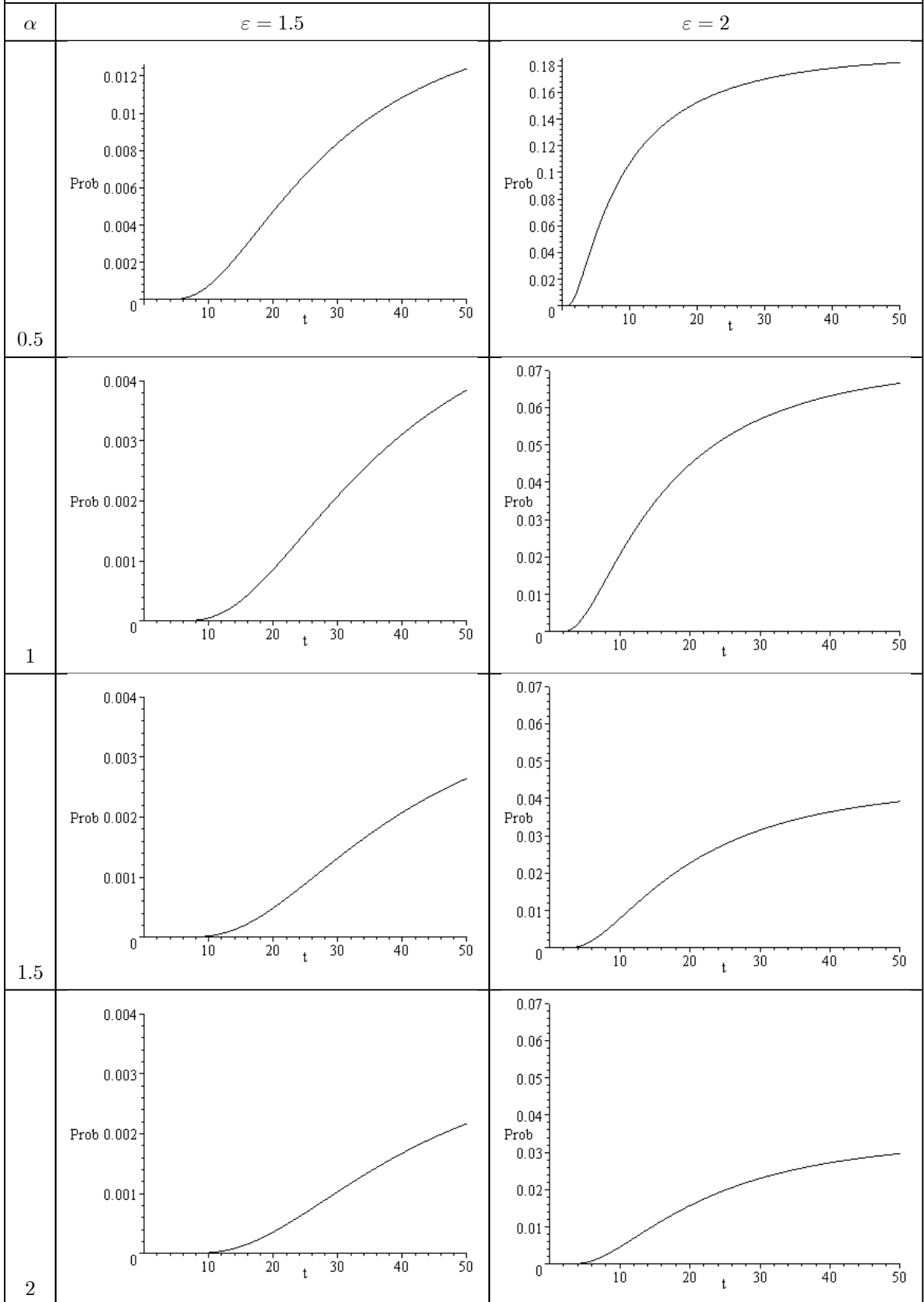


Figure 2: Probability of retirement for $\bar{\mu} = 0.03$, parameters' values: $\rho = 0.05$, $\sigma = 0.1$



but this is explained by the fall of retirement threshold, resulting from the fact that income effect decreases benefits while retired. When we compare the probabilities for positive and negative drifts, it is clear that, as expected, with negative drift the probability of retirement is much bigger than for positive drift.

When we compute the limits of probability at infinity the results show that probability of retirement converges from below to a constant. For the positive drift these constants are in each case close to the values reached by the probabilities at $t = 50$. Therefore with the positive drift most of the workers do not retire. For the negative drift, for all values of the coefficient of relative risk aversion, the probability of retirement is converging to one, showing that when individuals expect fall in wages sooner or latter everybody retires. These results contradict those of Teulings and van der Ende (2000), who show that the separation rate for workers is hump shaped: first the separation rate increases since a sufficient number of negative shocks of productivity have been accumulated. Latter it declines, since the trajectories with a large number of negative shocks have been already eliminated.

4 Retirement with the increasing probability of death

As in section 2 we turn now to a much more interesting scenario than the one with the infinite lifetime. We introduce an increasing probability of death, as a linear function of time δt , where $0 < \delta < 1$. The retirement system is the same as in the previous section, with perfect insurance giving benefits while working and while retired in exchange for the lifetime output. In such a setup the lifetime utility is:

$$U(Y_W, Y_R) = E \int_0^T \frac{1}{1-\alpha} Y_W^{1-\alpha} e^{-(\rho+\frac{\delta}{2})t} dt + E \int_T^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-(\rho+\frac{\delta}{2})(t-T)} dt \quad (45)$$

where the only difference with comparison to (18) is that the discount rate is increased by the probability of death. It can be rewritten as:

$$U(Y_W, Y_R) = E \int_0^T \frac{1}{1-\alpha} (Y_W^{1-\alpha} - \varepsilon^{1-\alpha} Y_R^{1-\alpha}) e^{-(\rho+\frac{\delta}{2})t} dt + \int_0^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-(\rho+\frac{\delta}{2})t} dt \quad (46)$$

In the same manner the budget constraint looks as follows:

$$E \int_0^T (P_t - Y_W + Y_R) e^{-(\rho+\frac{\delta}{2})t} dt = \int_0^\infty Y_R e^{-(\rho+\frac{\delta}{2})t} dt \quad (47)$$

In order to maximize utility subject to budget constraint we form a Lagrangian:

$$U(Y_W, Y_R, \lambda) = \int_0^\infty \left[\frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} - \lambda Y_R \right] e^{-(\rho+\frac{\delta}{2})t} dt + G(P, t) \quad (48)$$

where

$$G(P, t) = E \int_0^T \left[\frac{1}{1-\alpha} (Y_W^{1-\alpha} - \varepsilon^{1-\alpha} Y_R^{1-\alpha}) + \lambda (P_t - Y_W + Y_R) \right] e^{-(\rho+\frac{\delta}{2})t} dt$$

is stochastic part of the utility.

To simplify the problem we first maximize the Lagrangian to find the values of income while working (Y_W) and Lagrange multiplier (λ) in terms of income while retired (Y_R). From first order conditions we

have:

$$Y_W^{-\alpha} = \lambda \quad (49)$$

$$\varepsilon^{1-\alpha} Y_R^{-\alpha} = \lambda \quad (50)$$

As we can see this is the same result as in the infinite lifetime case, and it gives identical relation between Y_W and Y_R :

$$Y_W = \varepsilon^{\frac{\alpha-1}{\alpha}} Y_R \quad (51)$$

Thus the relation between levels of incomes in the case of increasing probability of death is the same as in the case of the infinite lifetime, although the levels themselves are different. This results from the insurance setup we assumed.

Using the results from (49), (50) and (51) we can rewrite the Lagrangian as a function of only income while retired:

$$U(Y_R, t) = \int_t^\infty \frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-(\rho+\frac{\delta}{2}s)s} ds + G(P, t) \quad (52)$$

where

$$G(P, t) = E \int_t^T \left[\frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) + \varepsilon^{1-\alpha} Y_R^{-\alpha} P \right] e^{-(\rho+\frac{\delta}{2}s)s} ds$$

We are looking for expected value of $G(P, t)$ as a function of productivity and time. With Ito's lemma and the law of motion of productivity (1) we can write the differential equation defining $G(P, t)$:

$$0 = A + \varepsilon^{1-\alpha} Y_R^{-\alpha} P - (\rho + \delta t) G(P, t) + \bar{\mu} P G_P(P, t) + \frac{\sigma^2}{2} P^2 G_{PP}(P, t) + G_t(P, t) \quad (53)$$

where

$$A = \frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right)$$

The method of solving equation (53) is presented in Appendix 1. The general solution is:

$$\begin{aligned} G(P, t) = & \int_{-\infty}^{\beta_2} B_0(x) P^x \exp \left[\frac{1}{2} \delta t^2 + \left(\rho - \bar{\mu} x + \frac{1}{2} \sigma^2 x \right) t - \frac{1}{2} \sigma^2 x^2 t \right] dx \\ & + \varepsilon^{1-\alpha} Y_R^{-\alpha} P e^{\frac{1}{2} \delta t^2 + (\rho - \bar{\mu}) t} \int_t^\infty e^{-\frac{1}{2} \delta s^2 - (\rho - \bar{\mu}) s} ds + A e^{\frac{1}{2} \delta t^2 + \rho t} \int_t^\infty e^{-\frac{1}{2} \delta s^2 - \rho s} ds \end{aligned} \quad (54)$$

In order to find the work-quitting threshold of productivity P^* and the value of $B_0(x)$ we must ensure that the value matching and smooth pasting conditions hold. The value matching condition (equation (55) below) shows that at the moment of retirement the utility from continued work with the possibility of future retirement must be equal to the utility from retiring now. We cannot, as we did in the infinite lifetime with constant probability of death case, just equalize the stochastic part $G(P, t)$ of Lagrangian to zero, since the increasing probability of death makes timing of the retirement decision important. The worker cannot change the level of benefits but she/he can influence the utility after the retirement - the higher the probability of death, the lower the utility. The utility after retirement is given by:

$$U_R(Y_R, t) = \int_t^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-(\rho+\frac{\delta}{2}s)(s-t)} dt$$

and it must be equalized with the Lagrangian (52), after the substitution of (54). The smooth pasting condition (equation (56) below) guarantees the continuity of the utility at the threshold P^* :

$$\begin{aligned} & \int_t^\infty \frac{\alpha}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-(\rho+\delta s)(s-t)} ds + \int_{-\infty}^{\beta_2} B_0(x) P^{*x} e^{\frac{1}{2}\delta t^2 + (\rho - \bar{\mu}x + \frac{1}{2}\sigma^2 x)t - \frac{1}{2}\sigma^2 x^2 t} dx \\ & + \varepsilon^{1-\alpha} Y_R^{-\alpha} P^* e^{\frac{1}{2}\delta t^2 + (\rho - \bar{\mu})t} \int_t^\infty e^{-\frac{1}{2}\delta s^2 - (\rho - \bar{\mu})s} ds + A e^{\frac{1}{2}\delta t^2 + \rho t} \int_t^\infty e^{-\frac{1}{2}\delta s^2 - \rho s} ds \\ & = \int_t^\infty \frac{1}{1-\alpha} \varepsilon^{1-\alpha} Y_R^{1-\alpha} e^{-(\rho+\delta s)(s-t)} ds \end{aligned} \quad (55)$$

$$\begin{aligned} & \int_{-\infty}^{\beta_2} B_0(x) x P^{*x-1} e^{\frac{1}{2}\delta t^2 + (\rho - \bar{\mu}x + \frac{1}{2}\sigma^2 x)t - \frac{1}{2}\sigma^2 x^2 t} dx \\ & + \varepsilon^{1-\alpha} Y_R^{-\alpha} e^{\frac{1}{2}\delta t^2 + (\rho - \bar{\mu})t} \int_t^\infty e^{-\frac{1}{2}\delta s^2 - (\rho - \bar{\mu})s} ds = 0 \end{aligned} \quad (56)$$

Solution of these two equations would yield the retirement threshold P^* and function $B_0(x)$. However, since there is no analytical solution, it has to be solved numerically. In such a case we need the third equation, which would define income while retired (Y_R). We can get it from the Lagrangian (52), after substitution of the functional form of $G(P, t)$ from formula (54). Then the first order condition with respect to Y_R is given by:

$$\begin{aligned} & \alpha \varepsilon^{1-\alpha} Y_R^{-\alpha} \int_t^\infty e^{-\frac{\delta}{2}s^2 - \rho s} ds \left[1 + \left(\varepsilon^{\frac{\alpha-1}{\alpha}} - 1 \right) e^{\frac{1}{2}\delta t^2 + \rho t} \right] \\ & + \int_{-\infty}^{\beta_2} \frac{\partial B_0(x)}{\partial Y_R} P^x \exp \left[\frac{1}{2}\delta t^2 + \left(\rho - \bar{\mu}x + \frac{1}{2}\sigma^2 x \right) t - \frac{1}{2}\sigma^2 x^2 t \right] dx \\ & - \alpha \varepsilon^{1-\alpha} Y_R^{-\alpha-1} P e^{\frac{1}{2}\delta t^2 + (\rho - \bar{\mu})t} \int_t^\infty e^{-\frac{1}{2}\delta s^2 - (\rho - \bar{\mu})s} ds = 0 \end{aligned} \quad (57)$$

In this equation everything is defined in terms of the known parameters and the three unknown variables. However we do not know the value of the derivative $\frac{\partial B_0(x)}{\partial Y_R}$. It may well be that function $B_0(x)$ depends on income while retired, and thus we cannot just assume that $\frac{\partial B_0(x)}{\partial Y_R} = 0$.

We have now a set of three equations (55), (56) and (57), and three unknowns P^* , $B_0(x)$ and Y_R . Nevertheless we are still unable to solve it, mainly because of the presence of $\frac{\partial B_0(x)}{\partial Y_R}$.

5 Conclusions

In this paper we have found the critical retirement level of worker's productivity under the perfect insurance retirement scheme. The crucial assumption of the model was that productivity follows geometric Brownian motion with drift. In perfect insurance scheme in exchange for the output of the whole lifetime worker is paid constant benefits chosen to maximize her/his lifetime utility. After solving the differential equation defining the stochastic part of the lifetime utility and using value matching and smooth pasting conditions, we calculated the expected lifetime utility and the retirement threshold of productivity in the case of the constant probability of death for both positive and negative drifts in productivity.

The results show that the retirement threshold of productivity is increasing with the individual's dislike of work, but it is decreasing with the rise of risk aversion. However in the risk neutral case there is no retirement - a risk neutral individual is willing to postpone consumption till it is more efficient, i.e.

for after retirement. With the infinite lifetime it means that the decision to retire is never taken and only the substitution effect matters. Although our model is theoretical the probabilities of retirement it yields are very reasonable. Probability is increasing with time and the disutility from working, but falling with the rise of risk aversion. It is also much higher for the negative drift than for the positive one.

Unfortunately we were not able to solve the model for the increasing probability of death. We found the functional form of the expected value of the stochastic part of lifetime utility, but it was impossible for us to find the retirement threshold as a function of time. Clearly this is the path for further research to follow. It may lead to interesting results and produce valuable tools for other research.

Appendix 1

The solution to partial differential equation (53) has a form:

$$G(P, t) = \int_{C_1}^{C_2} B(t, x) P^x dx + C(t)P + D(t)$$

The derivatives of this function are equal:

$$\begin{aligned} G_P &= \int_{C_1}^{C_2} x B(t, x) P^{x-1} dx + C \\ G_{PP} &= \int_{C_1}^{C_2} x(x-1) B(t, x) P^{x-2} dx \\ G_t &= \int_{C_1}^{C_2} B_t(t, x) P^x dx + C_t P + D_t \end{aligned}$$

Variable $x \in [C_1, C_2]$, where C_1 and C_2 are constants to be determined, is an equivalent of β in the infinite lifetime case. Since we want to study an option to quit it must be an equivalent of β_2 . Thus x should be negative. When an individual has very high probability of death the value of the option to retire is very high (what is the point in working if I am likely to die in a moment?). Therefore $C_1 = -\infty$. On the other hand, at the beginning of life/working career the whole life is before the worker and to approximate it we can use the infinite lifetime case. Hence $C_2 = \beta_2$.

- P^x : For each x , after division by $B(t, x)$ and rearranging terms:

$$\begin{aligned} b(t, x) &\equiv \ln B(t, x) \Rightarrow b_t = \frac{B_t(t, x)}{B(t, x)} \\ b_t(t, x) &= \rho + \delta t - \bar{\mu}x - \frac{1}{2}\sigma^2 x(x-1) \\ b(t, x) &= \frac{1}{2}\delta t^2 + \left(\rho - \bar{\mu}x + \frac{1}{2}\sigma^2 x\right)t - \frac{1}{2}\sigma^2 x^2 t + b_0(x) \\ \int_{-\infty}^{\beta_2} B(t, x) P^x dx &= \int_{-\infty}^{\beta_2} B_0(x) \exp \left[\frac{1}{2}\delta t^2 + \left(\rho - \bar{\mu}x + \frac{1}{2}\sigma^2 x\right)t - \frac{1}{2}\sigma^2 x^2 t + x \ln P \right] dx \end{aligned}$$

where $b_0(x)$ is an integration constant and for simplicity $B_0(x) \equiv e^{b_0(x)}$. Note that this integral can be written as a normal distribution function, since the log integrand is a parabola in x with a negative coefficient for the second order term.

- P :

$$\begin{aligned} C_t &= (\rho - \bar{\mu} + \delta t) C - \lambda \\ C(t) &= \lambda \Theta(t) e^{\frac{1}{2}\delta t^2 + (\rho - \bar{\mu})t} \\ \Theta(t) &\equiv \int_t^{C_3} e^{-\frac{1}{2}\delta s^2 - (\rho - \bar{\mu})s} ds \end{aligned}$$

$C(t)$ is the net discounted value of a flow of income with value 1 with a discount rate $\rho - \bar{\mu}$ and a death rate of δs for $s > t$, measured a time t . Note that $\Theta(t)$ can be written as a normal distribution function again. C_3 is a constant to be determined. Economic interpretation suggest that it is equal to infinity, since this part of the solution together with the constant represent the value of continued work and, with the increasing probability of death and no definite end of life, there is a possibility of working for the infinite time.

- constant:

$$\begin{aligned} D_t &= (\rho + \delta t) D - A \\ D(t) &= A \Psi(t) e^{\frac{1}{2}\delta t^2 + \rho t} \\ \Psi(t) &\equiv \int_t^{C_3} e^{-\frac{1}{2}\delta s^2 - \rho s} ds \end{aligned}$$

$D(t)$ has a similar interpretation as $C(t)$, but now with a discount rate ρ .

These three elements form the general solution of equation (53) as shown in formula (54).

Appendix 2

- Infinite lifetime:

Utility for $\alpha = 1$:

$$U(Y_W, Y_R) = \int_0^\infty [\ln Y_R + \ln \varepsilon] e^{-\rho t} dt + E \int_0^T [\ln Y_W - (\ln Y_R + \ln \varepsilon)] e^{-\rho t} dt \quad (19')$$

The solutions to the set of three first order conditions are:

$$\lambda = \frac{1}{Y_R} \quad (24')$$

$$Y_W = Y_R \quad (25')$$

Lagrangian:

$$\rho U(Y_W, Y_R, \lambda) = \ln Y_R + \ln \varepsilon - 1 + \rho G(P) \quad (26')$$

where

$$G(P) = E \int_0^T \left(\frac{1}{Y_R} P - \ln \varepsilon \right) e^{-\rho t} dt$$

The critical threshold of productivity P^* and constant A_2 are equal:

$$P^* = \frac{\rho - \bar{\mu}}{\rho} \frac{\beta_2}{\beta_2 - 1} Y_R \ln \varepsilon \quad (34')$$

$$A_2 = \frac{(\ln \varepsilon)^{1-\beta_2}}{\rho (\beta_2 - 1) \left[\frac{\rho - \bar{\mu}}{\rho} \frac{\beta_2}{\beta_2 - 1} Y_R \right]^{\beta_2}} \quad (35')$$

The equation, which can be solved numerically to find Y_R , has a form:

$$\frac{-\beta_2 Y_R^{1-\beta_2} \ln \varepsilon}{(\beta_2 - 1) \left[-\frac{\rho - \bar{\mu}}{\rho} \frac{\beta_2}{1-\beta_2} \ln \varepsilon \right]^{\beta_2}} P_0^{\beta_2} + \frac{\rho P_0}{\rho - \bar{\mu}} - Y_R = 0 \quad (37')$$

Increasing probability of death:

Utility for $\alpha = 1$:

$$U(Y_W, Y_R) = \int_0^\infty [\ln Y_R + \ln \varepsilon] e^{-(\rho + \frac{\delta}{2}t)t} dt + E \int_0^T [\ln Y_W - (\ln Y_R + \ln \varepsilon)] e^{-(\rho + \frac{\delta}{2}t)t} dt \quad (45')$$

Results from the optimization of utility:

$$\frac{1}{Y_W} = \lambda \quad (49')$$

$$\frac{1}{Y_R} = \lambda \quad (50')$$

and therefore for $\alpha = 1$ income while working is equal to income while retired, as in the infinite lifetime case.

Lagrangian as a function of only income while retired:

$$U(Y_R) = \int_0^\infty [\ln \varepsilon + \ln Y_R - 1] e^{-(\rho + \frac{\delta}{2}t)t} dt + G(P, t) \quad (52')$$

where

$$G(P, t) = E \int_0^T \left[-\ln \varepsilon + \frac{1}{Y_R} P \right] e^{-(\rho + \frac{\delta}{2}t)t} dt$$

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