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## Harsanyi Solutions in Line-graph Games

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# Harsanyi Solutions in Line-graph Games

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# Harsanyi solutions in line-graph games<sup>1</sup>

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## **Abstract**

Recently, applications of cooperative game theory to economic allocation problems have gained popularity. To understand these applications better, economic theory studies the similarities and differences between them. The purpose of this paper is to investigate a special class of cooperative games that generalizes some recent economic applications with a similar structure. These are so-called line-graph games being cooperative TU-games in which the players are linearly ordered. Examples of situations that can be modeled like this are sequencing situations, water distribution situations and political majority voting. The main question in cooperative game models of economic situations is how to allocate the earnings of coalitions among the players. We apply the concept of Harsanyi solution to line-graph games. We define four properties that each selects a unique Harsanyi solution from the class of all Harsanyi solutions. One of these solutions is the well-known Shapley value which is widely applied in economic models. We apply these solutions to the economic situations mentioned above.

Keywords: TU-game, Harsanyi dividends, Shapley value, sharing system, Harsanyi solution, line-graph game.

# 1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair  $(N, v)$ , where  $N = \{1, \dots, n\}$  is a finite set of  $n$  players and  $v: 2^N \rightarrow \mathbb{R}$  is a *characteristic function* on  $N$  such that  $v(\emptyset) = 0$ . For any coalition  $S \subseteq N$ , the real number  $v(S)$  is the *worth* of coalition  $S$ , i.e. the members of coalition  $S$  can obtain a total payoff of  $v(S)$  by agreeing to cooperate.

A *payoff vector* of an  $n$ -person TU-game is an  $n$ -dimensional vector giving a payoff to any player  $i \in N$ . A *point-valued solution* is a function  $f$  that assigns a single payoff vector  $f(v) \in \mathbb{R}^n$  to any game  $(N, v)$ . A point-valued solution  $f$  is *efficient* if it for any game  $(N, v)$  precisely distributes the worth  $v(N)$  of the grand coalition. An example of an efficient point-valued solution is the famous *Shapley value*, see Shapley (1953), being the average of the so-called *marginal value vectors*.

A *set-valued solution* is a mapping  $F$  that assigns to every game  $(N, v)$  a set of solution vectors  $F(v) \subset \mathbb{R}^n$ . A well-known set-valued solution is the Core. The *Core* of a game, introduced in game theory by Gillies (1953) is the set of all efficient and undominated payoff vectors, i.e. any payoff vector in the Core is efficient and at any payoff vector each coalition gets at least its own worth. Another set-valued solution is the *Selectope* (see Derks, Haller and Peters (2000)), also called the *Harsanyi Set* (see Vasil'ev and van der Laan (2002)), independently introduced by Hammer, Peled and Sorensen (1977) and Vasil'ev (1978b), respectively. This set is the collection of all payoff vectors obtained by distributing the (Harsanyi) *dividend* of each coalition  $S$  over the players in  $S$  in any possible way. To be more precisely, a *sharing system* assigns to any player in any coalition a nonnegative number, namely the share of that player in that coalition, with for any coalition the sum of the shares of its players in the coalition equal to one. Then a payoff vector is in the Harsanyi Set if there exists a sharing system such that any player gets a payoff equal to the sum of its shares in the dividends. For any game the Core is contained in the Harsanyi Set, with equality when the game is almost positive. Moreover, for any game the Harsanyi set is non-empty since it always contains the Shapley value which is obtained by applying the equal sharing system within each coalition.

In this paper we consider the class of Harsanyi solutions. This concept has been proposed in Vasil'ev (1982) and has recently been discussed extensively in Vasil'ev (2003). For a given sharing system, a point-valued solution is a *Harsanyi solution* when there is a fixed sharing system such that the solution assigns to any player a payoff equal to the sum of its corresponding shares in the Harsanyi dividends. So, the concept of a Harsanyi solution selects for any game the payoff vector in the Harsanyi set corresponding to the chosen sharing system.

In this paper we will consider the concept of Harsanyi solution on the class of line-graph games. In a standard TU-game players only differ with respect to their marginal contributions to the worths of the coalitions. Examples of models in which players not only differ with respect to their marginal contributions, but also are part of some relational structure (which possibly affects the cooperation possibilities) are *games in coalition structure* (see e.g. Aumann and Drèze (1974) or Owen (1977)) and *games with limited communication structure*. In the latter model players can cooperate only if they are connected within in a (communication) graph on the set of players (see e.g. Myerson (1977)). In this paper we consider a special type of games with limited communication structure, called *line-graph games*, in which the communication (graph) structure is given by a linear ordering on the set of players. Following Myerson (1977) and Greenberg and Weber (1986), in such a line-graph game, only consecutive players can communicate with each other. For this class of graph games we will define four different properties to be satisfied by a Harsanyi solution. Each of these four properties uniquely selects a Harsanyi solution from the class of all Harsanyi solutions, i.e. on the class of line-graph games, each of the four properties uniquely determines a sharing system yielding the corresponding Harsanyi solution. One of these solutions appears to be the well-known Shapley value. The concept of Harsanyi solution on line-graph games will be applied to sequencing games, see e.g. Curiel (1988) and Hamers (1995), the water distribution problem, see Ambec and Sprumont (2002) and to analyse political power in restricted majority games. In a sequencing game a set of jobs is in a queue to be processed on one machine. Depending on the waiting costs and processing times of each job, cost savings can be realized by jobs switching positions. Making switches so that we obtain an efficient queue (i.e. a queue that minimizes total costs), the main question then is how jobs that switch to positions later in the queue have to be compensated. In the water distribution problem agents are located along a river from upstream to downstream, and water flows into the river between each pair of agents. Each agent can consume its own water inflow, but can also decide to let water stream through to downstream agents. Depending on the utilities of the different agents for water, efficiency gains can be realized if an agent does not consume all its water. Main question then is how agents that let water flow downstream should be compensated for not consuming their water. In the political power majority games that we consider, the political parties are ordered on a line according to their political preferences concerning e.g. economic policy, ethics, environmental issues and so on. We measure power of those parties taking account of their position on this line.

This paper is organised as follows. Section 2 is a preliminary section containing the concepts of TU-game and Harsanyi solutions for TU-games. In Section 3 we discuss the concept of graph game, in particular the special class of line-graph games. In Section 4

we define four properties and we prove that each of these properties uniquely determines a Harsanyi solution on this class. These results are applied to the class of sequencing games in Section 5, to the water distribution problem in Section 6 and to majority games in Section 7. Finally, Section 8 contains a summary and some concluding remarks.

## 2 Harsanyi solutions for TU-games

In this paper we assume that  $N$  is a fixed set of players, allowing to denote a TU-game  $(N, v)$  shortly by its characteristic function  $v$ . We denote the collection of all TU-games on  $N$  by  $\mathcal{G}$ . We first recall some properties of TU-games. A TU-game  $v$  is *superadditive* if  $v(S \cup T) \geq v(S) + v(T)$  for any pair of subsets  $S, T \subseteq N$  such that  $S \cap T = \emptyset$ . Further, a TU-game  $v$  is *convex* if  $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$  for all  $S, T \subseteq N$ . A special class of convex games are unanimity games. For each nonempty  $T \subseteq N$ , the *unanimity* game  $u^T$  is given by  $u^T(S) = 1$  if  $T \subseteq S$ , and  $u^T(S) = 0$  otherwise. It is well-known that the unanimity games form a basis for  $\mathcal{G}$ . Moreover, denoting the collection of all nonempty subsets of  $N$  by  $\Omega$ , we have that

$$v = \sum_{S \in \Omega} \Delta^S(v) u^S,$$

where the Harsanyi dividends  $\Delta^S(v)$  (see Harsanyi, 1959) are given by

$$\Delta^S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T), \quad S \in \Omega. \quad (2.1)$$

Equivalently, by applying the Möbius transformation, we have that

$$v(S) = \sum_{T \subseteq S} \Delta^T(v), \quad S \in \Omega. \quad (2.2)$$

So, the worth of a coalition  $S$  is equal to the sum of the dividends off all its subcoalitions. This also gives a recursive definition of the dividends. The dividend of every one-player coalition is equal to its worth while, recursively, the dividend of every coalition with at least two players is equal to its worth minus the sum of the dividends of all its proper subcoalitions. In this sense the dividend of a coalition  $S$  can be interpreted as the extra contribution of the cooperation among the players in  $S$  that they did not already realize by cooperating in smaller coalitions.

If  $\Delta^S(v) \geq 0$  for all  $S \subseteq N$  with  $|S| \geq 2$  then we call game  $v$  *almost positive*. If, moreover,  $v(\{i\}) \geq 0$  for all  $i \in N$  then game  $v$  is called *totally positive*. It is well-known that every almost positive game is convex.



A set-valued solution  $F$  assigns a set  $F(v) \subset \mathbb{R}^n$  of payoff vectors to every TU-game  $v \in \mathcal{G}$ . A well-known set-valued solution is the *Core*, assigning to every game  $v$  the (possibly empty) set

$$C(v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = v(N), \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subseteq N\}.$$

It is well-known that  $C(v)$  is nonempty if and only if  $v$  is balanced (see Bondareva (1963)). Another set-valued solution is the *Selectope*, see Derks, Haller and Peters (2000), also called the *Harsanyi Set*, see Vasil'ev and van der Laan (2002), independently introduced by Hammer, Peled and Sorensen (1977) and Vasil'ev (1978b), respectively. This set is the collection of all payoff vectors obtained by distributing the dividend of each coalition  $S$  over the players in  $S$  in any possible way. To state this precisely, a sharing system is a system  $p = (p_i^S)_{S \in \Omega, i \in S}$ , assigning for each  $S \in \Omega$  a nonnegative share  $p_i^S$  to every player  $i \in S$ , summing up to 1, for each coalition  $S \in \Omega$ . For given  $N$ , the collection of sharing systems is given by

$$P = \{p = (p_i^S)_{S \in \Omega, i \in S} \mid p \geq 0, \sum_{j \in S} p_j^S = 1, \text{ for each } S \in \Omega\}.$$

For a game  $v$  and sharing system  $p \in P$ , let the payoff vector  $\phi^p(v) \in \mathbb{R}^n$  be given by

$$\phi_i^p(v) = \sum_{\{S \in \Omega \mid i \in S\}} p_i^S \Delta^S(v), \quad i \in N,$$

i.e. the payoff  $\phi_i^p(v)$  to player  $i \in N$  is the sum over all coalitions  $S \in \Omega$ , containing  $i$ , of the share  $p_i^S \Delta^S(v)$  of player  $i$  in the Harsanyi dividend of coalition  $S$ . We therefore call the payoff vector  $\phi^p(v)$  a *Harsanyi payoff vector*. Observe that, due to the equality  $v(N) = \sum_{S \in \Omega} \Delta^S(v)$ , for each sharing system  $p \in P$  it holds that  $\sum_{i \in N} \phi_i^p(v) = v(N)$ , and thus each Harsanyi payoff vector is efficient. The Harsanyi Set or Selectope is the set of all Harsanyi payoff vectors of the game  $v$ , i.e.

$$H(v) = \{\phi^p(v) \mid p \in P\}.$$

Clearly, by definition we have that  $H(v) \neq \emptyset$  for all  $v \in \mathcal{G}$ . Furthermore, it has been shown by Vasil'ev (1978a,b) that  $C(v) \subseteq H(v)$  with equality if and only if  $v$  is almost positive (for more details, see also Derks *et.al.* (2000) and Vasil'ev and van der Laan (2002)).

Now, a point-valued solution  $f$  on  $\mathcal{G}$  is called a *Harsanyi solution* when there exists a fixed sharing system  $p \in P$  such that

$$f(v) = \phi^p(v), \quad v \in \mathcal{G}.$$

The Harsanyi solutions were introduced by Vasil'ev (1982) (see also Vasil'ev (2003)). An example of this type of solution is the famous *Shapley value*,  $\psi(v)$ , defined by

$$\psi(v) = \phi^p(v), \text{ with } p_i^S = \frac{1}{|S|}, \quad i \in S, \quad S \in \Omega,$$

i.e. the Shapley value assigns to any game  $v \in \mathcal{G}$  the Harsanyi payoff vector which equally distributes the Harsanyi dividend of coalition  $S$  over the players in  $S$ .

For a permutation  $\pi: N \rightarrow N$ , assigning rank number  $\pi(i) \in N$  to any player  $i \in N$ , we define  $\pi^i = \{j \in N | \pi(j) \leq \pi(i)\}$ , i.e.  $\pi^i$  is the set of all players with rank number at most equal to the rank number of  $i$ , including  $i$  itself. Then the *marginal value vector*  $m^\pi(v) \in \mathbb{R}^n$  of game  $v$  and permutation  $\pi$  is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus \{i\}), \quad i \in N,$$

and thus assigns to player  $i$  its marginal contribution to the worth of the coalition consisting of all its predecessors in  $\pi$ . It is well-known that the Shapley value is equal to the average of the marginal value vectors over all permutations. Moreover, in case  $v$  is convex, the Core of  $v$  is equal to the convex hull of all marginal value vectors. For a permutation  $\pi$  on  $N$  and a coalition  $S \subseteq N$ , let  $i(S)$  be the player in  $S$  with the highest rank number, i.e.  $\pi(j) \leq \pi(i(S))$  for all  $j \in S$ . Further, let the sharing system  $p \in P$  be defined by  $p_j^S(\pi) = 1$  if  $j = i(S)$  and  $p_j^S(\pi) = 0$  for all  $j \in S, j \neq i(S)$ . Then it follows straightforward (see e.g. Derks *et.al.*, 2002) that

$$m^\pi(v) = \phi^{p(\pi)}(v),$$

i.e. the marginal value vector  $m^\pi(v)$  equals the Harsanyi payoff vector obtained by giving the dividend of any coalition  $S$  to the last player in  $S$  according to the permutation  $\pi$ . Hence, the point-valued solution  $f$  on  $\mathcal{G}$  assigning  $f(v) = m^\pi(v)$  to any  $v \in \mathcal{G}$  is a Harsanyi solution with fixed sharing system  $p = p(\pi)$ .

### 3 Harsanyi dividends in line-graph games

In order to apply Harsanyi solutions to line-graph games, we first have to discuss some notions and results for games with communication structure. In standard cooperative game theory it is assumed that any coalition of players may form. On the other hand, in many situations the collection of possible coalitions is restricted by some social, hierarchical, economical, communicational or technical structure. Such situations can often be modelled by a (directed) graph reflecting the structure. We refer to e.g. Myerson (1977, 1980), Owen (1986) and Borm *et.al.* (1994) for games with communication structures, Gilles *et.al.* (1992), van den Brink and Gilles (1996) and van den Brink (1997) for games with

permission structure, Algaba *et.al.* (2000) and Bilbao (1998) for games endowed with some abstract structure.

In this paper we consider cooperative games  $(N, v)$  in which the communication structure between the players is given by a *graph*  $(N, A)$  with the player set  $N$  as the set of nodes and with  $A \subseteq \{\{i, j\} \mid i, j \in N, i \neq j\}$  a collection of unordered pairs as the set of edges reflecting the communication possibilities. We shortly denote the game  $(N, v)$  with graph  $(N, A)$  as the *graph game*  $(v, A)$  and the collection of all graph games by  $\mathcal{G} \times \mathcal{E}$ , where  $\mathcal{E}$  is the set of all graphs on  $N$ . In a graph game  $(v, A) \in \mathcal{G} \times \mathcal{E}$ , players can only cooperate when they are able to communicate with each other. That means, a coalition  $S \subseteq N$  can only realise its worth  $v(S)$  when  $S$  is *connected* in the graph  $(N, A)$ , i.e. when for any two players  $i$  and  $j$  in  $S$ , there is a subset  $\{\{i_k, i_{k+1}\} \mid k = 1, \dots, t\} \in A$  of edges such that  $i_1 = i$ ,  $i_{t+1} = j$ , and  $\{i_2, \dots, i_t\} \subseteq S$ . When  $S$  is not connected, the players in  $S$  can only realise the sum of the worths of the components of the subgraph<sup>1</sup>  $(S, A(S))$ . For  $S \subseteq N$ , let  $C_A(S)$  be the collection of components (maximally connected subsets) of  $S$ , i.e.  $T \in C_A(S)$  if and only if (i)  $T \subseteq S$  is connected in the subgraph  $(S, A(S))$  and (ii)  $T \cup \{i\}$  is not connected in  $(S, A(S))$  for all  $i \in S \setminus T$ . Observe that the collection  $C_A(S)$  of components of  $S$  forms a partition of  $S$ .

Following Myerson (1977), for given graph game  $(v, A)$ , the so-called restricted game  $v^A \in \mathcal{G}$  induced by the graph  $(N, A)$ , is defined by

$$v^A(S) = \sum_{T \in C_A(S)} v(T), S \subseteq N. \quad (3.3)$$

Observe that  $S$  is the unique component of the subgraph  $(S, A(S))$  if and only if  $S$  is connected in  $(N, A)$ . Hence,  $v^A(S) = v(S)$  if  $S$  is connected in  $(N, A)$  (including the empty set), otherwise  $v^A(S)$  is the sum of the worths of the components in  $(S, A(S))$ .

For graph games, the number of terms on the right-hand side of formula (2.1) expressing the Harsanyi dividends as a sum of the worths of coalitions may reduce considerably when the graph is *cycle-complete*<sup>2</sup>. For these graphs, Bilbao (1998, Proposition 3) proves that

$$\Delta^S(v^A) = \begin{cases} 0, & \text{if } S \text{ is not connected,} \\ \sum_{\{T \subseteq S \mid (S \setminus Ex(S)) \subseteq T\}} (-1)^{|S| - |T|} v(T), & \text{if } S \text{ is connected,} \end{cases} \quad (3.4)$$

with  $Ex(S) = \{i \in S \mid S \setminus \{i\} \text{ is connected}\}$ . So, in the restricted game, the dividend of any not connected coalition is zero and the dividend of a connected coalition  $S$  is equal

<sup>1</sup>Given graph  $(N, A)$  and  $S \subseteq N$ , the subgraph  $(S, A(S))$  is defined by  $A(S) = \{\{i, j\} \in A \mid i, j \in S\}$ .

<sup>2</sup>A graph  $(N, A)$  is cycle-complete if  $(S, A(S))$  is a complete graph whenever the players in  $S$  form a cycle in  $A$ , i.e.  $A(S) = \{\{i, j\} \mid i, j \in S, i \neq j\}$  whenever the players in  $S$  can be labeled by  $S = \{i_1, \dots, i_t\}$  such that for  $i_{t+1} = i_1$  we have that  $\{i_k, i_{k+1}\} \in A$  for all  $k \in \{1, \dots, t\}$ . Examples of cycle-complete graphs are complete graphs and cycle free graphs.

to the alternating sum of the worths of the subsets containing all non-extreme nodes of  $S$ . In this paper we restrict attention to a subclass of cycle-free (and thus cycle-complete) graphs, the so-called line-graphs.

As mentioned before, some interesting economic situations, such as sequencing situations, water distribution problems and majority voting of political parties in parliament, can be modelled as cooperative games in which the communication structure is given by a linear ordering on the set of players. In such games the structure on the set of players is given by a line-graph, i.e. for some permutation  $\pi$  on  $N$  the communication structure is given by the graph  $(N, A(\pi))$ , with  $N$  the set of players and  $A(\pi) = \{\{\pi(i), \pi(i+1)\} \mid i = 1, \dots, n-1\}$  the set of edges. By renumbering the players we may assume without loss of generality that  $\pi(i) = i$ ,  $i = 1, \dots, n$ , so that the line-graph reflects the natural ordering from 1 to  $n$ . To simplify notation, in the remaining we denote  $L = \{\{i, i+1\} \mid i = 1, \dots, n-1\}$ . Clearly, for the line-graph  $(N, L)$ , the set  $\mathcal{L}$  of (nonempty) connected coalitions is given by

$$\mathcal{L} = \{S \subseteq N \mid S = [i, j], 1 \leq i \leq j \leq n\},$$

where  $[i, j]$  denotes the set  $\{i, i+1, \dots, j-1, j\} \subseteq N$  of consecutive players.

Following Myerson (1977) and Greenberg and Weber (1986), in the line-graph game  $(v, L)$ , players in a coalition  $S$  can only realise the worth  $v(S)$  when  $S = [i, j]$ . So, the restricted game  $v^L \in \mathcal{G}$  is given by

$$v^L(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{L}, \\ \sum_{T \in \mathcal{C}_L(S)} v(T), & \text{if } S \notin \mathcal{L}. \end{cases} \quad (3.5)$$

We now have the following theorem, where  $v[i, j]$  denotes  $v(\{i, i+1, \dots, j\})$ ,  $j \geq i$ .

**Theorem 3.1** *Let  $(v, L)$  be a line-graph game. Then the dividends of the restricted game  $v^L$  are given by*

$$\Delta^S(v^L) = \begin{cases} 0, & \text{if } S \notin \mathcal{L}, \\ v[i, j] - v[i+1, j] - v[i, j-1] + v[i+1, j-1], & \text{if } S = [i, j]. \end{cases} \quad (3.6)$$

**Proof** Since a line-graph does not contain any cycle, the theorem follows immediately from formula (3.4) by observing that  $S$  is not connected when  $S \notin \mathcal{L}$  and that  $Ex(S) = \{i, j\}$  when  $S = [i, j]$ .  $\square$

We say that a line-graph game  $(v, L)$  is *linear-convex* if for all  $i, j$  with  $j > i$  it holds that

$$v[i, j] - v[i+1, j] - v[i, j-1] + v[i+1, j-1] \geq 0.$$

Clearly, the line-graph game is linear-convex if  $(N, v)$  is convex, since linear-convexity only requires the convexity conditions  $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$  when  $S = [i + 1, j]$  and  $T = [i, j - 1]$  for some  $j > i$ . Recall that a game is *almost positive* (respectively *totally positive*) if the dividends of any coalition containing at least two players (respectively all coalitions) are non-negative. Hence, the next corollary follows immediately from Theorem 3.1.

**Corollary 3.2** *Let  $(v, L)$  be a linear-convex line-graph game. Then the restricted game  $v^L$  is almost positive. If also  $v(i) \geq 0$  for all  $i \in N$ , then  $v^L$  is totally positive.*

Recall that an almost (totally) positive game is convex. So, for convexity of the restricted game  $v^L$  it is sufficient that the line-graph game  $(v, L)$  is linear-convex. As we will see in the next sections, in many applications linear-convexity of a line-graph game is more easy to prove than convexity or superadditivity. Moreover, by Corollary 3.2 and coincidence of the Harsanyi Set and the Core of an almost positive game (established in Vasil'ev (1978b)) it follows that in case  $(v, L)$  is linear-convex, the Core of the restricted game  $v^L$  is equal to its Harsanyi Set, implying that any Harsanyi payoff vector of the restricted game  $v^L$  is a payoff vector in the Core of  $v^L$ .

It is well known that the restricted game of superadditive line-graph games is balanced, see e.g. Le Breton, Owen and Weber (1992) and Potters and Reynierse (1995). This result follows immediately from Granot and Huberman (1982), who showed that a permutational convex game is balanced. More precisely, let  $u$  and  $\ell$  be the two permutations on  $N$  defined by  $u(i) = i$ ,  $i = 1, \dots, n$ , respectively  $\ell(i) = n + 1 - i$ ,  $i = 1, \dots, n$ . Then it follows that when  $v$  is superadditive, the restricted game  $v^L$  satisfies the permutational convexity condition of Granot and Huberman for the two permutations  $u$  and  $\ell$ , from which it follows that the two marginal vectors  $m^u(v^L)$  and  $m^\ell(v^L)$  are in the Core of  $v^L$ .

## 4 Harsanyi solutions for line-graph games

We now consider Harsanyi solutions of the restricted game  $v^L$  of the line-graph game  $(v, L)$ . In particular we discuss some properties which select a unique Harsanyi solution. Recall from Theorem 3.1 that for any line-graph game  $(v, L)$  the dividend of a coalition  $S$  in the restricted game is equal to zero when  $S$  is not in the set  $\mathcal{L}$  of connected coalitions. In the following, let  $\mathcal{G}^L \subset \mathcal{G}$  be the set of all games  $w$  satisfying  $\Delta^S(w) = 0$  when  $S \notin \mathcal{L}$ . So, any restricted game  $v^L$  is in the subset  $\mathcal{G}^L$ . Reversely, it follows straightforward that for any  $w \in \mathcal{G}^L$  the line-graph game  $(w, L)$  satisfies  $w^L = w$ . Further, by defining  $f(v, L) = g(v^L)$ , a point-valued solution  $g$  on  $\mathcal{G}^L$  assigning a vector  $g(w) \in \mathbb{R}^n$  to any game  $w \in \mathcal{G}^L$  induces a solution  $f$  on the subclass  $\mathcal{G} \times \{L\}$  of  $\mathcal{G} \times \mathcal{E}$  of all line-graph games. So, considering

solutions for the class of line-graph games  $\mathcal{G} \times \{L\}$  reduces to considering solutions on the subclass  $\mathcal{G}^L$  of  $\mathcal{G}$ . To introduce some properties of solutions on the class  $\mathcal{G}^L$ , let, for  $i = 1, \dots, n-1$ ,  $(N, L(i))$  be the graph on  $N$  with  $L(i) = L \setminus \{(i, i+1)\}$  as the set of edges obtained by deleting the edge  $(i, i+1)$  from  $L$ . Although the induced graph game  $(v, L(i))$  is not a line-graph game, the corresponding restricted game  $v^{L(i)}$  is well defined by formula (3.3) and belongs to the class  $\mathcal{G}^L$ , since any coalition  $S$  that is not connected in  $(N, L)$  is also not connected in  $(N, L(i))$ . We now state the following properties.

- Definition 4.1** 1. A point-valued solution  $f$  on  $\mathcal{G}^L$  is called **upper equivalent** if for any  $i = 1, \dots, n-1$  and any  $v \in \mathcal{G}$  it holds that  $f_j(v^{L(i)}) = f_j(v^L)$ ,  $j = 1, \dots, i$ .
2. A point-valued solution  $f$  on  $\mathcal{G}^L$  is called **lower equivalent** if for any  $i = 1, \dots, n-1$  and any  $v \in \mathcal{G}$  it holds that  $f_j(v^{L(i)}) = f_j(v^L)$ ,  $j = i+1, \dots, n$ .
3. A point-valued solution  $f$  on  $\mathcal{G}^L$  is said to have the **equal loss property** if for any  $i = 1, \dots, n-1$  and any  $v \in \mathcal{G}$  it holds that  $\sum_{j=1}^i (f_j(v^L) - f_j(v^{L(i)})) = \sum_{j=i+1}^n (f_j(v^L) - f_j(v^{L(i)}))$ .
4. A point-valued solution  $f$  on  $\mathcal{G}^L$  is called **fair** if for any  $i = 1, \dots, n-1$  and any  $v \in \mathcal{G}$  it holds that  $f_i(v^L) - f_i(v^{L(i)}) = f_{i+1}(v^L) - f_{i+1}(v^{L(i)})$ .

The last property is the fairness property introduced already by Myerson (1977) and states that deleting the edge between  $i$  and  $i+1$  hurts both players  $i$  and  $i+1$  equally. The equal loss property is also some kind of fairness, but instead of individual payoffs for the players on the edge that is deleted, it states that the total payoff of the players at both sides of the deleted edge change by the same amount. Upper equivalence means that the payoff of a player does not depend on the presence of downward edges, while lower equivalence means that the payoff of a player does not depend on the presence of upward edges. Which property is most appropriate depends on the application that is in mind and will be discussed after the next theorem and in the applications in the next sections.

The next theorem says that each of these four properties uniquely determines a Harsanyi solution on the class  $\mathcal{G}^L$  of restricted games. To state the theorem, let  $f^u$ ,  $f^\ell$ ,  $f^e$  and  $f^s$  be the point-valued solutions on  $\mathcal{G}$  defined by  $f^u(v) = m^u(v)$ ,  $f^\ell(v) = m^\ell(v)$ ,  $f^e(v) = \frac{1}{2}(m^u(v) + m^\ell(v))$  and  $f^s(v) = \psi(v)$  for all  $v \in \mathcal{G}$ , i.e.  $f^u$  assigns the marginal value vector  $m^u(v)$  to each game  $v$ ,  $f^\ell$  the marginal value vector  $m^\ell(v)$ ,  $f^e$  the average of these two vectors<sup>3</sup> and  $f^s$  the Shapley value  $\psi(v)$ .

**Theorem 4.2** Let  $f: \mathcal{G}^L \rightarrow \mathbb{R}^n$  be a Harsanyi solution on the class  $\mathcal{G}^L$  of restricted games. Then,

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<sup>3</sup>For sequencing games the function  $f^e$  has been introduced in Curiel *et. al.* (1993, 1994) as the  $\beta$ -rule, see also the next section.

1.  $f$  is upper equivalent if and only if  $f = f^u$ .
2.  $f$  is lower equivalent if and only if  $f = f^\ell$ .
3.  $f$  satisfies the equal loss property if and only if  $f = f^e$ .
4.  $f$  is fair if and only if  $f = f^s$ .

## Proof

1. First, recall from Section 2 that  $f^u$ , being the marginal value vector with respect to permutation  $u(i) = i$ ,  $i = 1, \dots, n$ , is the Harsanyi solution  $\phi^{\bar{p}}$  with fixed sharing system  $\bar{p}_k^S = 1$  if  $k = i(S) = \max\{h \mid h \in S\}$  and  $\bar{p}_k^S = 0$  otherwise, for any  $S \in \Omega$ . That this indeed is the case can be easily verified since for any  $j$ ,

$$\sum_{i=1}^j \phi_i^{\bar{p}}(v) = \sum_{i=1}^j \sum_{\{S \in \Omega \mid i=i(S)\}} \Delta^S(v) = \sum_{\{S \subseteq [1,j]\}} \Delta^S(v) = v[1,j],$$

where the last equality follows from formula (2.2). Hence  $f_j^u(v) = m_j^u(v) = v[1,j] - v[1,j-1] = \phi_j^{\bar{p}}(v)$ ,  $j = 1, \dots, n$ .

Second, to show that  $f^u$  is upper equivalent note that for any  $v \in \mathcal{G}$  and for all  $i = 1, \dots, n-1$  we have by definition of the restricted games that  $v^L[1,j] = v^{L(i)}[1,j]$ ,  $j = 1, \dots, i$ . Hence  $f_j^u(v^L) = m_j^u(v^L) = v^L[1,j] - v^L[1,j-1] = v^{L(i)}[1,j] - v^{L(i)}[1,j-1] = m_j^u(v^{L(i)}) = f_j^u(v^{L(i)})$  for  $j = 1, \dots, i$ . So,  $f^u$  is upper equivalent.

It remains to show that  $f = f^u$  for any upper equivalent Harsanyi solution  $f$  on  $\mathcal{G}^L$ . So, for some  $p \in P$ , let  $f = \phi^p$  be upper equivalent. Since for any  $w \in \mathcal{G}^L$ ,  $\Delta^S(w) = 0$  if  $S \notin \mathcal{L}$ , it is sufficient to consider the shares  $p_j^S$ ,  $j \in S$  when  $S = [h,k]$  for some  $1 \leq h \leq k \leq n$ . When  $h = k$  we have that  $S = \{h\}$  and  $p_h^S = 1$  by definition. So, consider  $k > h$  and suppose that  $p_i^{[h,k]} > 0$  for some  $i$ ,  $h \leq i \leq k-1$ . Now, take  $w = u^{[h,k]} \in \mathcal{G}$  being the unanimity game of coalition  $[h,k]$ . Then  $w^L = w = u^{[h,k]}$  and thus  $\Delta^S(w^L) = 1$  if  $S = [h,k]$  and zero otherwise. Hence  $\phi_i^p(w^L) = p_i^{[h,k]} > 0$ . However, for  $w = u^{[h,k]}$ , the restricted game  $w^{L(k-1)}$  is equal to the zero game (the game assigning zero worth to any coalition  $S$ ) with all dividends equal to zero. Hence  $\phi_i^p(w^{L(k-1)}) = 0$ , contradicting the upper equivalence property. It follows that  $p_i^{[h,k]} = 0$  for any  $i$ ,  $h \leq i \leq k-1$ , implying that  $p_k^{[h,k]} = 1$  for any coalition  $[h,k] \in \mathcal{L}$ . So, the dividend of any coalition  $[h,k]$  is assigned to the last player  $k$ , implying that  $f = f^u$ . This proves the first statement.

2. The proof of this case is analogous to the case above. First, from Section 2 we know that  $f^\ell$ , being the marginal value vector w.r.t the permutation  $\ell(i) = n + 1 - i$ ,  $i = 1, \dots, n$ , is the Harsanyi solution  $\phi^p$  with fixed sharing system  $\underline{p}_k^S = 1$  if  $k = j(S) = \min\{h | h \in S\}$  and  $\underline{p}_k^S = 0$  otherwise, for any  $S \in \Omega$ . In a similar way as under 1 for  $f^u$ , this can be easily verified since for any  $j$ , it follows that  $f_j^\ell(v) = v[j, n] - v[j + 1, n] = \phi_j^p(v)$ ,  $j = 1, \dots, n$ .

Second, for any  $v \in \mathcal{G}$  and for all  $i = 1, \dots, n - 1$ , we have that  $v^L[j, n] = v^{L(i)}[j, n]$ ,  $j = i + 1, \dots, n$  and thus  $f_j^\ell(v^L) = f_j^\ell(v^{L(i)})$  for  $j = i + 1, \dots, n$ . Hence  $f^\ell$  is a lower equivalent Harsanyi solution.

Next, for some  $p \in P$ , let  $f = \phi^p$  be a lower equivalent Harsanyi solution on  $\mathcal{G}^L$ . To show that  $f = f^\ell$ , again we only have to consider the shares  $p_j^S$ ,  $j \in S$  when  $S = [h, k]$  for some  $1 \leq h < k \leq n$ . Now, suppose that  $p_i^{[h, k]} > 0$  for some  $i$ ,  $h + 1 \leq i \leq k$  and let  $w = u^{[h, k]} \in \mathcal{G}$  be the unanimity game of  $[h, k]$ . Then  $\phi_i^p(w^L) = p_i^{[h, k]} > 0$  and  $\phi_i^p(w^{L(h)}) = 0$ , contradicting the lower equivalence property. It follows that  $p_i^{[h, k]} = 0$  for any  $i$ ,  $h + 1 \leq i \leq k$ , implying that  $p_h^{[h, k]} = 1$  for any coalition  $S = [h, k]$ . So, the dividend of any coalition  $[h, k]$  is assigned to the first player  $h$ , implying that  $f = f^\ell$ . This proves the second statement.

3. From the first two cases it follows immediately that  $f_j^e(v) = \frac{1}{2}(\phi_j^{\bar{p}}(v) + \phi_j^p(v)) = \widehat{\phi}_j^{\hat{p}}(v)$ ,  $j = 1, \dots, n$ , where  $\hat{p} = \frac{1}{2}(\bar{p} + p)$ . Hence,  $f^e$  is a Harsanyi solution. Further,  $\sum_{j=1}^n f_j^u(v) = \sum_{j=1}^n m_j^u(v) = \sum_{j=1}^n m_j^\ell(v) = \sum_{j=1}^n f_j^\ell(v) = v(N)$  for any  $v \in \mathcal{G}$ . Applying this to  $v^L$  and  $v^{L(i)}$  for some  $i = 1, \dots, n$ , it follows from the upper and lower equivalency of  $f^u$ , respectively  $f^\ell$  that

$$\begin{aligned} \sum_{j=1}^i (m_j^u(v^L) - m_j^u(v^{L(i)})) &= 0 = \sum_{j=i+1}^n (m_j^\ell(v^L) - m_j^\ell(v^{L(i)})) \quad \text{and} \\ \sum_{j=1}^i (m_j^\ell(v^L) - m_j^\ell(v^{L(i)})) &= v^L(N) - v^{L(i)}(N) = \sum_{j=i+1}^n (m_j^u(v^L) - m_j^u(v^{L(i)})). \end{aligned}$$

Summing up these two equations gives  $2 \sum_{j=1}^i (f_j^e(v^L) - f_j^e(v^{L(i)})) = 2 \sum_{j=i+1}^n (f_j^e(v^L) - f_j^e(v^{L(i)}))$ , showing that  $f^e$  satisfies the equal loss property.

Next, for some  $p \in P$ , let  $f = \phi^p$  be a Harsanyi solution on  $\mathcal{G}^L$  satisfying the equal loss property. To show that  $f = f^e$ , again we only have to consider the shares  $p_j^S$ ,  $j \in S$  when  $S = [h, k]$  for some  $1 \leq h < k \leq n$ . Let  $w = u^{[h, k]} \in \mathcal{G}$  be the unanimity game of  $[h, k]$ . Then, according to the equal loss property, we must have that

$$\sum_{j=1}^i (\phi_j^p(w^L) - \phi_j^p(w^{L(i)})) = \sum_{j=i+1}^n (\phi_j^p(w^L) - \phi_j^p(w^{L(i)})), \text{ for any } i, h \leq i \leq k - 1.$$



Since  $w^L = w = u^{[h,k]}$  and  $w^{L(i)}$  is a zero game, it must hold that

$$\sum_{j=h}^i p_j^{[h,k]} = \sum_{j=i+1}^k p_j^{[h,k]}, \text{ for each } i, h \leq i \leq k-1.$$

Solving these  $k-h$  equations together with  $\sum_{j=h}^k p_j^{[h,k]} = 1$ , it follows that  $p_h^{[h,k]} = p_k^{[h,k]} = \frac{1}{2}$  and  $p_j^{[h,k]} = 0$  for all  $j, h < j < k$ . Hence for any  $S = [h, k] \in \mathcal{L}$  it follows that  $p^{[h,k]} = \frac{1}{2}(\bar{p}^{[h,k]} + \underline{p}^{[h,k]})$ . Hence  $f = \phi^p = f^e$ . This proves the third statement.

4. We know already that the Shapley value  $f^s$  is the Harsanyi solution with  $p_j^S = \frac{1}{|S|}$  for all  $j \in S, S \in \Omega$ . Next, observe that for any  $v \in \mathcal{G}$  and each  $i = 1, \dots, n-1$ , it holds that  $\Delta^S(v^L) = \Delta^S(v^{L(i)})$  when  $S = [h, k]$  with  $1 \leq h \leq k \leq i$  or  $i+1 \leq h \leq k \leq n$ , and that  $\Delta^S(v^{L(i)}) = 0$  when  $S = [h, k]$  with  $1 \leq h \leq i$  and  $i+1 \leq k \leq n$ . Define  $w = v^L - v^{L(i)}$ . Then  $\Delta^S(w) = 0$  for all  $S = [h, k]$  with  $1 \leq h \leq k \leq i$  or  $i+1 \leq h \leq k \leq n$  and  $\Delta^S(w) = \Delta^S(v^L)$  when  $S = [h, k]$  with  $1 \leq h \leq i$  and  $i+1 \leq k \leq n$ . Hence  $i$  and  $i+1$  are symmetric players in  $w$ . It follows from the linearity and symmetric player property<sup>4</sup> of the Shapley value that  $f_i(v^L) - f_i(v^{L(i)}) = f_i(w) = f_{i+1}(w) = f_{i+1}(v^L) - f_{i+1}(v^{L(i)})$ , showing that  $f^s$  is fair.

To show that  $f = f^s$  for any fair Harsanyi solution  $f$  on  $\mathcal{G}^L$ , for some  $p \in P$  suppose  $f = \phi^p$  is fair. Again, we only have to consider the shares  $p_j^S, j \in S$  when  $S = [h, k]$  for some  $1 \leq h < k \leq n$ . Let  $w = u^{[h,k]} \in \mathcal{G}$  be the unanimity game of  $[h, k]$ . Then the fairness property requires that

$$\phi_i^p(w^L) - \phi_i^p(w^{L(i)}) = \phi_{i+1}^p(w^L) - \phi_{i+1}^p(w^{L(i)}), \text{ for any } h \leq i < k.$$

Since  $w^L = w = u^{[h,k]}$  and  $w^{L(i)}$  is a zero game, it must hold that

$$p_i^{[h,k]} = p_{i+1}^{[h,k]}, \text{ for each } i, h \leq i < k.$$

Solving these  $k-h$  equations together with  $\sum_{j=h}^k p_j^{[h,k]} = 1$ , it follows that  $p_j^{[h,k]} = \frac{1}{k-h+1}$  for all  $j, h \leq j \leq k$ . Hence for any  $S = [h, k] \in \mathcal{L}$  and  $j \in S$ , it follows that  $p_j^S = \frac{1}{|S|}$ , implying that  $f = \phi^p = f^s$ . This proves the last statement.

□

The first three assertions of Theorem 4.2 show that any of the functions  $f^u, f^\ell, f^e$  is characterized as a Harsanyi solution satisfying one additional property. Observe that

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<sup>4</sup>A point-valued solution  $f$  on  $\mathcal{G}$  is linear if  $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$  for all  $v, w \in \mathcal{G}$  and  $\alpha, \beta \in \mathbb{R}$ , where  $(\alpha v + \beta w) \in \mathcal{G}$  is defined by  $(\alpha v + \beta w)(S) = \alpha v(S) + \beta w(S)$  for all  $S \subseteq N$ . A point-valued solution  $f$  satisfies the symmetric player property on  $\mathcal{G}$  if  $f_i(v) = f_j(v)$  whenever  $v(S \cup \{i\}) = v(S \cup \{j\})$  for all  $S \subseteq N \setminus \{i, j\}$ .

a Harsanyi solution satisfies efficiency by definition. Myerson (1977) showed already that on the class of graph-restricted games the Shapley value  $f^s$  is characterised by component efficiency and fairness, see van den Brink (2001) for a related result on the class of cooperative games<sup>5</sup>. So, the first part of the proof of assertion 4 also follows immediately from Myerson. However, the second part of this proof shows that also on the (smaller) subclass of line-graph games, the fairness property uniquely determines the Shapley distribution of the dividends within the set of Harsanyi solutions.

The theorem also shows that the upper equivalence property implies that in any coalition  $[h, k]$  the dividend is fully given to the last player  $k$ , whereas the lower equivalent property implies that the dividend of  $[h, k]$  goes to the first player  $h$ . Which of these rules is the most appropriate one, may depend on the underlying situation. To give an example, suppose that any edge  $(i, i + 1)$  is under control of player  $i$ ,  $i = 1, \dots, n - 1$ , i.e. player  $i$  has the power to decide on whether to keep the edge or to delete the edge from the sequence of edges. In case  $i$  deletes the edge, no cooperation between the players in front of the edge and the players after the edge is possible anymore. In this situation the lower equivalent property seems to be the most appropriate solution. First, because any profit of cooperation is distributed upwards to the first player in a coalition, being in control of whether or not to keep the edge. Second, the lower equivalent property says that, when  $i$  deletes the edge between  $i$  and  $i + 1$ , the players after  $i$  are not hurt. So, only players  $j$  in front of  $i$  and  $i$  itself will suffer from deleting the edge  $(i, i + 1)$ , giving player  $i$  a strong incentive not to delete the edge. We will illustrate this with some specific examples in the next sections. Similarly, the upper equivalent property and thus the solution  $f^u$  seems to be more appropriate when player  $i$  is in control of the edge  $(i - 1, i)$ ,  $i = 2, \dots, n$ . Consequently, the function  $f^e$  satisfying the equal loss property may be appropriate when both  $i$  and  $i + 1$  have equal control on the edge  $(i, i + 1)$ . The Shapley value indicates that all players share equally the control on any edge. Note that according to the Shapley value all players share equal control over the edges, but fairness only equalizes the change in payoffs of the two players on the deleted edge. According to  $f^e$  the two players on an edge have equal control over that edge, but the equal loss property equalizes the change in total payoffs of all players at both sides of the deleted edge.

Finally, recall from the previous section that  $v^L$  is permutationally convex with respect to the permutations  $u$  and  $\ell$  when  $v$  is superadditive. As a result it follows that for any superadditive line graph game  $(v, L)$ , both the lower equivalent Harsanyi solution and the upper equivalent Harsanyi solution are in the Core of the game and hence also the equal loss property Harsanyi solution  $f^e$  is in the Core. The Shapley value may be outside

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<sup>5</sup>An non-cooperative implementation of the Shapley value can be found in Pérez-Castillo and Wettstein (2001).

the Core, for an example see Section 7.

## 5 Sequencing games

A one-machine sequencing situation, see e.g. Curiel (1988) or Hamers (1995) is described as a triple  $(N, p, q)$ , where  $N = \{1, \dots, n\}$  is the set of jobs in a queue to be processed,  $p \in \mathbb{R}_+^n$  is an  $n$ -vector with  $p_i$  the processing time of job  $i$  and  $q = (q_i)_{i \in N}$  is a collection of cost functions  $q_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , specifying the costs  $q_i(t)$  where  $t$  is the total time needed to complete job  $i$ . For a permutation<sup>6</sup>  $\rho$  on  $N$  describing the positions of the jobs in the queue, the completion time of job  $i$  is given by  $T_i(\rho) = \sum_{\{j | \rho(j) \leq \rho(i)\}} p_j$ , i.e. the sum of its waiting time and its own processing time, and the costs of processing  $i$  are given by  $C_i(\rho) = q_i(T_i(\rho))$ . The total costs of a coalition  $S \subseteq N$  given a permutation  $\rho$  are given by  $C_S(\rho) = \sum_{i \in S} C_i(\rho)$ . In the sequel we assume without loss of generality that the initial positions of the jobs in the queue are given by the permutation  $\rho^0$  with  $\rho^0(i) = i$  for all  $i \in N$ , so that the costs of a coalition  $S$  of jobs according to  $\rho^0$  are given by

$$C_S(\rho^0) = \sum_{i \in S} C_i(\rho^0) = \sum_{i \in S} q_i \left( \sum_{\{j | j \leq i\}} p_j \right), \quad S \subseteq N.$$

Now, each coalition  $S$  of jobs can obtain cost savings by rearranging the jobs amongst the members of  $S$ . Then the minimal cost of the grand coalition is given by

$$C_N = \min_{\rho} C_N(\rho).$$

However, members of any other coalition  $S$  can only rearrange their positions under the condition that the members of  $S$  are not allowed to ‘jump’ over jobs outside  $S$ . So, a permutation  $\rho$  is admissible for  $S$  if for any  $j \notin S$  the set of its predecessors does not change with respect to the initial situation, i.e. if for any  $j \notin S$  it holds that  $\{k \in N \mid \rho(k) < \rho(j)\} = \{k \in N \mid k < j\}$ . Let  $\mathcal{A}(S)$  be the set of admissible permutations for  $S$ . Then the minimal cost of  $S$  is given by

$$C_S = \min_{\rho \in \mathcal{A}(S)} C_S(\rho).$$

This gives the cost savings *sequencing* game  $(N, v)$  with the set of players  $N$  representing the jobs<sup>7</sup> and characteristic function  $v$  given by

$$v(S) = C_S(\rho^0) - C_S, \quad S \subseteq N.$$

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<sup>6</sup>In this section we denote permutations by  $\rho$  to distinguish an order in a queue from an order  $\pi$  in marginal value vectors.

<sup>7</sup>Alternatively, we can think of each player having one job to be done and in the following game situation these players are negotiating over the side payments.

From now on, in this section we refer to the jobs as players. Obviously, since only permutations in  $\mathcal{A}(S)$  are admissible, only connected coalitions (i.e. coalitions of consecutive players) can realise cost reductions. So, taking the line-graph  $(N, L)$  as defined in the previous section, it follows immediately that the characteristic functions  $v$  of the line-graph game  $(v, L)$  and the restricted game  $v^L$  are equal to each other and are given by

$$v(S) = v^L(S) = \sum_{T \in \mathcal{C}_L(S)} v(T), \quad S \subseteq N.$$

By definition,  $v$  is superadditive and thus  $v^L$  is permutationally convex with respect to  $u$  and  $\ell$ . Hence, each of the solutions  $f^u$ ,  $f^\ell$  and  $f^e$  provide a Core solution for distributing the worth  $v(N)$ , see also Curiel *et.al.* (1993, 1994), where this has been shown for  $f^e$  (in sequencing games known as the  $\beta$ -rule).

We now consider the case of linear cost, i.e.  $q_i(t) = \alpha_i t$  for all  $t \geq 0$  with  $\alpha_i > 0$ , in more detail. In this case it is well-known that for each coalition  $[i, j] \in \mathcal{L}$  holds

$$v[i, j] = \sum_{\{k, h \in [i, j] | k < h\}} g_{kh},$$

where  $g_{kh} = \max(0, \alpha_h p_k - \alpha_k p_h)$  is the gain of a switch between player  $k$  and  $h$  in any permutation such that player  $k$  is directly in front of  $h$ , see e.g. Curiel (1988) or Hamers (1995). Applying Theorem 3.1 we obtain for any  $[i, j] \in \mathcal{L}$ ,  $i < j$ , that

$$\Delta^{[i, j]}(v) = \Delta^{[i, j]}(v^L) = v[i, j] - v[i + 1, j] - v[i, j - 1] + v[i + 1, j - 1] = g_{ij} \geq 0.$$

Further, observe that  $\Delta^{\{i\}}(v) = v(\{i\}) = 0$ ,  $i \in N$ , so that all dividends are nonnegative and thus the game  $v$  is totally positive. Hence, in this case also the Shapley value is in the Core. In fact, recall that for totally positive games the Harsanyi set equals the Core and therefore any Harsanyi solution gives a payoff vector in the Core.

In the previous section we have seen that the Harsanyi solution  $f^e$  distributes the dividend  $g_{ij}$  of any coalition  $[i, j]$  equally over the first and the last player of the coalition. Hence, it follows immediately that

$$f_h^e(v) = \frac{1}{2} \left( \sum_{j > h} g_{hj} + \sum_{i < h} g_{ih} \right), \quad h \in N,$$

i.e.  $f^e(v)$  gives the well-known so-called Equal Gain Splitting (EGS) payoff vector (which thus also coincides with the  $\beta$ -rule for general sequencing situations). Characterisations of the EGS-solution for linear costs sequencing situations have been given by Curiel (1988) and Hamers (1995). For instance, the latter author shows that the EGS solution is the

unique solution satisfying efficiency, the equivalence property<sup>8</sup> and the switch property<sup>9</sup>. Here we have shown that the EGS-solution is the unique Harsanyi solution satisfying the equal loss property.

We now consider the upper equivalent Harsanyi solution  $f^u$ , giving the dividend of any coalition  $[i, j]$  to the last player  $j$  in the coalition. The net-costs of player  $h$  resulting from this solution are given by the costs of the waiting time in the initial order minus the savings obtained from cooperation, i.e. the net-costs  $c_h^u(N, v^L)$  of player  $h$  in the linear cost sequencing situation  $(N, p, q)$  induced by  $f^u(v^L)$  is given by

$$c_h^u(N, v^L) = C_h(\rho^0) - f_h^u(v^L) = C_h(\rho^0) - \sum_{j < h} g_{jh}, \quad h \in N. \quad (5.7)$$

In Fernández, Borm, Hendricks and Tijs (2002) it is shown that this cost-assignment rule is the unique solution being *stable* and satisfying the so-called property of *Drop Out Monotonicity* (DOM). Stability means that  $c^u(N, v^L)$  is in the Core of the cost-game for any linear cost sequencing situation  $(N, p, q)$ . Clearly, it corresponds to the fact that  $f^u(v^L)$  is in the Core of the cost-savings game. To state DOM, let  $(N_{-k}, p_{-k}, q_{-k})$  with player set  $N_{-k} = N \setminus \{k\}$  be the  $(n - 1)$ -player sequencing situation obtained when player  $k$  leaves the queue (i.e. job  $k$  is cancelled) and let  $L_{-k}$  be the line graph on  $N_{-k}$  given by  $L_{-k} = \{(k-1, k+1)\} \cup \{(i, i+1) \mid i = 1, \dots, k-2, k+1, \dots, n-1\}$ . Then a cost assignment rule  $r$  assigning costs  $r_h(N, v^L)$  for all  $h \in N$  satisfies DOM if for any linear cost situation  $(N, p, q)$  it holds that

$$r_h(N_{-k}, v^{L_{-k}}) \leq r_h(N, v^L), \quad h \in N_{-k},$$

i.e. if one of the players leaves the queue, for each of the remaining players the costs are nonincreasing.

To show that cost rule  $c^u$  induced by  $f^u$  indeed satisfies DOM, let  $\rho_{-k}^0$  be the initial order  $\rho^0$  restricted to  $N_{-k}$ . Then

$$\begin{aligned} c_h^u(N_{-k}, v^{L_{-k}}) &= C_h(\rho_{-k}^0) - \sum_{j < h} g_{jh} \\ &= C_h(\rho^0) - \sum_{j < h} g_{jh} = c_h(N, v^L), \quad h = 1, \dots, k-1, \end{aligned}$$

and

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<sup>8</sup>The equivalence property states that when two initial orders only differ with respect to the mutual positions of the players in front of some player  $h$ , then this player gets the same payoff in both situations.

<sup>9</sup>The switch property states that if two consecutive players switch position, then both players get the same change of their payoffs in the new cost savings game when compared with their payoffs in the original game.

$$\begin{aligned}
c_h^u(N_{-k}, v^{L-k}) &= C_h(\rho_{-k}^0) - \sum_{\{j < h | j \neq k\}} g_{jh} \\
&= C_h(\rho^0) - \alpha_h p_k - \left( \sum_{j < h} g_{jh} - g_{kh} \right) \\
&= c_h(N, v^L) - (\alpha_h p_k - \max(0, \alpha_h p_k - \alpha_k p_h)) \\
&= c_h(N, v^L) - \min(\alpha_h p_k, \alpha_k p_h) < c_h(N, v^L), \quad h = k + 1, \dots, n.
\end{aligned}$$

So, for the players in front of  $k$  there is no change in the net-costs (this reflects the upper equivalent property of  $f^u$ ), whereas for any player  $h$  after  $k$  the decrease  $\alpha_h p_k$  in initial costs when  $k$  leaves the queue is bigger than the loss of the dividend  $g_{kh}$ .

The DOM property advocated in Fernández *et al.* (2002) seems to be very appealing and reasonable: when player  $k$  drops out, the players in front of  $k$  are not affected, while for the players after  $k$  the costs are decreasing. However, any medal has two sides. To highlight the other side, suppose that player  $i$  is not willing to accept any permutation  $\rho$  that places her after a player  $k > i$ . This refusal of  $i$  to cooperate with the players after her in the queue results in the graph game  $(v, L(i))$  in which the edge  $(i, i + 1)$  has been deleted from the line-graph  $L$ . As a consequence of this refusal of  $i$ , the dividends  $g_{jh}$  of the coalitions  $[j, h]$ ,  $j \leq i < h$  can not be realised anymore. So, according to the upper equivalent solution  $f^u$  the net-cost becomes equal to

$$c_h^u(N, v^{L(i)}) = \begin{cases} C_h(\rho^0) - \sum_{j < h} g_{jh} = c_h(N, v^L), & h = 1, \dots, i, \\ C_h(\rho^0) - \sum_{i+1 \leq j < h} g_{jh}, & h = i + 1, \dots, n. \end{cases}$$

Comparing this with equation (5.7), it follows again that the costs do not change for the players before the edge  $(i, i + 1)$ , whereas a player  $h$  after the edge loses all dividends  $g_{jh}$ ,  $j \leq i$ , and therefore suffers from an increase in the costs with  $\sum_{j \leq i} g_{jh}$ . So, the upper equivalent solution has the serious drawback that it does not give any incentive to a player  $i$  to cooperate with its successors in the queue: not cooperating does not hurt her. It only hurts players that come after  $i$  in the queue.

To make the point clearly, we just consider a two player sequencing situation. Of course, nothing happens if the initial order of 1 before 2 is optimal already. So, suppose it is optimal to reverse the initial order and to place 2 in front of 1 generating the decrease  $g_{12}$  of the costs. However, according to the upper equivalence (drop out monotonic) rule  $f^u$ , this decrease is fully assigned to player 2. Why should player 1 be willing to cooperate? On the contrary, player 1 has the power to play the noncooperative ultimatum game and to offer the first place in the queue to player 2 if player 2 is willing to give all the gains of this change to player 1, i.e. player 1 can sell his place against a price equal to  $\alpha_1 p_2$  (the additional costs of waiting for player 1) plus gains  $g_{12}$  of this trade. Since player 2 is indifferent to accepting this offer or not, there is no reason to refuse, and certainly if player 1 offers his place against a slightly lower price it is beneficial for player 2 to accept the offer.

In fact, player 2 needs the cooperation of player 1 to become the first player in the queue, or in words of the previous section, player 1 is in control of the edge  $(1, 2)$ . Extending this reasoning we could say that any player  $i < n$  is in control of the edge  $(i, i + 1)$ . As argued in the previous section, instead of using  $f^u$  as solution rule, this asks for the lower equivalent solution  $f^\ell$  assigning the full dividend  $g_{ij}$  of any coalition  $[i, j]$  to its first player  $i$ . The resulting costs induced by  $f^\ell$  are given by

$$c_h^\ell(N, v^L) = C_h(\rho^0) - f_h^\ell(v^L) = C_h(\rho^0) - \sum_{j>h} g_{hj}, \quad h \in N.$$

Of course, this cost rule does not satisfy DOM, but any player  $i$  is willing to cooperate with its successors. Considering the structure of the sequencing situation and the dominance of player  $i$  over the edge  $(i, i + 1)$ , any convex combination of the lower equivalent solution  $f^\ell$  and the equal loss property solution  $f^e$ , giving player  $i$  at least half of any dividend  $g_{ij}$ ,  $j > i$  seems to be a reasonable solution rule.

We conclude this section by considering shortly  $m$ -machine ( $m \geq 2$ ) sequencing situations and in particular the case  $m = 2$ . Following Hamers (1995), an  $m$ -machine situation is given by a tuple  $(M, N, p, q)$ , with  $M = \{1, \dots, m\}$  a set of  $m$  (identical) machines and  $N = \{1, \dots, n\}$  a set of  $n$ -players, each player  $i \in N$  having one job with processing time  $p_i$  to be processed on one of the machines and with cost function  $q_i$ . Without loss of generality we may assume that  $n > m$ , otherwise all jobs can be processed simultaneously without any waiting time. Initially, there is an ordering described by the map  $b: N \rightarrow M \times N$ , saying that job  $j$  is in position  $k$  on machine  $h$  when  $b(j) = (h, k)$ . So, according to the initial schedule  $b$  the completion time of job  $j$  is given by  $C_j(b) = \sum_{\{i \in N | b_1(i) = b_1(j), b_2(i) \leq b_2(j)\}} p_i$ . For any machine  $h$ , it is assumed that the starting time of the last job on machine  $h$  is not later than the completion time of the last job on any other machine, so that no last job can make any profit by switching to the end of the queue on another machine. By rearranging the jobs a cost savings game is obtained. For some coalition of players  $S \subseteq N$  representing the jobs, a rearrangement is *admissible* if: (i) two players  $i$  and  $j$  on the same machine can only switch if all players on positions between  $i$  and  $j$  are in the coalition, and (ii) two players  $i$  and  $j$  on different machines  $h$  and  $k$  can only switch if all players having a position on  $h$  after  $i$  and all players having a position on  $k$  after  $j$  are in the coalition. Note that these requirements imply that each job outside  $S$  has the same starting time as in the initial order  $b$ .

For a 2-machine situation this can again modelled as a line-graph game. Let in the initial order  $n_1$  be the number of players on machine 1 and  $n_2 = n - n_1$  the number of players on machine 2. Then index the players on machine 1 from the first to last successively by  $1, 2, \dots, n_1$ , and on machine 2 in reverse order from the last to the first by  $n_2, \dots, n$ .

Then again  $\mathcal{L} = \{[i, j] \mid i \leq j\}$  is the set of feasible coalitions: if  $i$  and  $j$  are on the same machine, the coalition  $[i, j]$  contains all players between  $i$  and  $j$ , if  $i$  is on machine 1 and  $j$  on machine 2, then  $[i, j]$  contains all players on 1 after  $i$  and all players on 2 after  $j$ . So, for the 2-machine situation all earlier results still hold. Although the game does not need to be convex (see Hamers (1995) for an example of a non-convex game), by definition we have that the resulting cost-savings line-graph game  $(v, L)$  is superadditive and hence  $v^L$  is permutationally convex with respect to the permutations  $u$  and  $\ell$ . Hence, again each of the functions  $f^u$ ,  $f^\ell$  and  $f^e$  provides a Core solution for any cost structure (see also Hamers (1995) in which the nonemptiness of the Core has been shown if all cost functions are linear). Clearly, for the 2-machine situation the functions  $f^u$  and  $f^\ell$  do not seem to be very appropriate because of the asymmetric treatment of the players on the different machines. For example, since  $f^u$  assigns the dividend of any coalition  $[i, j]$  to player  $j$ , it assigns the dividend to player  $j$  being in a position after player  $i$  when  $i$  and  $j$  are both on machine 1, whereas it assigns the dividend to player  $j$  being in a position before player  $i$  when  $i$  and  $j$  are both on machine 2. Also  $f^u$  assigns the dividend of a coalition  $[i, j]$  with  $i$  on machine 1 and  $j$  on machine 2 to the player  $j$  on the second machine. Therefore, in this situation the equal loss Harsanyi solution  $f^e$  seems to be more appropriate.

Finally, it should be observed that for  $m > 2$  this  $m$ -machine sequencing situation can also be modelled as a graph game, but that it is not a line-graph game anymore. Because of the admissibility condition (ii) stated above, the underlying graph has to contain an edge between any pair of last players. So, let  $(N, A)$  be the graph. Then the subgraph  $(S, A(S))$  is a line graph when  $S$  is the set of players on precisely one of the machines, but  $(S, A(S))$  is the complete graph when  $S$  is the subset of all last players. As a result the game does not need to be balanced. Moreover, the equivalent and equal loss properties are not well-defined anymore, which makes the functions  $f^u$ ,  $f^\ell$  and  $f^e$  inappropriate. Although these functions are still well-defined, assigning respectively the marginal vectors  $m^u(v)$ ,  $m^\ell(v)$  and the average of these two, a reasonable successive numbering of the players does not exist anymore (unless also the machines are put in some order). Of course, the fairness property is well-defined on any graph game, making the Shapley value  $f^s$  a reasonable choice. (Note that the graph of an  $m$ -machine sequencing situation ( $m \geq 3$ ) is still cycle-complete, so that Proposition 3 of Bilbao (1998) concerning the dividends, see equation (3.4), is still valid.)

## 6 The water distribution problem

In their paper ‘Sharing a river’, Ambec and Sprumont (2002) consider the problem of the optimal distribution of water to agents located along a river from upstream to downstream.



Let  $N = \{1, \dots, n\}$  be the set of players representing the agents on the river, numbered successively from upstream to downstream and let  $e_i \geq 0$  be the flow of water entering the river between player  $i - 1$  and  $i$ ,  $i = 1, \dots, n$ , with  $e_1$  the inflow before the most upstream player 1. Further it is assumed that each player has a quasi-linear utility function given by  $u^i(x_i, t_i) = b^i(x_i) + t_i$  where  $t_i$  is a monetary compensation to player  $i$ ,  $x_i$  is the amount of water allocated to player  $i$  and  $b^i: \mathbb{R}_+ \rightarrow \mathbb{R}$  is a continuous nondecreasing function yielding the benefit  $b^i(x_i)$  to player  $i$  of the consumption  $x_i$  of water. An allocation is a pair  $(x, t) \in \mathbb{R}_+^n \times \mathbb{R}^n$  of water distribution and compensation scheme, satisfying

$$\sum_{i=1}^n t_i \leq 0 \quad \text{and} \quad \sum_{i=1}^j x_i \leq \sum_{i=1}^j e_i, \quad j = 1, \dots, n.$$

The first condition is a budget condition and says that the total amount of compensations is nonpositive, i.e. the compensations only redistribute the total welfare. The second condition reflects that any player can use the water that entered upstream, but that the water inflow downstream of some player can not be allocated to this player. So, for any  $j$ , the sum of the water uses  $x_1, \dots, x_j$  is at most equal to the inflows  $e_1, \dots, e_j$ .

Because of the quasi-linearity and the possibility of making money transfers, an allocation is Pareto optimal (efficient) if and only if the distribution of the water streams maximizes the total benefits, i.e. the water distribution  $x^* \in \mathbb{R}_+^n$  solves the following maximization problem:

$$\max_{x_1, \dots, x_n} \sum_{i=1}^n b^i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^j x_i \leq \sum_{i=1}^j e_i, \quad j = 1, \dots, n, \quad \text{and} \quad x_i \geq 0, \quad i = 1, \dots, n. \quad (6.8)$$

A welfare distribution distributes the total benefits of an optimal water distribution  $x^*$  over the players, i.e. it is a vector  $z \in \mathbb{R}^n$  assigning utility  $z_i$  to player  $i$  and satisfying  $\sum_{i=1}^n z_i = \sum_{i=1}^n b^i(x_i^*)$ . Clearly, any welfare distribution  $z$  can be implemented by the allocation  $(x, t)$  with  $x_i = x_i^*$  and  $t_i = z_i - b^i(x_i^*)$ ,  $i = 1, \dots, n$ .

The problem to find a reasonable welfare distribution can be modelled as a line-graph game  $(v, L)$ . Obviously, for any pair of players  $i, j$  with  $j > i$  it holds that water inflow entering the river before the upstream player  $i$  can only be allocated to the downstream player  $j$  if all players between  $i$  and  $j$  cooperate, otherwise any player between  $i$  and  $j$  can take the flow from  $i$  to  $j$  for its own use. Hence, only coalitions of consecutive players are admissible. To define the characteristic function  $v$  of the line-graph game  $(v, L)$ , put  $v(N) = \sum_{i=1}^n b^i(x_i^*)$  with  $x^* \in \mathbb{R}_+^n$  a solution of the maximization problem (6.8). As to the coalitions from  $\mathcal{L}$ , for any connected coalition  $S = [i, j]$  its worth  $v(S)$  is given by

$$v(S) = \sum_{h=i}^j b^h(x_h^{*S}) \quad \text{where} \quad x^{*S} = (x_h^{*S})_{h=i}^j \quad \text{solves}$$

$$\max_{x_i, \dots, x_j} \sum_{h=i}^j b^h(x_h) \quad \text{s.t.} \quad \sum_{k=i}^h x_k \leq \sum_{k=i}^h e_k, \quad h = i, \dots, j, \quad \text{and} \quad x_k \geq 0, \quad k = i, \dots, j. \quad (6.9)$$

Without loss of generality we may normalize the benefit functions by assuming  $b^i(e_i) = 0$  (otherwise replace  $b^i(x^i)$  by  $\widehat{b}^i(x_i) = b^i(x^i) - b^i(e_i)$ ), implying that  $v(\{i\}) = b^i(e_i) = 0$ ,  $i = 1, \dots, n$ , so that the values  $v(S)$ ,  $|S| \geq 2$ , represent the net-gains of cooperating. Again, the characteristic function  $v$  of the line-graph game  $(v, L)$  and the restricted game  $v^L$  are equal to each other and are given by

$$v^L(S) = v(S) = \sum_{T \in \mathcal{C}_L(S)} v(T), \quad S \subseteq N.$$

We refer to this game as the *river game*. Clearly, the game  $v$  is superadditive and hence it follows from Granot and Huberman (1982) that  $v = v^L$  is permutationally convex for the permutations  $u$  and  $\ell$ . Hence, each of the functions  $f^u$ ,  $f^\ell$  and  $f^e$  provides a Core solution. In case all functions  $b^i$  are differentiable with derivative going to infinity as  $x_i$  tends to zero, strictly increasing and strictly concave, Ambec and Sprumont have shown that the game is convex and hence the Core contains all marginal vectors. In fact, recall from Section 3 that  $v = v^L$  is totally positive and thus convex if  $v$  is linear convex. Hence it is sufficient to prove linear convexity of  $v$ , which may simplify the rather complicated proof given by Ambec and Sprumont. Without confusion, in the remaining of this section we denote for simplicity the characteristic function  $v^L$  of the restricted game by  $v$ .

Under the conditions for convexity, Ambec and Sprumont have shown that the marginal vector  $m^u(v)$  corresponding to the permutation  $u$  is the unique element in the Core of the game satisfying a so-called fairness condition. This condition (quite different from the fairness condition of Myerson to characterize the Shapley value) says that any coalition gets at most its *aspiration level*, defined as the highest utility which it can obtain when it can use all the water of all the players  $1, \dots, \widehat{s}$ , where  $\widehat{s} = \max\{s \mid s \in S\}$ . Clearly, this implies that any coalition  $[1, j]$  can get at most  $v[1, j]$ ,  $j = 1, \dots, n$ , so that it follows trivially that indeed the marginal vector  $m^u(v)$  assigning  $m_i^u(v) = v[1, i] - v[1, i-1]$ ,  $i = 1, \dots, n$ , is the unique candidate in the Core satisfying the aspiration requirements. For the proof that it indeed satisfies the requirements we refer to Ambec and Sprumont (2002).

As we have seen in Section 4, the marginal vector  $m^u(v)$  is assigned to  $v$  by the Harsanyi solution  $f^u(v)$  and thus is upper equivalent: when a player  $k$  does not want to cooperate, the players in front of  $k$ , including  $k$  itself are not hurt. However, as in the sequencing game, this is a very counterintuitive outcome. Although any upstream coalition  $[1, i]$  can prevent that coalition  $[i+1, n]$  gets more than  $v[i+1, n]$  by using all flows  $e_1, \dots, e_i$  by itself, all benefits from cooperating go to the coalition  $[i+1, n]$ . Again

the upper equivalent solution has the serious drawback that it does not give any incentive to a player  $i$  to cooperate with its successors in the queue.

Repeating the reasoning once more, again we consider a two player situation. In this case there is no gain of cooperation when in the optimal solution player 1 fully consumes its upstream inflow  $e_1$ . However, suppose it is optimal to allocate a part of  $e_1$  to the second player. The upper equivalent solution discussed above requires that player 1 is just compensated by player 2 for its loss of utility, i.e. player 1 receives a compensation  $t_1 = b^1(e_1) - b^1(x_1^*) = -b^1(x_1^*) \geq 0$ , giving her utility  $b^1(x_1^*) + t_1 = b^1(e_1) = v(\{1\}) = 0$  (with normalized benefit functions). So, as in the sequencing game, there is no reason for player 1 to cooperate. However, again player 1 has the power to play the noncooperative ultimatum game and to pass the stream  $e_1 - x_1^*$  to player 2 if this latter player is willing to give up all the gains of cooperation, i.e. player 1 can sell this stream against a price (or compensation) equal to  $t_1 = -b^1(x_1^*) + v[1, 2]$ , where the first term is to compensate player 1 for its decrease in water consumption and the second term is equal to the total gain of cooperation. Again player 2 is indifferent to accepting this offer or not and therefore is willing to accept the offer (or any slightly lower price). Also in this river game we may argue that player 1 is in control of the edge  $(1, 2)$  and in general, any player  $i < n$  is in control of the edge  $(i, i + 1)$ . Therefore, it seems to be much more reasonable to apply the lower equivalent Harsanyi solution  $f^\ell$  to this game, yielding the marginal vector  $m^\ell(v)$  as the outcome of the game. Again any convex combination of the lower equivalent solution  $f^\ell$  and the equal loss property solution  $f^e$  seems to be a reasonable solution rule.

In the remaining of this section we consider the particular case of linear benefit functions, which has interesting properties in itself, in more detail. So, with the normalization that  $b^i(e_i) = 0$ , the benefit functions are given by  $b^i(x_i) = b_i(x_i - e_i)$  with  $b_i > 0$ ,  $i = 1, \dots, n$ , the constant marginal benefit of one additional unit of water consumption. To avoid technical difficulties, we assume the regularity assumption that  $b_i \neq b_j$  for any pair  $j \neq i$ . Let  $b_k = \max_{j \in N} b_j$ . Then, for each coalition  $S = [i, j]$  containing  $k$  and any solution  $x^{*S}$  of the maximization problem (6.9), it follows straightforwardly that  $x_k^{*S} = \sum_{h=i}^k e_h$ , i.e. at the optimal solution the inflow of coalition  $S$  upstream of  $k$  is fully allocated to player  $k$ . Hence, there is no gain of cooperation between coalitions  $[j, k]$ ,  $j < k$  and coalitions  $[k + 1, h]$ ,  $h > k + 1$ . In the efficient allocation player  $k$  uses all its upstream inflow and a new game starts after  $k$  with player  $k + 1$  as its first player. So, to analyse this river game with linear benefit functions, without loss of generality we may assume that  $k = n$ , i.e. the last player consumes all the water. The problem is to decide on the compensations to be paid by player  $n$  to the upstream players.

Although the assumption that  $b_n = \max_{j \in N} b_j$  is harmless and implies that all water is consumed by player  $n$ , of course we can not assume that  $b_i$  is increasing in  $i$ . It may happen that there exist  $j < n - 1$  such that  $b_{j+1} < b_j$ . Now, consider a connected coalition  $S = [i, j]$ . To give its worth  $v(S)$ , let  $B(S) = \{i_1, \dots, i_t\} \subseteq S$  be the ordered subset of  $S$  satisfying (i)  $i_r < i_t$  if  $r < t$ , (ii)  $i_t = j$ , (iii)  $b_{i_1} = \max_{h \in [i, j]} b_h$  and (iv)  $b_{i_{k+1}} = \max_{h \in [i_k+1, j]} b_h$  for all  $k = 1, \dots, t-1$ . So,  $i_1$  is the player with the highest marginal benefit in  $S$  and  $i_{k+1}$  is the player with the highest marginal benefit of the successors of  $i_k$  in  $S$ ,  $k = 1, \dots, t-1$ . When within  $S = [i, j]$  the coefficients are increasing in  $i$ , then  $t = 1$  and  $i_1 = j$ . We call  $B(S)$  the  $B$ -structure of  $S = [i, j]$ .

**Example 6.1** Let  $N = \{1, \dots, 6\}$  with marginal benefit given by the vector  $b = (3, 5, 1, 4, 3, 7)$ . Then  $B([1, 5]) = \{2, 4, 5\}$  with  $b_2 = 5$  the highest marginal benefit in  $S = [1, 5]$ ,  $b_4 = 4$  the highest benefit of the successors of player 2 in  $S$  and  $b_5 = 3$  the highest benefit of the successors of player 4 in  $S$ . Further  $B([1, 6]) = \{6\}$  with  $b_6 = 7$  the highest marginal benefit in  $[1, 6]$ .

Next, for  $S = [i, j]$ , we define the subsets  $[i, j]^k$  by

$$[i, j]^k = [i_{k-1} + 1, i_k], \quad k = 1, \dots, t, \quad (6.10)$$

where  $i_0 + 1 = i$ . So,  $b_{i_k} = \max_{h \in [i, j]^k} b_h$ . Clearly, any set  $[i, j]^k$  is connected and the collection of sets  $[i, j]^k$ ,  $k = 1, \dots, t$ , forms a partition of  $[i, j]$ . Finally, for a connected coalition  $T$ , define  $e(T) = \sum_{h \in T} e_h$  as the total water inflow of the coalition  $T$ . Clearly, under the regularity assumption ( $b_i \neq b_j$  for any pair  $i \neq j$ ), for any coalition  $S = [i, j]$  the unique solution  $x^{*S}$  of the maximization problem (6.9) is given by

$$x_h^{*S} = \begin{cases} 0, & \text{if } h \notin B(S), \\ e([i, j]^k), & \text{if } h = i_k, \quad k = 1, \dots, t, \end{cases} \quad (6.11)$$

i.e. within coalition  $S$  the inflow of the members of  $S$  is allocated to the players  $h = i_k \in B(S)$ , endowing them with the amounts  $x_{i_k}^{*S}$  being equal to the total inflows  $\sum_{r \in [i, j]^k} e_r$  of  $i_k$  and their predecessors from  $i_{k-1} + 1$  to  $i_k - 1$ .

To check intuitively the straightforward formula (6.11), we first mention that with  $b^i(x_i) = b_i(x_i - e_i)$ ,  $i = 1, \dots, n$ , the dual to the maximization problem (6.9) is of the form

$$\min_{y_i, \dots, y_j} \sum_{h=i}^j e([i, h])y_h \quad \text{s.t.} \quad \sum_{k=h}^j y_k \geq b_h, \quad h = i, \dots, j, \quad \text{and } y_h \geq 0, \quad h = i, \dots, j. \quad (6.12)$$

Second, one can quite easily verify that the vector  $y^{*S} = (y_i^{*S}, \dots, y_j^{*S})$ , defined by

$$y_h^{*S} = \begin{cases} 0, & \text{if } h \notin B(S), \\ b_{i_k} - b_{i_{k+1}}, & \text{if } h = i_k, \quad k = 1, \dots, t, \end{cases} \quad (6.13)$$

(with  $b_{i_{t+1}} = 0$ ), is a feasible solution of the dual problem (6.12). Moreover, it is not very hard to check that  $y^{*S}$  and  $x^{*S}$  (as given in formula (6.11)) satisfy the equality

$$\sum_{h=i}^j e([i, h])y_h^{*S} = \sum_{h=i}^j b_h x_h^{*S}.$$

Consequently, by duality theorem it follows that  $x^{*S}$  and  $y^{*S}$  are the optimal solutions of the optimization problems (6.9) and (6.12), respectively. So, taking  $x_h = x_h^{*S}$ ,  $h \in S = [i, j]$ , in the benefit functions  $b_h(x_h) = b_h(x_h - e_h)$ , we obtain the following lemma.

**Lemma 6.2** *For  $S = [i, j]$ , let  $B(S) = \{i_1, \dots, i_t\} \subseteq S$  be the  $B$ -structure of  $S$ . Then*

$$v[i, j] = \sum_{k=1}^t b_{i_k} e([i, j]^k) - \sum_{h=i}^j b_h e_h.$$

We now apply Theorem 3.1 to get the dividends of the connected coalitions  $[i, j] \in \mathcal{L}$  with  $i < j$  (recall that all single player dividends are zero because of the normalisation and that also all not connected coalitions have zero dividends).

**Theorem 6.3** *Let  $v$  be the (restricted) characteristic function of the linear river game with benefit functions  $b_i(x_i) = b_i(x_i - e_i)$ ,  $i = 1, \dots, n$ . Then, for any  $S = [i, j] \in \mathcal{L}$ ,  $i < j$ , the dividend of  $S$  is given by*

$$\Delta^{[i, j]}(v) = [b^*([i, j]) - b^*([i, j - 1])]e_i,$$

where  $b^*(T) = \max_{h \in T} b_h$ ,  $T \in \Omega$ .

**Proof** From Theorem 3.1 we know that  $\Delta^{[i, j]}(v) = v[i, j] - v[i + 1, j] - v[i, j - 1] + v[i + 1, j - 1]$ . Let  $B([i, j]) = \{i_1, \dots, i_t\}$  be the  $B$ -structure of  $[i, j]$ . Then either  $B([i + 1, j]) = B([i, j])$  (in case  $i_1 \neq i$ ), or  $B([i + 1, j]) = \{i_2, \dots, i_t\}$  (in case  $i_1 = i$ ). In the first case it follows from Lemma 6.2 that

$$\begin{aligned} v[i, j] - v[i + 1, j] &= \left[ \sum_{k=1}^t b_{i_k} e([i, j]^k) - \sum_{h=i}^j b_h e_h \right] - \left[ \sum_{k=1}^t b_{i_k} e([i + 1, j]^k) - \sum_{h=i+1}^j b_h e_h \right] \\ &= b_{i_1} e_i - b_i e_i = [b^*([i, j]) - b_i] e_i, \end{aligned}$$

since by definition of the  $B$ -structure  $b^*([i, j]) = \max_{h \in [i, j]} b_h = b_{i_1}$ . Also in the second case it follows that

$$\begin{aligned} v[i, j] - v[i + 1, j] &= \left[ \sum_{k=1}^t b_{i_k} e([i, j]^k) - \sum_{h=1}^j b_h e_h \right] - \left[ \sum_{k=2}^t b_{i_k} e([i + 1, j]^k) - \sum_{h=i+1}^j b_h e_h \right] \\ &= b_{i_1} e_i - b_i e_i = [b^*([i, j]) - b_i] e_i. \end{aligned}$$

Hence, in both cases we have that  $v[i, j] - v[i+1, j] = [b^*([i, j]) - b_i]e_i$ . Analogously it follows that  $v[i, j-1] - v[i+1, j-1] = [b^*([i, j-1]) - b_i]e_i$ . Hence  $\Delta^{[i, j]}(v) = [b^*([i, j]) - b^*([i, j-1])]e_i$ .  $\square$

Theorem 6.3 shows that the dividend of any coalition  $[i, j]$  only depends on the water inflow  $e_i$  of its first player and the  $B$ -structure of  $[i, j]$ . Since by definition,  $b^*([i, j]) \geq b^*([i, j-1])$ , we have the following corollary.

**Corollary 6.4** *Any (normalized) linear river game is totally positive and thus convex.*

In particular we also have that (if  $e_i > 0$ ) the dividend of a coalition  $[i, j]$  is positive if and only if  $b_j = \max_{h \in [i, j]} b_h$  and thus  $B([i, j]) = \{j\}$  and zero otherwise.

**Corollary 6.5** *For each  $i$  with  $e_i > 0$ , the dividend of any coalition  $S = [i, j]$  is positive if and only if  $b_j > b_h$ , for any  $h$ ,  $i \leq h < j$ . In particular it holds that*

- (i)  $\Delta^{[i, n]}(v) > 0$  (Recall that  $b_n = \max_{h \in N} b_h$ ),
- (ii) for any  $j$  such that  $b_j < b_{j-1}$ :  $\Delta^{[i, j]}(v) = 0$  for all  $i < j$ .

To conclude this section we discuss shortly the Harsanyi solutions for the linear river game. First, recall that because of the assumption that  $b_n > \max_h b_h$ , all water is consumed by the last player. This player has to compensate the others. The upper equivalent solution assigns the dividend of any coalition  $[i, j]$  to its last player  $j$ . So, it follows immediately from the second property of the corollary above that any player  $j$  such that  $b_j < b_{j-1}$  gets payoff equal to zero in the game  $v$ , that means the last player  $n$  only pays such a player  $t_j = b_j e_j$  to compensate her for the loss of utility by not consuming its own water flow  $b_j$ . In particular in the extreme case that  $b_n > b_1 > b_2 > \dots > b_{n-1}$ , all the dividends go to the last player, yielding the solution  $f^u(v) = (0, \dots, 0, v(N))^\top$ . This does not seem to be very reasonable. Any player  $j$  with  $b_j < b_{j-1}$  has to agree that all its upstream water flow  $\sum_{h=1}^j e_h$  goes through, without getting any payoff from this cooperative behavior. To give the payoff to player  $j$  in case  $b_j > b_{j-1}$ , let  $B([1, j-1]) = \{i_1, \dots, i_t\}$  be the  $B$ -structure of  $[1, j-1]$ . Recall that  $i_t = j-1$ . Since  $f_j^u(v) = v[1, j] - v[1, j-1]$ , it follows from Lemma 6.2 by some straightforward calculations (or simple observations on the structure of the game) that  $f_j^u(v) = \sum_{i=1}^{j-1} [b_j - b^*([i, j-1])]^+ e_i$ : player  $j$  gets the value she adds to the water inflow of her upstream players (as usual, we put  $[c - d]^+ = \max(c - d, 0)$ ). Summarizing, any player  $j < n$  is minimally compensated by player  $n$ . It gives these players no incentive to cooperate and let water stream downward.

The lower equivalent solution assigns the dividend of any coalition  $[i, j]$  to its first player  $i$ , yielding payoffs  $f_i^\ell(v) = v[i, n] - v[i+1, n] = [b([i, n]) - b_i]e_i = (b_n - b_i)e_i$   $i = 1, \dots, n$  (see proof of Theorem 6.3 for the second equality). This seems to be very

attractive. First the compensation scheme is simple: to any player  $i < n$ , player  $n$  has to pay  $b_n e_i = f_i^\ell(v) + b_i e_i$ , being the last player's value of the water inflow  $e_i$ . In fact, the price  $b_n$  that player  $n$  has to pay for one unit of water is just equal to its marginal benefit. Second, it means that any player  $i$  (with  $e_i > 0$ ) gets a positive payoff from cooperation. We may conclude that the compensation schedule is very transparent and that each player gets a reasonable payoff from cooperation and therefore is willing to cooperate, maybe except player  $n$ , which can be resolved easily by setting the price to be paid slightly below  $b_n$ . As a drawback it can be argued that player  $j$  is not rewarded for letting pass the water inflows of its upstream players  $1, \dots, j-1$  to its downstream player  $j+1$  (and finally to player  $n$ ). Outcomes that do this are provided by the Shapley value and (partially) the equal loss property Harsanyi solution  $f^e$ . The latter solution still does not compensate player  $j$  for letting pass the water of its upstream players when  $b_j < b_{j-1}$ , because in this case the dividend of any coalition  $[i, j]$ ,  $i < j$  is equal to zero. The Shapley value gives any player  $j > i$  a share  $\frac{1}{n-i+1}$  in the positive dividend of coalition  $[i, n]$  (when  $e_i > 0$ ) and therefore any player  $j$  between  $i$  and  $n$  is rewarded for letting pass to water inflow  $e_i$  from  $i$  to  $n$ . Finally recall from Corollary 6.4 that  $v$  is convex and thus the Shapley value is in the Core of the linear water game.

## 7 Political power in line-graph majority games

As a last application we consider majority games between parties in parliament. Let the set of players  $N = \{1, \dots, n\}$  represent the set of parties in parliament,  $s_i$  the number of seats (votes) of party (player)  $i$ ,  $i \in N$  and  $w = \sum_{i \in N} s_i$  the total number of seats and the quota  $q$ ,  $\frac{1}{2}w < q < w$ , the minimum number of seats needed to pass a ballot. The corresponding majority game is given by the characteristic function  $v$  with  $v(S) = 1$  if  $S$  is winning, i.e.  $\sum_{i \in S} s_i \geq q$  and  $v(S) = 0$  otherwise. A winning coalition  $S$  is called a Minimal Winning Coalition (MWC) if  $v(S \setminus \{i\}) = 0$  for all  $i \in S$ . It is well-known that a simple game  $v$  with  $v(S) \in \{0, 1\}$  for all  $S$  and  $v(N) = 1$  has a non-empty core if and only if there is at least one veto player (player  $i$  is a veto player if  $i \in S$  when  $v(S) = 1$ ) and that the Core distributes the worth  $v(N) = 1$  amongst the veto-players. In the remaining we assume that  $s_i \leq w - q$  for all  $i \in N$ , so that the majority game has no veto-players and thus an empty Core. Two well-known solutions assigning a nonnegative power vector  $f(v) \in \mathbb{R}^n$  to any majority game  $v$  are the Shapley value and the (normalized) Banzhaf value, see e.g. Banzhaf (1967) and Van den Brink and Van der Laan (1998). These values give a positive power to any non-zero player ( $i$  is a zero player if  $v(S \setminus \{i\}) - v(S) = 0$  for any  $S$  containing  $i$ ).

We now consider the situation that the parties can be ordered linearly according

to their political preferences concerning e.g. economic policy, ethics, environmental issues and so on. Then the parties can be indexed successively from player 1 for the most left-wing party to player  $n$  for the most right-wing party. In such a political structure it is reasonable to suppose that only connected coalitions will form, i.e. this situation can be modelled by the line-graph game  $(v, L)$  with  $\mathcal{L} = \{S \subseteq N \mid S = [i, j] \text{ for some } i < j\}$  the collection of feasible coalitions. Since  $v$  is superadditive, it follows again that the restricted game  $v^L$  is permutationally convex with respect to the permutations  $u$  and  $\ell$ . Hence the solutions  $f^u(v^L)$ ,  $f^\ell(v^L)$  and  $f^e(v^L)$  are in the Core of the restricted game  $v^L$ . So, the Core is nonempty, implying that  $v^L$  has at least one veto-player. Indeed, since only coalitions of successive parties can form, there is at least one player needed in both the most left-wing MWC and the most right-wing MWC and thus in any majority coalition. Moreover, for  $h \leq k$ , let  $[1, k]$  be the most left-wing MWC and  $[h, n]$  the most right-wing MWC. Then it follows immediately that  $[h, k]$  is the set of veto-players. Observe that  $s_i \leq w - q$  for all  $i \in N$  implies that  $h > 1$  and  $k < n$ . It should also be noticed that then any player is in at least one MWC of the restricted game  $v^L$  and therefore  $v^L$  does not contain any zero player (although  $v$  may have zero players).

**Example 7.1** Take  $N = 7$ ,  $w = 100$  with  $s_1 = s_7 = 25$ ,  $s_i = 10$ ,  $i = 2, \dots, 6$  and  $q = 60$ . In the restricted game  $v^L$ ,  $[1, 5]$  is the most left-wing MWC,  $[3, 7]$  the most right-wing MWC and  $[3, 5] = \{3, 4, 5\}$  the set of veto-players.

Applying the Harsanyi solutions, when  $h = k$  it follows that  $f^u(v^L) = f^\ell(v^L) = f^e(v^L) = e(h)$  with  $e(h)$  the unique Core element, where  $e(i) \in \mathbb{R}^n$  is given by  $e_i(i) = 1$  and  $e_j(i) = 0$  for all  $j \neq i$ . When  $h < k$ , then  $f^u(v^L) = e(h)$ ,  $f^\ell(v^L) = e(k)$  and  $f^e(v^L) = \frac{1}{2}(e(k) + e(h))$ . So,  $f^u(v^L)$  gives all power to the most right-wing veto-player,  $f^\ell(v^L)$  to the most left-wing veto-player and  $f^e(v^L)$  divides the power between the most extreme veto-players. Observe that according to these solutions no power is assigned to any other player, including veto-players between the two most extreme veto-players. Indeed, the two most extreme veto-players  $h$  and  $k$  can be considered to be critical. When the most left-wing coalition  $[1, k]$  is formed, the most right-wing veto-player  $k$  has the highest incentive (or lowest objection) to break away and form another MWC. So, if this player is willing to cooperate in  $[1, k]$ , then it can be expected that any other player in  $[1, k]$  is willing to cooperate in  $[1, k]$ , including any other veto-player. Similarly this holds for  $h$  in the MWC  $[h, n]$ . The equal loss Harsanyi solution, giving both players a payoff of  $\frac{1}{2}$ , seems to be an appropriate power index for this political situation. Note that both the Shapley value and the (normalized) Banzhaf power index give positive power to any player and thus are not in the Core.

Finally it is interesting to consider the dividends in  $v^L$ . The next lemma says that



each MWC in  $\mathcal{L}$  has dividend equal to 1. Each other coalition in  $\mathcal{L}$  has either dividend 0 or  $-1$ .

**Lemma 7.2** *Let  $v^L$  be the restricted game of a majority line-graph game  $(v, L)$ . Then, for  $S \in \mathcal{L}$ ,  $\Delta^S(v^L) = 1$  if  $S$  is an MWC and  $\Delta^S(v^L) \in \{-1, 0\}$  otherwise.*

**Proof** First, observe that  $\Delta^T(v^L) = 0$  for any  $T = [i, j]$  with  $v[i, j] = 0$ . Next, let  $S = [i, j]$  be an MWC, i.e.  $v[i, j] = 1$  and  $v(T) = 0$  for all  $T \subset S$ ,  $T \neq S$ . From Theorem 3.1 it follows that  $\Delta^{[i, j]}(v^L) = v[i, j] - v[i, j-1] - v[i+1, j] + v[i+1, j-1] = 1 - 0 - 0 + 0 = 1$ . Next, for MWC  $[i, j]$ , consider any coalition  $[i, k]$  with  $k > j$ . Then  $k - 1 \geq j$  and thus  $v[i, k] = v[i, k-1] = 1$ , whereas  $v[i+1, k] = 1$  if  $v[i+1, k-1] = 1$  and  $v[i+1, k] \in \{0, 1\}$  if  $v[i+1, k-1] = 0$ . Hence  $\Delta^{[i, k]}(v^L) = 0$  if  $v[i+1, k-1] = 1$  and  $\Delta^{[i, k]}(v^L) \in \{-1, 0\}$  if  $v[i+1, k-1] = 0$ . Similarly, this holds for any coalition  $[h, j]$  with  $[i, j]$  an MWC and  $h < i$ . Finally, when  $S = [h, k]$  with  $[i, j]$  a MWC and  $h < i < j < k$ , then  $v[h, k] = v[h+1, k] = v[h, k-1] = v[h+1, k-1] = 1$  and thus  $\Delta^S(v^L) = 0$ .  $\square$

Since the worth  $v^L(N) = 1$  is equal to the sum of all dividends, the lemma implies that the number of coalitions with dividend equal to  $-1$  is one less than the number of MWC's. When  $s_i \leq w - q$  for all  $i \in N$ , then there are at least two MWC's and thus at least one coalition with negative dividend.

Observe that also in a standard majority game each MWC  $S$  has dividend equal to one. Clearly, since  $v(S) = 1$  and  $v(T) = 0$  for any  $T \subset S$ ,  $T \neq S$ , it follows from formula (2.1) that  $\Delta^S(v) = 1$  if  $S$  is an MWC. However, the next example shows that in a standard majority game  $v$  also other (winning) coalitions may have positive dividend and even bigger than one.

**Example 7.3** Take  $N = 5$ ,  $s_i = 1$  for all  $i$  and  $q = 3$ , so that  $v(T) = 1$  if and only if  $|T| \geq 3$ . Hence any coalition of precisely 3 players is an MWC and has dividend 1. Further, any coalition of 4 players contains 4 subcoalitions of 3 players. Applying formula (2.2) it follows that  $\Delta^T(v) = -3$  when  $|T| = 4$ . Finally, the grand coalition  $N$  contains 10 subcoalitions of 3 players, each with dividend  $+1$ , and 5 subcoalitions of 4 players, each with dividend  $-3$ . Hence the total dividend of its subcoalitions is equal to  $10 + 5(-3) = -5$ , implying that  $\Delta^N(v) = +6$ .

## 8 Summary and concluding remarks

In this paper we applied Harsanyi solutions to the special class of line-graph games. We introduced four properties that each characterize one unique solution in the set of Harsanyi solutions. We extensively discussed three applications and argued which Harsanyi solution

seems most appropriate in what circumstance. We introduced linear-convexity as a condition on line-graph games that guarantees that the corresponding restricted game is almost positive, and thus convex. Since this condition is rather easy to verify (at least more easy than superadditivity or convexity of a game), we obtain new results and more easy proofs of old results with respect to certain solutions belonging to the Core of a game. In particular, all Harsanyi solutions belong to the Core of a linear-convex line-graph game.

Our first application were sequencing games in which a set of jobs in a queue can change position (under certain rules) and negotiate about the distribution of cost savings made by the switch. We argued that it seems not reasonable to give the full dividend of a switch to the player who was last since this gives the player in front no incentive to switch position. By an ultimatum game argument we even argued that giving the dividend to the first player is much more reasonable. But also the equal split of the dividends among the two players is possible, and by linear-convexity of the sequencing game also the Shapley value (which gives also all intermediate players a share in the dividend of a switch between two players) is a Core solution. The Shapley value is also characterized in the sequencing situation of Maniquet (2003) which is closely related to the model discussed in this paper. The main difference is that Maniquet does not assume an initial order of the players, but looks for a fair allocation of utility (consisting of the waiting cost that can be compensated by monetary transfers) depending only on the waiting cost of the players. His results can be applied when the players negotiate about their position in the queue and monetary transfers before they enter the queue. Since usually players negotiate after they arrive in the queue, we studied a model where the initial queue is given and players negotiate over the cost savings they can realize by switching positions<sup>10</sup>.

We also applied our results to the water distribution problem recently discussed in Ambec and Sprumont (2002). These authors consider the problem of the optimal distribution of water to agents located along a river. Using our general approach we showed that the solution proposed by Ambec and Sprumont boils down to giving the full dividend to the downstream player. Again using an ultimatum game argument we argued that this does not seem reasonable because the upstream player clearly has the control whether to let water run through or consume it himself and thus should get a decent share in the dividend from cooperation. Also a positive share for the upstream player is necessary to give him an incentive to let water run through to downstream players. Again by showing linear-convexity of the river game we saw that all four solutions are Core solutions.

Finally, we applied our results to majority games between political parties in parliament, where the parties can be ordered linearly according to their political preferences.

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<sup>10</sup>Maniquet's sequencing game is not a line-graph game since any two-player coalition can have a positive dividend. On the other hand, all other coalitions have a zero dividend in his game, while in a line-graph game also coalitions of more than two players can have a positive dividend.

The corresponding simple majority game is not linear-convex and the Shapley value, in general, will not be a Core element. For these games we obtained the interesting property that dividends of minimal winning coalitions are equal to 1, while the dividends of other coalitions are either 0 or 1.

We conclude that many economic situations can be modeled as line-graph games. Therefore this is an interesting class of games to study. We showed that this modelling of economic situations as line-graph games can simplify the analysis of these situations considerably, and moreover gives more results. Also it can help to get more insight in these situations. For example, in the sequencing situations we saw that, in principle, nothing changes when there are two queues since the corresponding games are still line-graph games. However, our results do no longer straightforward hold for more than two queues since then the underlying graph is no longer a line-graph.

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