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Cycles and Multiple Equilibria in the Market for Durable Lemons

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Cycles and Multiple Equilibria in the Market for Durable Lemons

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Abstract: We investigate the nature of market failure in a dynamic version of Akerlof (1970) where identical cohorts of a durable good enter the market over time. In the dynamic model, equilibria with qualitatively different properties emerge. Typically, in equilibria of the dynamic model, sellers with higher quality wait in order to sell and wait more than sellers of lower quality. Our main result is that *for any distribution of quality* that there exist an *infinite number* of cyclical equilibria where *all goods are traded within a certain number of periods after entering the market*.

Key Words: Dynamic Trading, Asymmetric Information, Entry, Durable Goods.

JEL Classification: D82

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1. Introduction

Since the pioneering work of Akerlof (1970) it is a commonplace in the economics profession asymmetric information is an important source of market failures in competitive markets. The standard model of adverse selection considers a static market with atomistic agents whose valuations depend on quality and a standard result is that only low quality goods are traded (if at all) even if the buyers are willing to pay more than the reservation price of sellers for each individual quality (see also, Wilson, 1979, 1980). This so-called *lemons problem* affects a large spectrum of markets, including the classic example of second-hand car markets, insurance markets, labor markets, financial markets and even the market for thoroughbred yearlings (Chezum and Wimmer (1996). In many cases, the good under consideration is a durable good.

Durability introduces two complicating factors in the used goods markets: goods not traded in any period can be offered for sale in the future and, in addition, new cohorts of potential sellers may enter the market over time. Janssen and Roy (1999a, 1999b) have investigated some of the issues that arise when durability is explicitly taken into account in a dynamic model. Janssen and Roy (1999a) address the issue whether a *given stock of goods* can be traded over time. They show that in any dynamic competitive equilibrium all goods eventually will be traded. The main idea behind this result is that (due to a lower use value of the good) low quality sellers have less incentives to wait (before selling) compared to high quality sellers. Once certain (low) qualities are sold, only relatively high qualities remain in the market. Consumers can predict that sellers of different qualities will sort themselves into different time periods and, hence, they are willing to pay higher prices in later periods. The equilibrium is thus one in which higher qualities are sold in later periods at higher prices.

Janssen and Roy (1999b) address the same issue in the context of markets where identical cohorts of goods with *uniformly distributed* quality enter the market over time. In such markets, the infinite repetition of the static equilibrium under adverse selection is an equilibrium in the dynamic model. In fact, it is the unique stationary equilibrium and also the only equilibrium where prices and average quality traded are (weakly)

monotonic over time. They show that there exists at least one other equilibrium, however, where *all* goods are traded within finite time after they have entered the market. This equilibrium is cyclical in prices and quantities in the sense that once all goods are traded, prices and quantities will fall. Up to the moment all goods are sold, prices and expected quality monotonically increase.

In this paper we extend the analysis of Janssen and Roy (1999b) in a number of ways. First, we relax the assumption that in every period a cohort of *uniformly distributed* qualities enters the market. Instead, we allow for any arbitrary distribution, which satisfies some mild regularity condition. Second, our results are stronger in the sense that we show the existence of an *infinite* number of equilibria, where *all* goods are traded within finite time after they have entered the market. Hence, there is a strong sense in which a coordination problem is present in such dynamic markets. Finally, we show the extent to which the uniform distribution is special. It turns out that for a set of values of the model's parameters and a set of distributions, which have relatively little probability mass in the neighborhood of the static equilibrium, it is impossible to construct a dynamic equilibrium with monotonically increasing prices and quantities up to the moment everything is sold. We provide an example where this is the case. Hence, the equilibrium construction for the uniform distribution does not extend naturally to the class of all distributions.

The main economic insight provided by this paper is to give a different perspective on the adverse selection problem. In the static Akerlof-Wilson model, the adverse selection problem manifests itself in the fact that relatively high quality goods cannot be traded despite the potential gains from trade. In the dynamic market for durable goods, the *lemons problem* is not so much the impossibility of trading relatively high quality goods, but rather that sellers with relatively high quality goods need to wait in order to trade.[1] So, the cost of waiting becomes an important factor in the welfare loss arising due to asymmetric information. Also, as there exists an infinite

[1] There are certain situations in which the fact that a seller has waited for a long time might indicate low rather than high quality. This would be true, for example, when the buyers can inspect quality - high valuation buyers are more likely to inspect and select the relatively high quality houses - leaving unsold goods of relatively low quality for later periods (Taylor, 1998). A paper with a similar spirit is that of

number of equilibria, there is a serious coordination problem present in dynamic markets with adverse selection.

Our specific model is as follows. We consider a competitive market for a perfectly durable good where potential sellers are privately informed about the quality of the goods they own. Each period, a cohort of sellers of equal size and with an identical, but arbitrary, distribution of quality enters the market. The demand side is modeled in the following simple way. Buyers are identical, have unit demand and for any given quality, a buyer's willingness to pay exceeds the reservation price of a seller for that quality. As buyers do not know the quality, their willingness to pay in a period equals the expected valuation of goods traded in that period. Moreover, there are more buyers than sellers in each period so that in equilibrium, prices equal the expected valuation. Once traded, goods are not re-sold in the same market.[2]

The Akerlof-Wilson model can be considered the static version of our model. The adverse selection problem implies that in equilibrium only a certain range of low qualities is traded. The infinitely repeated version of a static equilibrium outcome is also an equilibrium in our dynamic model. Hence, the issue of existence of dynamic equilibria is easily resolved. In this dynamic equilibrium high quality goods remain unsold forever.

We concentrate on the existence of other equilibria with more interesting properties - where prices and average quality traded fluctuate over time. We provide a characterization result saying that in all such equilibria the range of quality, which is eventually traded in the market, exceeds that in the stationary (static) outcome. Moreover, sellers of different qualities within each cohort of entrants separate themselves out over time. As the use value of low quality goods is lower than that of high quality goods, low quality sellers sell earlier than high quality sellers, the owner of a good with lower quality trades earlier, owners of higher quality goods wait longer. In order to highlight the waiting aspect of the adverse selection problem and also to

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Vettas (1997). As stated earlier, our model is designed to understand the nature of the lemons problem and so we do not allow for any technology which can directly modify the information structure.

[2] Our analysis bears some resemblance to that by Sobel (1991) of a durable goods monopoly where new cohorts of consumers enter the market over time. Unlike our framework, there is no correlation between the valuations of buyers and sellers in his model.

sharpen the contrast between the properties of equilibria of our dynamic model with those of the static model, the main part of the analysis is devoted to proving the existence of an equilibrium where *every* potential seller entering the market trades within a certain finite number of periods after entering the market. We show in fact that an infinite number of these equilibria exist.

There are three important intertemporal factors in the market which determine the market dynamics in all the non-stationary equilibria of our model. First, once a certain range of quality is traded, only sellers of higher quality goods are left in the market, which tends to improve the distribution of quality of potentially tradable goods in the future. Second, the entry of a new cohort of potential sellers with goods of all possible quality dilutes the average quality of potentially tradable goods - as they cannot be distinguished by buyers from higher quality sellers left over from the past. Finally, as time progresses and stocks of untraded goods accumulate from the past, the new cohort of traders entering the market in any period becomes increasingly less significant in determining the distribution of quality of tradable goods.[3]

Other recent literature[4] on adverse selection has focused on various processes (such as signaling and screening) through which the difficulties of trading under asymmetric information may be resolved and has emphasized the role of non-market institutions in this context (such as certification intermediaries and leasing). The present paper, in contrast, is motivated by a more basic issue which also underlies the original Akerlof paper viz., the functioning of the price mechanism in a perfectly competitive market when traders have private information. It is important to understand the nature of market failures due to adverse selection before analyzing the role of institutions in mitigating these failures.

The paper is organized as follows. Section 2 sets out the model, the equilibrium concept and some preliminary results. Section 3 provides a characterization result.

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[3] If there is no entry of sellers after the initial period, or equivalently, if buyers can distinguish the period of entry of sellers in the market, then only the first factor is relevant. In that case, it has been shown earlier for fairly general distributions of quality (see, Janssen and Roy (1998)) that in *every* equilibrium all goods are traded in finite time. Vincent (1990) analyzes a dynamic auction game with similar features.

4 See, for instance, Guha and Waldman (1997), Hendel and Lizzeri (1999a,b), Lizzeri (1999) and Waldman, (1999).

The main result of the paper relating to the existence of an infinite number of equilibria where all goods are traded within finite time after entry into the market are outlined in section 4. Section 5 concludes. Proofs are contained in the Appendix.

2. The Model

Consider a Walrasian market for a perfectly durable good whose quality, denoted by q, varies between $\underline{q}$ and $\bar{q}$, where $0 < \underline{q} < \bar{q} < \infty$. Time is discrete and is indexed by $t = 1,2,\ldots\infty$. Each time period t a set of sellers I_t enters the market, I is the set of all sellers, $I = \bigcup_{t=1}^{\infty} I_t$ and t_i is the period of entry of seller $i \in I$. Each seller is endowed with one unit of the durable good of quality q_i. Let the total Lebesgue measure of sellers from the set I_t who own a good of quality less than or equal to q be a function $m(\{i | i \in I_t, q_i \le q\}) \equiv m(q)$, which is independent of t. We assume that $m(q)$ is strictly increasing and absolutely continuous with respect to the Lebesgue measure.

The measure of all sellers who enter the market in each period is strictly positive, so $m(\bar{q}) > 0$. Each seller i knows the quality q_i of the good he is endowed with and derives flow utility from ownership of the good until he sells it. Therefore, the seller's reservation price is the discounted sum of gross surplus due to ownership and we assume that it is exactly equal to q_i. Sthis implies that the per period gross surplus is $(1-\delta)q_i$.

Each time period t a set of buyers, with measure larger than $m(\bar{q})$, enters the market. All buyers are identical and have unit demand. A buyer's valuation of quality q is equal to vq, where $v > 1$. Thus, under full information, a buyer's valuation exceeds the seller's. All buyers know the *ex ante* distribution of the sellers with respect to qualities but do not know the quality of the good offered by a particular seller. When a buyer buys a good he leaves the market forever. All players discount the future with common discount factor δ, $0 < \delta < 1$. They are risk neutral and rational agents.

We will denote expected quality of the good from seller i conditional on the fact that he belongs to a certain subset $I' \subset I$ as $\theta(\{I'\})$, this value is defined for all $I' \subset I$ such that $\mu(\{I'\}) > 0$ and it follows that

$$\theta(I') \equiv \frac{1}{\mu(I')} \int_{i \in I'} q_i \, d\mu(I').$$

In order to have an adverse selection problem we assume $vE(q) < \bar{q}$, where $E(q)$ is the unconditional expected quality of all goods, $E(q) = \theta(\{I_t\}) = \theta(\{I\})$. This assumption implies that the static Akerlof-Wilson version of the model has a largest equilibrium quality, which we will denote by $q_S \in (\underline{q}, \bar{q})$:

$$q_S = \max_q \{q \mid v\theta(\{i \mid i \in I, q_i \in [\underline{q}, q]\}) = q\}.$$

To simplify our analysis we introduce the following two regularity assumptions. Throughout this paper, we assume that these assumptions hold. Basically assumptions 2.1 and 2.2 assure that the distribution of quality is sufficiently well-behaved for some left-neighborhood of q_S.

Assumption 2.1. The measure function $\mu(q)$ is strictly increasing and absolutely continuous with respect to the Lebesgue measure on $[q_S - \varepsilon_\mu, \bar{q}]$ for some $\varepsilon_\mu > 0$. Moreover, there exist numbers m_μ, M_μ such that for any q', q'', $q_S - \varepsilon_\mu \le q' < q'' \le \bar{q}$:

$$0 < m_\mu < \frac{\mu(q'') - \mu(q')}{(q'' - q')} < M_\mu.$$

Assumption 2.2. The measure function $\mu(q)$ is a differentiable function at $q = q_S$ and

$$\frac{d\mu(q)}{dq}(q_s) = f \ge m_\mu > 0.$$

Given a sequence of market prices $\mathbf{p} = \{p_t\}_{t=1}^{\infty}$ each seller i chooses whether or not to sell and if he chooses to sell, the time period in which to sell. If he chooses not to sell his gross surplus is equal to q_i and therefore his net surplus equals zero, while if he decides to sell in period $t \ge t_i$ his gross surplus is

$$\sum_{t=t_i}^{t-1}(1-\delta)q_i\delta^{t-t_i}+\delta^{t-t_i}p_t=(1-\delta)q_i\frac{1-\delta^{t-t_i}}{(1-\delta)}+\delta^{t-t_i}p_t=q_i(1-\delta^{t-t_i})+\delta^{t-t_i}p_t,$$

and, therefore, his net surplus equals

$$s_i=q_i(1-\delta^{t-t_i})+\delta^{t-t_i}p_t-q_i=(p_t-q_i)\delta^{t-t_i}.$$

The set of time periods in which it is optimal to sell for a seller i is given by

$$T_i(\mathbf{p})\equiv\arg\max_{t\geq t_i}\{s_i|s_i\geq 0\}=\arg\max_{t\geq t_i}\{(p_t-q_i)\delta^{t-t_i}|(p_t-q_i)\geq 0\}.$$

If $p_t-q_i<0$ for all $t\geq t_i$ then $T_i(\mathbf{p})=\varnothing$.

Each potential seller i chooses a time period $\tau_i\in T_i$ in which to sell. Let $\hat{\boldsymbol{\tau}}=\{\tau_i\}_{i\in I}$ be a set of all selling decisions. We will denote a set of the sellers who choose time period t for trade as J_t, and it follows that $J_t\equiv\{i\in I|\tau_i=t\}$. This generates a certain distribution of qualities over all time periods and the expected quality of the goods offered for sale in time period t is $\theta_t=\theta(\{J_t\})$ when $\mu(\{J_t\})>0$.

In the sections that follow we will use the following additional notation: We denote by $\mu(q_{t-1},q_t)$ the measure of sellers from I_t whose goods are of quality from the range $[q_{t-1},q_t]$: $\mu(q_{t-1},q_t)=\mu(\{i|i\in I_t,q_i\in[q_{t-1},q_t]\})$. It follows that $\mu(q_{t-1},q_t)=\mu(q_t)-\mu(q_{t-1})$ and $\mu(q_t,q_{t-1})=-\mu(q_{t-1},q_t)$. Moreover, $\theta(q_{t-1},q_t)$ is used for the expected quality of goods from sellers who belong to I_t whose goods are of quality from the range $[q_{t-1},q_t]$: $\theta(q_{t-1},q_t)=\theta(\{i|i\in I_t,q_i\in[q_{t-1},q_t]\})$. It follows that for all $q_S-\varepsilon_m\leq q_{t-1}\leq q_t\leq\bar{q}$,

$$\theta(q_{t-1},q_t)=\begin{cases}\dfrac{1}{\mu(q_{t-1},q_t)}\displaystyle\int_{q_{t-1}}^{q_t}q\,d\mu, & \text{if } q_t\neq q_{t-1}\\ q_{t-1}, & \text{if } q_t=q_{t-1}\end{cases}$$

and $\theta(q_t,q_{t-1})=\theta(q_{t-1},q_t)$.

The following lemma assures that $\theta(q_{t-1},q_t)$ is continuous in its arguments.

Lemma 2.1. For all $\theta_S - \varepsilon_m \le \theta_{t-1} \le \theta_t \le \bar{\theta}$ the function $h(\theta_{t-1}, \theta_t)$ is a strictly increasing continuous function. Moreover, $\exists m_h, M_h$, such that

$$0 < m_h < \frac{h(\theta_{t-1}, \theta_t) - h(\theta_{t-1}, \theta_{t-1})}{(\theta_t - \theta_{t-1})} < M_h .$$

A dynamic equilibrium is an equilibrium where all players rationally maximize their objectives, expectations are fulfilled and markets clear in every period. On the equilibrium path, buyers' expectations of quality in a period where a strictly positive measure of goods is offered for sale must equal the expected quality in that time period. As all buyers are identical, we assume that their expectations of quality in period t are symmetric and denoted by E_t.

Definition 2.1. A dynamic equilibrium is described in terms of a sequence of prices $\mathbf{p} = \{p_t\}_{t=1}^{\infty}$, a set of selling decision $\hat{\mathbf{o}} = \{\tau_i\}_{i \in I}$ and a sequence of buyers' quality expectations $\mathbf{E} = \{E_t\}_{t=1}^{\infty}$ such that:

a) **Seller maximize:** $\tau_i \in T_i(\mathbf{p})$ for all $i \in I$, i.e., seller i chooses time period τ_i to trade optimally.

b) **Buyers maximize and market clear**: If $m(\{J_t\}) > 0$ then $p_t = vE_t$, i.e., if there is a strictly positive amount of trade in time period t, then each buyer earns zero net surplus so that he is indifferent between buying and not buying and market clears. If $m(\{J_t\}) = 0$ then $p_t \ge vE_t$, i.e., if zero measure of trade occurs in time period t then each buyer can earn at most zero net surplus. Hence, not buying is optimal for him in that period.

c) **Expectations are fulfilled when trade occurs**: If $m(\{J_t\}) > 0$ then $E_t = \theta_t$.

d) **Expectations are reasonable even if no trade occurs:** For all t $E_t \ge \underline{\theta}$.

Given the set-up described above, conditions (a)-(c) are quite standard. Condition (d) is introduced for the formal reason that expected quality is not defined when no trade occurs. The condition says that even in periods in which (at most) zero measure

of seller intends to sell, buyers should believe that the expected quality is larger than the apriori lowest possible quality. This condition assures that autarky, i.e., no trade in any period, cannot be sustained in an equilibrium of the dynamic model. Given the condition, the willingness to pay, hence the price in any period, is restricted from below by $v\underline{q}$ and sellers with low enough qualities prefer to sell against this price rather than not sell.

3. Characterization of equilibrium.

We start the analysis characterizing the properties of any dynamic equilibrium. In the Proposition 3.1 below we first argue that if a good of certain quality sells in period t, then all goods with lower qualities that have entered the market in and before period t will also sell in that period. This fact allows us to define for each period a marginal seller q_t as the seller of the highest quality in period t. It also allows us to define s_t as the surplus of the marginal seller in period t, i.e., $s_t = p_t - q_t$. This part of the Proposition 3.1 basically follows from the fact that the use value of low qualities is lower than the use value of high qualities so that low qualities are more ready to sell.

The second part of the Proposition 3.1 argues that the marginal seller in any period makes non-negative net surplus. This implies that the other sellers in that period make strictly positive surplus.

The third part of the Proposition 3.1 argues that the marginal seller in period t is indifferent between selling in period t and selling in the first future period in which a quality larger than his own quality is sold. Prices in that future period will be higher, reflecting higher average quality, but the discounted surplus is such that the seller is indifferent.

The last part of the Proposition 3.1 says that if it exists the highest quality that will ever be sold in any dynamic equilibrium is either equal to $\bar{q}$ or it is such that the seller makes zero surplus. It is clear that if a seller makes zero net surplus, prices in all future periods cannot be higher as then this seller will have an incentive to wait and

sell in that future period. This part also says that if the highest quality sold in a dynamic equilibrium makes strictly positive surplus, then it must be equal to $\bar{q}$.

Proposition 3.1. Any dynamic equilibrium has the following properties.

a) For all t and all $q_t \in [\underline{q}, \bar{q}]$: if $q_t \in J_t$, then $J_t = \{i | q_i \in [\underline{q}, q_t], t_i \le t\}$, i.e., in every period t in which trade occurs the set of qualities traded is a range $[\underline{q}, q_t]$, where q_t is the marginal quality traded in period t. All sellers who are in the market at t and own goods of quality not larger than q_t prefer to trade in that time period.

b) $p_t \ge q_t$.

c) Let $\tilde{t}(t) = \min_{t>t} \{t | q_t > q_t\}$, i.e., $\tilde{t}(t)$ is the first period after t where $q_t > q_t$. Then $p_t - q_t = \delta^{\tilde{t}(t)-t}\left(p_{\tilde{t}(t)} - q_t\right)$, i.e., the marginal seller in period t is just indifferent between selling in that period and in the first next period where the marginal quality is larger than his own quality.

d) If $\hat{t}(t) = \min\left(\arg\max_{t>t} q_t\right)$ then $\left(p_{\hat{t}(t)} - q_{\hat{t}(t)}\right)\left(\bar{q} - q_{\hat{t}(t)}\right) = 0$, i.e., if $q_{\hat{t}(t)}$ is the first largest marginal quality for all periods after period t, then either this is the highest possible quality $\bar{q}$ or the surplus of the corresponding marginal seller equals zero.

It is easily seen that the infinitely repeated outcome of the static model is a dynamic equilibrium of our model. Hence, existence of equilibrium is not really an issue. In the next section, we will show that in the dynamic model there are infinitely many other equilibria, each one starting from a certain neighborhood of the largest static equilibrium quality.

4. Equilibria Trading all Goods.

We will now show that for any measure function $m(q)$ which satisfies Assumptions 2.1 and 2.2 and for all generic values of the parameters v, δ, $\underline{q}$ and $\bar{q}$ there exist an infinite number of dynamic equilibria covering all qualities up to $\bar{q}$. As we already

know that our model has at least one equilibrium, a general existence proof is trivial. That is why we use a constructive proof showing how to find an equilibrium sequence of marginal qualities that is such that all qualities up to $\bar{q}$ are traded. The fact that there are infinitely many dynamic equilibria follows from the fact that if there exists a dynamic equilibrium covering all qualities up to $\bar{q}$ starting from some $q_1 < q_S$, then we can show that there also exists a dynamic equilibrium covering all qualities up to $\bar{q}$ starting from a q_1', with $q_1 < q_1' < q_S$.

Before we will go into the details of the analysis, we first introduce an important parameter. Assumption 2.2 allows us to define a parameter a, which describes the relation between the distribution of quality over the range $[\underline{q}, q_S]$ and the marginal distribution at q_S itself:

$$a \equiv \frac{1}{m(q_S)}(v-1)q_S \frac{dm(q)}{dq}(q_s) = \frac{(v-1)q_S f}{m(q_S)}.$$

Obviously, a is strictly positive. We will provide an economic interpretation of the parameter a and argue that generically, it must be that $a < 1$. To this end, consider the surplus of the marginal seller in the static model as a function of q:

$$s(q) \equiv p(q) - q = vh(q) - q = v\frac{1}{m(q)}\int_{\underline{q}}^{q} q\,dm(q) - q, \text{ and}$$

$$\frac{ds(q)}{dq}(q_S) = v\frac{d}{dq}\left(\frac{1}{m(q)}\int_{\underline{q}}^{q} q\,dm(q)\right)(q_S) - 1 = a - 1.$$

Hence, a-1 can be interpreted as the way in which the surplus of the marginal seller changes in the neighborhood of the largest static equilibrium quality. Thus, the surplus of the marginal seller can be written as

$$s(q) \equiv s(q_S) + \frac{ds(q)}{dq}(q_S)(q - q_S) + o(q - q_S) = (a-1)(q - q_S) + o(q - q_S). \qquad (1)$$

Suppose then that $a > 1$.[5] This would imply that $s(\theta) > 0$ in some right neighborhood of θ_S. But this contradicts the assumption that θ_S is the highest static equilibrium quality.

In the uniform case, we have $a = \frac{1}{2}v$[6]. As in the uniform case adverse selection implies that $1 < v < 2$, the uniform distribution is a special case of the case when $a \in (\frac{1}{2},1)$. In subsection 4.1 we will start with this simplest case, which generalizes the analysis in Janssen and Roy (1999b). We show that one can construct a "monotonic" sequence of marginal qualities θ_t that are strictly increasing over time until all goods are sold. The main reason why the case $a \in (\frac{1}{2},1)$ is to be distinguished from other cases can be seen by looking at equation (1). If we choose $\theta_1 = \theta_S$, then in the second period, the measure of qualities above θ_S that are not yet sold is two times as high as the original measure. If $a \in (\frac{1}{2},1)$, the distribution of qualities in the second period is such that a new "static" equilibrium emerges that is larger than θ_S. As in the second period we can write $s_2(\theta) = (2a-1)(\theta_2 - \theta_S) - a(\theta_1 - \theta_S) + o((\theta_2 - \theta_S),(\theta_1 - \theta_S))$ it possible to choose θ_1 close enough to θ_S such that $\theta_2 > \theta_1$ and $s_2 > 0$.

If $a < \frac{1}{2}$, however, it may not be possible to construct such a "monotonic" equilibrium and we show this by example. In subsection 4.2 we show that dynamic equilibria nevertheless exist if $a \in (0,\tilde{a}(\delta))$, where $\tilde{a}(\delta)$ is some decreasing function of δ. The kind of equilibrium we obtain has marginal qualities θ_t strictly decreasing for some initial time periods after which they strictly increase until all goods are sold. The general theorem covering all values of a and δ is provided in subsection 4.3. As the equilibrium construction here becomes quite complicated, subsections 4.1 and 4.2 are also provided for didactical reasons.

The construction of equilibrium uses an "equilibrium sequence" which is defined below.

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[5] The case where $a = 1$ is a non-generic case.

[6] This easily follows from the fact that $\theta_S = \frac{v}{2-v}\theta$ and $m(\theta_S) = f(\theta_S - \theta)$.

Definition 4.1. An equilibrium sequence $\Theta_T(U)$ is a finite sequence of marginal qualities θ_t as functions of θ_1, the latter being defined over some range $U=(\underline{\theta_1},\overline{\theta_1})$, i.e., $\Theta_T(U)=\{\theta_t(\theta_1)\}_{t=1}^{T}$, such that all equilibrium conditions in Definition 2.1 hold for all $t=1,\ldots,T-1$. Moreover, for all $\theta_1 \in U$:

a) $\theta_t(\theta_1)$ is continuous for all $t=1,\ldots,T$;

b) $\theta_T(\theta_1) > \theta_t(\theta_1)$ for all $t=1,\ldots,T-1$;

c) $p_t = v h_t = p_t(\theta_1)$, the price in period t is continuous;

The above definition does not imply the existence of an equilibrium sequence. However, it easy to see that there exists at least one equilibrium sequence, namely $\Theta_1((\theta_S-\varepsilon_m,\theta_S))=\{\theta_1\}$, such that all mentioned above conditions are trivially satisfied.

The main property of an equilibrium sequence we use is that if there is a dynamic equilibrium with marginal qualities $\{\theta_t\}_{t=1}^{\infty}$ such that for $t=1,\ldots,T$ it can be described by a certain equilibrium sequence $\Theta_T(U)=\{\theta_t(\theta_1)\}_{t=1}^{T}$, then there is only one indifference equation, namely

$$p_T-\theta_T=\delta^{(\tilde{t}(T)-T)}\left(p_{\tilde{t}(T)}-\theta_T\right), \tag{2}$$

which relates prices p_t and marginal qualities θ_t for $t=T+1,\ldots\infty$ to prices p_k and marginal qualities θ_k for $k=1,\ldots,T$ (this follows from (b) above). Intuitively, θ_T summarizes all the relevant properties of the sequence of marginal qualities up to time period T. Our purpose, therefore, is to find an equilibrium sequence such that $\theta_T(\theta_1)=\overline{\theta}$ for some T and θ_1.

4.1 The case where $a > \frac{1}{2}$.

In this subsection we prove the existence of an increasing sequence $\{\theta_t\}_{t=1}^{T}$, where $\theta_T=\overline{\theta}$ when $a>\frac{1}{2}$. As the uniform distribution is a special case, the result obtained in this section shows to what extent the results obtained in Janssen and Roy (1999b) can

be generalized to allow for other types of distribution functions. The following theorem contains a statement of the formal result.

Theorem 4.1. For any $a \in \left(\frac{1}{2},1\right)$ and for any generic value of $\bar{q}$, there exist an infinite number of dynamic equilibria such that all goods are sold within T periods after entering the market. The sequence $\{q_t\}_{t=1}^{T}$ is monotonically increasing.

The proof consists of three steps. In Proposition 4.1 we prove that it is possible to construct an equilibrium sequence of an arbitrary length where marginal qualities $\{q_t\}$ are strictly increasing and very close to the static equilibrium quality q_S. Under these circumstances the main indifference equation (2) takes the following form:

$$p_t - q_t = d(p_{t+1} - q_t). \tag{3}$$

In other words, the marginal seller in period t is just indifferent between selling in that period and in the next period. We will denote such monotonic equilibrium sequences as $\Theta^1(U)$ and call a dynamic equilibrium, which is based on them, as "dynamic equilibrium of type I".

Proposition 4.1. If $a \in \left(\frac{1}{2},1\right)$, then there exist an infinite number of $\Theta^1(U)$. Moreover, for $\forall e_g \geq \frac{1}{2a-1}$ $\exists T_0$ such that for all $t > T_0$ $\exists U_t^0 = \left(q_1^0(t), q_S\right)$ and $\exists \Theta^1{}_t\left(U_t^0\right)$ such that:

a) for all $t = 1,\ldots,t$ $q_t(q_1)$ is differentiable at $q_1 = q_S$ and $q_t(q_S) = q_S$;

b) for all $q_1 \in U_t^0$ $0 < q_t(q_1) - q_S < e_g \frac{1}{d} s_t(q_1)$.

Proposition 4.1 implies that if $a \in \left(\frac{1}{2},1\right)$, we can construct an equilibrium sequence of an arbitrarily long length t such that in period $t+1$ there will be more sellers with high quality ($q_i > q_S$) goods than the number of sellers with low quality ($q_i < q_S$). This allows us to expand the equilibrium sequence Θ_t for some more periods.

Next, in Proposition 4.2, we prove that when we are able to construct an equilibrium sequence of an arbitrary length where all marginal qualities belong to a certain neighborhood of q_S, then we can expand it in such a way that the surplus of the

last marginal quality θ_t could be made any value between 0 and $(v-1)\theta_t$. More precisely, given any equilibrium sequence Θ_t with $0 < \theta_t(\theta_1) - \theta_S < \varepsilon_g \frac{1}{d} s_t(\theta_1)$ we can construct another sequence $\Theta_{t'}$, where $t' > t$, such that $\Theta_t \subset \Theta_{t'}$ and $p_t(\theta_1)$ covers the whole interval $(\theta_t(\theta_1), v\theta_t(\theta_1))$. The conditions under which the Proposition 4.2 holds are the same as the conclusion reached in Proposition 4.1. These conclusions are replicated here as in later subsections we will also make use of it.

Proposition 4.2. If there exist $\varepsilon_g > 0$ and T_0 such that for all $t > T_0$ $\exists U_t^0 = (\theta_1^0(t), \theta_S)$ and $\exists \Theta_t(U_t^0)$ such that for all $\theta_1 \in U_t^0$ $\theta_t(\theta_1) - \theta_S < \varepsilon_g \frac{1}{d} s_t(\theta_1)$, then for any $\varepsilon_S > 0$ and $\varepsilon_q > 0$ $\exists T_S$ such that for all $t \geq T_S$ $\exists U_t^S = (\theta_1^S(t), \theta_S) \subset U_t^0$ and $\exists \Theta_t(U_t^S)$ such that:

a) for any $\theta_1 \in U_t^S$ $|\theta_t(\theta_1) - \theta_S| < \varepsilon_q$;

b) $s_t(\theta_S) = 0$;

c) $s_t(\theta_1^S) > (v-1)\theta_t(\theta_1^S) - \varepsilon_S$.

Proposition 4.2 tells us that if we could trade goods for many time periods and, therefore, accumulate "high quality sellers", then we can organize trade in such a way that in the last time period of the equilibrium sequence "almost" all sellers who prefer to sell in that period will have goods of quality very close to θ_S.

Finally, in Proposition 4.3 we prove that if we are able to trade goods along an equilibrium path from a certain range of qualities such that the price in the last period of the equilibrium sequence can be made any value between the marginal quality and buyer's valuations of the marginal quality, then we can expand that equilibrium sequence in such a way that wider range of qualities could be traded with the same properties. Doing so, after a finite number of iterations we generically can construct an equilibrium sequence where $\theta_T = \bar{\theta}$, i.e., all goods are traded by period *T*.

Proposition 4.3. If $\exists \boldsymbol{q}^{(k)} \in \left[\boldsymbol{q}_S, \bar{\boldsymbol{q}}\right)$ such that for any $\boldsymbol{e}_S > 0$ and $\boldsymbol{e}_q > 0$ $\exists T_S^{(k)}$ such that for all $t > T_S^{(k)}$ $\exists U_t^{(k)} = \left(\underline{\boldsymbol{q}}_1(t,k), \bar{\boldsymbol{q}}_1(t,k)\right)$[7] and $\exists \Theta_t\left(U_t^{(k)}\right)$ such that:

a) for all $\boldsymbol{q}_1 \in U_t^{(k)}$ $\left|\boldsymbol{q}_t(\boldsymbol{q}_1) - \boldsymbol{q}^{(k)}\right| < \boldsymbol{e}_q$;

b) $0 \le s_t(\underline{\boldsymbol{q}}_1(t,k)) < \boldsymbol{e}_S$;

c) $s_t\left(\bar{\boldsymbol{q}}_1(t,k)\right) > (v-1)\boldsymbol{q}_t\left(\bar{\boldsymbol{q}}_1(t,k)\right) - \boldsymbol{e}_S$;

then either

a) $\exists \bar{\boldsymbol{e}}_S > 0$ such that for any T $\exists \bar{t} > T$, $\exists \tilde{\boldsymbol{q}}_1(\bar{t})$ and $\exists \Theta_{\bar{t}}(\tilde{\boldsymbol{q}}_1)$ such that $\boldsymbol{q}_{\bar{t}}(\tilde{\boldsymbol{q}}_1) = \bar{\boldsymbol{q}}$ and $s_{\bar{t}}(\tilde{\boldsymbol{q}}_1) > \bar{\boldsymbol{e}}_S$, or

b) for any $\bar{\boldsymbol{e}}_S > 0$ and $\bar{\boldsymbol{e}}_q > 0$ $\exists \boldsymbol{q}^{(k+1)} \in \left(v\boldsymbol{q}^{(k)}, \bar{\boldsymbol{q}}\right]$ and $\exists T_S^{(k+1)}$ such that for all $t > T_S^{(k+1)}$ $\exists U_t^{(k+1)} = \left[\underline{\boldsymbol{q}}_1(t,k+1), \bar{\boldsymbol{q}}_1(t,k+1)\right] \subset U_t^{(k)}$ and $\exists \Theta_t\left(U_t^{(k+1)}\right)$ such that:

- for all $\boldsymbol{q}_1 \in U_t^{(k+1)}$ $\left|\boldsymbol{q}_t(\boldsymbol{q}_1) - \boldsymbol{q}^{(k+1)}\right| < \bar{\boldsymbol{e}}_q$;
- $0 \le s_t(\underline{\boldsymbol{q}}_1(t,k+1)) < \bar{\boldsymbol{e}}_S$;
- $s_t\left(\bar{\boldsymbol{q}}_1(t,k+1)\right) > (v-1)\boldsymbol{q}_t\left(\bar{\boldsymbol{q}}_1(t,k+1)\right) - \bar{\boldsymbol{e}}_S$.

Proposition 4.3 basically says that if we have constructed an equilibrium sequence for a sufficiently large number of periods, then we can either make sure that after some more time periods the next marginal quality can be chosen relatively far from the present marginal quality and such that all desirable properties are kept (case (b)) or we can reach $\bar{\boldsymbol{q}}$ (case (a)).

Propositions 4.1, 4.2 and 4.3 taken together give us a large part of the proof of Theorem 4.1.

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[7] Here we don't make a distinction between $\underline{\boldsymbol{q}}_1 < \bar{\boldsymbol{q}}_1$ and $\underline{\boldsymbol{q}}_1 > \bar{\boldsymbol{q}}_1$. All we need is $U_t^{(k)}$ to be a nonempty open set while $\underline{\boldsymbol{q}}_1$ and $\bar{\boldsymbol{q}}_1$ are its boundary points.

4.2 The case of small a and d.

In this section, we construct an equilibrium sequence for the case when a and δ are small. We first provide an example showing why the analysis of the previous subsection does not continue to be valid. The example shows that when a and δ are small there does not exist a $q_2 > q_1$ such that $s_2 \geq 0$ and such that q_1 is indifferent between selling in period 1 and selling in period 2.

Example 4.1 Let us take $v = 1.2$, $\delta = 0.1$, $\underline{q} = 10$, $\bar{q} = 13$ and a measure function $m(q)$ such that $m(10) = 0$ and

$$\frac{d}{dq} m(q) = \begin{cases} 101, & \text{if } \underline{q} < q < 10.1 \\ 1, & \text{if } 10{,}1 < q < \bar{q} \end{cases}.$$

The static equilibrium quality for this case is unique and equals $q_S = \sqrt{151.5} \approx 12.31$.

In any dynamic equilibrium we must have $q_1 \in [\underline{q}, q_S]$ (otherwise we would have $s_1 < 0$). The following picture 4.1 shows the graph of functions $X2 = q_2(X1)$ and $S2 = s_2(X1)$ where $X1 = q_1$.

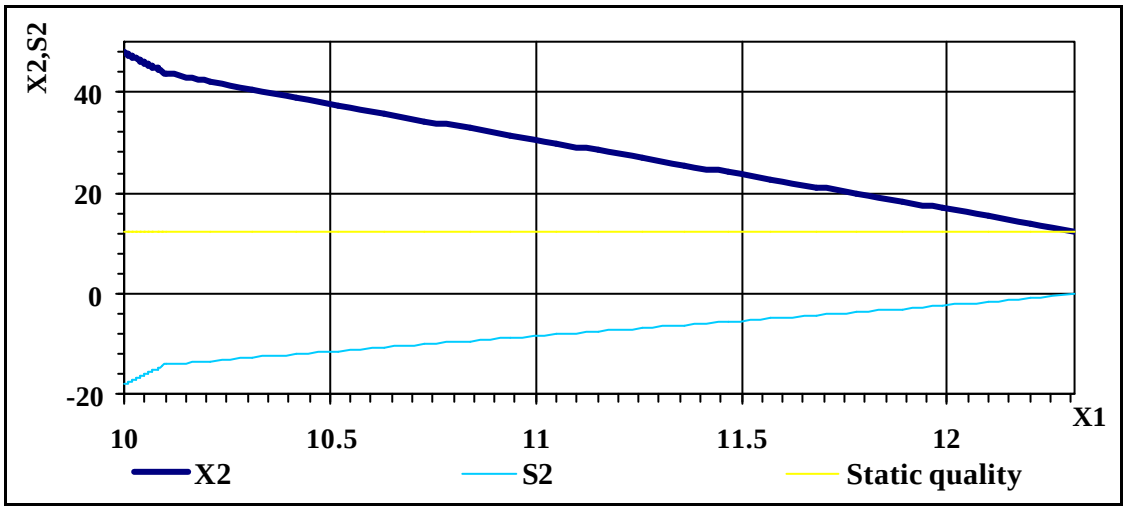

Figure 4.1.

It is easy to see that for any value of q_1 we get $q_2(q_1)$ above q_S and the surplus in the second period is negative. //

We will now prove that if a is relatively small, particularly if $a \in \left(0,(1-\delta)^2\right)$, then we are still able to construct infinitely many dynamic equilibria such that all goods from the range $\left[\underline{q},\bar{q}\right]$ are traded. The equilibrium sequence is non-monotonic. Note that the parameter configuration analyzed here partially overlaps with the parameter configuration analyzed in the previous subsection. The result we will prove is formally stated in Theorem 4.2 below.

Theorem 4.2. For any $a \in \left(0,(1-\delta)^2\right)$, and for any generic value of $\bar{q}$, there exist an infinite number of dynamic equilibria such that all goods are sold in finite time after entering the market.

In order to prove this theorem we only need to show that when $a \in \left(0,(1-\delta)^2\right)$ it is also possible to construct an equilibrium sequence of an arbitrary large length t where marginal qualities $\{q_t\}$ are very close to the static equilibrium quality q_S. We will construct a sequence that is strictly decreasing for some time, $q_{t+1} < q_t$, and only the last marginal quality q_T exceeds all previous ones. We denote such a sequence as "equilibrium sequence of type II" and write $\Theta^2(U)$.

In this case our indifference equation (2) becomes the following system:

$$p_t - q_t = \delta^{T-t}\left(p_t - q_t\right),\ t = 1,\ldots,T\ . \tag{4}$$

Proposition 4.4. If $a \in \left(0,(1-\delta)^2\right)$, then there exist an infinite number of $\Theta^2(U)$. Moreover, for $\forall \varepsilon_g > 0$ $\exists T_0$ such that for all $t > T_0$ $\exists U_t^0 = \left(q_1^0(t), q_S\right)$ and $\exists \Theta^2{}_t\left(U_t^0\right)$ such that:

a) for all $t = 1,\ldots,t$ $q_t(q_1)$ is differentiable at $q_1 = q_S$ and $q_t(q_S) = q_S$;

b) for all $q_1 \in U_t^0$ $0 < q_t(q_1) - q_S < \varepsilon_g \frac{1}{d} s_t(q_1)$.

Note that the conclusions reached in Proposition 4.4 are identical to the conclusions reached in Proposition 4.1 so that we can make use of Propositions 4.2 and 4.3 to get the proof of Theorem 4.2.

4.3 The General case.

Finally, we prove that for any value of a, we are able to construct infinitely many dynamic equilibria such that all goods in $\left]\underline{q},\bar{q}\right[$ are traded. The structure of the corresponding equilibrium sequences becomes a mixture of the equilibrium sequences of type I and type II.

Theorem 4.3. For any generic value of $\bar{q}$, there exist an infinite number of dynamic equilibria such that all goods are sold in finite time after entering the market.

Again, like in subsection 4.2, the only thing we need to prove is that it is possible to construct an equilibrium sequence of an arbitrary large length where marginal qualities $\{q_t\}$ are very close to the static equilibrium quality q_S. This is the content of Proposition 4.5.

Proposition 4.5. There exist an infinite number of $\Theta(U)$. Moreover, for $\forall \varepsilon_g > 0$ $\exists T_0$ and $\exists k_m \geq 1$ such that for all $t > T_0$ $\exists U^0_{tk_m+1} = \left(q_1^0(tk_m+1), q_S\right)$ and $\exists \Theta_{tk_m+1}\left(U^0_{tk_m+1}\right)$ such that:

a) for all $t = 1,\ldots,tk_m+1$ $q_t(q_1)$ is differentiable at $q_1 = q_S$ and $q_t(q_S) = q_S$;

b) for all $q_1 \in U^0_{tk_m+1}$ $0 < q_{tk_m+1}(q_1) - q_S < \varepsilon_g \frac{1}{d} s_{tk_m+1}(q_1)$.

The main difference with the related Propositions 4.1 and 4.4 is that here the equilibrium sequence constructed around q_S is partly composed of increasing subsequences and partly composed of decreasing subsequences. Therefore, we need two indices (t and k_m) to keep track of the whole equilibrium sequence.

Note that the conclusions reached in Proposition 4.5 are again identical to the conclusions reached in Proposition 4.1 so that we can make use of Propositions 4.2 and 4.3 to get the proof of Theorem 4.3.

5. Conclusions

In this paper, we have provided a different perspective on the way the adverse selection problem manifests itself in durable good markets, where entry takes place in the same market. In the static Akerlof-Wilson model, adverse selection results in high quality goods not being able to trade despite the potential gains from trade. The infinite repetition of this static equilibrium is also an equilibrium in the dynamic model where a durable good is traded in a competitive market. Our main result in this paper, is that there are infinitely many other equilibria where all goods are sold within finite time after entering the market. In each of these dynamic equilibria, the marginal quality that is sold in the first period lies in a small neighborhood of the static equilibrium. This result holds true for all generic values of the parameters governing the behavior of buyers and sellers and the distribution of qualities in the population of sellers.

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Appendix.

In the appendix we will use the following additional notation.

a) $K_t = \dfrac{\eta(q_{t-1}, q_t) - \eta(q_{t-1}, q_{t-1})}{(q_t - q_{t-1})}$; (5)

b) $F_t = \dfrac{\mu(q_{t-1}, q_t)}{(q_t - q_{t-1})}$; (6)

c) $y_t = q_t - q_{t-1}$; (7)

d) $z_t = y_t - y_{t-1}$; (8)

e) $g_t = \frac{1}{d} s_t$;

f) $g(q) = \begin{cases} \frac{1}{q - q_S}(q - \nu\eta(q, q)), \text{ if } q \neq q_S \\ \lim\limits_{q \to q_S} g(q) = 1 - a, \text{ if } q = q_S \end{cases}$; (9)

g) $\bar{g}(q) = \max\limits_{q \in [q_S, \bar{q}]} |g(q)|$;

h) $\gamma_{t-1} = q_{t-1}(\nu - 1) - g_{t-1}$. (10)

Proof of Lemma 2.1 is on request.

Proof of Proposition 3.1. We prove all statements of the proposition sequentially.

a) Let us take any period t of positive amount of trade J_t so that $\mu(J_t) > 0$ and take any $i \in J_t$. By the definition of dynamic equilibrium we can write:

$t \in \arg\max\limits_{t \geq t_i} \{(p_t - q_i)\delta^{t - t_i} \mid (p_t - q_i) \geq 0\}$.

This implies $(p_t - q_i)\delta^{t - t_i} \geq (p_t - q_i)\delta^{t - t_i}$ for all $t > t \geq t_i$. Now take any $q < q_i$:

$$(p_t - q)\delta^{t - t_i} - (p_t - q)\delta^{t - t_i} = (p_t - q_i)\delta^{t - t_i} - (p_t - q_i)\delta^{t - t_i} + (q_i - q)(\delta^t - \delta^t)\delta^{-t_i} \geq$$
$$\geq (q_i - q)(1 - \delta^{t - t})\delta^{t - t_i} > 0.$$

So, for all sellers with a good of quality less then q_i who are still in the market in a certain period and have not yet traded it is optimal to trade in that period. Thus,

we can define q_t as $q_t = \sup_i \{q_i | i \in J_t\}$ and then it is easy to see that $J_t = \{i | q_i \in [\underline{q}, q_t], t_i \le t\}$. Finally, if $m(J_t) = 0$ for some t, then we set $q_t = \underline{q}$.

b) By the equilibrium definition, for all t $E_t \ge \underline{q}$ and $p_t \ge vE_t$ so that $p_t \ge v\underline{q}$. Thus, if $m(J_t) = 0$, we have $p_t \ge v\underline{q} > \underline{q} = q_t$. If $m(J_t) > 0$, it is optimal for the marginal seller q_t to trade in period t and a necessary condition is $p_t - q_t \ge 0$. So, $p_t \ge q_t$ for all t.

c) Suppose $p_t - q_t - \delta^{(\tilde{t}-t)}(p_{\tilde{t}} - q_t) = \sigma > 0$. Then, we can find a seller i of quality $q_i = q_t + \frac{1}{2}\sigma$ such that $q_t < q_i < q_{\tilde{t}}$ and $t_i \le t$, i.e., he is in the market by period t. By definition of $\{q_t\}$ he will trade in period $\tilde{t}$. But it can be shown that this is not optimal:

$\delta^{(\tilde{t}-t_i)}(p_{\tilde{t}} - (q_t + \frac{1}{2}\sigma)) - \delta^{(t-t_i)}(p_t - (q_t + \frac{1}{2}\sigma)) = \delta^{t-t_i}\left[-\sigma + \frac{1}{2}\sigma(1 - \delta^{(\tilde{t}-t)})\right] < 0$.

So, it is not possible that $p_t - q_t > \delta^{(\tilde{t}(t)-t)}(p_{\tilde{t}(t)} - q_t)$. A similar argument shows that it is impossible to have $p_t - q_t < \delta^{(\tilde{t}(t)-t)}(p_{\tilde{t}(t)} - q_t)$.

d) Suppose $\bar{q} - q_{\hat{t}} = \sigma > 0$. We will show that in this case $p_{\hat{t}} - q_{\hat{t}} = 0$. Suppose not. Then it must be $p_{\hat{t}} - q_{\hat{t}} = \varepsilon > 0$. Let us take a seller i of quality $q_i = q_{\hat{t}} + \frac{1}{2}\min\{\varepsilon, \sigma\}$ such that $q_{\hat{t}} < q_i < \bar{q}$ and $t_i = \hat{t}$. By definition of $\{q_t\}$ he will never trade because for all $t \ge t_i$ $q_t \le \max_{t \ge \hat{t}}\{q_t\} = q_{\hat{t}} < q_i$. If he, however, traded in period $\hat{t}$ he would get

$p_{\hat{t}} - q_i = p_{\hat{t}} - q_{\hat{t}} - \frac{1}{2}\min\{\varepsilon, \sigma\} = \varepsilon - \frac{1}{2}\min\{\varepsilon, \sigma\} \ge \frac{1}{2}\varepsilon > 0$, which is a contradiction.

So, it must be the case that $(p_{\hat{t}(t)} - q_{\hat{t}(t)})(\bar{q} - q_{\hat{t}(t)}) = 0$.

Proof of Proposition 4.1. Using the fact that $q_t(q_1) > q_{t-1}(q_1)$ we express the expected quality sold in period t in terms of $m(q_{t-1}, q_t)$ and $h(q_{t-1}, q_t)$:

$$h_t(q_{t-1}, q_t) = \frac{th(q_{t-1}, q_t)m(q_{t-1}, q_t) + h(\underline{q}, q_{t-1})m(\underline{q}, q_{t-1})}{tm(q_{t-1}, q_t) + m(\underline{q}, q_{t-1})},\ t \ge 1,\ q_0 \equiv \underline{q}. \qquad (11)$$

Now we consider the indifference condition (3) with $p_t = \nu h_t$. It can be written for $t \geq 2$ as $h_t(q_{t-1}, q_t) - h_{t-1}(q_{t-2}, q_{t-1}) = \frac{1-d}{\nu d} s_{t-1}$. (12)

The main part of the proof is by induction. At first we will prove that if all conditions to be proved (except $q_t(q_1) > q_S$) are true for some $t > 2$, then $\exists q_{t+1}(q_1) > q_t(q_1)$ such that those conditions are also true for $t+1$. Next, we will show that there exist q_1 and q_2 such that those conditions are satisfied. Finally, we will show that for some T_0 we get $q_{T_0}(q_1) > q_S$ and, therefore, $q_t(q_1) > q_S$ for all $t > T_0$ and all $q_1 \in U_t^0 \subset U_{T_0}^0$.

Suppose that for some $t > 2$ $\exists U_{t-1}^0 = (q_1^0(t-1), q_S)$, $\exists \{q_t(q_1)\}_{t=1}^{t-1}$, and $\exists \{p_t(q_1)\}_{t=1}^{t-1}$ such that for all $q_1 \in U_{t-1}^0$:

a) $q_t(q_1) > q_{t-1}(q_1)$, $s_t(q_1) > 0$; $q_t(q_S) = q_S$, $s_t(q_S) = 0$;

b) q_t and s_t are continuous functions differentiable at $q_1 = q_S$ so that we can write:

$$q_t = q_S + \frac{dq_t}{dq_1}(q_S)(q_1 - q_S) + o(q_1 - q_S), \text{ and } s_t = \frac{ds_t}{dq_1}(q_S)(q_1 - q_S) + o(q_1 - q_S);$$

c) $\frac{ds_t}{dq_1}(q_S) < 0$, $\frac{dy_t}{dq_1}(q_S) < 0$, $\frac{dz_t}{dq_1}(q_S) < 0$;

d) for all $t = 1, \ldots, t-2$ $p_t - q_t = d(p_{t+1} - q_t)$;

e) for all $t = 1, \ldots, t-2$ $\frac{d(s_t - bs_{t-1})}{dq_1}(q_S) \leq 0$, where $b = \frac{2a - (1-d)}{2ad}$.

We will prove that then $\exists U_t^0 = (q_1^0(t), q_S) \subset U_{t-1}^0$ such that for all $q_1 \in U_t^0$ equation (12) determines a unique value of q_t as a function of q_1 and

a) $q_t(q_1) > q_{t-1}(q_1)$, $s_t(q_1) > 0$; $q_t(q_S) = q_S$, $s_t(q_S) = 0$;

b) q_t and s_t are continuous functions differentiable at $q_1 = q_S$ so that we can write:

$$q_t = q_S + \frac{dq_t}{dq_1}(q_S)(q_1 - q_S) + o(q_1 - q_S), \text{ and } s_t = \frac{ds_t}{dq_1}(q_S)(q_1 - q_S) + o(q_1 - q_S);$$

c) $\frac{ds_t}{d\theta_1}(\theta_S) < 0$, $\frac{dy_t}{d\theta_1}(\theta_S) < 0$, $\frac{dz_t}{d\theta_1}(\theta_S) < 0$;

d) $\frac{d(s_t - \beta s_{t-1})}{d\theta_1}(\theta_S) < 0$, where $\beta = \frac{2a-(1-d)}{2ad}$, and $\theta_t(\theta_1) - \theta_S < s_t(\theta_1)\frac{2}{2a-1}$.

Let us first consider the left-hand side of the equation (12) as a function $G_t(\theta_t, \theta_{t-1}, \theta_{t-2}) \equiv h_t(\theta_{t-1}, \theta_t) - h_{t-1}(\theta_{t-2}, \theta_{t-1})$. It is easily seen that as $\theta_{t-1} \geq \theta_{t-2} \geq \underline{\theta}$ evaluating $G_t(\theta_t, \theta_{t-1}, \theta_{t-2})$ at $\theta_t = \theta_{t-1}$ yields $G_t(\theta_{t-1}, \theta_{t-1}, \theta_{t-2}) \leq 0$. So, $G_t(\theta_{t-1}(\theta_1), \theta_{t-1}(\theta_1), \theta_{t-2}(\theta_1)) \leq 0 < \frac{1-d}{vd} s_{t-1}(\theta_1)$ for any $\theta_1 \in U_{t-1}^0$. Also, for any small $\varepsilon > 0$ such that $\theta_S + \varepsilon < \bar{\theta}$, $G_t(\theta_S + \varepsilon, \theta_{t-1}(\theta_S), \theta_{t-2}(\theta_S)) > 0$. As $G_t(\theta_S + \varepsilon, \theta_{t-1}(\theta_S), \theta_{t-2}(\theta_S)) > \frac{1-d}{vd} s_{t-1}(\theta_S) = 0$ and as $G_t(\theta_1)$ and $s_{t-1}(\theta_1)$ are both continuous functions, there must exist a neighborhood $\overline{U_t^0} = \left(\overline{\theta_1^0(t)}, \theta_S\right) \subset U_{t-1}^0$ such that for any $\theta_1 \in \overline{U_t^0}$ $G_t(\theta_S + \varepsilon, \theta_{t-1}(\theta_1), \theta_{t-2}(\theta_1)) > \frac{1-d}{vd} s_{t-1}(\theta_1)$. Finally, $G_t(\theta_t, \theta_{t-1}, \theta_{t-2})$ is a strictly increasing continuous function w.r.t. θ_t.

Taking all these facts together and applying the intermediate point theorem we can draw the following conclusion. For all $\theta_1 \in \overline{U_t^0}$ there exists a unique continuous function $\theta_t(\theta_1)$ such that $\theta_{t-1}(\theta_1) < \theta_t(\theta_1) < \theta_S + \varepsilon$. $s_t(\theta_1) = v h_t(\theta_{t-1}, \theta_t) - \theta_t$ is also continuous function and $\theta_t(\theta_S) = \theta_S$, $s_t(\theta_S) = 0$.

To prove the rest of step (b) of our induction step, we will now show that $\theta_t(\theta_1)$ and $s_t(\theta_1)$ both are differentiable functions at $\theta_1 = \theta_S$. For all $t = 2, \ldots, t$ the indifference condition (3) can be written as $h_t(\theta_{t-1}, \theta_t) = \frac{1}{d} h_{t-1}(\theta_{t-2}, \theta_{t-1}) - \frac{1-d}{vd}\theta_{t-1}$. Taking the first differentials of this identity w.r.t. θ_1 at $\theta_1 = \theta_S$, and using (11), we get:

$$d_S\left(\frac{t h(\theta_{t-1}, \theta_t) m(\theta_{t-1}, \theta_t) + h(\underline{\theta}, \theta_{t-1}) m(\underline{\theta}, \theta_{t-1})}{t m(\theta_{t-1}, \theta_t) + m(\underline{\theta}, \theta_{t-1})}\right) =$$

$$= \frac{1}{d} d_S\left(\frac{(t-1) h(\theta_{t-2}, \theta_{t-1}) m(\theta_{t-2}, \theta_{t-1}) + h(\underline{\theta}, \theta_{t-2}) m(\underline{\theta}, \theta_{t-2})}{(t-1) m(\theta_{t-2}, \theta_{t-1}) + m(\underline{\theta}, \theta_{t-2})}\right) - \frac{1-d}{vd} d_S\theta_{t-1},$$

where $d_S\theta_t = d\theta_t\big|_{\theta_1 = \theta_S}$. Taking the derivatives explicitly into account yields

$$\frac{\tau(f_{q_S} d_S q_t - f_{q_S} d_S q_{t-1}) + f_{q_S} d_S q_{t-1}}{\tau m(q_S, q_S) + m(q, q_S)} - h_t(q_S, q_S)\frac{\tau(f d_S q_t - f d_S q_{t-1}) + f d_S q_{t-1}}{\tau m(q_S, q_S) + m(q, q_S)} =$$

$$= \frac{1}{d}\left(f_{q_S} \frac{(\tau-1) d_S q_{t-1} - (\tau-2) d_S q_{t-2}}{(\tau-1) m(q_S, q_S) + m(q, q_S)} - f h_{t-1}(q_S, q_S) \frac{(\tau-1) d_S q_{t-1} - (\tau-2) d_S q_{t-2}}{(\tau-1) m(q_S, q_S) + m(q, q_S)} \right) -$$

$$- \frac{1-d}{vd} d_S q_{t-1}.$$

Rewriting gives

$$\delta a(\tau d_S q_t - (\tau-1) d_S q_{t-1}) = a((\tau-1) d_S q_{t-1} - (\tau-2) d_S q_{t-2}) - (1-\delta) d_S q_{t-1}, \text{ and}$$

$$\tau\delta d_S q_t = d_S q_{t-1}\left((1+\delta)(\tau-1) - \frac{1-d}{a}\right) - (\tau-2) d_S q_{t-2}. \tag{13}$$

So we can write $\frac{dq_t}{dq_1}(q_S) = \frac{(1+\delta)(\tau-1) - \frac{1-d}{a}}{\tau\delta} \frac{dq_{t-1}}{dq_1}(q_S) - \frac{\tau-2}{\tau\delta} \frac{dq_{t-2}}{dq_1}(q_S)$. As, by assumption, q_{t-1} and q_{t-2} both are differentiable at $q_1 = q_S$ so is $q_t(q_1)$. Also, as surplus is defined by $s_t = p_t - q_t = v h_t(q_t, q_{t-1}) - q_t$, s_t is also differentiable at $q_1 = q_S$.

Next, we prove part (c) of the induction argument. To this end, we can rewrite (13), using (7) and (8) in the following way:

$$\tau\delta d_S y_t = d_S y_{t-1}\left(\tau - 1 - \delta - \frac{1-d}{a}\right) - \frac{(1-d)(1-a)}{a} d_S q_{t-2}, \text{ and}$$

$$\tau\delta d_S z_t = d_S z_{t-1}\left(\tau - 3 + \frac{(1-d)(2a-1)}{a}\right) + d_S y_{t-2} \frac{(1-d)(2a-1)}{a}. \tag{14}$$

Using the induction assumptions, we have $\frac{dy_{t-2}}{dq_1}(q_S) < 0$, $\frac{dy_{t-1}}{dq_1}(q_S) < 0$ and $\frac{dz_{t-1}}{dq_1}(q_S) < 0$ so it follows from (14) that $\frac{dz_t}{dq_1}(q_S) < 0$, and $\frac{dy_t}{dq_1}(q_S) = \frac{dy_{t-1}}{dq_1}(q_S) + \frac{dz_t}{dq_1}(q_S) < 0$.

Similarly, we can get the following expressions for the first differentials of the surpluses $d_S s_t$:

$$d_S s_t = a(\tau-1) d_S y_t - (1-a) d_S q_t, \text{ and} \tag{15}$$

$$d_S(s_t - s_{t-1}) = d_S z_t (a\tau - 1) + d_S y_{t-1}(2a-1). \tag{16}$$

Again, by our induction assumptions we have $\frac{ds_{t-1}}{dq_1}(q_S) < 0$ and it follows that $\frac{d(s_t - s_{t-1})}{dq_1}(q_S) < 0$, and $\frac{ds_t}{dq_1}(q_S) = \frac{ds_{t-1}}{dq_1}(q_S) + \frac{d(s_t - s_{t-1})}{dq_1}(q_S) < 0$.

The only parts of the induction argument we still have to prove are $\frac{d(s_t - \beta s_{t-1})}{dq_1}(q_S) < 0$ and for all t $\exists U_t^0 = (q_1^0(t), q_S) \in \overline{U_t^0}$ such that for all $q_1 \in U_t^0$ $s_t(q_1) > \beta s_{t-1}(q_1)$, where $\beta = \frac{2\alpha - (1-\delta)}{2\alpha\delta}$.

Subtracting (15) for $t-1$ from (15) for t we can write

$d_S s_t - d_S s_{t-1} = \alpha(t-1) d_S y_t - \alpha(t-2) d_S y_{t-1} - (1-\alpha)(d_S q_t - d_S q_{t-1})$, or

$$d_S s_t - d_S s_{t-1} = (\alpha t - 1) d_S y_t - \alpha(t-2) d_S y_{t-1}. \quad (17)$$

We can write the indifference condition (3) as $s_{t-1} = \delta s_t + \delta(q_t - q_{t-1})$, the first differentials of which w.r.t. q_1 at $q_1 = q_S$ becomes $\delta d_S y_t = d_S s_{t-1} - \delta d_S s_t$. Substituting it into (17) yields $d_S s_t - d_S s_{t-1} = (\alpha t - 1)(d_S s_{t-1} - \delta d_S s_t) - \alpha(t-2)(d_S s_{t-2} - \delta d_S s_{t-1})$. Rewriting gives

$$t\delta d_S s_t = \left((t-1)(1+\delta) + \tfrac{(1-\delta)(\alpha-1)}{\alpha}\right) d_S s_{t-1} - (t-2) d_S s_{t-2}. \quad (18)$$

We can express the above equation in terms of $d_S(s_t - \beta s_{t-1})$ instead of $d_S s_t$, where $\beta = \frac{2\alpha - (1-\delta)}{2\alpha\delta}$. It can be shown that

$$t\delta d_S(s_t - \beta s_{t-1}) = (t-2)\left[d_S(s_{t-1} - \beta s_{t-2})\left(\delta + \tfrac{(1-\delta)}{2\alpha}\right) + \tfrac{(1-\delta)}{2\alpha}\tfrac{(2\alpha-1)(1-\delta)}{2\alpha\delta} d_S s_{t-2}\right].$$

By our induction assumptions we have $\frac{d(s_{t-1} - \beta s_{t-2})}{dq_1}(q_S) \le 0$, and $\frac{ds_{t-2}}{dq_1}(q_S) < 0$ so it follows that $\frac{d(s_t - \beta s_{t-1})}{dq_1}(q_S) < 0$, and, therefore,

$$\frac{ds_t}{dq_1}(q_S) < \beta \frac{ds_{t-1}}{dq_1}(q_S) \le \cdots \le \beta^{t-t} \frac{ds_t}{dq_1}(q_S) \le \cdots \le \beta^{t-1} \frac{ds_1}{dq_1}(q_S) < 0. \quad (19)$$

Now again let us consider the indifference equation (3) as $\delta(q_t - q_{t-1}) = s_{t-1} - \delta s_t$. Summing it up from $t = 2$ to $t = t$ and rewriting gives

$$q_t = q_1 + \tfrac{1}{\delta}(s_1 - s_t) + \tfrac{1-\delta}{\delta} \sum_2^t s_t.$$

Taking the first differential w.r.t. q_1 at q_S and using (19), we get:

$$\begin{aligned}
\frac{dq_t}{dq_1}(q_S) &= 1+\tfrac{1}{d}\left(\frac{ds_1}{dq_1}(q_S)-\frac{ds_t}{dq_1}(q_S)\right)+\tfrac{1-d}{d}\sum_2^t\frac{ds_t}{dq_1}(q_S)> \\
&> 1+\tfrac{1}{d}\left(b^{-(t-1)}\frac{ds_t}{dq_1}(q_S)-\frac{ds_t}{dq_1}(q_S)\right)+\tfrac{1-d}{d}\sum_2^t b^{-(t-t)}\frac{ds_t}{dq_1}(q_S)= \\
&= 1+\tfrac{1}{d}\left(\left(b^{-(t-1)}-1\right)+(1-d)b^{-t}\,\tfrac{b^{\,2}\left(b^{\,t-1}-1\right)}{b-1}\right)\frac{ds_t}{dq_1}(q_S)= \\
&= 1+\tfrac{\left(1-b^{-(t-1)}\right)}{d}\left((1-d)\tfrac{b}{b-1}-1\right)\frac{ds_t}{dq_1}(q_S)>1+\tfrac{1}{d}\left((1-d)\tfrac{b}{b-1}-1\right)\frac{ds_t}{dq_1}(q_S)= \\
&= 1+\tfrac{1}{d}\left(\tfrac{2a-(1-d)-2a+1}{2a-1}\right)\frac{ds_t}{dq_1}(q_S)=1+\tfrac{1}{2a-1}\frac{ds_t}{dq_1}(q_S)>\tfrac{1}{2a-1}\frac{ds_t}{dq_1}(q_S).
\end{aligned}$$

As q_t and s_t are differentiable at $q_1=q_S$ they can be written as follows:

$$q_t(q_1)-q_S-\tfrac{2}{2a-1}s_t(q_1)=\left(\frac{dq_t}{dq_1}(q_S)-\tfrac{2}{2a-1}\frac{ds_t}{dq_1}(q_S)\right)(q_1-q_S)+o(q_1-q_S).$$

The above inequality implies that $\exists U_t^0=(q_1^0(t),q_S)\in\overline{U_t^0}$ such that for all $q_1\in U_t^0$

$q_t(q_1)-q_S-\tfrac{2}{2a-1}s_t(q_1)<-\tfrac{1}{2a-1}\frac{ds_t}{dq_1}(q_S)(q_1-q_S)+o(q_1-q_S)<0$, or

$q_t(q_1)-q_S<e_g\,s_t(q_1)$, where $e_g=\tfrac{1}{2a-1}$.

Now we will show that all induction assumptions are valid for $t=1,2$. Let us first consider the function $s_1(q_1)=p_1-q_1=vh(q,q_1)-q_1$. In terms of the function m it can be written as $s_1(q_1)=\dfrac{v}{m(q,q_1)}\int_q^{q_1}q\,dm-q_1$. $s_1(q_1)$ is a continuous function over $U_0^0\equiv(q_S-e_m,q_S)$ and differentiable at $q_1=q_S$. From the definition of a it follows that $\dfrac{ds_1}{dq_1}(q_S)=-(1-a)$ and $s_1(q_1)=-(1-a)(q_1-q_S)+o(q_1-q_S)$. Hence, there exists a neighborhood, namely $U_1^0=(q_1^0(1),q_S)\subset U_0^0$, such that for all $q_1\in U_1^0$ $s_1(q_1)>0$. Obviously, q_1 itself is continuous and differentiable at $q_1=q_S$. Then, using the definitions of y_t and z_t, we get $\dfrac{dy_2}{dq_1}(q_S)=\dfrac{dq_2}{dq_1}(q_S)-1=-\tfrac{(1-d)(1-a)}{2da}<0$, and $\dfrac{dz_2}{dq_1}(q_S)=-\tfrac{(1-d)(1-a)}{2da}-1<0$.

Finally, using (13), (15) and (16) yields $\frac{d(s_2 - s_1)}{d\theta_1}(\theta_S) = -\frac{(1-\delta)(1-\alpha)(2\alpha-1)}{2\delta\alpha} < 0$, and, consequently, $\frac{ds_2}{d\theta_1}(\theta_S) < 0$ and $\frac{d(s_2 - \beta s_1)}{d\theta_1}(\theta_S) = 0$. That ends the proof of the induction argument.

We finish the proof by showing that for some T_0 we must have $\theta_{T_0}(\theta_1) > \theta_S$, hence, for all $t > T_0$ and all $\theta_1 \in U_t^0 \subset U_{T_0}^0$: $\theta_t(\theta_1) > \theta_S$. To see this consider the sequence $\left\{\frac{d\theta_t}{d\theta_1}(\theta_S)\right\}_{t=1}^{\infty}$. The first term of this sequence equals 1. Moreover, the sequence is decreasing with strictly negative increment as

$$\frac{d\theta_t}{d\theta_1}(\theta_S) - \frac{d\theta_{t-1}}{d\theta_1}(\theta_S) = \frac{dy_t}{d\theta_1}(\theta_S) < \frac{dy_2}{d\theta_1}(\theta_S) = -\frac{(1-\delta)(1-\alpha)}{2\delta\alpha} < 0.$$

Thus, there exists a first negative term of $\left\{\frac{d\theta_t}{d\theta_1}(\theta_S)\right\}_{t=1}^{\infty}$, which can be denoted by $\frac{d\theta_{T_0}}{d\theta_1}(\theta_S) < 0$, where $T_0 \le \left[\frac{2\delta\alpha}{(1-\delta)(1-\alpha)} + 1\right]$. It implies that for all $\theta_1 \in U_{T_0}^0$ $\theta_S < \theta_{T_0}(\theta_1)$.

Proof of Proposition 4.2.

In case when $\theta_t > \theta_{t-1}$ we have $\tilde{t}(t-1) = t$ and our indifference equation (2) can be written as (3), or $p_t = \theta_{t-1} + \frac{1}{\delta}(p_{t-1} - \theta_{t-1}) = \frac{1}{\delta}s_{t-1} + \theta_{t-1}$. Rewriting yields $\nu h_t(\theta_{t-1}, \theta_t) = \frac{1}{\delta}s_{t-1} + \theta_{t-1}$.

Function $h_t(\theta_{t-1}, \theta_t)$ strictly increases w.r.t. θ_t for all θ_{t-1}, so there exists an inverse function which determines θ_t as a function of $\theta_{t-1}(\theta_1)$ and $s_{t-1}(\theta_1)$, $\theta_t = \theta_t\left(\theta_{t-1}, \frac{1}{\delta}s_{t-1}\right)$. This function is defined for all θ_1 as long as $\nu h_t\left(\theta_{t-1}(\theta_1), \bar{\theta}\right) \ge \frac{1}{\delta}s_{t-1}(\theta_1) + \theta_{t-1}(\theta_1)$. Using (11) we can write:

$$\nu \frac{t h(\theta_{t-1}, \theta_t) m(\theta_{t-1}, \theta_t) + h(\underline{\theta}, \theta_{t-1}) m(\underline{\theta}, \theta_{t-1})}{t m(\theta_{t-1}, \theta_t) + m(\underline{\theta}, \theta_{t-1})} = \gamma_{t-1} + \theta_{t-1}, \text{ where } \gamma_t = \frac{1}{\delta}s_t.$$

Then

$$y_t\left(\nu\eta(q_{t-1},q_t)-(g_{t-1}+q_{t-1})\right)=\frac{m(\underline{q},q_{t-1})}{t\dfrac{m(q_{t-1},q_t)}{y_t}}\left(g_{t-1}+\frac{q_{t-1}-\nu\eta(\underline{q},q_{t-1})}{q_{t-1}-q_S}(q_{t-1}-q_S)\right),\text{ and}$$

$$y_t\left(\nu\eta(q_{t-1},q_t)-q_{t-1}-g_{t-1}\right)=\frac{m(\underline{q},q_{t-1})}{tF_t}\left(g_{t-1}+g(q_{t-1})(q_{t-1}-q_S)\right).$$

where g and F_t were defined in (9) and (6) correspondingly. Then we get:

$$\nu y_t^{\,2}\frac{\eta(q_{t-1},q_t)-q_{t-1}}{y_t}+y_t\left((\nu-1)q_{t-1}-g_{t-1}\right)=\frac{m(\underline{q},q_{t-1})}{tF_t}\left(g_{t-1}+g(q_{t-1})(q_{t-1}-q_S)\right),\text{ and}$$

$$\nu y_t^{\,2}K_t+y_t\jmath_{t-1}-\frac{m(\underline{q},q_{t-1})}{tF_t}\left(g_{t-1}+g(q_{t-1})(q_{t-1}-q_S)\right)=0, \tag{20}$$

where K_t and $\jmath_{t-1}$ were defined in (5) and (10) correspondingly.

Now let us take any $\varepsilon_S>0$ and any $\varepsilon_q\in\left(0,\min\left\{\varepsilon_g,\tfrac{1}{2}(\bar{q}-q_S)\right\}\right)$. Then we take a small $\varepsilon>0$ such that $\varepsilon<\frac{1-d}{\nu d}\min\left\{1,\dfrac{\varepsilon_S}{2(\nu-1)\bar{q}},\dfrac{1}{3\bar{q}}\varepsilon_q\right\}$, a small $\varepsilon_1>0$ such that $\varepsilon_1<\tfrac{1}{2}\min\left\{\varepsilon_S,(\nu-1)q_S\right\}$, and a large T such that $T>\dfrac{(1+\bar{g}\varepsilon_g)m(\underline{q},\bar{q})}{m_m}\max\left\{\dfrac{1}{d\varepsilon\varepsilon_1},\dfrac{4\nu M_h}{\varepsilon_1^{\,2}}(\nu-1)\bar{q},\dfrac{4(\bar{q}-\underline{q})}{m_h(\bar{q}-q_S)^2}\right\}$. By the assumption of the Proposition 4.2 for that T there exist corresponding $U_T^0=\left(q_1^0(T),q_S\right)$ and $\Theta_T\left(U_T^0\right)$.

Now let us take the subset $\hat{U}_T^0=\left(\hat{q}_1^0(T),q_S\right)\subset U_T^0$ such that for all $q_1\in\hat{U}_T^0$:

a) $q_1-q_S+\frac{1}{d}s_1(q_1)<\frac{1}{3}\varepsilon_q$;

b) $q_T-q_S<\varepsilon_q$;

c) $\max\limits_{t=1,\ldots,T}\left\{\sup\limits_{q_1\in\hat{U}_T^0}s_t(q_1)\right\}<\dfrac{\varepsilon_q}{3}\dfrac{d}{(1-d)(T-1)}$;

d) $\jmath_T(q_1)>\varepsilon_1$ (it is always possible as $\jmath_T(q_S)=(\nu-1)q_S>\varepsilon_1$).

Now we will prove that if for all $q_1\in\hat{U}_T^0$ and some $t\ge T+1$ $\jmath_{t-1}(q_1)>\varepsilon_1$, then there exist well-defined functions $q_t(q_1)$ and $s_t(q_1)$ such that $y_t=q_t-q_{t-1}$ is determined by (20) and $s_t(q_1)>s_{t-1}\left(\frac{1}{d}-\varepsilon\right)>0$.

At first we prove the existence of $\theta_t(\theta_1)$ showing that $\nu h_t\left(\theta_{t-1}(\theta_1),\bar{\theta}\right) \geq \frac{1}{d} s_{t-1}(\theta_1) + \theta_{t-1}(\theta_1)$ if $\gamma_{t-1}(\theta_1) \geq 0$ and $\theta_{t-1} < \theta_S + \varepsilon_\theta$:

$$\nu h_t\left(\theta_{t-1},\bar{\theta}\right) - \tfrac{1}{d} s_{t-1} + \theta_{t-1} = \nu h_t\left(\theta_{t-1},\bar{\theta}\right) + \gamma_{t-1} - \nu\theta_{t-1} \geq \nu\left(h_t\left(\theta_{t-1},\bar{\theta}\right) - \theta_{t-1}\right) =$$
$$= \nu\left(\frac{t h\left(\theta_{t-1},\bar{\theta}\right) m\left(\theta_{t-1},\bar{\theta}\right) + h\left(\underline{\theta},\theta_{t-1}\right) m\left(\underline{\theta},\theta_{t-1}\right)}{t m\left(\theta_{t-1},\bar{\theta}\right) + m\left(\underline{\theta},\theta_{t-1}\right)} - \theta_{t-1}\right) >$$
$$> \nu \frac{t m_h m_m \left(\bar{\theta} - \theta_{t-1}\right)^2 - \left(\bar{\theta} - \underline{\theta}\right) m\left(\underline{\theta},\bar{\theta}\right)}{t m\left(\theta_{t-1},\bar{\theta}\right) + m\left(\underline{\theta},\theta_{t-1}\right)} > \nu \frac{\frac{1}{4} T m_h m_m \left(\bar{\theta} - \theta_S\right)^2 - \left(\bar{\theta} - \underline{\theta}\right) m\left(\underline{\theta},\bar{\theta}\right)}{t m\left(\theta_{t-1},\bar{\theta}\right) + m\left(\underline{\theta},\theta_{t-1}\right)} > 0,$$

thus there exist $\theta_t(\theta_1)$ and $s_t(\theta_1)$ such that $y_t = \theta_t - \theta_{t-1}$ is determined by (20).

Using the fact that for $t = T+1$ $\gamma_{t-1}(\theta_1) > \varepsilon_1$ we can solve (20) w.r.t. y_t [8]:

$$y_t = \frac{\gamma_{t-1}}{2\nu K_t}\left[-1 + \sqrt{1 + \frac{4\nu K_t m\left(\underline{\theta},\theta_{t-1}\right)}{t F_t \gamma_{t-1}^2}\left(g_{t-1} + g(\theta_{t-1})(\theta_{t-1} - \theta_S)\right)}\right]. \qquad (21)$$

It can be shown that the expression under the square root above is positive and, therefore, y_t is uniquely defined by (21).

Now we will show that $s_t > s_{t-1}\left(\frac{1}{d} - \varepsilon\right)$. Using the well-known inequality that for all $x > 0$ $\sqrt{1+x} < 1 + \frac{1}{2}x$ we get:

$$y_t = \frac{\gamma_{t-1}}{2\nu K_t}\left[-1 + \sqrt{1 + \frac{4\nu K_t m\left(\underline{\theta},\theta_{t-1}\right)}{t F_t \gamma_{t-1}^2}\left(g_{t-1} + g(\theta_{t-1})(\theta_{t-1} - \theta_S)\right)}\right] <$$
$$< \frac{m\left(\underline{\theta},\theta_{t-1}\right)}{t F_t \gamma_{t-1}}\left(g_{t-1} + g(\theta_{t-1})(\theta_{t-1} - \theta_S)\right) < \frac{m\left(\underline{\theta},\bar{\theta}\right)}{T m_m \varepsilon_1}\left(1 + \bar{g}\varepsilon_g\right) g_{t-1} < \delta g_{t-1},$$

and, therefore, $s_t = \frac{1}{d} s_{t-1} - y_t > \frac{1}{d} s_{t-1} - \varepsilon s_{t-1} = s_{t-1}\left(\frac{1}{d} - \varepsilon\right)$.

Now we will prove that $\exists T_S > T$, $\exists U_{T_S}^S = \left(\theta_1^S(T_S),\theta_S\right) \subset \hat{U}_T^0$ and $\exists \Theta_{T_S}$ such that for all $t = T+1,\ldots,T_S - 1$ and for all $\theta_1 \in U_{T_S}^S$ $s_t > \left(\frac{1}{d} - \varepsilon\right) s_{t-1}$, $\gamma_{t-1} > \varepsilon_1$ and $\gamma_{T_S - 1}\left(\theta_1^S\right) = \varepsilon_1$.

Suppose not, then for all $t \geq T+1$ and for all $\theta_1 \in \hat{U}_T^0$ $\gamma_{t-1}(\theta_1) > \varepsilon_1$. But in this case we have an induction: for all $t \geq T+1$ $\exists \theta_t(\theta_1) \in \left(\theta_{t-1}(\theta_1),\bar{\theta}\right)$ and $\exists s_t(\theta_1) > \left(\frac{1}{d} - \varepsilon\right) s_{t-1}(\theta_1) > 0$. Let fix any $\theta_1 \in \hat{U}_T^0$ and consider the following sequences

[8] Another solution is always negative and doesn't satisfy $\theta_t > \theta_{t-1}$.

of numbers: $\{\theta_t\}_{t=T+1}^{\infty}$ and $\{s_t\}_{t=T+1}^{\infty}$. The former increases and is bounded, so $\exists \lim_{t\to\infty} \theta_t = \theta_\infty \le \bar{\theta}$. The later also increases and $\exists \lim_{t\to\infty} s_t = +\infty$ as

$$s_t > \left(\tfrac{1}{d} - \varepsilon\right)s_{t-1} > \left(\tfrac{1}{d} - \tfrac{1-d}{vd}\right)s_{t-1} = \left(1 + \tfrac{(v-1)(1-d)}{dv}\right)s_{t-1}.$$

But if we take a limit of γ_{t-1} we get a contradiction:

$$\lim_{t\to\infty} \gamma_t = \lim_{t\to\infty} \left((v-1)\theta_{t-1} - \tfrac{1}{d}s_{t-1}\right) = (v-1)\theta_\infty - \tfrac{1}{d}\lim_{t\to\infty} s_{t-1} = -\infty,$$

as $\gamma_{t-1} > \varepsilon_1$ for all $t \ge T+1$.

So, it must be the case that $\exists T_S > T$, $\exists U_{T_S}^S = (\theta_1^S, \theta_S)$ and $\exists \Theta_{T_S}$ such that for all $t = T+1, \ldots, T_S - 1$ and for all $\theta_1 \in U_{T_S}^S$ $s_t > \left(\tfrac{1}{d} - \varepsilon\right)s_{t-1} > 0$, $\gamma_{t-1} > \varepsilon_1$ and $\gamma_{T_S-1}(\theta_1^S) = \varepsilon_1$.

Now we will prove by induction that for all $t \ge T_S$ $\exists U_t^S = (\theta_1^S(t), \theta_S) \subset U_{t-1}^S$ and $\exists \Theta_t \supset \Theta_{t-1}$ such that $\gamma_{t-1}(\theta_1^S(t)) = \varepsilon_1$ and $s_t(\theta_1) > \left(\tfrac{1}{d} - \varepsilon\right)s_{t-1}(\theta_1)$ for all $\theta_1 \in U_t^S$. Suppose that for some $t \ge T_S$ $\exists U_t^S = (\theta_1^S(t), \theta_S) \subset U_{T_S}^S$ and $\exists \Theta_t$ such that for all $\tau = T+1, \ldots, t-1$ and for all $\theta_1 \in U_t^S$ $\gamma_{t-1}(\theta_1) > \varepsilon_1$, $s_t(\theta_1) > \left(\tfrac{1}{d} - \varepsilon\right)s_{t-1}(\theta_1) > 0$ and $\gamma_{t-1}(\theta_1^S(t)) = \varepsilon_1$. It implies that

$$(v-1)\theta_t(\theta_1^S(t)) - s_t(\theta_1^S(t)) = (\theta_{t-1} + y_t)(v-1) - (g_{t-1} - y_t) = (v-1)\theta_{t-1} - g_{t-1} + vy_t =$$

$$= \varepsilon_1 + vy_t < \tfrac{1}{2}\varepsilon_S + v\varepsilon d g_{t-1} = \tfrac{1}{2}\varepsilon_S + v\varepsilon d\left((v-1)\theta_{t-1} - \gamma_{t-1}\right) = \tfrac{1}{2}\varepsilon_S + v\varepsilon d(v-1)\theta_{t-1} <$$

$$< \tfrac{1}{2}\varepsilon_S + v\varepsilon d(v-1)\bar{\theta} < \tfrac{1}{2}\varepsilon_S + \tfrac{1}{2}\varepsilon_S = \varepsilon_S.$$

So, $s_t(\theta_1^S) > (v-1)\theta_t(\theta_1^S) - \varepsilon_S$.

Summing up the indifference equation (3) in a form $d(\theta_t - \theta_{t-1}) = s_{t-1} - ds_t$ from $\tau = 2$ to $\tau = t$ we get $d(\theta_t - \theta_1) = \sum_{t=2}^{t} s_{t-1} - d\sum_{t=2}^{t} s_t = (s_1 - s_t) + (1-d)\sum_{t=2}^{t} s_t$, or

$$\begin{aligned}
\theta_t - \theta_S &= \left(\theta_1 - \theta_S + \tfrac{1}{\delta} s_1\right) + \tfrac{1-\delta}{\delta}\sum_{t=2}^{t} s_t - \tfrac{1}{\delta} s_t = \tfrac{1}{3}\varepsilon_\theta + \tfrac{1}{\delta}\left((1-\delta)\sum_{t=2}^{t} s_t - s_t\right) = \\
&= \tfrac{1}{3}\varepsilon_\theta + \tfrac{1}{\delta}\left((1-\delta)\sum_{t=2}^{T} s_t + (1-\delta)\sum_{t=T+1}^{t} s_t - s_t\right) < \\
&< \tfrac{1}{3}\varepsilon_\theta + \tfrac{1}{\delta}\left((1-\delta)(T-1)\max_{t=1,\dots,t}\{\sup s_t\} + (1-\delta)\sum_{t=T+1}^{t}\left(s_t\left(\tfrac{1}{\delta}-\varepsilon\right)^{t-t}\right) - s_t\right) < \\
&< \tfrac{1}{3}\varepsilon_\theta + \tfrac{1}{\delta}\left(\tfrac{\delta}{3}\varepsilon_\theta + (1-\delta)s_t\sum_{t=-\infty}^{t}\left(\tfrac{1}{\delta}-\varepsilon\right)^{t-t} - s_t\right) = \tfrac{2}{3}\varepsilon_\theta + \tfrac{1}{\delta}\left(\tfrac{(1-\delta)}{1-\left(\frac{1}{\delta}-\varepsilon\right)^{-1}} - 1\right)s_t < \\
&< \tfrac{2}{3}\varepsilon_\theta + \tfrac{\delta\varepsilon}{1-\delta-\delta\varepsilon}\gamma_{t-1} = \tfrac{2}{3}\varepsilon_\theta + \tfrac{\delta\varepsilon}{1-\delta-\delta\varepsilon}\left((\nu-1)\theta_{t-1} - \pi_{t-1}\right) < \tfrac{2}{3}\varepsilon_\theta + \frac{\varepsilon(\nu-1)\theta_{t-1}}{\frac{1-\delta}{\delta}-\varepsilon} < \\
&< \tfrac{2}{3}\varepsilon_\theta + \frac{\varepsilon(\nu-1)\bar{\theta}}{\frac{1-\delta}{\delta}-\frac{1-\delta}{\nu\delta}} = \tfrac{2}{3}\varepsilon_\theta + \frac{\varepsilon(\nu-1)\bar{\theta}}{\frac{1-\delta}{\nu\delta}(\nu-1)} = \tfrac{2}{3}\varepsilon_\theta + \frac{\nu\delta\varepsilon\bar{\theta}}{1-\delta} < \tfrac{2}{3}\varepsilon_\theta + \tfrac{1}{3}\varepsilon_\theta = \varepsilon_\theta .
\end{aligned}$$

So, $\theta_t - \theta_S < \varepsilon_\theta$ for all $\theta_1 \in U_t^S$.

Finally, let us consider $\pi_t\left(\theta_1^S(t)\right)$:

$$\begin{aligned}
&\pi_t\left(\theta_1^S(t)\right) = (\nu-1)\theta_t\left(\theta_1^S(t)\right) - \tfrac{1}{\delta}s_t\left(\theta_1^S(t)\right) = (\theta_{t-1} + y_t)(\nu-1) - \tfrac{1}{\delta}(\gamma_{t-1} - y_t) = \\
&= \pi_{t-1} + y_t\left(\nu - 1 + \tfrac{1}{\delta}\right) - \tfrac{1-\delta}{\delta}\gamma_{t-1} < \varepsilon_1 + \delta\varepsilon\gamma_{t-1}\left(\nu - 1 + \tfrac{1}{\delta}\right) - \tfrac{1}{\delta}\gamma_{t-1}(1-\delta) < \\
&< \varepsilon_1 + \left(\delta\nu\varepsilon - (1-\delta)\right)\tfrac{1}{\delta}\gamma_{t-1} < \varepsilon_1 + \left(\delta\nu\tfrac{1-\delta}{\nu\delta} - (1-\delta)\right)\tfrac{1}{\delta}\gamma_{t-1} = \varepsilon_1,
\end{aligned}$$

so $\pi_t\left(\theta_1^S(t)\right) < \varepsilon_1$. On the other hand $\pi_t(\theta_S) = (\nu-1)\theta_S > \varepsilon_1$ and $\pi_t(\theta_1)$ is continuous, so $\exists\theta_1^S(t+1) \in U_t^S = \left(\theta_1^S(t), \theta_S\right)$ such that $\pi_t\left(\theta_1^S(t+1)\right) = \varepsilon_1$, that ends the induction.

But if $\pi_t\left(\theta_1^S(t+1)\right) = \varepsilon_1$ then we have $s_{t+1}\left(\theta_1^S(t+1)\right) > (\nu-1)\theta_{t+1}\left(\theta_1^S(t+1)\right) - \varepsilon_S$ as was shown above, and let us take $U_{t+1}^S = \left(\theta_1^S(t+1), \theta_S\right) \subset U_t^S$ so that $\theta_{t+1} - \theta_S < \varepsilon_\theta$ for all $\theta_1 \in U_{t+1}^S$, that ends the proof.

Proof of Proposition 4.3.

So far we were considering θ_t as a function of θ_1, $\theta_t = \theta_t(\theta_1)$. Now we will consider θ_t as a function of θ_{t-1} and γ_{t-1}, $\theta_t = \theta_t(\theta_{t-1}, \gamma_{t-1})$, where $\gamma_{t-1} = \gamma_{t-1}(\theta_1)$ and $\theta_{t-1} = \theta_{t-1}(\theta_1)$. We define the following limit function:

$$\hat{\theta}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \lim_{t\to\infty}\theta_t\left(\hat{\theta}_0, \hat{\gamma}_0\right). \tag{22}$$

In the same spirit as before we introduce functions $\hat{y}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \hat{\theta}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right) - \hat{\theta}_0$, $\hat{s}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \hat{\gamma}_0 - \hat{y}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right)$, $\hat{\gamma}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \tfrac{1}{\delta}\hat{s}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right)$, $\hat{\pi}_0\left(\hat{\theta}_0, \hat{\gamma}_0\right) = (\nu-1)\hat{\theta}_0 - \hat{\gamma}_0$.

Taking the limit (22) explicitly[9] yields that the limit actually exists for all $\hat{q}_0 \in [q_S, \bar{q} - \varepsilon)$ and $\hat{g}_0 \in (\varepsilon, \nu h(\hat{q}_0, \bar{q}) - \hat{q}_0 - \varepsilon)$ where $\varepsilon > 0$ is an arbitrarily small number. Convergence is uniform, hence $\hat{q}_1(\hat{q}_0, \hat{g}_0)$ is continuous and it follows that

$$\hat{q}_1(\hat{q}_0, \hat{g}_0) = \begin{cases} \hat{q}_0, \text{ if } \varepsilon < \hat{g}_0 \le (\nu - 1)\hat{q}_0 \\ \hat{q}_1(\hat{q}_0, \hat{g}_0) : \nu h(\hat{q}_0, \hat{q}_1) = \hat{q}_0 + \hat{g}_0, \text{ if } (\nu - 1)\hat{q}_0 \le \hat{g}_0 < \nu h(\hat{q}_0, \bar{q}) - \hat{q}_0 - \varepsilon \end{cases}.$$

Then we define $\hat{q}_1(\hat{q}_0, \hat{g}_0)$ on a boundary where $\hat{g}_0 = 0$, $\hat{g}_0 = \nu h(\hat{q}_0, \bar{q}) - \hat{q}_0$ or $\hat{q}_0 = \bar{q}$ by taking corresponding limits of the function $\hat{q}_1(\hat{q}_0, \hat{g}_0)$ when $\varepsilon \to +0$, that yields $\hat{q}_1(\hat{q}_0, 0) = \hat{q}_0$, $\hat{q}_1(\hat{q}_0, \nu h(\hat{q}_0, \bar{q}) - \hat{q}_0) = \bar{q}$ and $\hat{q}_1(\bar{q}, \hat{g}_0) = \bar{q}$[10].

Finally we define $\hat{q}_{t+1}(\hat{q}_0, \hat{g}_0)$ for all $t > 1$ as follows. If for some $\hat{q}_0 \in [q_S, \bar{q}]$, $\hat{g}_0 \in [0, \frac{\nu - 1}{d}\hat{q}_0]$ and for all $t = 0, \ldots, t$ there exist functions $\hat{q}_t(\hat{q}_0, \hat{g}_0)$ and $\hat{g}_t(\hat{q}_0, \hat{g}_0)$ such that $0 \le \hat{g}_t \le \nu h(\hat{q}_t, \bar{q}) - \hat{q}_t$ and $\hat{q}_t(\hat{q}_0, \hat{g}_0) \le \bar{q}$ then we take $\hat{q}_{t+1}(\hat{q}_0, \hat{g}_0) = \hat{q}_1(\hat{q}_t, \hat{g}_t)$.

It can be easily seen that if $0 < \hat{g}_t < \nu h(\hat{q}_t, \bar{q}) - \hat{q}_t$ and $\hat{q}_t(\hat{q}_0, \hat{g}_0) < \bar{q}$ then $\hat{q}_{t+1}(\hat{q}_0, \hat{g}_0)$ has the following limit representation[11]:

$$\hat{q}_{t+1}(\hat{q}_0, \hat{g}_0) = \lim_{t \to \infty} q_{t+t+1}(q_{t+t}(\ldots(\hat{q}_0, \hat{g}_0)), g_{t+t}(\ldots(\hat{q}_0, \hat{g}_0))).$$

The main use of that trick is to substitute complex functions $q_t(q_{t-1}(\ldots(q_{t-t}, g_{t-t})), g_{t-1}(\ldots(q_{t-t}, g_{t-t})))$ by theirs limit analogs for very large t when the measure of "low quality goods" becomes negligible compare to the measure of "high quality goods". Limit functions $\hat{q}_t(\hat{q}_0, \hat{g}_0)$ would have been exactly the same as $q_t(q_{t-1}(\ldots(q_{t-t}, g_{t-t})), g_{t-1}(\ldots(q_{t-t}, g_{t-t})))$ if there had been no entry of new sellers[12].

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[9] This can be done by considering two cases, namely $\hat{\jmath}_0 \ge 0$ and $\hat{\jmath}_0 \le 0$. The former yields $\lim_{t \to \infty} y_t = 0$ while the later yields $\lim_{t \to \infty} \nu K_{t+1} y_{t+1} = \hat{\jmath}_0$. The final result then is straightforward.

[10] The expression $\hat{q}_1(\bar{q}, \hat{g}_0)$ is defined only if $\hat{\jmath}_0 \ge 0$, i.e., $\hat{g}_0 \le (\nu - 1)\hat{q}_0$.

[11] If $\hat{g}_t(\hat{q}_0, \hat{g}_0) = 0$, $\hat{g}_t(\hat{q}_0, \hat{g}_0) = \nu h(\hat{q}_t(\hat{q}_0, \hat{g}_0), \bar{q}) - \hat{q}_t(\hat{q}_0, \hat{g}_0)$ or $\hat{q}_t(\hat{q}_0, \hat{g}_0) = \bar{q}$ for some t the function $\hat{q}_{t+1}(\hat{q}_0, \hat{g}_0)$ is not a limit function any more, but after all derivations have been made those functions will never be evaluated at such points.

[12] No entry case is described in Janssen and Roy (1999a).

Now let us fix $\hat{\boldsymbol{q}}_0 = \boldsymbol{q}^{(k)}$ and take any $\hat{g}_0 \in \left(0, \frac{\nu-1}{d}\boldsymbol{q}^{(k)}\right)$. If for some $\boldsymbol{t} \geq 0$ we have obtained the functions $\hat{\boldsymbol{q}}_t\left(\hat{\boldsymbol{q}}_0, \hat{g}_0\right) = \hat{\boldsymbol{q}}_t\left(\hat{g}_0\right)$ and $\hat{g}_t\left(\hat{\boldsymbol{q}}_0, \hat{g}_0\right) = \hat{g}_t\left(\hat{g}_0\right)$, and at the same time $0 < \hat{g}_t \leq \nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_t, \bar{\boldsymbol{q}}\right) - \hat{\boldsymbol{q}}_t$ then there exists the next function, namely $\hat{\boldsymbol{q}}_{t+1}\left(\hat{g}_0\right) = \hat{\boldsymbol{q}}_1\left(\hat{\boldsymbol{q}}_t\left(\hat{g}_0\right), \hat{g}_t\left(\hat{g}_0\right)\right)$ such that $\hat{\boldsymbol{q}}_{t+1}\left(\hat{g}_0\right) \in \left[\hat{\boldsymbol{q}}_t, \bar{\boldsymbol{q}}\right]$ for all $\hat{g}_0 \in \left(0, \frac{\nu-1}{d}\boldsymbol{q}^{(k)}\right)$.

We will show that $\exists \hat{t} \geq 0$ and $\exists \hat{g} \in \left(0, \frac{\nu-1}{d}\boldsymbol{q}^{(k)}\right)$ such that either $\hat{g}_{\hat{t}}\left(\hat{g}\right) = 0$, or $\hat{g}_{\hat{t}}\left(\hat{g}\right) > \nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_{\hat{t}}\left(\hat{g}\right), \bar{\boldsymbol{q}}\right) - \hat{\boldsymbol{q}}_{\hat{t}}\left(\hat{g}\right)$.

Suppose not, that means that for any $t \geq 0$ and any $\hat{g}_0 \in \left(0, \frac{\nu-1}{d}\boldsymbol{q}^{(k)}\right)$ $\exists \hat{\boldsymbol{q}}_t\left(\hat{g}_0\right)$ and $\exists \hat{g}_t\left(\hat{g}_0\right)$ such that $0 < \hat{g}_t \leq \nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_t, \bar{\boldsymbol{q}}\right) - \hat{\boldsymbol{q}}_t$ and $\boldsymbol{q}^{(k)} \leq \hat{\boldsymbol{q}}_t \leq \bar{\boldsymbol{q}}$. Let fix any $\hat{g} \in \left(0, \frac{\nu-1}{d}\boldsymbol{q}^{(k)}\right)$ and get infinite sequences $\left\{\hat{\boldsymbol{q}}_t\left(\hat{g}\right)\right\}_{t=0}^{\infty}$ and $\left\{\hat{g}_t\left(\hat{g}\right)\right\}_{t=0}^{\infty}$. The former is weakly increasing and bounded so $\exists \lim_{t\to\infty} \hat{\boldsymbol{q}}_t = \hat{\boldsymbol{q}}_\infty \leq \bar{\boldsymbol{q}}$. But this implies that the later has a limit either $\lim_{t\to\infty} \hat{g}_t = \lim_{t\to\infty}\left\{\nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_t, \hat{\boldsymbol{q}}_{t+1}\right) - \hat{\boldsymbol{q}}_t\right\} = \nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_\infty, \hat{\boldsymbol{q}}_\infty\right) - \hat{\boldsymbol{q}}_\infty = (\nu - 1)\hat{\boldsymbol{q}}_\infty > 0$. Taking a limit of the indifference equation $\boldsymbol{d}\hat{g}_{t+1} = \hat{g}_t - \left(\hat{\boldsymbol{q}}_{t+1} - \hat{\boldsymbol{q}}_t\right)$ gives rise to a contradiction:

$$\boldsymbol{d}(\nu - 1)\hat{\boldsymbol{q}}_\infty = \lim_{t\to\infty} \boldsymbol{d}\hat{g}_{t+1} = \lim_{t\to\infty}\left\{\hat{g}_t - \left(\hat{\boldsymbol{q}}_{t+1} - \hat{\boldsymbol{q}}_t\right)\right\} = \lim_{t\to\infty} \hat{g}_t = (\nu - 1)\hat{\boldsymbol{q}}_\infty .$$

So, only two possibilities are left:

a) Case 1. $\exists \hat{t}$ and $\exists \hat{g} \in \left(0, \frac{\nu-1}{d}\boldsymbol{q}^{(k)}\right)$ such that for all $t = 1, \ldots, \hat{t} - 1$ and all $\hat{g}_0 \in (0, \hat{g})$ $\hat{g}_t > 0$, $\nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_{t-1}, \bar{\boldsymbol{q}}\right) \geq \hat{\boldsymbol{q}}_{t-1} + \hat{g}_{t-1}$, $\hat{g}_{\hat{t}} > 0$ while $\nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_{\hat{t}-1}\left(\hat{g}\right), \bar{\boldsymbol{q}}\right) < \hat{\boldsymbol{q}}_{\hat{t}-1}\left(\hat{g}\right) + \hat{g}_{\hat{t}-1}\left(\hat{g}\right)$; and

b) Case 2. $\exists \hat{t}$ and $\exists \hat{g} \in \left(0, \frac{\nu-1}{d}\boldsymbol{q}^{(k)}\right)$ such that for all $t = 1, \ldots, \hat{t}$ and all $\hat{g}_0 \in (0, \hat{g})$ $\hat{g}_t > 0$ and $\nu \boldsymbol{h}\left(\hat{\boldsymbol{q}}_{t-1}, \bar{\boldsymbol{q}}\right) \geq \hat{\boldsymbol{q}}_{t-1} + \hat{g}_{t-1}$ while $\hat{g}_{\hat{t}}\left(\hat{g}\right) = 0$.

The detailed proof of the Cases is on request. We prove that in the Case 1 $\exists \overline{\boldsymbol{e}_S} > 0$ such that for any T $\exists \bar{t} > T$, $\exists \tilde{\boldsymbol{q}}_1\left(\bar{t}\right) \in \left(\boldsymbol{q}_S - \boldsymbol{e}_m, \boldsymbol{q}_S\right)$ and $\exists \Theta_{\bar{t}}\left(\tilde{\boldsymbol{q}}_1\right)$ such that $\boldsymbol{q}_{\bar{t}}\left(\tilde{\boldsymbol{q}}_1\right) = \bar{\boldsymbol{q}}$ and $s_{\bar{t}}\left(\tilde{\boldsymbol{q}}_1\right) > \overline{\boldsymbol{e}_S}$. In other words in this case there exist infinite number equilibrium sequences such that all goods are traded in the last period.

In the Case 2 we define $q^{(k+1)}$ as $q^{(k+1)} = \hat{q}_{\hat{t}}(\hat{g})$. We show that $q^{(k+1)} \in \left(\nu q^{(k)}, \bar{q}\right]$. Then we prove that either we have the same result as in the Case 1, or for any $\bar{e}_S > 0$ and $\bar{e}_q > 0$ $\exists T_S^{(k+1)} > T_S^{(k)}$ such that for all $t > T_S^{(k+1)}$ $\exists U_t^{(k+1)} = \left(\underline{q}_1(t,k+1), \bar{q}_1(t,k+1)\right) \subset U_t^{(k)}$ and $\exists \Theta_t\left(U_t^{(k+1)}\right)$ such that for all $q_1 \in U_t^{(k+1)}$ $\left|q_t(q_1) - q^{(k+1)}\right| < \bar{e}_q$, $0 \le s_t(\underline{q}_1) < \bar{e}_S$ and $s_t\left(\bar{q}_1\right) > (\nu - 1)q_t\left(\bar{q}_1\right) - \bar{e}_S$.

Proof of Theorem 4.1.

Consequently applying Propositions 4.1 and 4.2 we get the following result: for any $e_S > 0$ and $e_q > 0$ $\exists T_S$ such that for all $t \ge T_S$ $\exists U_t^S = \left(\underline{q}_1^S(t), q_S\right)$ and $\exists \Theta_t\left(U_t^S\right)$ such that for any $q_1 \in U_t^S$ $q_t(q_1) \in (q_S, q_S + e_q)$ and $s_t\left(\underline{q}_1^S\right) > (\nu - 1)q_t\left(\underline{q}_1^S\right) - e_S$. Now we can see that we are under the conditions of Proposition 4.3 if we take $q^{(1)} = q_S < \bar{q}$. Here we distinguish three cases.

a) Case 1. For any $k = 1, \ldots \infty$ there exists $q^{(k+1)} \in \left(\nu q^{(k)}, \bar{q}\right)$ such that for any $\bar{e}_S > 0$ and $\bar{e}_q > 0$ $\exists T_S^{(k+1)}$ such that for all $t > T_S^{(k+1)}$ $\exists U_t^{(k+1)} = \left[\underline{q}_1(t,k+1), \bar{q}_1(t,k+1)\right] \subset U_t^{(k)}$ and $\exists \Theta_t\left(U_t^{(k+1)}\right)$ such that:

- for all $q_1 \in U_t^{(k+1)}$ $\left|q_t(q_1) - q^{(k+1)}\right| < \bar{e}_q$;
- $0 \le s_t\left(\underline{q}_1(t,k+1)\right) < \bar{e}_S$;
- $s_t\left(\bar{q}_1(t,k+1)\right) > (\nu - 1)q_t\left(\bar{q}_1(t,k+1)\right) - \bar{e}_S$.

But in this case we get infinite sequence $\left\{q^{(k)}\right\}_{k=1}^{\infty}$ where $q^{(k+1)} > \nu q^{(k)}$, that contradicts with $q^{(k)} < \bar{q}$ as $\lim_{k \to \infty} q^{(k)} = +\infty$. So after some steps $\hat{k}$ we must meet either the Case 2 or Case 3.

b) Case 2. There exists $q^{(\hat{k})}$ such that there does not exist $q^{(k+1)} \in \left(\nu q^{(k)}, \bar{q}\right]$. In accordance with Proposition 4.3 we can make a conclusion: $\exists \bar{e}_S > 0$ such that for any T $\exists \bar{t} > T$, $\exists \tilde{q}_1\left(\bar{t}\right)$ and $\exists \Theta_{\bar{t}}\left(\tilde{q}_1\right)$ such that $q_{\bar{t}}\left(\tilde{q}_1\right) = \bar{q}$ and $s_{\bar{t}}\left(\tilde{q}_1\right) > \bar{e}_S$. In other

words there are infinite number of equilibrium sequences such that all goods are sold in the last period and the last marginal surplus is strictly positive and separated from zero, $s_t(\tilde{q}_1) > \overline{e_S} > 0$.

In this case we can construct infinite number of dynamic equilibria by concatenating equilibrium sequences, e.g. we take $\Theta_t = \{q_t\}_{t=1}^t$ and let a dynamic equilibrium be the following sequence of marginal sellers: $\{\bar{q}_t\}_{t=1}^{\infty}$ such that $\bar{q}_t = q_t$ if $t \le t$ and $\bar{q}_t = q_{t-t}$ if $t > t$.

c) Case 3. There exists $q^{(\hat{k})}$ such that there exists $q^{(k+1)} = \bar{q}$.

Note here that $q^{(k+1)}$ is determined in terms of the previous point $q^{(\hat{k})}$, the measure function $m(q)$ and parameters $\underline{q}$, v and δ. In other words $q^{(k+1)} = \Omega\left(q^{(k)}, m(q), \underline{q}, v, \delta\right)$, where $\Omega\left(q^{(k)}, m(q), \underline{q}, v, \delta\right)$ is some operator. Therefore the case when $\Omega\left(q^{(k)}, m(q), \underline{q}, v, \delta\right) = \bar{q}$ is non-generic.

Proof of Proposition 4.4.

We begin with solving the system of indifference equations (4) w.r.t. p_t:

$$p_t - q_t\left(1-\delta^{t-t}\right) = \frac{p_1 - q_1\left(1-\delta^{t-1}\right)}{\delta^{t-1}},\ t = 1,\ldots,t-1. \qquad (23)$$

We will look for such a sequence of functions $\{q_t(q_1)\}_{t=1}^t$ satisfying (4) that $q_{t+1} < q_t$ for all $t = 1,\ldots,t-2$. In this case we have $p_t(q_t) = \frac{v}{m(\underline{q}, q_t)}\int_{\underline{q}}^{q_t} q\,dm$, and $\frac{dp_t}{dq_t}(q_S) = a$.

Substituting this into the first differential of (23) we get $\frac{dq_t}{dq_1}(q_S) = \frac{1}{\delta^{t-1}}\frac{\left(1-a-\delta^{t-1}\right)}{\left(1-a-\delta^{t-t}\right)}$.

Hence, we can write $q_t(q_1)$ and $s_t(q_1)$ as $q_t(q_1) = q_S - \frac{dq_t}{dq_1}(q_S)(q_S - q_1) + o(q_1 - q_S)$

and $s_t(q_1) = -(1-a)\frac{dq_t}{dq_1}(q_S)(q_1 - q_S) + o(q_1 - q_S)$. Then

$$\theta_{t+1}-\theta_t=\left(-\frac{1}{\delta^t}\frac{\left(1-\alpha-\delta^{t-1}\right)}{\left(1-\alpha-\delta^{t-t-1}\right)}+\frac{1}{\delta^{t-1}}\frac{\left(1-\alpha-\delta^{t-1}\right)}{\left(1-\alpha-\delta^{t-t}\right)}\right)(\theta_S-\theta_1)+o(\theta_1-\theta_S)=$$
$$=-\frac{\left(1-\alpha-\delta^{t-1}\right)(1-\delta)(1-\alpha)}{\delta^t\left(1-\alpha-\delta^{t-t}\right)\left(1-\alpha-\delta^{t-t-1}\right)}(\theta_S-\theta_1)+o(\theta_1-\theta_S).$$

It follows that if $\alpha<1-\delta$ then $\frac{d\theta_t}{d\theta_1}(\theta_S)>1$. Thus there exists a neighborhood, namely $U^0_{t-1}=\left(\theta_1^0(t-1),\theta_S\right)$, such that for all $\theta_1\in U^0_{t-1}$ $\theta_t<\theta_S$ and, therefore, $\theta_{t+1}<\theta_t$, and $s_t>0$. Therefore there exists a sequence of functions $\{\theta_t(\theta_1)\}_{t=1}^{t-1}$ such that all conditions to be proved are satisfied except the last one and we only have to show that if $\alpha\in(0,\tilde{\alpha}(\delta))$ then there exists $\theta_t(\theta_1)$ and $U^0_t=\left(\theta_1^0(t),\theta_S\right)\in U^0_{t-1}$ such that for all $\theta_1\in U^0_t$ $0<\theta_t(\theta_1)-\theta_S<\varepsilon_g\frac{1}{d}s_t(\theta_1)$.

Given the structure of $\{\theta_t(\theta_1)\}_{t=1}^{t}$ we can write:

$$p_t(\theta_1)=p_t(\theta_1(\theta_1),\theta_2(\theta_1),\ldots,\theta_t(\theta_1))=v\frac{t\int_{\underline{\theta}}^{\theta_t(\theta_1)}\theta dm-\sum_{t=1}^{t-1}\int_{\underline{\theta}}^{\theta_t(\theta_1)}\theta dm}{tm(\underline{\theta},\theta_t(\theta_1))-\sum_{t=1}^{t-1}m(\underline{\theta},\theta_t(\theta_1))},$$

and, consequently,

$$d_S p_t=\alpha\left(td_S\theta_t-\sum_{t=1}^{t-1}d_S\theta_t\right)=\alpha\left(td_S\theta_t-d_S\theta_1\sum_{t=1}^{t-1}\frac{1}{\delta^{t-1}}\frac{\left(1-\alpha-\delta^{t-1}\right)}{\left(1-\alpha-\delta^{t-t}\right)}\right)=$$
$$=\alpha\left(td_S\theta_t-d_S\theta_1\frac{1-\alpha-\delta^{t-1}}{\delta^{t-1}}\sum_{t=1}^{t-1}\frac{\delta^t}{1-\alpha-\delta^t}\right).$$

Substituting this into the first differential of (4) yields

$$\frac{d\theta_t}{d\theta_1}(\theta_S)=\frac{1-\alpha-\delta^{t-1}}{\alpha t\delta^{t-1}}\left(\alpha\sum_{t=1}^{t-1}\frac{\delta^t}{1-\alpha-\delta^t}-1\right),$$

and, therefore,

$$\theta_t=\theta_S+\frac{d\theta_t}{d\theta_1}(\theta_S)(\theta_1-\theta_S)+o(\theta_1-\theta_S).$$

This implies that there exists a neighborhood $\overline{U^0_t}=\left(\overline{\theta_1^0(t)},\theta_S\right)\in U^0_{t-1}$, such that for all $\theta_1\in\overline{U^0_t}$ $\theta_t>\theta_S$ as long as $\alpha\sum_{t=1}^{t-1}\frac{\delta^t}{1-\alpha-\delta^t}<1$. So the condition $\alpha\sum_{t=1}^{\infty}\frac{\delta^t}{1-\alpha-\delta^t}\le 1$ actually is a sufficient one. Note here that

$$\sum_{t=1}^{\infty}\frac{a\delta^t}{1-a-\delta^t}<\sum_{t=1}^{\infty}\frac{a\delta^t}{1-a-\delta}=\frac{a\delta}{1-a-\delta}\sum_{t=1}^{\infty}\delta^{t-1}=\frac{a\delta}{(1-a-\delta)(1-\delta)}=$$

$$=1+\frac{a-(1-\delta)^2}{(1-a-\delta)(1-\delta)},$$

so if $a\le(1-\delta)^2$ then $a\sum_{t=1}^{\infty}\frac{\delta^t}{1-a-\delta^t}<1$ and $a\le(1-\delta)^2$ is a sufficient condition.

Then we check whether $s_t>0$:

$$d_S s_t = d_S p_t - d_S q_t = -\frac{1-a-\delta^{t-1}}{\delta^{t-1}}\left(1-\frac{1}{at}\left(1-a\sum_{t=1}^{t-1}\frac{\delta^t}{1-a-\delta^t}\right)\right)d_S q_1 .$$

Hence, $\frac{ds_t}{dq_1}(q_S)<-\frac{1-a-\delta^{t-1}}{\delta^{t-1}}\left(1-\frac{1}{at}\right)$ and it follows that $\frac{ds_t}{dq_1}(q_S)<0$ when $t>\frac{1}{a}$.

So there exists $\overline{\overline{T_0}}=\left[\frac{1}{a}\right]+1$ such that for all $t>\overline{\overline{T_0}}$ $\exists\overline{\overline{U_t^0}}=\left(\overline{\overline{q_1^0(t)}},q_S\right)\subset\overline{U_t^0}$ and $\exists\Theta^2{}_t\left(\overline{\overline{U_t^0}}\right)$ such that for all $q_1\in\overline{\overline{U_t^0}}$ $q_t(q_1)>q_S$, $s_t(q_1)>0$. $q_t(q_S)=q_S$ and $s_t(q_S)=0$ for all $\tau=1,\dots,t$ by construction.

Finally we will prove that if t is taken sufficiently large than $\exists\varepsilon_g>0$ such that $0<q_t(q_1)-q_S<\varepsilon_g\frac{1}{\delta}s_t(q_1)$.

Let us consider the ratio $\frac{s_t(q_1)}{q_t(q_1)-q_S}$. It can be written as

$$\frac{s_t(q_1)}{q_t(q_1)-q_S}=\frac{\frac{ds_t}{dq_1}(q_S)(q_1-q_S)+o(q_1-q_S)}{\frac{dq_t}{dq_1}(q_S)(q_1-q_S)+o(q_1-q_S)}=\frac{at+o\left((q_1-q_S)^0\right)}{1-a\sum_{t=1}^{t-1}\frac{\delta^t}{1-a-\delta^t}}-1.$$ Therefore

$$\lim_{t\to\infty}\left[\lim_{q_1\to q_S-0}\frac{s_t(q_1)}{q_t(q_1)-q_S}\right]=\lim_{t\to\infty}\frac{at}{1-a\sum_{t=1}^{t-1}\frac{\delta^t}{1-a-\delta^t}}-1=+\infty .$$

This implies that we actually can take any $\varepsilon_g>0$ such that $\exists T_0>\overline{\overline{T_0}}$ such that for all $t>T_0$ $\exists U_t^0=\left(q_1^0(t),q_S\right)$ and $\exists\Theta^2{}_t\left(U_t^0\right)$ such that $0<q_t(q_1)-q_S<\varepsilon_g\frac{1}{\delta}s_t(q_1)$.

Proof of Proposition 4.5.

Suppose we have obtained an equilibrium sequence $\Theta_{k_1}\left(U^0_{k_1}\right)$, where $U^0_{k_1}=\left(\theta_1^0(k_1),\theta_S\right)$ such that for all $t=1,\ldots,k_1$ θ_t and s_t can be represented as $\theta_t=\theta_S+\frac{d\theta_t}{d\theta_1}(\theta_S)(\theta_1-\theta_S)+o(\theta_1-\theta_S)$, and $s_t=\frac{ds_t}{d\theta_1}(\theta_S)(\theta_1-\theta_S)+o(\theta_1-\theta_S)$ where $\frac{ds_t}{d\theta_1}(\theta_S)<0$ and $\frac{d\theta_{k_1}}{d\theta_1}(\theta_S)>0$, and for all $\theta_1\in U^0_{k_1}$ $\theta_t(\theta_1)<\theta_{k_1}(\theta_1)<\theta_S$.

We introduce the following new variable:

$$\alpha_{k_1}\equiv-\frac{ds_{k_1}}{d\theta_{k_1}}(\theta_S)=-\left(\frac{d\theta_{k_1}}{d\theta_1}(\theta_S)\right)^{-1}\frac{ds_{k_1}}{d\theta_1}(\theta_S)>0. \tag{24}$$

In terms of α_{k_1}, surplus s_{k_1} can be represented as $s_{k_1}(\theta_1)=\alpha_{k_1}(\theta_{k_1}-\theta_S)+o(\theta_{k_1}-\theta_S)$. There exists at least one of such a sequence, namely $\{\theta_1\}$, where $k_1=1$.

Now we will construct a new equilibrium sequence $\Theta_t(\Theta_{k_1})$ in the following way. We will repeat the whole structure of Θ_{k_1} t times. In other words, for all $t=1,\ldots,t-1$ and for all $l=1,\ldots,k_1-1$ we put $\theta_{(t-1)k_1+l+1}(\theta_1)<\theta_{(t-1)k_1+l}(\theta_1)$ if $\theta_{l+1}(\theta_1)<\theta_l(\theta_1)$ and vice versa. Another rule is that for all $t=2,\ldots,t$ $\theta_{tk_1}(\theta_1)<\theta_{(t-1)k_1}(\theta_1)$.

Having done this we can see that each of the sequences $\{\theta_l(\theta_1)\}_{l=(t-1)k_1+1}^{l=tk_1}$ for all $t=1,\ldots,t$ is an equilibrium sequence Θ_{k_1}. Now we have to find $\theta_{tk_1+1}(\theta_1)$ such that for all $t=1,\ldots,t$ $\delta^{tk_1-1}\left(p_{tk_1}-\theta_{tk_1}\right)=\delta^{tk_1}\left(p_{tk_1+1}-\theta_{tk_1}\right)$, in other words, the seller of quality θ_{tk_1} must be indifferent between selling in time period tk_1 and tk_1+1. Loosely speaking we try to construct a sort of equilibrium sequence of type II using Θ_{k_1} as single components instead of θ_t.

We will show that if $\alpha_{k_1} > \delta$ and $ak_1 \sum_{t=1}^{\infty} \frac{\delta^{(t-1)k_1+1}}{\alpha_{k_1} - \delta^{(t-1)k_1+1}} \le 1$ then it can be done for any t and if $t > \frac{1-a}{ak_1}$ then $s_{tk_1+1} > 0$. Applying the same procedure as in the proof of the Proposition 4.4 we get the following indifference equations:

$$\delta^{(t-1)k_1}\left(p_{tk_1} - \left(1-\delta^{(t-t)k_1+1}\right)q_{tk_1}\right) = p_{k_1} - \left(1+\delta^{(t-1)k_1+1}\right)q_{k_1}. \quad (25)$$

Taking the first differentials of (25) at $q_t = q_S$ and using definition of α_{k_1} (24) we get:

$$\delta^{(t-1)k_1}\left(-\alpha_{tk_1} d_S q_{tk_1} + \delta^{(t-t)k_1+1} d_S q_{tk_1}\right) = -\alpha_{k_1} d_S q_{k_1} + \delta^{(t-1)k_1+1} d_S q_{k_1}.$$

Using the fact that $\alpha_{tk_1} = \alpha_{k_1}$ yields $\left(\alpha_{k_1} - \delta^{(t-t)k_1+1}\right) d_S q_{tk_1} = \frac{\alpha_{k_1} - \delta^{(t-1)k_1+1}}{\delta^{(t-1)k_1}} d_S q_{k_1}$, or $\frac{dq_{tk_1}}{dq_{k_1}}(q_S) = \frac{1}{\delta^{(t-1)k_1}} \frac{\alpha_{k_1} - \delta^{(t-1)k_1+1}}{\alpha_{k_1} - \delta^{(t-t)k_1+1}}$. Hence, we can write $q_{tk_1}(q_1)$ and $s_{tk_1}(q_1)$ as:

$$q_{tk_1}(q_1) = q_S - \frac{dq_{tk_1}}{dq_{k_1}}(q_S) \cdot \frac{dq_{k_1}}{dq_1}(q_S)(q_S - q_1) + o(q_1 - q_S), \text{ and}$$

$$s_{tk_1}(q_1) = \alpha_{tk_1}\left(q_{tk_1} - q_S\right) + o\left(q_{tk_1} - q_S\right) = \alpha_{k_1} \frac{dq_{tk_1}}{dq_{k_1}} \frac{dq_{k_1}}{dq_1}(q_S)(q_1 - q_S) + o(q_1 - q_S). \text{ Then}$$

$$q_{(t+1)k_1} - q_{tk_1} = \left(- \frac{dq_{(t+1)k_1}}{dq_{k_1}}(q_S) + \frac{dq_{tk_1}}{dq_{k_1}}(q_S) \right) \frac{dq_{k_1}}{dq_1}(q_S)(q_S - q_1) + o(q_1 - q_S) =$$

$$= -\frac{\left(\alpha_{k_1} - \delta^{(t-1)k_1+1}\right)\left(1-\delta^{k_1}\right)\alpha_{k_1}}{\delta^{tk_1}\left(\alpha_{k_1} - \delta^{(t-t-1)k_1+1}\right)\left(\alpha_{k_1} - \delta^{(t-t)k_1+1}\right)} \frac{dq_{k_1}}{dq_1}(q_S)(q_S - q_1) + o(q_1 - q_S).$$

It follows that for any $t = 1, \ldots, t$ $\frac{dq_{tk_1}}{dq_{k_1}}(q_S) \ge 1$ as long as $\alpha_{k_1} > \delta$[13]. Thus there exists a neighborhood, namely $\overline{U^0_{tk_1}} = \left(\overline{q_1^0(tk_1)}, q_S\right) \subset \overline{U^0_{k_1}}$, such that for all $q_1 \in \overline{U^0_{tk_1}}$ $q_{tk_1} < q_S$ and, therefore, $q_{(t+1)k_1} < q_{tk_1}$, and $s_{tk_1} > 0$. Therefore there exists a system of functions $\{q_l(q_1)\}_{l=1}^{l=tk_1}$ which satisfies all the indifference equations (25) such that all conditions of $\Theta_t\left(\Theta_{k_1}\right)$ are satisfied except the last one and now we have to find $q_{tk_1+1}(q_1) \ge q_{tk_1}(q_1)$ such that $s_{tk_1+1}(q_1) > 0$.

[13] Note that this condition is trivially satisfied for $t = 1$.

Given the structure of $\{q_l(q_1)\}_{l=1}^{l=tk_1}$ we can write:

$$p_{tk_1+1}\left(q_{k_1}(q_1),\ldots,q_{tk_1}(q_1),q_{tk_1+1}(q_1)\right)=v\frac{(tk_1+1)\int_{\underline{q}}^{q_{tk_1+1}(q_1)}qdm-k_1\sum_{t=1}^{t}\int_{\underline{q}}^{q_{tk_1}(q_1)}qdm}{(tk_1+1)m\left(q,q_{tk_1+1}(q_1)\right)-k_1\sum_{t=1}^{t}m\left(q,q_{tk_1}(q_1)\right)},$$

and, consequently,

$$d_S\,p_{tk_1+1}=a\left((tk_1+1)d_Sq_{tk_1+1}-k_1\sum_{t=1}^{t}d_Sq_{tk_1}\right)=$$
$$=a\left((tk_1+1)d_Sq_{tk_1+1}-d_Sq_{k_1}\frac{k_1\left(a_{k_1}-d^{(t-1)k_1+1}\right)}{d^{tk_1}}\sum_{t=1}^{t}\frac{d^{(t-1)k_1+1}}{a_{k_1}-d^{(t-1)k_1+1}}\right).$$

Substituting this into the first differential of (25) yields

$$\frac{dq_{tk_1+1}}{dq_{k_1}}(q_S)=\frac{a_{k_1}-d^{(t-1)k_1+1}}{a(tk_1+1)d^{(t-1)k_1+1}}\left(ak_1\sum_{t=1}^{t}\frac{d^{(t-1)k_1+1}}{a_{k_1}-d^{(t-1)k_1+1}}-1\right),$$

and, therefore,

$$q_{tk_1+1}=q_S+\frac{dq_{tk_1+1}}{dq_{k_1}}(q_S)\cdot\frac{dq_{k_1}}{dq_1}(q_S)(q_1-q_S)+o(q_1-q_S).$$

This implies that there exists a neighborhood $\overline{U^0_{tk_1+1}}=\left(\overline{q_1^0(tk_1+1)},q_S\right)\in\overline{U^0_{tk_1}}$, such that $q_{tk_1+1}>q_S$ as long as $\sum_{t=1}^{t}\frac{ak_1d^{(t-1)k_1+1}}{a_{k_1}-d^{(t-1)k_1+1}}<1$. So the condition $\sum_{t=1}^{\infty}\frac{ak_1d^{(t-1)k_1+1}}{a_{k_1}-d^{(t-1)k_1+1}}\le 1$ actually is a sufficient one. Now we will check whether $s_{tk_1+1}>0$:

$$d_Ss_{tk_1+1}=-\frac{a_{k_1}-d^{(t-1)k_1+1}}{d^{(t-1)k_1+1}}\left(\frac{1}{a(tk_1+1)}\left(ak_1\sum_{t=1}^{t}\frac{d^{(t-1)k_1+1}}{a_{k_1}-d^{(t-1)k_1+1}}-1\right)+1\right)d_Sq_{k_1},$$

and, therefore,

$$\frac{ds_{tk_1+1}}{dq_1}(q_S)<-\frac{a_{k_1}-d^{(t-1)k_1+1}}{d^{(t-1)k_1+1}}\left(1-\frac{1}{a(tk_1+1)}\right)\frac{dq_{k_1}}{dq_1}(q_S).$$

It follows that $\frac{ds_{tk_1+1}}{dq_1}(q_S)<0$ when $t>\frac{1-a}{ak_1}$.

Now we will construct the desired equilibrium sequence Θ and our objective is to get Θ_{k_m} such that $ak_m\sum_{t=1}^{\infty}\frac{d^{(t-1)k_m+1}}{a_{k_m}-d^{(t-1)k_m+1}}\le 1$. We start from $k_1=1$ and $\Theta_{k_1}=\Theta_1$.

Suppose that $\alpha_{k_1} \le \delta$ and we cannot take $t > 1$. In this case we take $t = 1$, in other words we look for $\theta_{k_1+1} > \theta_{k_1}$. It follows that

$$\frac{d\theta_{k_1+1}}{d\theta_1}(\theta_S) = \frac{ak_1\delta - \alpha_{k_1} + \delta}{a(k_1+1)\delta} = \frac{ak_1\delta + (\delta - \alpha_{k_1})}{a(k_1+1)\delta} > 0,$$

and $\theta_{k_1+1} < \theta_S$, hence $s_{k_1+1} > 0$, in some left neighborhood of θ_S. We take $\Theta_{k_2} \equiv \Theta_1(\Theta_{k_1})$, where $k_2 = k_1 + 1$, as an initial equilibrium sequence and repeat the described procedure again.

Now we will prove that after some n steps we get $\alpha_{k_n} > \delta$. To do so we take a lower limit of $\frac{\alpha_{k_{n+1}}}{\alpha_{k_n}}$ with $n \to \infty$. In this case $k_n \overset{n\to\infty}{\to} \infty$ as $k_{n+1} = k_n + 1$ and we have:

$$\lim_{n\to\infty} \frac{\alpha_{k_{n+1}}}{\alpha_{k_n}} = \lim_{n\to\infty} \left(\frac{1}{\alpha_{k_n}} \frac{-\frac{ds_{k_n+1}}{d\theta_{k_n}}}{\frac{d\theta_{k_n+1}}{d\theta_{k_n}}} \right) = \lim_{n\to\infty} \left(\frac{-\frac{\alpha_{k_n} - \delta}{\delta} \left(\frac{1}{a(k_n+1)} \left(\frac{\delta a k_n}{\alpha_{k_n} - \delta} - 1 \right) + 1 \right)}{\alpha_{k_n} \frac{ak_n\delta - \alpha_{k_n} + \delta}{a(k_n+1)\delta}} \right) =$$

$$= \lim_{k_n\to\infty} \frac{(\delta - \alpha_{k_n})(1-a) + ak_n\alpha_{k_n}}{\alpha_{k_n}(ak_n\delta - \alpha_{k_n} + \delta)} = \lim_{k_n\to\infty} \frac{1 + \left(\frac{\delta}{\alpha_{k_n}} - 1 \right) \frac{1-a}{ak_n}}{\delta + \frac{\delta - \alpha_{k_n}}{ak_n}}.$$

Suppose $\alpha_{k_n} \le \delta$ for any n, then

$$\lim_{n\to\infty} \frac{\alpha_{k_{n+1}}}{\alpha_{k_n}} > \lim_{k_n\to\infty} \frac{1}{\delta + \frac{d}{ak_n}} = \frac{1}{\delta} > 1.$$

But this implies that $\lim_{n\to\infty} \alpha_{k_n} = +\infty$ that contradicts with $\alpha_{k_n} \le \delta$.

Hence, there exists some number n such that we get $\Theta_{k_n} = \{\theta_t(\theta_1)\}_{t=1}^{t=k_n}$ and $\alpha_{k_n} > \delta$.

Then we check whether $\sum_{t=1}^{\infty}\frac{ak_n\delta^{(t-1)k_n+1}}{\alpha_{k_1}-\delta^{(t-1)k_n+1}}\le 1$ or not. If it is then we are done. In the case $\sum_{t=1}^{\infty}\frac{ak_n\delta^{(t-1)k_n+1}}{\alpha_{k_1}-\delta^{(t-1)k_n+1}}>1$ we will show that it is possible to find τ_n such that in equilibrium sequence $\Theta_{k_{n+1}}=\Theta_{\tau_n}\left(\Theta_{k_n}\right)$ we get $q_{\tau_n k_n+1}<q_S$.

We take $\tau_n=1$ and if $\frac{dq_{k_n+1}}{dq_{k_n}}(q_S)>0$ then $\Theta_{k_{n+1}}=\Theta_1\left(\Theta_{k_n}\right)$ and $q_{k_n+1}<q_S$ in some neighborhood of q_S. If $\frac{dq_{tk_n+1}}{dq_{k_n}}(q_S)\le 0$ for some t, then it follows that

$$\frac{dq_{tk_n+1}}{dq_{k_n}}(q_S)=\frac{\alpha_{k_n}-\delta^{(t-1)k_n+1}}{a(tk_n+1)\delta^{(t-1)k_n+1}}\left(\sum_{t=1}^{t}\frac{ak_n\delta^{(t-1)k_n+1}}{\alpha_{k_n}-\delta^{(t-1)k_n+1}}-1\right)<0,\quad\text{or}\quad\sum_{t=1}^{t}\frac{ak_n\delta^{(t-1)k_n+1}}{\alpha_{k_n}-\delta^{(t-1)k_n+1}}<1.$$

On the other hand we have made an assumption that $\sum_{t=1}^{\infty}\frac{ak_n\delta^{(t-1)k_n+1}}{\alpha_{k_n}-\delta^{(t-1)k_n+1}}>1$. Therefore there exists τ_n, such that $\sum_{j=1}^{\tau_n-1}\frac{ak_n\delta^{(j-1)k_n+1}}{\alpha_{k_n}-\delta^{(j-1)k_n+1}}<1$ while $\sum_{j=1}^{\tau_n}\frac{ak_n\delta^{(j-1)k_n+1}}{\alpha_{k_n}-\delta^{(j-1)k_n+1}}\ge 1$, and we have for $\Theta_{k_{n+1}}=\Theta_{\tau_n}\left(\Theta_{k_n}\right)$:

$$\frac{dq_{\tau_n k_n+1}}{dq_{k_n}}(q_S)=\frac{\alpha_{k_n}-\delta^{(\tau_n-1)k_n+1}}{a(\tau_n k_n+1)\delta^{(\tau_n-1)k_n+1}}\left(\sum_{j=1}^{\tau_n-1}\frac{ak_n\delta^{(j-1)k_n+1}}{\alpha_{k_n}-\delta^{(j-1)k_n+1}}-1\right)+\frac{ak_n}{a(\tau_n k_n+1)}=$$
$$=\frac{1}{a(\tau_n k_n+1)}\left(\frac{\alpha_{k_n}-\delta^{(\tau_n-1)k_n+1}}{\delta^{(\tau_n-1)k_n+1}}\left(ak_n\sum_{j=1}^{\tau_n-1}\frac{ak_n\delta^{(j-1)k_n+1}}{\alpha_{k_n}-\delta^{(j-1)k_n+1}}-1\right)+ak_n\right)<\frac{ak_n}{a(\tau_n k_n+1)}<1.$$

In other words, $q_{k_n}<q_{k_{n+1}}=q_{\tau_n k_n+1}<q_S$ and $s_{k_{n+1}}=s_{\tau_n k_n+1}>0$, and we can continue with $\Theta_{k_{n+1}}=\Theta_{\tau_n}\left(\Theta_{k_n}\right)$.

Now we will show that at some stage m we will get $\sum_{t=1}^{\infty}\frac{ak_m\delta^{(t-1)k_m+1}}{\alpha_{k_m}-\delta^{(t-1)k_m+1}}\le 1$.

Suppose not, that means that for all m $\sum_{t=1}^{\infty}\frac{ak_m\delta^{(t-1)k_m+1}}{\alpha_{k_m}-\delta^{(t-1)k_m+1}}>1$. Then we can evaluate $\alpha_{k_{n+1}}=\alpha_{\tau_n k_n+1}$ taking into account that $\alpha_{k_n}>\delta$:

$$\alpha_{k_{n+1}} = -\frac{\dfrac{ds_{t_n k_n+1}}{dq_{k_m}}(q_S)}{\dfrac{ds_{t_n k_n+1}}{dq_{k_n}}(q_S)} = \frac{\dfrac{\alpha_{k_n}-\delta^{(t_n-1)k_n+1}}{\delta^{(t_n-1)k_n+1}}\left(\dfrac{1}{a(\tau_n k_n+1)}\left(\sum_{j=1}^{t_n}\dfrac{ak_n\delta^{(j-1)k_n+1}}{\alpha_{k_n}-\delta^{(j-1)k_n+1}}-1\right)+1\right)}{\dfrac{\alpha_{k_n}-\delta^{(t_n-1)k_n+1}}{a(\tau_n k_n+1)\delta^{(t_n-1)k_n+1}}\left(\sum_{j=1}^{t_n}\dfrac{ak_n\delta^{(j-1)k_n+1}}{\alpha_{k_n}-\delta^{(j-1)k_n+1}}-1\right)} =$$

$$= 1+\frac{a(\tau_n k_n+1)}{\sum_{j=1}^{t_n-1}\dfrac{ak_n\delta^{(j-1)k_n+1}}{\alpha_{k_n}-\delta^{(j-1)k_n+1}}-1+\dfrac{ak_n\delta^{(t_n-1)k_n+1}}{\alpha_{k_n}-\delta^{(t_n-1)k_n+1}}} > 1+\frac{a\tau_n k_n}{\dfrac{ak_n\delta^{(t_n-1)k_n+1}}{\alpha_{k_n}-\delta^{(t_n-1)k_n+1}}} =$$

$$= 1+\frac{\tau_n\left(\alpha_{k_n}-\delta^{(t_n-1)k_n+1}\right)}{\delta^{(t_n-1)k_n+1}} > 1+\frac{\alpha_{k_n}-\delta}{\delta^{(t_n-1)k_n+1}} > 1.$$

So, for all large n $\alpha_{k_n} > 1 > \delta$, and, therefore

$$\alpha_{k_{n+1}} = 1+\frac{\tau_n\left(\alpha_{k_n}-\delta^{(t_n-1)k_n+1}\right)}{\delta^{(t_n-1)k_n+1}} > \frac{\tau_n\left(1-\delta^{(t_n-1)k_n+1}\right)+\delta^{(t_n-1)k_n+1}}{\delta^{(t_n-1)k_n+1}} >$$

$$> \frac{\tau_n-(\tau_n-1)\delta^{(t_n-1)k_n+1}}{\delta^{(t_n-1)k_n+1}} = \frac{(\tau_n-1)-(\tau_n-1)\delta^{(t_n-1)k_n+1}+1}{\delta^{(t_n-1)k_n+1}} > \frac{1}{\delta^{(t_n-1)k_n+1}}. \qquad (26)$$

Then, it follows that $\lim_{n\to\infty}\tau_n \neq 1$ because otherwise we would have $\tau_n = 1$ for all large n and, consequently,

$$\lim_{n\to\infty}\frac{\alpha_{k_{n+1}}}{\alpha_{k_n}} = \lim_{k_n\to\infty}\frac{1+\left(\dfrac{\delta}{\alpha_{k_n}}-1\right)\dfrac{1-a}{ak_n}}{\delta+\dfrac{\delta-\alpha_{k_n}}{ak_n}} > \lim_{k_n\to\infty}\frac{1-\dfrac{1-a}{ak_n}}{\delta} = \frac{1}{\delta},$$

and finally,

$$\overline{\lim_{l\to\infty}}\sum_{t=1}^{\infty}\frac{ak_{n+l}\delta^{(t-1)k_{n+l}+1}}{\alpha_{k_{n+l}}-\delta^{(t-1)k_{n+l}+1}} \le \overline{\lim_{l\to\infty}}\sum_{t=1}^{\infty}\frac{ak_{n+l}\delta^{(t-1)k_{n+l}+1}}{\alpha_{k_{n+l}}-\delta} = \overline{\lim_{l\to\infty}}\frac{ak_{n+l}\delta}{\alpha_{k_{n+l}}-\delta}\sum_{t=1}^{\infty}\delta^{(t-1)k_{n+l}} =$$

$$= \overline{\lim_{l\to\infty}}\frac{ak_{n+l}\delta}{\left(\alpha_{k_{n+l}}-\delta\right)\left(1-\delta^{k_{n+l}}\right)} = \overline{\lim_{l\to\infty}}\frac{ak_{n+l}\delta}{\left(\alpha_{k_{n+l}}-\delta\right)} = \overline{\lim_{l\to\infty}}\frac{\delta a\dfrac{k_{n+l}}{\alpha_{k_{n+l}}}}{1-\dfrac{\delta}{\alpha_{k_{n+l}}}} = \delta a\overline{\lim_{l\to\infty}}\frac{k_{n+l}}{\alpha_{k_{n+l}}} = 0,$$

that contradicts with our assumption that for all m $\sum_{t=1}^{\infty}\dfrac{ak_m\delta^{(t-1)k_m+1}}{\alpha_{k_m}-\delta^{(t-1)k_m+1}} > 1$.

So, $\lim_{n\to\infty}\tau_n \neq 1$ and there exists a subsequence n_l such that $n_l \overset{l\to\infty}{\to} \infty$ and $\tau_{n_l} \geq 2$.

Taking an upper limit along that subsequence yields:

$$\overline{\lim_{l\to\infty}}\left(\sum_{t=1}^{\infty}\frac{ak_{n_l+1}d^{(t-1)k_{n_l+1}+1}}{a_{k_{n_l+1}}-d^{(t-1)k_{n_l+1}+1}}\right)\le\overline{\lim_{n_l\to\infty}}\left(ak_{n_l+1}d\sum_{t=1}^{\infty}\frac{d^{(t-1)k_{n_l+1}}}{a_{k_{n_l+1}}-d}\right)=\overline{\lim_{n_l\to\infty}}\left(\frac{ak_{n_l+1}d}{a_{k_{n_l+1}}-d}\sum_{t=0}^{\infty}d^{tk_{n_l+1}}\right)=$$

$$=\overline{\lim_{n_l\to\infty}}\left(\frac{ak_{n_l+1}d}{a_{k_{n_l+1}}-d}\cdot\frac{1}{1-d^{k_{n_l+1}}}\right)=\overline{\lim_{n_l\to\infty}}\frac{ak_{n_l+1}d}{a_{k_{n_l+1}}-d}=\overline{\lim_{n_l\to\infty}}\frac{a\left(t_{n_l}k_{n_l}+1\right)d}{a_{k_{n_l+1}}-d}.$$

Using the inequality (26) we get:

$$\overline{\lim_{l\to\infty}}\left(ak_{n_l+1}\sum_{t=1}^{\infty}\frac{\delta^{(t-1)k_{n_l+1}+1}}{\alpha_{k_{n_l+1}}-\delta^{(t-1)k_{n_l+1}+1}}\right)=\overline{\lim_{n_l\to\infty}}\frac{a\left(\tau_{n_l}k_{n_l}+1\right)\delta}{\alpha_{k_{n_l+1}}-\delta}\le\overline{\lim_{n_l\to\infty}}\frac{a\left(\tau_{n_l}k_{n_l}+1\right)\delta}{\frac{1}{\delta^{(t_{n_l}-1)k_{n_l}+1}}-\delta}=$$

$$=\overline{\lim_{n_l\to\infty}}\frac{a\left(\tau_{n_l}k_{n_l}+1\right)\delta^{(t_{n_l}-1)k_{n_l}+2}}{1-\delta^{(t_{n_l}-1)k_{n_l}+2}}=a\delta^2\overline{\lim_{\substack{n_l\to\infty\\ t_{n_l}\ge 2}}}\left(\tau_{n_l}k_{n_l}+1\right)\delta^{(t_{n_l}-1)k_{n_l}}=$$

$$=a\delta^2\overline{\lim_{\substack{n_l\to\infty\\ t_{n_l}\ge 2}}}\left(\left(\tau_{n_l}-1\right)k_{n_l}+1\right)\delta^{(t_{n_l}-1)k_{n_l}}+a\delta^2\overline{\lim_{\substack{n_l\to\infty\\ t_{n_l}\ge 2}}}k_{n_l}\delta^{(t_{n_l}-1)k_{n_l}}\le$$

$$\le a\delta^2\overline{\lim_{(t_{n_l}-1)k_{n_l}\to\infty}}\left(\left(\tau_{n_l}-1\right)k_{n_l}+1\right)\delta^{(t_{n_l}-1)k_{n_l}}+a\delta^2\overline{\lim_{k_{n_l}\to\infty}}k_{n_l}\delta^{k_{n_l}}=a\delta^2\lim_{x\to\infty}(x+1)\delta^x=0.$$

Thus, we have proved the following statement: $\exists m$ and, therefore $\exists k_m\ge 1$, such that $ak_m\sum_{t=1}^{\infty}\frac{\delta^{(t-1)k_m+1}}{\alpha_{k_m}-\delta^{(t-1)k_m+1}}\le 1$. But this implies that for any t:

$$\frac{d\theta_{tk_m+1}}{d\theta_{k_m}}(\theta_S)=\frac{\alpha_{k_m}-\delta^{(t-1)k_m+1}}{a(tk_m+1)\delta^{(t-1)k_m+1}}\left(ak_m\sum_{t=1}^{t}\frac{\delta^{(t-1)k_m+1}}{\alpha_{k_m}-\delta^{(t-1)k_m+1}}-1\right)<0\text{, and}$$

$$\frac{ds_{tk_m+1}}{d\theta_{k_m}}(\theta_S)=<-\frac{\alpha_{k_m}-\delta^{(t-1)k_m+1}}{\delta^{(t-1)k_m+1}}\left(1-\frac{1}{a(tk_m+1)}\right)<0\text{ as long as }t>\frac{1-a}{ak_m}.$$

So there exists $\overline{\overline{T_0}}=\left[\frac{1-a}{ak_m}\right]+1$ such that for all $t>\overline{\overline{T_0}}$ $\exists\overline{\overline{U^0_{tk_m+1}}}=\left(\overline{\overline{\theta_1^0(tk_m+1)}},\theta_S\right)\subset\overline{U^0_{k_m}}$ and $\exists\Theta_t\left(\overline{\overline{U^0_{tk_m+1}}}\right)$ such that for all $\theta_1\in\overline{\overline{\overline{U^0_{tk_m+1}}}}$ $\theta_{tk_m+1}(\theta_1)>\theta_S$, $s_{tk_m+1}(\theta_1)>0$. $\theta_t(\theta_S)=\theta_S$ and $s_t(\theta_S)=0$ for all $t=1,\ldots,tk_m+1$ by construction.

Finally we will prove that if t is taken sufficiently large than $\exists \varepsilon_g > 0$ such that $\theta_{tk_m+1}(\theta_1) - \theta_S < \varepsilon_g \frac{1}{\delta} s_{tk_m+1}(\theta_1)$. Let us consider the ratio $\frac{s_{tk_m+1}(\theta_1)}{\theta_{tk_m+1}(\theta_1) - \theta_S}$. It can be written as follows:

$$\frac{s_{tk_m+1}(\theta_1)}{\theta_{tk_m+1}(\theta_1) - \theta_S} = \frac{\frac{ds_{tk_m+1}}{d\theta_1}(\theta_S)(\theta_1 - \theta_S) + o(\theta_1 - \theta_S)}{\frac{d\theta_{tk_m+1}}{d\theta_1}(\theta_S)(\theta_1 - \theta_S) + o(\theta_1 - \theta_S)} = \frac{a(tk_m + 1) + o\left((\theta_1 - \theta_S)^0\right)}{1 - ak_m \sum_{t=1}^{t} \frac{\delta^{(t-1)k_m+1}}{\alpha_{k_m} - \delta^{(t-1)k_m+1}}} - 1.$$

Therefore

$$\lim_{t\to\infty}\left[\lim_{\theta_1 \to \theta_S - 0} \frac{s_{tk_m+1}(\theta_1)}{\theta_{tk_m+1}(\theta_1) - \theta_S}\right] = \lim_{t\to\infty} \frac{a(tk_m + 1)}{1 - ak_m \sum_{t=1}^{t} \frac{\delta^{(t-1)k_m+1}}{\alpha_{k_m} - \delta^{(t-1)k_m+1}}} - 1 = +\infty.$$

This implies that we actually can take any $\varepsilon_g > 0$ such that $\exists T_0 > \overline{\overline{T_0}}$ such that for all $t > T_0$ $\exists U^0_{tk_m+1} = \left(\theta_1^0(tk_m + 1), \theta_S\right)$ and $\exists \Theta_{tk_m+1}\left(U^0_{tk_m+1}\right)$ such that for all $\theta_1 \in U^0_{tk_m+1}$ $0 < \theta_{tk_m+1} - \theta_S < \varepsilon_g \frac{1}{\delta} s_{tk_m+1}$.