Leitner, Johannes

Working Paper

Utility Maximization and Duality

CoFE Discussion Paper, No. 00/34

Provided in Cooperation with:
University of Konstanz, Center of Finance and Econometrics (CoFE)

Suggested Citation: Leitner, Johannes (2000) : Utility Maximization and Duality, CoFE Discussion Paper, No. 00/34, University of Konstanz, Center of Finance and Econometrics (CoFE), Konstanz, https://nbn-resolving.de/urn:nbn:de:bsz:352-opus-6176

This Version is available at:
https://hdl.handle.net/10419/85171

UTILITY MAXIMIZATION AND DUALITY

JOHANNES LEITNER

JOHANNES.LEITNER@UNI-KONSTANZ.DE

CENTER OF FINANCE AND ECONOMETRICS (COFE)

UNIVERSITY OF KONSTANZ

ABSTRACT. In an arbitrage free incomplete market we consider the problem of maximizing terminal isoelastic utility. The relationship between the optimal portfolio, the optimal martingale measure in the dual problem and the optimal value function of the problem is described by an BSDE. For a *totally unhedgeable price for instantaneous risk*, isoelastic utility of terminal wealth can be maximized using a portfolio consisting of the locally risk-free bond and a *locally efficient* fund only. In a markovian market model we find a non-linear PDE for the logarithm of the value function. From the solution we can construct the optimal portfolio and the solution of the dual problem.

Keywords: Utility, Optimal Portfolios, Duality Theory.

AMS 91 Classifications: 90A09, 90A10

JEL Classifications: G11

I would like to thank Professor M. Kohlmann for his suggestions and support. I am also thankful to Professor M. Schweizer and Professor S. Tang.

Introduction

We study the problem of maximizing expected isoelastic utility of terminal wealth in an incomplete continuous time market with continuous price process. The isoelastic utility of exponent $p \neq 0, 1$ is defined as $u^{(p)}(x) := \operatorname{sgn}(1-p)\frac{|x|^p}{p}$ and for $p = 0$ by $u^{(0)}(x) := \ln(|x|)$. The two cases $p < 1$ and $p > 1$ are very different in there economic interpretation, but can be treated to some extend by the same mathematical methods. Solving the optimization problem for $p < 1$ is a plausible approach to find portfolios of optimal expected growth. There are several papers on this topic: See, e.g. Merton (1990), Pliska (1986), He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), Karatzas and Shreve (1999), Kramkov and Schachermayer (1999).

For $p = 2$ the problem is related to the mean-variance hedging problem, see Gourieroux, Laurent and Pham (1998), (GLP98), Pham, Rheinländer and Schweizer (1998) and Laurent and Pham (1999).

The theory of stochastic duality, which goes back to Bismut (1973, 1975), is the central tool for solving these problems. This theory allows to formulate an optimization problem over a set of martingale measures, being dual to the original optimization problem over a set of self-financing hedging-strategies. Under quite general conditions, the solution of one of the problems can be transformed into a solution of the corresponding dual problem.

Another important approach, is to try to solve the optimization problem locally, i.e. by so-called myopic strategies which maximize in some sense the expected growth rate of the portfolio at every instant of time. In some important cases these strategies turn out to be globally optimal too. See, e.g., Mossin (1968), Leland (1972), Aase (1984, 1986, 1987, 1988), Foldes (1991), Goll and Kallsen (2000). This approach is related to the risk-sensitive stochastic control approach, see Bielecki and Pliska (1999, 2000).

We consider an arbitrage-free (in a sense to be specified later) continuous time market model with unrestricted trading. We use the modern equivalent martingale measure approach, see Harrison and Pliska (1981), Delbaen and Schachermayer (1994). After some technical preparations in Section 1 and specification of the model in Section 2, we formulate the optimization problem and its corresponding dual Problem in Section 3. We show a representation property (formula (3.8)), relating the terminal value V_T^{opt} of a portfolio to a martingale

Research supported by the Center of Finance and Econometrics, Project Mathematical Finance.

measure Z_T^{opt}, to be sufficient for the optimality of V_T^{opt} for the utility maximization problem and the optimality of Z_T^{opt} for the dual problem. The optimal values of the two problem are related by a simple formula. In Section 4 we introduce the notion of a *totally unhedgeable price for instantaneous risk*. In this situation we can explicitly solve the utility optimization problem. The optimal portfolio is a *locally efficient portfolio*, a notion we introduce in Section 5. In Section 6 we give an existence result for the solutions of the two optimization problems. In Section 7 we derive a backward stochastic differential equation, (BSDE), such that from the solution the optimal portfolio, the optimal value function and the solution of the dual optimization problem can be constructed. See Yong and Zhou (1999) for an introduction to BSDEs. In Section 8 we consider a markovian market model. We transform the BSDE into a non-linear PDE for the logarithm of the value function. From the partial derivatives of the solution, we can construct under additional assumptions the optimal portfolio and the solution of the dual optimization problem.

1. Self-financing Hedging Strategies

Let a filtered probability space $\Omega_\infty := (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P)$, satisfying the usual conditions be given. For simplicity we assume $\mathcal{F}_0$ to be trivial up to sets of measure 0 with respect to P and $\mathcal{F}_{\infty-} = \mathcal{F}_\infty := \mathcal{F}$. For an adapted process X set $X_{0-} := X_0$, $X_{t-} := \lim_{h\searrow 0} X_{t-h}$ for $t > 0$ if the limit exists and define the processes $X_- := (X_{t-})_{0\leq t<\infty}$ and $\Delta X := X - X_-$ if X_{t-} exists for all $t > 0$. The components of X are denoted as X^i, $1 \leq i \leq d$. For a process X and a map $\tau : \Omega \to \bar{\mathbb{R}}_+$, denote the stopped process at time τ by X^τ. We will often restrict a semimartingale X on Ω_∞ to an interval $[t, T]$, $0 \leq t \leq T < \infty$, resp. to $[t, \infty)$. Therefore we introduce the following filtered probability space (again satisfying the usual conditions), $\Omega_{[t,T]} := \left(\Omega, \mathcal{F}_T, \left(\mathcal{F}_s^{[t,T]}\right)_{s\geq 0}, P_{|\mathcal{F}_T}\right)$ for all $0 \leq t \leq T \leq \infty$, $t < \infty$, where $\mathcal{F}_s^{[t,T]} := \mathcal{F}_{t\vee s\wedge T}$ for $0 \leq s < \infty$. The process $X_s^{[t,T]} := X_{t\vee s\wedge T}$ is then a semimartingale on $\Omega_{[t,T]}$. However, on $[t, T]$ we often write X instead of $X^{[t,T]}$. Set $\Omega_T := \Omega_{[0,T]}$.

For $q > 1$ define $L^q(\Omega_{[t,T]})$, respectively $L_t^q(\Omega_{[t,T]})$, as the set of $\mathcal{F}_T$-measurable random variables X, such that $E[X] < \infty$ a.s., respectively $E_t[X] < \infty$ a.s., where $E_t[\cdot] := E[\cdot|\mathcal{F}_t]$ denotes the generalized conditional expectation. Denote the conditional variance by $\mathrm{Var}_t(\cdot)$. The stochastic exponential of a semimartingale X is denoted as $\mathcal{E}(X)$ and we set $\mathcal{E}_t(X) := \mathcal{E}(\mathbf{1}_{[t,\infty)}X)$. As a general references we cite Jacod and

Shiryaev (1987), (J&S 87), and Jacod (1979). Denote the set of predictable processes which are locally integrable, resp. locally Riemann-Stieltjes integrable, with respect to a local martingale M, resp. with respect to a process A of finite variation, by $L^1_{loc}(M)$, resp. by $L^1_{loc}(A)$. If the semimartingale X admits a decomposition $X = X_0 + A + M$, where M is a local martingale and A is a process of finite variation then $L^1_{loc}(X) := L^1_{loc}(M) \cap L^1_{loc}(A)$.

We can now define the market model: Let $S = (S_t)_{0 \leq t < \infty}$ be a $\mathbb{R}^d$-valued semimartingale. $\mathcal{M} := (\Omega_\infty, S) = ((\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, P), S)$ is a model for a market, where S describes the price processes of d assets. We will often consider such a market on an interval $[t, T]$, $0 \leq t < T < \infty$. This is equivalent to work with the following market model $\mathcal{M}_{[t,T]}$ defined by $\mathcal{M}_{[t,T]} := \left(\Omega_{[t,T]}, S^{[t,T]}\right)$. Set $\mathcal{M}_T := \mathcal{M}_{[0,T]}$. We want to model the economic activity of investing money into a portfolio of assets and changing the number of assets held over time according to a certain hedging strategy. This is achieved with the following definition:

Definition 1.1. A *hedging strategy* in the market $\mathcal{M}$ is a $H \in L^1_{loc}(S)$.

The corresponding *value process* V^H of H is defined as $V^H := HS$.

The *gains process* of H is defined as the semimartingale $G^H := H \cdot S$.

H is called *self-financing* if $V^H = V^H_0 + G^H$, i.e. $H_t S_t = H_0 S_0 + \int_0^t H_s dS_s$, $\forall\, t \geq 0$. Denote the space of all self-financing hedging strategies in $\mathcal{M}$ by $\mathcal{SF}(\mathcal{M})$.

Note that for $H \in \mathcal{SF}(\mathcal{M})$, we have $H^{[t,T]} \in \mathcal{SF}(\mathcal{M}_{[t,T]})$. The idea of a self-financing hedging strategy is that the changes over time of the corresponding value process are solely caused by the changes of the value of the assets held in the portfolio and not by withdrawing money from or adding money to the portfolio.

Definition 1.2. A semimartingale B such that B and B_- are strictly positive is called a *numéraire* for the market $\mathcal{M}$. The market discounted with respect to B is then defined as $\mathcal{M}^B := \left(\Omega_\infty, S^B\right)$, where $S^B := \frac{S}{B}$.

For $0 \leq t \leq T < \infty$, the market restricted to the interval $[t,T]$ is defined as $\mathcal{M}^B_{[t,T]} := \left(\mathcal{M}^B\right)_{[t,T]} = \left(\Omega_{[t,T]}, \left(S^B\right)^{[t,T]}\right)$.

Note that for a numéraire B, B^{-1} is a numéraire too and S^B is a semimartingale.

Usually there is in addition to the market $\mathcal{M}$ a numéraire B given and the market $\bar{\mathcal{M}} := (\Omega_\infty, \bar{S}), \bar{S} := (S,B)$ is considered. Often B is the price process of a locally risk-free bond. If the numéraire is traded, i.e. the value process of a portfolio in $\mathcal{M}$, one can try to extend a hedging strategy in $\mathcal{M}$ to a self-financing hedging strategy in $\bar{\mathcal{M}}$. Define the discounted market $\bar{\mathcal{M}}^B = (\Omega_\infty, (S^B, 1))$. The idea is to extend H to a self-financing hedging strategy $\bar{H} = (H, \hat{H}) \in \mathcal{SF}(\bar{\mathcal{M}}^B)$ by defining the process $\hat{H} := H_0 S^B_0 + H \cdot S^B - HS^B$ and then to show that $\bar{H}$ is a self-financing hedging strategy in $\bar{\mathcal{M}}$ too, see Geman, El Karui and Rochet (1995) and Goll and Kallsen (2000). (Note that $H \cdot S^B - HS^B = (H \cdot S^B)_- + \Delta(H \cdot S^B) - HS^B = (H \cdot S^B)_- + H\Delta S^B - HS^B = (H \cdot S^B)_- - HS^B_-$ is predictable, hence $\bar{H}$ as well.)

Proposition 1.3. *Let B be a numéraire for the market $\mathcal{M}$. Then $\mathcal{SF}(\mathcal{M}^B) = \mathcal{SF}(\mathcal{M})$ holds.*

Proof. Let $H \in \mathcal{SF}(\mathcal{M}^B)$. Set $V^B = HS^B$. First, we have to show $H \in L^1_{loc}(S)$. Since $S = S^B B = S_0 + S^B_- \cdot B + B_- \cdot S^B + [S^B, B]$ this follows if we show that $H \in L^1_{loc}(S^B_- \cdot B) \cap L^1_{loc}(B_- \cdot S^B) \cap L^1_{loc}([S^B, B])$. Note that $HS^B_- = H(S^B - \Delta S^B) = V^B - \Delta(H \cdot S^B) = V^B_0 + H \cdot S^B - \Delta(H \cdot S^B) = V^B_0 + (H \cdot S^B)_- = V^B_-$, which is locally bounded. Since $[S^B_- \cdot B, S^B_- \cdot B] = (S^B_- \otimes S^B_-) \cdot [B,B]$ and $H(S^B_- \otimes S^B_-)H = (V^B_-)^2$ is locally integrable with respect to $[B,B]$, we find $H \in L^1_{loc}(S^B_- \cdot B)$.

That $H \in L^1_{loc}(B_- \cdot S^B) \cap L^1_{loc}([S^B, B])$ is easy to see. We calculate

$$\begin{aligned} H \cdot S &= H \cdot (S^B B) = H \cdot (S^B_- \cdot B + B_- \cdot S^B + [S^B, B]) \\ &= (H S^B_-) \cdot B + (B_- H) \cdot S^B + [H \cdot S^B, B] \\ &= V^B_- \cdot B + B_- \cdot (H \cdot S^B) + [V^B, B] \\ &= V^B B - V^B_0 B_0 = H S^B B - H_0 S^B_0 B_0 \\ &= HS - H_0 S_0. \end{aligned}$$

This implies $\mathcal{SF}(\mathcal{M}^B) \subseteq \mathcal{SF}(\mathcal{M})$. Now observe that $(\mathcal{M}^B)^{B^{-1}} = \mathcal{M}$, since B^{-1} is a numéraire. This implies the reverse inclusion. □

There is an alternative way to construct self-financing hedging strategies:

Lemma 1.4. *Let $H \in \mathcal{SF}(\mathcal{M})$ be such that $V^H \neq 0$ and $V^H_- \neq 0$ almost surely. Set $\tilde{H} := \frac{H}{V^H_-}$. Then $\tilde{H} \in L^1_{loc}(S)$, $\tilde{H} S_- = 1$ and*

$$V^H = V^H_0 + V^H_- \cdot (\tilde{H} \cdot S) = V^H_0 \mathcal{E}(\tilde{H} \cdot S), \tag{1.1}$$

holds. Conversely, let $\tilde{H} \in L^1_{loc}(S)$ with $\tilde{H} S_- = 1$ be given and set $H := v_0 \mathcal{E}(\tilde{H} \cdot S)_- \tilde{H}$ for a $\mathcal{F}_0$-measurable random variable v_0. Then $H \in \mathcal{SF}(\mathcal{M})$ and $V^H = v_0 \mathcal{E}(\tilde{H} \cdot S)$. We call $\tilde{H}$ a generator for the self-financing strategy H and define $V^{(\tilde{H})} := V^H$.

Proof. Since $\left(V^H_-\right)^{-1}$ is locally bounded we have $\tilde{H} \in L^1_{loc}(S)$. We have $V^H - V^H_0 = G^H = H \cdot S = (V^H_- \tilde{H}) \cdot S = V^H_- \cdot (\tilde{H} \cdot S)$. The second identity in (1.1) follows immediately from the uniqueness of the solution to the

Doléan-Dade SDE defining the stochastic exponential, see J&S 87, I.4f.

Conversely, we calculate

$$\begin{aligned} HS &= v_0\mathcal{E}(\tilde{H}\cdot S)_-\tilde{H}S = v_0\mathcal{E}(\tilde{H}\cdot S)_-(\tilde{H}S_- + \tilde{H}\Delta S) \\ &= v_0\mathcal{E}(\tilde{H}\cdot S)_-(1+\Delta(\tilde{H}\cdot S)) \\ &= v_0\left(\mathcal{E}(\tilde{H}\cdot S)_- + \mathcal{E}(\tilde{H}\cdot S)_-\Delta(\tilde{H}\cdot S)\right) \\ &= v_0\left(\mathcal{E}(\tilde{H}\cdot S)_- + \Delta\big(\mathcal{E}(\tilde{H}\cdot S)_-\cdot(\tilde{H}\cdot S)\big)\right) \\ &= v_0\left(\mathcal{E}(\tilde{H}\cdot S)_- + \Delta\big(\mathcal{E}(\tilde{H}\cdot S)-1\big)\right) \\ &= v_0\mathcal{E}(\tilde{H}\cdot S) = v_0 + v_0\mathcal{E}(\tilde{H}\cdot S)_-\cdot(\tilde{H}\cdot S) \\ &= v_0 + v_0\mathcal{E}(\tilde{H}\cdot S)_-\tilde{H}\cdot S = V_0^H + G^H. \end{aligned}$$

□

2. Arbitrage-free Markets

So far we did not worry about arbitrage. We consider in this section the market $\bar{\mathcal{M}} := (\Omega_\infty, \bar{S})$, where $\bar{S} := (S, B)$ is $\mathbb{R}^d\times\mathbb{R}$-valued and B is a numéraire, with $B_0 = 1$, which we assume to be uniformly bounded and uniformly bounded away from 0 on finite intervals. For $0 \le t \le T \le \infty, t < \infty$, denote the set of uniformly integrable, resp. local martingales, living on $\Omega_{[t,T]}$ by $\mathcal{L}^u(\Omega_{[t,T]})$, resp. by $\mathcal{L}(\Omega_{[t,T]})$. Define the following sets of local martingale measures:

(2.1)
$$\bar{\mathcal{D}}(\bar{\mathcal{M}}_{[t,T]}) := \left\{Z\in\mathcal{L}(\Omega_{[t,T]})|Z\mathbf{1}_{[0,t]} = 1, Z\ge 0, (S^B)^{[t,T]}Z\in\mathcal{L}(\Omega_{[t,T]})\right\},$$

(2.2)
$$\mathcal{D}(\bar{\mathcal{M}}_{[t,T]}) := \left\{Z\in\mathcal{L}(\Omega_{[t,T]})|Z\mathbf{1}_{[0,t]} = 1, Z> 0, (S^B)^{[t,T]}Z\in\mathcal{L}(\Omega_{[t,T]})\right\},$$

(2.3)
$$\mathcal{D}^{abs}(\bar{\mathcal{M}}_{[t,T]}) := \left\{Z\in\bar{\mathcal{D}}(\bar{\mathcal{M}}_{[t,T]})|Z \text{ uniformly integrable martingale}\right\},$$

and

(2.4)
$$\mathcal{D}^e(\bar{\mathcal{M}}_{[t,T]}) := \left\{ Z \in \mathcal{D}(\bar{\mathcal{M}}_{[t,T]}) | Z \text{ uniformly integrable martingale} \right\}.$$

We will work with the following No-Arbitrage condition:

$$\mathcal{D}^e(\bar{\mathcal{M}}_T) \neq \emptyset, \ \forall\, T < \infty. \tag{2.5}$$

This condition is known to be equivalent to the NFLVR-condition, see Delbaen and Schachermayer (1994). It implies that

$$\mathcal{D}^e(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset, \ \forall\, 0 \leq t \leq T < \infty. \tag{2.6}$$

We will often work with the following sets of equivalent, resp. absolutely continuous, local martingale measures, for $q > 1$:

$$\mathcal{D}^q(\bar{\mathcal{M}}_{[0,T]}) := \left\{ Z \in \mathcal{D}(\bar{\mathcal{M}}_{[0,T]}) | Z_T \in L^q(\Omega_{[0,T]}) \right\}, \tag{2.7}$$

$$\mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]}) := \left\{ \frac{Z_{t\vee\cdot}}{Z_t} | Z \in \mathcal{D}^q(\bar{\mathcal{M}}_{[0,T]}) \right\}, \tag{2.8}$$

$$\bar{\mathcal{D}}^q(\bar{\mathcal{M}}_{[0,T]}) := \left\{ Z \in \mathcal{D}^{abs}(\bar{\mathcal{M}}_{[0,T]}) | Z_T \in L^q(\Omega_{[0,T]}) \right\}, \tag{2.9}$$

$$\bar{\mathcal{D}}^q_t(\bar{\mathcal{M}}_{[t,T]}) := \left\{ \frac{Z_{t\vee\cdot}}{Z_t} | Z \in \bar{\mathcal{D}}^q(\bar{\mathcal{M}}_{[0,T]}), Z_t > 0 \right\}. \tag{2.10}$$

Note that $Z \in \mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]})$ implies $Z_T \in L^p_t(\Omega_{[t,T]})$. For $q < 1$ set $\mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]}) := \mathcal{D}(\bar{\mathcal{M}}_{[t,T]})$ and $\bar{\mathcal{D}}^q(\bar{\mathcal{M}}_{[t,T]}) := \bar{\mathcal{D}}^q_t(\bar{\mathcal{M}}_{[t,T]}) := \bar{\mathcal{D}}(\bar{\mathcal{M}}_{[t,T]})$. For $Z \in \mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]})$ and $t \leq t' \leq T' \leq T$, we have $\frac{Z^{[t',T']}}{Z_{t'}} \in \mathcal{D}^q_{t'}(\bar{\mathcal{M}}_{[t',T']})$. Note also that $\mathcal{D}^q_0(\bar{\mathcal{M}}_{[0,T]}) = \mathcal{D}(\bar{\mathcal{M}}_{[0,T]})$, since $\mathcal{F}_0$ was assumed to be trivial.

p will always denote a real number different from 1. We define $q := \frac{p}{p-1}$, such that for $p \neq 0, 1$, $p^{-1} + q^{-1} = 1$ holds, but for $p = 0$ we have $q = 0$.

Let $\mathcal{B} \subseteq \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]})$. We call a $H \in \mathcal{B}$ an $\mathcal{B}$-*arbitrage*, if $V^H_0 = 0$, $V^H_T \geq 0$ and $V^H_T \neq 0$ almost surely. If there exists no $\mathcal{B}$-*arbitrage*, then $\mathcal{B}$ is called arbitrage-free. In all probabilistic theories of financial markets allowing to trade at an infinitely large number of instances of time one has to exclude certain self-financing hedging strategies, e.g. doubling strategies, in order to avoid arbitrage opportunities. We will define several arbitrage-free subsets of $\mathcal{SF}(\bar{\mathcal{M}}_{[t,T]})$:

1. For $p > 1$ and $\mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset$, (see Delbaen and Schachermayer (1996), (DS96)):

$$\mathcal{SF}^p(\bar{\mathcal{M}}_{[t,T]}) := \Big\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) | V^H_T \in L^p(\Omega_{[t,T]}), \frac{V^H}{B^{[t,T]}} Z \in \mathcal{L}^u(\Omega_{[t,T]}), \forall\, Z \in \mathcal{D}^q(\bar{\mathcal{M}}_{[t,T]}) \Big\}, \tag{2.11}$$

resp.

$$\mathcal{SF}^p_t(\bar{\mathcal{M}}_{[t,T]}) := \Big\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) | V^H_T \in L^p_t(\Omega_{[t,T]}), \frac{V^H}{B^{[t,T]}} Z \in \mathcal{L}^u(\Omega_{[t,T]}), \forall\, Z \in \mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]}) \Big\}. \tag{2.12}$$

Note that

$$\mathcal{SF}^p_t(\bar{\mathcal{M}}_{[t,T]}) = \Big\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) | V^H_T \in L^p_t(\Omega_{[t,T]}), \frac{V^H}{B^{[t,T]}} Z \in \mathcal{L}^u(\Omega_{[t,T]}), \forall\, Z \in \bar{\mathcal{D}}^q_t(\bar{\mathcal{M}}_{[t,T]}) \Big\}, \tag{2.13}$$

since for $Z \in \bar{\mathcal{D}}^q_t(\bar{\mathcal{M}}_{[t,T]})$ we can find a $\bar{Z} \in \bar{\mathcal{D}}^q(\bar{\mathcal{M}}_{[0,T]})$ with $Z = \frac{\bar{Z}_{t\vee\cdot}}{\bar{Z}_t}$ and for $Z' \in \mathcal{D}^q(\bar{\mathcal{M}}_{[0,T]})$, we have $\tilde{Z} := \frac{\bar{Z}+Z'}{2} \in \mathcal{D}^q(\bar{\mathcal{M}}_{[0,T]})$, which implies $\hat{Z} := \frac{\tilde{Z}_{t\vee\cdot}}{\tilde{Z}_t} \in \mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]})$ and for $H \in \mathcal{SF}^p_t(\bar{\mathcal{M}}_{[t,T]})$ that $\frac{V^H}{B^{[t,T]}} Z = \frac{V^H}{B^{[t,T]}}((\bar{Z}_t + Z'_t)\hat{Z} - \frac{Z'_{t\vee\cdot}}{Z'_t})$ is a uniformly integrable martingale.

2. For $p < 1$

$$\mathcal{SF}^p(\bar{\mathcal{M}}_{[t,T]}) := \mathcal{SF}^p_t(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) | V^H \geq 0 \right\}. \tag{2.14}$$

3. For $p > 1$ and $\bar{S} \in \mathcal{S}^p_{loc}(\Omega_{[t,T]})$

$$\mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) | V^H \in \mathcal{S}^p(\Omega_{[t,T]}) \right\}, \tag{2.15}$$

where $\mathcal{S}^p(\Omega_{[t,T]})$ denotes the space of L^p-integrable semimartingales, see Delbaen, Monat, Schachermayer, Schweizer and Stricker (1997) (DMSSS97) for the case $p = 2$ and Grandits and Krawczyk (1998), (GK98), for the general case $p > 1$.

Lemma 2.1. *For $p > 1$ assume $\mathcal{D}^q_t(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset$ and $\bar{S}$ to be continuous. Then $\mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]}) \subseteq \mathcal{SF}^p_t(\bar{\mathcal{M}}_{[t,T]})$. In particular $\mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]})$ is arbitrage-free.*

Proof. For $H \in \mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]})$ set $\tau_n := \inf\left\{s \geq 0 \middle| \left|\frac{V_s^H}{B_s}\right| \geq n\right\}$, $H^n := H$ on $[0, \tau_n)$ and $\left(0, \frac{V_{\tau_n}^H}{B_{\tau_n}}\right) \in \mathbb{R}^d \times \mathbb{R}$ on $[\tau_n, T]$. Then $H^n \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$, since $\left|\frac{V^{H^n}}{B^{[t,T]}}\right| \leq n$. It follows $E\left[\frac{V_T^{H^n}}{B_T} Z_T | \mathcal{F}_s\right] = \frac{V_s^{H^n}}{B_s} Z_s$ for all $t \leq s \leq T$ and all $Z \in \mathcal{D}_t^q(\bar{\mathcal{M}}_{[t,T]})$. $V_s^{H^n}$ converges almost surely to V_s^H and $\left|\frac{V_T^{H^n}}{B_T} Z_T\right| \leq \left|\frac{\sup_{t \leq s \leq T}\left(V_s^H\right)}{B_T} Z_T\right| \in L^1(\Omega_{[t,T]})$, since $\sup_{t \leq s \leq T}\left(V_s^H\right) \in L^p(\Omega_{[t,T]})$ by Doob's maximal inequality, hence we find $E\left[\frac{V_T^H}{B_T} Z_T | \mathcal{F}_s\right] = \frac{V_s^H}{B_s} Z_s$ for all $t \leq s \leq T$. □

Define for $\mathcal{F}_t$-measurable v

$$\mathcal{A}_v^{*p}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ \frac{V_T^H}{B_T} \middle| H \in \mathcal{SF}^p(\bar{\mathcal{M}}_{[t,T]}), \frac{V_t^H}{B_t} = v \right\}, \tag{2.16}$$

and

$$\mathcal{G}_v^p(\bar{\mathcal{M}}_{[t,T]}) := \left\{ \frac{V_T^H}{B_T} \middle| H \in \mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]}), \frac{V_t^H}{B_t} = v \right\}. \tag{2.17}$$

For $p > 1$ and $\mathcal{D}^q(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset$, $\mathcal{SF}^p(\bar{\mathcal{M}}_{[t,T]})$ has an important property: $\mathcal{A}_1^{*p}(\mathcal{M}_{[t,T]})$ is known to be closed, if $\left(\frac{S}{B}\right)^{[t,T]}$ is locally in $L^p(\Omega_{[t,T]})$ in the sense, that there exists a sequence $U_n, n \in \mathbf{N}$ of localizing stopping times increasing to infinity such that for each n, the family $\{S_\tau^{[t,T]} | \tau \text{ stopping time}, \tau \leq U_n\}$ is bounded in $L^p(\Omega_{[t,T]})$, see DS96. This condition certainly holds if $\bar{S}$ is continuous. To work with the spaces $\mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]})$ is in some sense more natural, since its definition involves only the objective probability measure P and no equivalent martingale measures. Furthermore $\mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]})$ is stable under stopping, a desirable property from an economic point of view. However, this space has in general weaker properties than $\mathcal{SF}^p(\bar{\mathcal{M}}_{[t,T]})$, see DMSSS97 and GK98.

We will often work with a continuous price process S, resp. $\bar{S}$. In this case $L^1_{loc}(S) = L^2_{loc}(S)$ holds. The price process admits a representation

$$S = S_0 + \mu \cdot \alpha + M, \tag{2.18}$$

where $\mu = (\mu^i)_{1 \leq i \leq d}$ is predictable, α is a predictable, increasing, continuous, locally integrable process such that μ is locally integrable with respect to α. Furthermore, there exists a symmetric non-negative $d \times d$-matrix-valued predictable process $C = (C^{ij})_{1 \leq i,j \leq d}$, locally integrable

with respect to α, such that $[S^i, S^j] = [M^i, M^j] =< M^i, M^j >= C^{ij} \cdot \alpha$. α can be chosen such that $B = \mathcal{E}(r \cdot \alpha)$ for a predictable process r.

In the continuous case, $\mathcal{D}^e(\bar{\mathcal{M}}_{[0,T]}) \neq \emptyset$ implies $\mu = rS - C\lambda$, $d\alpha$-almost surely for a predictable process $\lambda \in L^2_{loc}(M)$ and every $Z \in \mathcal{D}^e(\bar{\mathcal{M}}_{[t,T]})$ is of the form $Z = \mathcal{E}_t(\lambda \cdot M + N)^T$, where N is a not necessarily continuous local martingale orthogonal to M with $[M, N] = 0$, see Ansel and Stricker (1992).

3. Optimal Portfolios

Consider the problem of maximizing expected utility from terminal wealth. We follow a stochastic duality approach, which goes back to Bismut (1973, 1975), see also Karatzas, Lehoczky, Shreve and Xu (1991), (KLSX91), and Karatzas and Shreve (1999), Kramkov and Schachermayer (1999) and Schachermayer (2000) for general results.

We have already defined the so-called *isoelastic* utility functions $u^{(p)}, p \neq 1$, with constant index of relative risk-aversion, see Pratt (1964) and Arrow (1976). For optimization multiplication of the utility function with a constant factor or adding a constant has no effect. We choose to normalize the utility function such that $|U^{(p)}(1)| = 1$ for all $p \neq 0, 1$ and define for $p < 1, p \neq 0$

$$U^{(p)}(x) := \operatorname{sgn}(p) x^p, \quad \forall\, x \geq 0, \tag{3.1}$$

$U^{(p)}(x) = -\infty$ for $x < 0$ and

$$U^{(0)}(x) := \ln(x), \quad \forall\, x > 0, \tag{3.2}$$

$U^{(0)}(x) = -\infty$ for $x \leq 0$. For $p > 1$ set

$$U^{(p)}(x) := -|x|^p, \quad \forall\, x \in \mathbb{R}, \tag{3.3}$$

We have for $p < 1, p \neq 0$

$$\frac{dU^{(p)}}{dx}(x) = |p| x^{p-1}, \quad \forall\, x > 0, \tag{3.4}$$

and

$$\frac{dU^{(0)}}{dx}(x) = \frac{1}{x}, \quad \forall\, x > 0, \tag{3.5}$$

and set $\frac{dU^{(p)}}{dx}(0) := \infty$ for $p < 1$.

We want to solve the following optimization problem for fixed $0 \leq t \leq T < \infty$ and $p \neq 1$:

$$\mathcal{V}(p, t, T, \mathcal{B}) := \operatorname{ess\,sup}_{\substack{H \in \mathcal{B} \\ H_t \tilde{S}_t = 1}} E_t \left[U^{(p)} \left(V_T^H \right) \right] \tag{3.6}$$

where $\mathcal{B} \in \{\mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]}), \{\mathcal{SF}^p(\bar{\mathcal{M}}_{[t,T]}), \mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]})\}$ for $p > 1$, resp. $\mathcal{B} = \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ for $p < 1$, and the dual problem

$$\mathcal{W}^*(q,t,T,\mathcal{C}) := \operatorname{ess\,inf}_{Z\in\mathcal{C}} E_t\left[-U^{(q)}\left(\frac{B_t Z_T}{B_T}\right)\right], \tag{3.7}$$

where $\mathcal{C} \in \{\mathcal{D}_t^q(\bar{\mathcal{M}}_{[t,T]}), \bar{\mathcal{D}}_t^q(\bar{\mathcal{M}}_{[t,T]})\}$. ($-U^{(q)}$ equals the convex dual to $U^{(p)}$ up to a constant factor, see Rockafellar (1970)). See Karatzas and Shreve (1999) for the definition of ess sup and ess inf. If for $H \in \mathcal{B}$ with $V_0^H = 1$ and $\mathcal{V}(p,t,T,\mathcal{B}) = E_t\left[U^{(p)}\left(V_T^H\right)\right]$, then we say V^H solves Problem (3.6) for $\mathcal{B}$. If for $Z \in \mathcal{C}$, $\mathcal{W}^*(q,t,T,\mathcal{C}) = E_t\left[-U^{(q)}\left(\frac{B_t Z_T}{B_T}\right)\right]$, then we say Z solves the dual Problem (3.7) for $\mathcal{C}$. For the moment we are interested in the Problem (3.6) for $\mathcal{B} = \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ and set $\mathcal{V}(p,t,T) := \mathcal{V}(p,t,T,\mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]}))$. For $p > 1$, we set $\mathcal{W}^*(q,t,T) := \mathcal{W}^*(q,t,T,\bar{\mathcal{D}}_t^q(\bar{\mathcal{M}}_{[t,T]}))$, respectively for $p < 1$, we define $\mathcal{W}^*(q,t,T) := \mathcal{W}^*(q,t,T,\mathcal{D}_t^q(\bar{\mathcal{M}}_{[t,T]}))$. (It will turn out later, that $\mathcal{W}^*(q,t,T) = \mathcal{W}^*(q,t,T,\bar{\mathcal{D}}_t^q(\mathcal{M}_{[t,T]}))$ for $p > 1$ and for $p < 1$ if $\mathcal{V}(p,0,T) < \infty$.)

The following proposition shows the close relation between these two problems and gives the key idea how to handle the incompleteness of the market.

Proposition 3.1. *Assume that there exists an $H \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ with $V_T^H \geq 0$ and a $Z^{opt*,q,t,T} \in \bar{\mathcal{D}}_t(\bar{\mathcal{M}}_{[t,T]})$ such that for some $\mathcal{F}_t$-measurable random variable $c > 0$*

$$Z_T^{opt*,q,t,T} = cB_T \operatorname{sgn}(1-p)\frac{dU^{(p)}}{dx}\left(V_T^H\right), \tag{3.8}$$

and such that $\frac{V^H}{B^{[t,T]}} Z^{opt*,q,t,T}$ is a uniformly integrable martingale. Then $V^{opt,p,t,T} := \frac{V^H}{V_0^H}$ solves Problem (3.6) for $\mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ and $Z^{opt*,q,t,T}$ solves for $p > 1$, resp. $p < 1$, the dual Problem (3.7) for $\bar{\mathcal{D}}_t(\bar{\mathcal{M}}_{[t,T]})$, resp. for $\mathcal{D}_t(\bar{\mathcal{M}}_{[t,T]})$ and $\bar{\mathcal{D}}_t(\bar{\mathcal{M}}_{[t,T]})$. There exists at most one such pair $(V^{opt,p,t,T}, Z^{opt*,q,t,T})$ with a representation (3.8). For $p \neq 0$ the

corresponding optimal values satisfy

$$|\mathcal{V}(p,t,T)|^{p^{-1}}|\mathcal{W}^*(q,t,T)|^{q^{-1}} = 1. \tag{3.9}$$

Proof. Note that for $p < 1$ (3.8) implies $V_T^H > 0$. For $H \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ with $V_0^H = 1$ and since $U^{(p)}$ is concave we have

(3.10)

$$U^{(p)}\left(V_T^H\right) \leq U^{(p)}\left(V_T^{opt,p,t,T}\right) + \frac{dU^{(p)}}{dx}\left(V_T^{opt,p,t,T}\right)(V_T^H - V_T^{opt,p,t,T}).$$

Taking conditional expectations we find

$$\begin{aligned} E_t\left[U^{(p)}\left(V_T^H\right)\right] &\leq E_t\left[U^{(p)}\left(V_T^{opt,p,t,T}\right) + Z^{opt*,q,t,T}\frac{V_T^H - V_T^{opt,p,t,T}}{\operatorname{sgn}(1-p)cB_T}\right] \\ &\leq E_t\left[U^{(p)}\left(V_T^{opt,p,t,T}\right)\right], \end{aligned}$$

since $E_t\left[\frac{V_T^H}{B_T}Z^{opt*,q,t,T}\right] \leq \frac{V_t^H}{B_t} = \frac{V_t^{opt,p,t,T}}{B_t}$ for $p < 1$, respectively since $Z^{opt*,q,t,T} \in \bar{\mathcal{D}}_t^q(\bar{\mathcal{M}}_{[t,T]})$ for $p > 1$. Let $Q \in \bar{\mathcal{D}}_t^q(\bar{\mathcal{M}}_{[t,T]})$ and calculate

$$\begin{aligned} -U^{(q)}\left(\frac{\frac{dQ_T}{dP_T}}{B_T B_t^{-1}}\right) &= -U^{(q)}\left(\frac{Z_T^{opt*,q,t,T} + \left(\frac{dQ_T}{dP_T} - Z_T^{opt*,q,t,T}\right)}{B_T B_t^{-1}}\right) \\ &\geq -U^{(q)}\left(\frac{Z_T^{opt*,q,t,T}}{B_T B_t^{-1}}\right) - \frac{dU^{(q)}}{dx}\left(\frac{B_t Z_T^{opt*,q,t,T}}{B_T B_t^{-1}}\right)\frac{\frac{dQ_T}{dP_T} - Z_T^{opt*,q,t,T}}{B_T B_t^{-1}} \\ &= -U^{(q)}\left(\frac{Z_T^{opt*,q,t,T}}{B_T B_t^{-1}}\right) - \operatorname{sgn}(1-p)k\frac{V_T^{opt,p,t,T}}{B_T}\left(\frac{dQ_T}{dP_T} - Z_T^{opt*,q,t,T}\right), \end{aligned}$$

for some $\mathcal{F}_t$-measurable random variable $k > 0$. Taking conditional expectations we find

$$E_t\left[-U^{(q)}\left(\frac{B_t Z_T^{opt*,q,t,T}}{B_T}\right)\right] \leq E_t\left[-U^{(q)}\left(\frac{B_t\frac{dQ_T}{dP_T}}{B_T}\right)\right], \tag{3.11}$$

since $E_t\left[\frac{V_T^{opt,p,t,T}}{B_T}\frac{dQ_T}{dP_T}\right] \leq \frac{V_t^{opt,p,t,T}}{B_t}$ for $p < 1$, resp. since for $p > 1$ $Z^{opt*,q,t,T} \in \bar{\mathcal{D}}_t^q(\bar{\mathcal{M}}_{[t,T]})$. The uniqueness of the pair $(V_T^{opt,p,t,T}, Z_T^{opt*,q,t,T})$ follows from the strict concavity of $U^{(p)}$. Since $\frac{V^{opt,p,t,T}}{B^{[t,T]}}Z^{opt*,q,t,T}$ is a uniformly integrable martingale, it is determined by $\frac{V_T^{opt,p,t,T}}{B_T}Z_T^{opt*,q,t,T}$. Let $H' \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ with $V_0^{H'} = 1$ and $Z' \in \mathcal{D}_t(\bar{\mathcal{M}}_{[t,T]})$ such that

$$Z'_T = c' B_T \operatorname{sgn}(1-p)\frac{dU^{(p)}}{dx}\left(V_T^{H'}\right), \tag{3.12}$$

holds for a $\mathcal{F}_t$-measurable random variable $c' > 0$. Then $V_T^{opt,p,t,T} = V_T^{H'}$, $Z_T^{opt*,q,t,T} = Z'_T$ and $\frac{V^{opt,p,t,T}}{B^{[t,T]}}Z^{opt*,q,t,T} = \frac{V^{H'}}{B^{[t,T]}}Z'$. Assume that there exists a $t \leq s \leq T$ with $A := \{V_s^{H'} > V_s^{opt,p,t,T}\} \neq \emptyset$. In this case we can change the self-financing hedging strategy H' on $A \times [s,T]$ to a $H'' \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ such that $V_T^{H''} \geq V_T^{opt,p,t,T}$ and $V_T^{H''} > V_T^{opt,p,t,T}$ on A. From this we conclude the uniqueness of the pair $(V^{opt,p,t,T}, Z^{opt*,q,t,T})$. For $p \neq 0$ we find

$$\begin{aligned} U^{(p)}(V_T^{opt,p,t,T}) &= -\operatorname{sgn}(q)\left(V_T^{opt,p,t,T}\right)^p \\ &= -\operatorname{sgn}(q)\left(V_T^{opt,p,t,T}\right)^{p-1} V_T^{opt,p,t,T} \\ &= \frac{Z_T^{opt*,p,t,T}}{cp\ \operatorname{sgn}(1-p)}\frac{V_T^{opt,p,t,T}}{B_T}, \end{aligned}$$

hence

$$\mathcal{V}(p,t,T) = E_t\left[U^{(p)}(V_T^{opt,p,t,T})\right] = \frac{1}{cp\ \operatorname{sgn}(1-p)B_t}. \tag{3.13}$$

For $q \neq 0$ we find

$$\begin{aligned} -U^{(q)}\left(\frac{B_t Z_T^{opt*,p,t,T}}{B_T}\right) &= \operatorname{sgn}(p) B_t^q \left(\frac{Z_T^{opt*,p,t,T}}{B_T}\right)^q \\ &= \operatorname{sgn}(p) B_t^q \left(\frac{Z_T^{opt*,p,t,T}}{B_T}\right)^{q-1} \frac{Z_T^{opt*,p,t,T}}{B_T} \\ &= \operatorname{sgn}(p) B_t^q \left(c|p|\left(V_T^{opt,p,t,T}\right)^{p-1}\right)^{\frac{1}{p-1}} \frac{Z_T^{opt*,p,t,T}}{B_T} \\ &= \operatorname{sgn}(p) B_t^q (c|p|)^{\frac{1}{p-1}} Z_T^{opt*,p,t,T} \frac{V_T^{opt,p,t,T}}{B_T}, \end{aligned}$$

hence

$$\mathcal{W}^*(q,t,T) = E_t\left[-U^{(q)}\left(\frac{B_t Z_T^{opt*,p,t,T}}{B_T}\right)\right] = \operatorname{sgn}(p)(B_t c|p|)^{\frac{1}{p-1}}. \tag{3.14}$$

(3.13) and (3.14) together imply (3.9). □

We call $(V^{opt,p,t,T}, Z^{opt*,q,t,T})$ the *optimal pair* for the market $\bar{\mathcal{M}}_{[t,T]}$ with respect to optimization in $\mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$. We have the following stability property for optimal pairs:

Proposition 3.2. *If the pair $(V^{opt,p,t,T}, Z^{opt*,q,t,T})$ admits a representation (3.8) with $V^{opt,p,t,T} = V^H$ for a $H \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ with $V_0^H = 1$ and $Z^{opt*,q,t,T} \in \mathcal{D}_t^q(\bar{\mathcal{M}}_{[t,T]})$, then the optimal pair for the market $\bar{\mathcal{M}}_{[t',T]}$ exists and is given by*

$$\left(V^{opt,p,t',T}, Z^{opt*,q,t',T}\right) = \left(\frac{V_{t'\vee\cdot}^{opt,p,t,T}}{V_{t'}^{opt,p,t,T}}, \frac{Z_{t'\vee\cdot}^{opt*,q,t,T}}{Z_{t'}^{opt*,q,t,T}}\right). \tag{3.15}$$

Proof. Note first, that by J&S87, Lemma III.3.6, we have $Z^{opt*,q,t,T} > 0$. For $Z \in \bar{\mathcal{D}}_{t'}^q(\bar{\mathcal{M}}_{[t',T]})$ set $A := \left\{E_{t'}\left[Z_T^q\right] < E_{t'}\left[\left(\frac{Z_T^{opt*,q,t,T}}{Z_{t'}^{opt*,q,t,T}}\right)^q\right]\right\}$. Define $\tilde{Z} := Z^{opt*,q,t,T}$ on $[t,t') \cup A^c \times [t',T]$, and $\tilde{Z} := Z_{t'}^{opt*,q,t,T} Z$ on $A \times [t',T]$.

We calculate

$$\begin{aligned} E_t\left[\tilde{Z}_T^q\right] &= E_t\left[E_{t'}\left[\tilde{Z}_T^q\right]\right] \\ &= E_t\left[\mathbf{1}_A\left(Z_{t'}^{opt*,q,t,T}\right)^q E_{t'}\left[Z_T^q\right] + \mathbf{1}_{A^c} E_{t'}\left[\left(Z_T^{opt*,q,t,T}\right)^q\right]\right] \\ &\leq E_t\left[E_{t'}\left[\left(Z_T^{opt*,q,t,T}\right)^q\right]\right] = E_t\left[\left(Z_T^{opt*,q,t,T}\right)^q\right], \end{aligned}$$

hence $\tilde{Z} \in \bar{\mathcal{D}}_t^q(\bar{\mathcal{M}}_{[t,T]})$, $\tilde{Z}_T = Z_T^{opt*,q,t,T}$ and $A = \emptyset$ and we conclude $Z^{opt*,q,t',T} = \frac{Z_{t'\vee\cdot}^{opt*,q,t,T}}{Z_{t'}^{opt*,q,t,T}}$. By assumption we have $\frac{V_T^{opt,p,t,T}}{B_T} Z_T^{opt*,q,t,T} > 0$ and since $\frac{V^{opt,p,t,T}}{B^{[t,T]}} Z^{opt*,q,t,T}$ is a non-negative supermartingale we have $V^{opt,p,t,T} > 0$. Hence $\frac{V_T^{opt,p,t,T}}{V_{t'}^{opt,p,t,T}} \in L_{t'}^p(\Omega_{[t',T]})$ and $\frac{V_{t'\vee\cdot}^{opt,p,t,T}}{V_{t'}^{opt,p,t,T}} Z$ is a uniformly integrable martingale for all $Z \in \mathcal{D}_{t'}^q(\bar{\mathcal{M}}_{[t',T]})$. Since

$$Z_T^{opt*,q,t',T} = c_{t'} B_T \operatorname{sgn}(1-p) \frac{dU^{(p)}}{dx}\left(\frac{V_T^{opt,p,t,T}}{V_{t'}^{opt,p,t,T}}\right), \tag{3.16}$$

where

$$c_{t'} := \frac{c\frac{dU^{(p)}}{dx}\left(V_{t'}^{opt,p,t,T}\right)}{Z_{t'}^{opt*,q,t,T}}, \tag{3.17}$$

we can apply Proposition 3.1. □

Lemma 3.3. *For $p > 1$, $H \in \mathcal{SF}_t^0(\bar{\mathcal{M}}_{[t,T]})$ with $V_0^H = 1$, $Z^{opt*,q,t,T} \in \bar{\mathcal{D}}_t(\bar{\mathcal{M}}_{[t,T]})$ and assume $(V^H, Z^{opt*,q,t,T})$ to admit a representation (3.8). If $V_T^H \in L_t^{p+\epsilon}(\Omega_{[t,T]})$, (or equivalently $Z_T^{opt*,q,t,T} \in L_t^{q+\epsilon}(\Omega_{[t,T]})$), for some $\epsilon > 0$, then $H \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$ and $(V^H, Z^{opt*,q,t,T})$ is the optimal pair for the market $\bar{\mathcal{M}}_{[t',T]}$.*

Proof. Observe that $\frac{V_T^H}{B_T} Z_T \in L_t^{1+\tilde{\epsilon}}(\Omega_{[t,T]})$ for some $\tilde{\epsilon} > 0$ for all $Z \in \bar{\mathcal{D}}_t^q(\bar{\mathcal{M}}_{[t,T]})$. Hence $\frac{V_T^H}{B_T} Z$ is a uniformly integrable martingale. Now apply Proposition 3.1. □

In the next two sections we look at an example and postpone an existence result for the optimal pair $(V^{opt,p,t,T}, Z^{opt*,q,t,T})$ until Section 6.

4. Totally Unhedgeable Price for Instantaneous Risk

Assume $\bar{S}$ to be continuous such that we have a representation (2.18). In this section we seek a sufficient condition ensuring certain self-financing hedging strategies to be optimal for problem 3.6. See Karatzas and Shreve (1999), Example 6.7.4 for a similar result and the notion of *totally unhedgeable coefficients*. This notion describes a market model where the uncertainty in the coefficients defining the model is in a certain sense orthogonal to the uncertainty of the local martingale M driving the price process, such that we can not hedge against this risk. Set $\beta := \sqrt{\lambda C \lambda} \in L^2_{loc}(\alpha)$.

Definition 4.1. For $0 \leq t \leq T < \infty$, $\beta^{[t,T]}$ is called the instantaneous price for risk process, or instantaneous Sharpe-ratio process, for the market $\bar{\mathcal{M}}_{[t,T]}$.

Definition 4.2. For $0 \leq t \leq T < \infty$, let an $\mathcal{F}_t$-measurable random variable $c > 0$ and a not necessarily continuous local martingale N orthogonal to M, (or equivalently with $[N, M] = 0$), be given such that

1.

$$\mathcal{E}_t\left(\left(pr - q\frac{\beta^2}{2}\right)\cdot\alpha\right)_T = c\mathcal{E}_t(N)_T. \tag{4.1}$$

2. For $p < 1$, $\mathcal{E}_t(q\lambda \cdot M + N)^T$ is a uniformly integrable martingale.

We then call the instantaneous price for risk $\beta^{[t,T]}$ in the market $\bar{\mathcal{M}}_{[t,T]}$ *totally p-unhedgeable*. resp. *strongly totally p-unhedgeable* if $\mathcal{E}_t(N)^T$ is a uniformly integrable martingale.

Remark 4.3. For $p = 0$ we have $q = 0$ and we find a unique representation (4.1) with $c = 1$, $N = 0$ for any $0 \leq t \leq T < \infty$, thus $\beta^{[t,T]}$ is totally 0-unhedgeable in $\bar{\mathcal{M}}_{[t,T]}$. For $p = 0$, the optimization problem (3.6) is also known as maximizing the Kelly-criterion, see Kelly (1956), Breiman (1960) and Karatzas and Shreve (1999). For general results see Aase (1986) and Goll and Kallsen (2000).

Lemma 4.4. *If $\beta^{[t,T]}$ is totally p-unhedgeable in $\bar{\mathcal{M}}_{[t,T]}$, then $\beta^{[t',T]}$ is totally p-unhedgeable in $\bar{\mathcal{M}}_{[t',T]}$ for all $t \leq t' \leq T$.*

Proof. For $t \leq t' \leq T$ set $c' := c\mathcal{E}_t\left(-\left(pr - q\frac{\beta^2}{2}\right) \cdot \alpha + N\right)_{t'}$. This gives us a representation (4.1) and for $p < 1$, $\mathcal{E}_{t'}(q\lambda \cdot M + N)^T$ is a uniformly integrable martingale. □

Proposition 4.5. *Assume $\beta^{[t,T]}$ to be totally p-unhedgeable in $\bar{\mathcal{M}}_{[t,T]}$ with a representation (4.1), then the optimal pair for the market $\bar{\mathcal{M}}_{[t',T]}$ for $t \leq t' \leq T$ is given by*

$$\left(V^{opt,p,t',T}, Z^{opt*,q,t',T}\right) = \left(\frac{V^{(H^p)}_{t' \vee \cdot}}{V^{(H^p)}_{t'}}, \mathcal{E}_{t'}(\lambda \cdot M + N)^T\right), \tag{4.2}$$

where $H^p := \left(\frac{\lambda}{p-1}, \frac{1-\frac{\lambda S}{p-1}}{B}\right)^{[t,T]}$ *generates the value process*

$$V^{(H^p)} := \mathcal{E}_t\left(\left(r - \frac{\beta^2}{p-1}\right)\cdot\alpha + \frac{\lambda}{p-1}\cdot M\right)^T. \tag{4.3}$$

Furthermore, for $p \neq 0$

$$\mathcal{V}(p,t',T) = -\mathrm{sgn}(q)\frac{E_{t'}\left[\mathcal{E}_{t'}\left(\left(pr - q\frac{\beta^2}{2}\right)\cdot\alpha\right)_T\right]}{E_{t'}\left[\mathcal{E}_{t'}(N)_T\right]}, \tag{4.4}$$

resp. if $\beta^{[t,T]}$ *is strongly totally* p*-unhedgeable in* $\bar{\mathcal{M}}_{[t,T]}$,

$$\mathcal{V}(p,t',T) = -\mathrm{sgn}(q)E_{t'}\left[\mathcal{E}_{t'}\left(\left(pr - q\frac{\beta^2}{2}\right)\cdot\alpha\right)_T\right], \tag{4.5}$$

and

$$\mathcal{W}^*(p,t',T) = \mathrm{sgn}(p)\left(\mathcal{V}(p,t',T)\right)^{\frac{1}{1-p}}. \tag{4.6}$$

Proof. For $p \neq 0$ calculate

$$\begin{aligned}
\frac{dU^{(p)}}{dx}\left(V_T^{(H^p)}\right) &= -p\ \mathrm{sgn}(q)\left(V^{(H^p)_T}\right)^{p-1} \\
&= -p\ \mathrm{sgn}(q)\mathcal{E}_t\left(\left(r - \frac{\beta^2}{p-1}\right)\cdot\alpha + \frac{\lambda}{p-1}\cdot M\right)_T^{p-1} \\
&= -p\ \mathrm{sgn}(q)\mathcal{E}_t\left(\left((p-1)r - q\frac{\beta^2}{2}\right)\cdot\alpha + \lambda\cdot M\right)_T \\
&= \frac{\mathrm{sgn}(p-1)|p|cB_t}{B_T}\mathcal{E}_t\left(\lambda\cdot M + N\right)_T,
\end{aligned}$$

resp. for $p = 0$

$$\begin{aligned}\frac{dU^{(0)}}{dx}\left(V_T^{(H^p)}\right) &= \left(V^{(H^p)_T}\right)^{-1} \\ &= \mathcal{E}_t\left(\left(r+\beta^2\right)\cdot\alpha - \lambda\cdot M\right)_T^{-1} \\ &= \mathcal{E}_t\left(-r\cdot\alpha + \lambda\cdot M\right)_T \\ &= \frac{B_t}{B_T}\mathcal{E}_t\left(\lambda\cdot M\right)_T,\end{aligned}$$

and find a representation (3.8), since $\mathcal{E}_t\left(\lambda\cdot M+N\right)_T > 0$. Set

$$Z^{opt*,q,t,T} := \mathcal{E}_t\left(\lambda\cdot M+N\right)^T. \tag{4.7}$$

$$\begin{aligned}\frac{V^{(H^p)}}{B^{[t,T]}}Z^{opt*,q,t,T} &= \frac{1}{B_t}\mathcal{E}_t\left(-\frac{\beta^2}{p-1}\cdot\alpha + \frac{\lambda}{p-1}\cdot M\right)^T\mathcal{E}_t\left(\lambda\cdot M+N\right)^T \\ &= \frac{1}{B_t}\mathcal{E}_t\left(q\lambda\cdot M+N\right)^T,\end{aligned}$$

which is a uniformly integrable martingale on $[t,T]$ for $p<1$ by assumption. For $p>1$ and $\epsilon>1$ observe

$$\begin{aligned}\left(V_T^{(H^p)}\right)^{\epsilon p} &= \mathcal{E}_t\left(\left(r-\frac{\beta^2}{p-1}\right)\cdot\alpha + \frac{\lambda}{p-1}\cdot M\right)_T^{\epsilon p} \\ &= \mathcal{E}_t\left(\left(\epsilon pr - \epsilon q\beta^2 + \frac{\beta^2}{2}\frac{\epsilon^2p^2-\epsilon p}{(p-1)^2}\right)\cdot\alpha + \epsilon q\lambda\cdot M\right)_T \\ &= \mathcal{E}_t\left(\left(\epsilon pr - \frac{\epsilon q\beta^2}{2}\frac{p(2-\epsilon)-1}{p-1}\right)\cdot\alpha + \epsilon q\lambda\cdot M\right)_T \\ &= \mathcal{E}_t\left(\left(\epsilon pr - \frac{\epsilon q\beta^2}{2}\left(1-q(\epsilon-1)\right)\right)\cdot\alpha + \epsilon q\lambda\cdot M\right)_T,\end{aligned}$$

hence we find $V_T^{(H^p)} \in L_t^{p+\tilde{\epsilon}}(\Omega_{[t,T]})$ for some $\tilde{\epsilon}>0$, since $(1-q(\epsilon-1))>0$ for ϵ close to 1. By Lemma 3.3 we find $\left((V^{(H^p)})^T, \mathcal{E}_t(\lambda\cdot M+N)^T\right)$ to

be the optimal pair. For $p \neq 0$ we calculate

$$\begin{aligned} U^{(p)}\left(V_T^{(H^p)}\right) &= -\mathrm{sgn}(q)\left(V_T^{(H^p)}\right)^p \\ &= -\mathrm{sgn}(q)\mathcal{E}_t\left(\left(r-\frac{\beta^2}{p-1}\right)\cdot\alpha+\frac{\lambda}{p-1}\cdot M\right)_T^p \\ &= -\mathrm{sgn}(q)\mathcal{E}_t\left(\left(pr-q\frac{\beta^2}{2}\right)\cdot\alpha+q\lambda\cdot M\right)_T \\ &= -\mathrm{sgn}(q)c\mathcal{E}_t\left(q\lambda\cdot M+N\right)_T. \end{aligned}$$

Since $\mathcal{E}_t\left(q\lambda\cdot M+N\right)^T$ is a uniformly integrable martingale we find

(4.8)

$$E_t\left[U^{(p)}\left(V_T^{(H^p)}\right)\right] = -\mathrm{sgn}(q)c = -\mathrm{sgn}(q)\frac{E_t\left[\mathcal{E}_t\left(\left(pr-q\frac{\beta^2}{2}\right)\cdot\alpha\right)_T\right]}{E_t\left[\mathcal{E}_t(N)_T\right]}.$$

The last equation follows from (3.9). □

In the next section we give an interpretation of the portfolios generated by H^p.

5. Locally Efficient Portfolios

From Lemma 1.4 and $\mu = rS - C\lambda$, $d\alpha$-a.s., we immediately find $V^{(\hat{H})} := \mathcal{E}_t\left(H\cdot S\right)^T = \mathcal{E}_t\left(\left(r-HC\lambda\right)\cdot\alpha + H\cdot M\right)^T$ for a process $\hat{H} = \left(H, \frac{1-HS}{B}\right) \in L^1_{loc}(S)$. From Cauchy-Schwarz inequality it follows that $|HC\lambda| \leq \sqrt{\lambda C\lambda}\sqrt{HCH} = \beta\sqrt{HCH}$. We have

$$[V^{(\hat{H})}, V^{(\hat{H})}] = \left(V^{(\hat{H})}\right)^2 HCH\cdot\alpha^{[t,T]}.$$

We interpret $\sqrt{HCH}$ as a measure for the relative instantaneous risk of the portfolio generated by $\hat{H}$ and $\hat{H}\mu = r - HC\lambda$ as a measure for the instantaneously expected relative return rate. For $\beta \neq 0$ and $\sqrt{HCH} \neq 0$, we find for the instantaneous Sharpe-ratio $\frac{\hat{H}\mu - r}{\sqrt{HCH}} = \frac{-HC\lambda}{\sqrt{HCH}}$ of instantaneously expected relative excess return over the instantaneously risk-free return rate and relative instantaneous risk, $-\beta \leq \frac{-HC\lambda}{\sqrt{HCH}} \leq \beta$ and $\frac{-HC\lambda}{\sqrt{HCH}} = \beta$ iff $H \in k\lambda + \mathrm{Ker}(C)$ for a predictable, strictly negative, process $k \in L^2_{loc}(\beta)$, resp. $\frac{-HC\lambda}{\sqrt{HCH}} = -\beta$ iff $H \in k\lambda + \mathrm{Ker}(C)$ for a predictable, strictly positive, process $k \in L^2_{loc}(\beta)$. We call these hedging strategies *locally efficient*. We have seen in the last section that in

the case of a totally p-unhedgeable price for risk the optimal portfolios generated by H^p are locally efficient. See Markowitz (1952, 1987) and Sharpe (1964, 2000).

Define the following quantities for $p \neq 0, 1$:

$$\mathcal{R}^{(p,t,T)} := \frac{1}{p}\frac{d\ln\left(\mathcal{V}(p,t,T)\right)}{dT}, \tag{5.1}$$

$$\mathcal{R}^{(p)} := \lim_{T\to\infty}\frac{1}{pT}\ln\left(\mathcal{V}(p,0,T)\right) \tag{5.2}$$

and

$$\mathcal{R}^{(0,t,T)} := \frac{d\mathcal{V}(0,t,T)}{dT} \tag{5.3}$$

$$\mathcal{R}^{(0)} := \lim_{T\to\infty}\frac{1}{T}\mathcal{V}(0,0,T). \tag{5.4}$$

Under some regularity conditions, these quantities exist. By Theorem 4.5 we find immediately

Proposition 5.1. *Under the assumptions, $\alpha_t = t$ and $pr - q\frac{\beta^2}{2}$ constant for $p \neq 0, 1$, resp. $r + \frac{\beta^2}{2}$ constant for $p = 0$, we have*

$$\mathcal{V}(p,t,T) = \exp\left(\left(pr - q\frac{\beta^2}{2}\right)(T-t)\right), \tag{5.5}$$

$$\mathcal{R}^{(p,t,T)} = \mathcal{R}^{(p)} = r + \frac{\beta^2}{2(1-p)}, \tag{5.6}$$

resp.

$$\mathcal{V}(0,t,T) = \left(r + \frac{\beta^2}{2}\right)(T-t), \tag{5.7}$$

$$\mathcal{R}^{(0,t,T)} = \mathcal{R}^{(0)} = r + \frac{\beta^2}{2}. \tag{5.8}$$

See Bielecki and Pliska (1999, 2000) for an interpretation of the quantities $\mathcal{R}^{(p)}$ and the *risk-sensitive stochastic control* approach. $\mathcal{R}^{(p,t,T)}$ can be interpreted as an implied forward growth rate of the expected utility of wealth under the optimal self-financing hedging strategy. As we will see in Section 7, another related quantity is the right one to look at: We define

$$\mathcal{Y}(p,t,T) := \ln\left(|\mathcal{V}(p,t,T)|\right). \tag{5.9}$$

6. Existence of Optimal Portfolios

Let $0 \leq T < \infty$ be fixed. In this section we will assume $\bar{S}$ to be continuous and $\mathcal{D}^q(\bar{\mathcal{M}}_{[0,T]}) \neq \emptyset$ for $p > 1$, resp. for $p < 1$, $\mathcal{D}^e(\bar{\mathcal{M}}_{[0,T]}) \neq \emptyset$ and $\mathcal{V}(p, 0, T, \mathcal{SF}^p(\bar{\mathcal{M}}_{[0,T]})) < \infty$. We assume for simplicity in this section that $B = 1$. The results can be generalized to the case of a B such that B and B^{-1} are uniformly bounded on $[0, T]$.

Theorem 6.1. *Under the above assumptions, the optimal pair* $(V^{opt,p,0,T}, Z^{opt*,q,0,T})$, *satisfying (3.8) and* $Z^{opt*,q,0,T} \in \mathcal{D}(\bar{\mathcal{M}}_{[0,T]})$, *exists for the market* $\bar{\mathcal{M}}_{[0,T]}$ *with respect to optimization in* $\mathcal{SF}^p(\bar{\mathcal{M}}_{[0,T]})$.

Proof. We first prove the case $p > 1$. Since $\mathcal{A}_1^{*p}(\bar{\mathcal{M}}_{[0,T]})$ is closed and convex and since $L^p(\Omega_{[0,T]})$ is reflexive there exists an element $V^{opt,p,0,T}$ with minimal norm. As in GLP98, Lemma 4.1 and Theorem 4.1, it is easily shown that $V^{opt,p,0,T} \geq 0$. Since $U^{(p)}$ is concave we have for all $Y \in \mathcal{A}_0^{*p}(\bar{\mathcal{M}}_{[0,T]})$

$$U^{(p)}\left(V_T^{opt,p,0,T} + Y\right) \leq U^{(p)}\left(V_T^{opt,p,0,T}\right) + \frac{dU^{(p)}}{dx}\left(V_T^{opt,p,0,T}\right) Y. \tag{6.1}$$

It follows from the optimality of $V_T^{opt,p,0,T}$ and since $\frac{dU^{(p)}}{dx}\left(V_T^{opt,p,0,T}\right) \in L^q(\Omega_{[0,T]})$ that

$$E\left[B_T \frac{dU^{(p)}}{dx}\left(V_T^{opt,p,0,T}\right)\frac{Y}{B_T}\right] = 0, \tag{6.2}$$

for all $Y \in \mathcal{A}_0^{*p}(\bar{\mathcal{M}}_{[0,T]})$. From $V_T^{opt,p,0,T} \in L^p(\Omega_{[0,T]})$ it follows that $\frac{dU^{(p)}}{dx}\left(V_T^{opt,p,0,T}\right) \in L^q(\Omega_{[0,T]})$. Since $V_T^{opt,p,0,T} \in \mathcal{A}_1^{*p}(\bar{\mathcal{M}}_{[0,T]})$ we have

$E\left[\left(V_T^{opt,p,0,T}\right)^{p-1}\right] > 0$. We therefore find

$$Z^{opt*p,0,T} := \frac{E\left[\frac{dU^{(p)}}{dx}\left(V_T^{opt,p,0,T}\right)\,|\mathcal{F}_\cdot\right]}{E\left[\frac{dU^{(p)}}{dx}\left(V_T^{opt,p,0,T}\right)\right]} \in \bar{\mathcal{D}}^q(\bar{\mathcal{M}}_{[0,T]}). \tag{6.3}$$

Optimality follows now from Proposition 3.1. It was shown in GK98, Lemma 4.4, that $Z^{opt*,q,0,T} \in \mathcal{D}^q(\bar{\mathcal{M}}_{[0,T]})$.

For $p < 1$ the results of Kramkov and Schachermayer (1999), (KS99), can be applied. There, existence and uniqueness of an optimal solution $V^{opt,p,0,T}$ with $V_T^{opt,p,0,T} > 0$ for problem (3.6) is proved. Furthermore, the existence and uniqueness of a strictly positive process Z^{opt}, such that $E\left[\left(Z_T^{opt}\right)^q\right] = \inf_{Z\in\mathcal{D}(\bar{\mathcal{M}}_{[0,T]})} E\left[\left(\frac{Z_T}{B_T}\right)^q\right]$, and with the following properties is shown: $Z_T^{opt} = -\mathrm{sgn}(q)U^{(p)}(V_T^{opt,p,0,T})$, $V^{opt,p,0,T}Z^{opt}$ is a uniformly integrable martingale and for an arbitrary self-financing hedging strategy with non-negative value process V, the process VZ^{opt} is a *supermartingale*. We will show in the next lemma, that for a continuous price process $Z^{opt} \in \mathcal{D}(\bar{\mathcal{M}}_{[0,T]})$. □

The worrying fact is of course that Z^{opt} is in general only a supermartingale. However, the given example (Example 5.1' in KS99), showing that Z^{opt} is in general not a local martingale, involves a non-continuous price process. We define the following set of semimartingales living on $\Omega_{[0,T]}$ for $0 < \infty$, see KS99:

$$\mathcal{Y}(\bar{\mathcal{M}}_{[0,T]}) := \Big\{Y \geq 0 | Y_0 = 1, \frac{V^H}{B^{[0,T]}}Y \text{ is a supermartingale for all } H \in \mathcal{SF}^0(\bar{\mathcal{M}}_{[0,T]})\Big\}.$$

Lemma 6.2. *Assume $\bar{S}$ to be continuous and let $Y \in \mathcal{Y}(\bar{\mathcal{M}}_{[0,T]})$ with $Y_T > 0$ be given. If there exists a $H' \in \mathcal{SF}^0(\bar{\mathcal{M}}_{[0,T]})$ with $V_0^{H'} = 1$ and $V_T^{H'} > 0$ and such that $V^{H'}Y$ is a uniformly integrable martingale, then $Y \in \mathcal{D}(\bar{\mathcal{M}}_{[0,T]})$.*

Proof. Since Y is a non-negative supermartingale, we have by J&S87, Lemma III.3.6, that $Y > 0$ and $Y_- > 0$ almost surely and hence $Y = \mathcal{E}(Z)$ for $Z := \frac{1}{Y_-} \cdot Y$. Since Y is a supermartingale it is a special semimartingale and therefore Z too. Z admits a representation $Z = A+L$, where $A = A^T$ is a predictable process of finite variation, $L = L^T$ is a local martingale and $A_0 = L_0 = 0$. By J&S87, Theorem III.4.11, we find a predictable process $K \in L^2_{loc}(M)$ and a local martingale N orthogonal to all components of M, with $[M, N] = 0$, such that $L = K \cdot M + N$ and the representation $Y = \mathcal{E}(A + K \cdot M + N)$.

Since $V^{H'} \geq 0$ is a local martingale with respect to any equivalent martingale measure, $V_T^{H'} > 0$ implies $V^{H'} > 0$. By Lemma 1.4 there exists a $\tilde{H}' \in L^2_{loc}\left(S^T\right)$ such that $\left(\tilde{H}', \frac{1-\tilde{H}'S}{B}\right)^T$ generates $V^{H'}$. By assumption and since M and $\tilde{H}'C\lambda \cdot \alpha$ are continuous, $\frac{V^{H'}}{B^T}Y = \frac{V^{(\tilde{H}')}}{B^T}Y = \mathcal{E}(\tilde{H}'C(K-\lambda)\cdot\alpha + A + (K+\tilde{H}')\cdot M + N)^T$ is a uniformly integrable martingale. The Doléon-Dade SDE implies that $\left(\frac{V^{H'}}{B^T}Y\right)_- \cdot (\tilde{H}'C(K-\lambda)\cdot\alpha + A) = \frac{V^{H'}}{B^T}Y - \left(\frac{V^{H'}}{B^T}Y\right)_- \cdot ((K+\tilde{H}')\cdot M + N)^T - 1$ is a predictable local martingale of finite variation on

$[0,T]$, hence constant on $[0,T]$ almost surely, see J&S87, Corollary I.3.16. We therefore find $\mathcal{E}(\tilde{H}'C(K-\lambda)\cdot\alpha+A)^T=1$. Now let $H\in L^2_{loc}\left(S^T\right)$, set $\bar{H}:=\left(H,\frac{1-HS}{B}\right)^T$ and consider the discounted value process $V^*:=\frac{V^{(\bar{H})}}{B^T}=\mathcal{E}(-HC\lambda\cdot\alpha+H\cdot M)^T$ generated by $\bar{H}$. We have $V^*Y=\mathcal{E}(HC(K-\lambda)\cdot\alpha+A+(K+H)\cdot M+N)^T=\mathcal{E}((H-\tilde{H}')C(K-\lambda)\cdot\alpha+(K+H)\cdot M+N)^T$ is a supermartingale for all $H\in L^2_{loc}\left(S^T\right)$ by assumption. For $H:=K-\lambda+\tilde{H}'$ we find $(V^*Y)_-\cdot((K-\lambda)C(K-\lambda)\cdot\alpha)=V^*Y-(V^*Y)_-\cdot((K+H)\cdot M+N)-1$ to be a non-decreasing local supermartingale on $[0,T]$. Therefore $(K-\lambda)C(K-\lambda)=0$, $d\alpha$-a.s. and from $1=\mathcal{E}(\tilde{H}'C(K-\lambda)\cdot\alpha+A)^T=\mathcal{E}(A)^T$ we conclude $A=0$ and $Y=\mathcal{E}(\lambda\cdot M+N)^T\in\mathcal{D}(\bar{\mathcal{M}}_{[0,T]})$. □

In DMSSS97, Theorem A-C, (for $p=2$), and GK98, Theorem 3.1 and Theorem 4.1, (for $p>1$), necessary and sufficient conditions are given ensuring $\mathcal{G}_0^p(\bar{\mathcal{M}}_{[0,T]})$ to be closed. These results imply

Proposition 6.3. *If $\mathcal{G}_0^p(\bar{\mathcal{M}}_{[0,T]})$ is closed, then*

$$\mathcal{V}(p,0,T,\mathcal{SF}^p(\bar{\mathcal{M}}_{[0,T]}))=\mathcal{V}(p,0,T,\mathcal{G}^p(\bar{\mathcal{M}}_{[0,T]})), \tag{6.4}$$

and $V^{opt,p,0,T}$ can be obtained by a self-financing hedging strategy in $\mathcal{G}^p(\bar{\mathcal{M}}_{[0,T]})$. Furthermore, for $0\le t\le T$, the optimal pair for the market $\bar{\mathcal{M}}_{[t,T]}$ is given by

$$\left(\frac{V^{opt,p,0,T}_{t\vee\cdot}}{V^{opt,p,0,T}_t},\frac{Z^{opt*,q,0,T}_{t\vee\cdot}}{Z^{opt*,q,0,T}_t}\right)\in\mathcal{G}^p(\bar{\mathcal{M}}_{[t,T]})\times\mathcal{D}^q(\bar{\mathcal{M}}_{[t,T]}). \tag{6.5}$$

7. The BSDE Approach

In this section we will put to use Proposition 3.1 in a general setting. Assume the existence of a continuous local martingale N orthogonal to M such that (M, N) has the local martingale representation property and $[N, N] = \tilde{C} \cdot \alpha$. Since the case $p = 0$ is already solved (see Remark 4.3) we assume in this section $p \neq 0, 1$. Let $0 \leq t \leq T$ be fixed.

Consider the following formal calculation for the optimal solution $V^{opt,p,t,T}$ for a maximization problem of terminal utility in the market $\bar{\mathcal{M}}_{[t,T]}$ and an arbitrary attainable $\frac{Y}{B_T} \in \mathcal{A}_0^{*p}(\mathcal{M}_{[t,T]})$:

$$U(V_T^{opt,p,t,T} + Y) \leq U(V_T^{opt,p,t,T}) + U'(V_T^{opt,p,t,T})Y, \tag{7.1}$$

implies

$$E_t\left[B_T U'(V_T^{opt,p,t,T})\frac{Y}{B_T}\right] = 0, \tag{7.2}$$

since kY is attainable for all $\mathcal{F}_t$-measurable random variables k. Hence $B_T U'(V_T^{opt,p,t,T})$ should define an absolutely continuous martingale measure up to normalization. In general this argument breaks down because of integrability problems. However, for isoelastic utility with exponent $p > 1$ this approach works. None the less, we can try the following ansatz:

$$c_t B_T U'(V_T^{opt,p,t,T}) = \mathcal{E}_t(\lambda \cdot M + \nu \cdot N)_T, \tag{7.3}$$

respectively

$$\ln\left(\left(c_t B_T U'\left(V_T^{opt,p,t,T}\right)\right)^{-1} \mathcal{E}_t(\lambda \cdot M + \nu \cdot N)_T\right) = 0, \tag{7.4}$$

where $V^{opt,p,t,T} = \left(V^{(\hat{H})}\right)^T$ for $\hat{H} = \left(H, \frac{1-HS}{B}\right)^{[t,T]} \in L^2_{loc}(\bar{S})$ and $\nu \in L^2_{loc}(N)$. Ansatz (7.3) leads to a FBSDE. For the isoelastic utility functions ansatz (7.4) will lead to a BSDE, where (H, ν) form part of the solution. For $t \leq s \leq T$ define the adapted process

$$Y_s^{p,t,T} := \ln\left(\left(c_t B_s \frac{dU^{(p)}}{dx}\left(V_s^{opt,p,t,T}\right)\right)^{-1} \mathcal{E}_t(\lambda \cdot M + \nu \cdot N)_s\right). \tag{7.5}$$

Applying Itô's formula and by the definition of the stochastic exponential we find

$$\begin{aligned} Y_T^{p,t,T} &= Y_t^{p,t,T} + \int_t^T dY_s^{p,t,T} \\ &= Y_t^{p,t,T} + \int_t^T \frac{(p-1)H_sC_sH_s - \nu_s\tilde{C}_s\nu_s - \lambda_sC_s\lambda_s}{2} d\alpha_s \\ &\quad + \int_t^T \left((p-1)H_sC_s\lambda_s - pr_s\right) d\alpha_s \\ &\quad + \int_t^T \left(\lambda_s - (p-1)H_s\right) dM_s + \int_t^T \nu_s dN_s. \end{aligned}$$

Because of Proposition 3.2 and the formulas (3.16) and (3.17) we expect $Y^{p,t,T}$ to be independent of t, hence we arrive at the following BSDE for $t \le t' \le T$:

$$\begin{aligned} Y_{t'}^{(p,T)} &= -\int_{t'}^T \frac{(p-1)H_sC_sH_s - \nu_s\tilde{C}_s\nu_s - \lambda_sC_s\lambda_s}{2} d\alpha_s \\ &\quad - \int_{t'}^T \left((p-1)H_sC_s\lambda_s - pr_s\right) d\alpha_s \\ &\quad - \int_{t'}^T \left(\lambda_s - (p-1)H_s\right) dM_s - \int_{t'}^T \nu_s dN_s. \end{aligned} \tag{7.6}$$

Conversely, given an adapted solution $(Y^{(p,T)}, H, \nu)$ to the BSDE (7.6) on $[t,T]$, we can define a self-financing hedging strategy in $\bar{\mathcal{M}}_{[t,T]}$ by using

$$\hat{H} := \left(H, \frac{1 - HS}{B}\right)^{[t,T]} \tag{7.7}$$

as a generator for

$$V^{(\hat{H})} := \mathcal{E}_t\left((r - HC\lambda)\cdot\alpha + H\cdot M\right)^T \in \mathcal{SF}^0(\bar{\mathcal{M}}_{[t,T]}). \tag{7.8}$$

We also have

$$Z^\nu := \mathcal{E}_t(\lambda\cdot M + \nu\cdot N)^T \in \mathcal{D}(\bar{\mathcal{M}}_{[t,T]}). \tag{7.9}$$

Lemma 7.1. *$V_T^{(\hat{H})}$ and Z_T^ν satisfy (3.8):*

$$Z_T^\nu = \frac{\exp\left(-Y_t^{(p,T)}\right)}{B_t|p|} B_T sgn(1-p) \frac{dU^{(p)}}{dx}\left(V_T^{(\hat{H})}\right), \tag{7.10}$$

Proof. Observe

$$\begin{aligned}
1 &= \exp\left(\int_t^T \frac{(p-1)H_sC_sH_s - \nu_s\tilde{C}_s\nu_s - \lambda_sC_s\lambda_s}{2}d\alpha_s\right) \\
&\quad \times \exp\left(\int_t^T (p-1)H_sC_s\lambda_s - pr_sd\alpha_s\right) \\
&\quad \times \exp\left(Y_t^{(p,T)} + \int_t^T \lambda_s - (p-1)H_sdM_s + \int_t^T \nu_sdN_s\right),
\end{aligned}$$

which implies

$$\begin{aligned}
&\mathcal{E}_t(\lambda\cdot M + \nu\cdot N)_T \\
&= \exp\left(\int_t^T \frac{(1-p)H_sC_sH_s}{2} + (1-p)H_sC_s\lambda_s + pr_sd\alpha_s\right) \\
&\quad \times \exp\left(-Y_t^{(p,T)} + \int_t^T (p-1)H_sdM_s\right) \\
&= \left(\exp\left(\int_t^T r_s - H_sC_s\lambda_s - \frac{H_sC_sH_s}{2}d\alpha_s\right)\right)^{p-1} \\
&\quad \times \mathcal{E}_t(r\cdot\alpha)_T \exp\left(-Y_t^{(p,T)}\right)\left(\exp\left(\int_t^T H_sdM_s\right)\right)^{p-1} \\
&= \frac{\exp\left(-Y_t^{(p,T)}\right)}{B_t}B_T\mathcal{E}_t\left((r - HC\lambda)\cdot\alpha + H\cdot M\right)_T^{p-1} \\
&= \frac{\exp\left(-Y_t^{(p,T)}\right)}{B_t|p|}B_T\ \mathrm{sgn}(1-p)\frac{dU^{(p)}}{dx}\left(V_T^{(\hat{H})}\right).
\end{aligned}$$

□

Proposition 7.2. *Assume* $(Y^{(p,T)}, (H,\nu))$ *to be a solution to the BSDE (7.6) on* $[t,T]$. *Define* $\hat{H}$, *resp.* $V^{(\hat{H})}$, Z^ν *by (7.7), resp. (7.8), (7.9). If for* $p<1$, $\mathcal{E}_t\left((H+\lambda)\cdot M + \nu\cdot N\right)^T$ *is a uniformly integrable martingale, respectively if for* $p>1$, $V^{(\hat{H})}\hat{H} \in \mathcal{SF}_t^p(\bar{\mathcal{M}}_{[t,T]})$, *then* $\left(V^{(\hat{H})}, Z^\nu\right)$

is the optimal pair for the market $\bar{\mathcal{M}}_{[t,T]}$ *with respect to optimization in* $\mathcal{SF}^p_t(\bar{\mathcal{M}}_{[t,T]})$. *Furthermore we have*

$$\mathcal{V}(p,t,T) = -\mathrm{sgn}(q)\exp(Y_t^{(p,T)}) = -\mathrm{sgn}(q)\exp(\mathcal{Y}(p,t,T)). \tag{7.11}$$

Proof. The first assertion follows from Proposition 3.1, (7.11) follows from (3.13). □

Conversely, the existence of an optimal pair for the market $\bar{\mathcal{M}}_{[t,T]}$ together with the local martingale representation property of (M,N), implies the existence of a solution $(Y^{(p,T)},(H,\nu))$ for the BSDE (7.6) on $[t,T]$ satisfying the assumption of Proposition 7.2.

8. Markovian Market Model

As an example, we will transform in this section the BSDE (7.6) for a (for simplicity time-homogeneous) markovian market model into a non-linear partial differential equation with boundary condition.

Consider the following market model: Assume the existence of a $(m+m')$-dimensional Brownian motion $W=(W^1,W^2)$ on Ω_∞ and assume $\mathcal{F}$ to be generated by W. For simplicity, let $\hat{\mu}=(\mu,\mu') : \mathbb{R}^{d+d'} \to \mathbb{R}^{d+d'}$ and $\sigma : \mathbb{R}^{d+d'} \to \mathbb{R}^{(d+d')\times(m+m')}$ be smooth uniformly bounded functions with uniformly bounded derivatives of all orders, such that for all $x \in \mathbb{R}^{d+d'}$, $\sigma\sigma^*(x) : \mathbb{R}^{d+d'} \to \mathbb{R}^{d+d'}$ is invertible with uniformly in x bounded inverse. Furthermore, assume $\sigma\sigma^*(x) = C(x)\times C'(x) : \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^d \times \mathbb{R}^{d'}$. Then there exists a $\mathbb{R}^{d+d'}$-valued Markov process $X=(S,S')$ solving the SDE for $x_0 \in \mathbb{R}^{m+m'}$

$$dX_t = \hat{\mu}(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0. \tag{8.1}$$

Denote by M the martingale part of S, and by N the martingale part of S'. Note that M and N are orthogonal. Assume the interest rate r to be a bounded function of X, and define $B_t := \exp\left(\int_0^t r(X_s)ds\right)$ for all $t \geq 0$. Set $\lambda := C^{-1}(\mu - rS)$ and $\beta^2 := \lambda C \lambda$. We now interpret $\bar{S} := (S,B)$ as a price process and S' as (non-traded) state variables.

Consider the following non-linear PDE for $Y : \mathbb{R}^d \times \mathbb{R}^{d'} \times [0,\infty) \to \mathbb{R}$, $(p \neq 1)$:

$$-\frac{\partial Y}{\partial t} + \mathcal{L}_1 Y + \mathcal{L}_2 Y = \mathcal{L}_3 Y + \mathcal{L}^{(p)} Y + q\frac{\beta^2}{2} - pr,$$

with boundary condition $Y(s,s',0)=0,\ \forall\, s,s'$, where

$$\begin{aligned}\mathcal{L}_1 &:= \sum_{i=1}^{d}\mu_i\frac{\partial}{\partial s_i}+\frac{1}{2}\sum_{i,j=1}^{d}C_{i,j}\frac{\partial^2}{\partial s_i\partial s_j}\\ \mathcal{L}_2 &:= \sum_{i=1}^{d'}\mu'_i\frac{\partial}{\partial s'_i}+\frac{1}{2}\sum_{i,j=1}^{d'}C'_{i,j}\frac{\partial^2}{\partial s'_i\partial s'_j},\end{aligned}$$

and for $f\in C^{1,1}(\mathbb{R}^d\times\mathbb{R}^{d'}\times[0,\infty))$,

$$\begin{aligned}\mathcal{L}_3 f &:= -\frac{1}{2}\sum_{i,j=1}^{d'}\frac{\partial f}{\partial s'_i}C'_{i,j}\frac{\partial f}{\partial s'_j}\\ \mathcal{L}^{(p)} f &:= \frac{1}{2(p-1)}\sum_{i,j=1}^{d}\frac{\partial f}{\partial s_i}C_{i,j}\frac{\partial f}{\partial s_j}-q\sum_{i,j=1}^{d}\frac{\partial f}{\partial s_i}C_{i,j}\lambda_j.\end{aligned}$$

Assume $Y^{(p)}\in C^{2,2,1}(\mathbb{R}^d\times\mathbb{R}^{d'}\times[0,\infty))$ to be a solution of the PDE (8.2), satisfying the boundary condition $Y^{(p)}(\cdot,\cdot,0)=0$. Applying Itô's formula to the process $Y_t^{(p,T)}:=Y^{(p)}(S_t,S'_t,T-t)$ we find

$$\begin{aligned}&\left(Y^{(p,T)},\left(H^{opt,p,T},\nu^{opt*,q,T}\right)\right):=\\ &\left(Y^{(p,T)},\left(\frac{\lambda(S_\cdot,S'_\cdot)-\frac{\partial Y^{(p)}}{\partial s}(S_\cdot,S'_\cdot,T-\cdot)}{p-1},\frac{\partial Y^{(p)}}{\partial s'}(S_\cdot,S'_\cdot,T-\cdot)\right)\right),\end{aligned}\tag{8.2}$$

to be a solution for the BSDE (7.6). We give (admittedly quite strong and not easy to check) conditions, ensuring $\left(Y^{(p,T)},(H,\nu)\right)$ to be a useful solution:

Theorem 8.1. *If for $p>1$, $\mathcal{E}(\lambda\cdot M+\nu^{opt*,q,T}\cdot N)_T\in L^{q+\epsilon}(\Omega_{[0,T]})$ for an $\epsilon>0$, resp. if for $p<1, p\neq 0$, $\mathcal{E}\left((\lambda+H^{opt,p,T})\cdot M+\nu^{opt*,p,T}\cdot N\right)^T$ is a uniformly integrable martingale, then for all $0\le t\le T$*

$$V^{opt,p,t,T}=\mathcal{E}_t\left(\left(r-H^{opt,p,T}C\lambda\right)\cdot\alpha+H^{opt,p,T}\cdot M\right)^T,\tag{8.3}$$

and

$$Z^{opt*,q,t,T}=\mathcal{E}_t(\lambda\cdot M+\nu^{opt*,q,T}\cdot N)^T.\tag{8.4}$$

Proof. The assertion follows directly from Lemma 3.3 and Proposition 7.2. □

Remark 8.2. Because of Theorem B in DMSSS97, resp. Theorem 4.1 in GK98, and Theorem 2.15 in DMSSS97, the condition for $p > 1$ in the above theorem is more natural than it might appear at the first moment.

Remark 8.3. For constant $q\frac{\beta^2}{2} - pr$ one easily finds $Y^{(p)}(s, s', t) := -t\left(q\frac{\beta^2}{2} - pr\right)$ to be a solution of the PDE (8.2) satisfying the conditions of Theorem 8.1. This also proves that in this case the q-optimal martingale measure equals the minimal martingale measure, $Z^{opt*,p,t,T} = \mathcal{E}_t(\lambda \cdot M)$, see Föllmer and Schweizer (1990).

Remark 8.4. For $p = 2$, under the assumption of Theorem 8.1, we can construct the hedging numéraire explicitly. In Laurent and Pham (1999), Section 6, this is achieved under stronger assumptions on $\beta^2 - 2r$. See also Leitner (2000) for an application to the mean-variance efficiency problem and the calculation of the intertemporal price for risk.

9. Conclusions

One of the problems in the probabilistic theory of finance is, that one has to make an ansatz for the probability law of the future price processes. In a constantly changing world it is not at all clear how such

a law could ever be determined. Without knowing the actual law (it is even difficult to argue that such a law exists, since it is not clear how we can speak of probabilities in experiments which can not be repeated) it is difficult to define what we mean by an optimal hedging strategy.

In a more pragmatic approach one could try to parameterize a class of reasonable laws and look for parameters such that the implied prices fit the observed prices best. If observed and implied prices differ, or if the estimated law differs from the law an investor believes prices to follow, then the investor should buy underpriced and sell overpriced stocks until risk aversion, the trust in the used model and the confidence to be more clever than the market, are in a balance. In our model this would involve solving parameterized (F)BSDEs and estimating good parameters. In a discrete time model this can be achieved using a backward iterative algorithm, but would be computational very expensive, since observed prices have to be compared with implied prices for a large number of parameter values. For a markovian model as in Section 8, we would still have to solve a (parameterized) non-linear PDE.

Another problem in the probabilistic theory of finance is, that whenever a new, i.e. non attainable, stock or security is introduced to the market, then the market data C, λ, β and the optimal pairs $V^{opt,p,t,T}$, $Z^{opt*,p,t,T}$ will change in general.

We have shown, that the problem of maximizing isoelastic utility from terminal wealth is significantly facilitated if the price for instantaneous risk is assumed to be totally unhedgeable. In this situation optimal portfolios can be constructed from locally efficient portfolios and there holds a two-fund-theorem for all investors maximizing isoelastic utility of terminal wealth. The big advantage of locally efficient portfolios is that they can be determined by estimating the local characteristics of the price process. We only have to assume that prices follow some probability law (that allows for successful estimation of local characteristics), but we don't have to decide for a specific law.

References

Aase, K. K. (1984): Optimum Portfolio Diversification in a General Continuous-Time Model. *Stochastic Processes and their Applications* 18, 81-98.

Aase K. K. (1986): Ruin Problems and Myopic Portfolio Optimization in Continuous Trading. *Stochastic Processes and their Applications* 21, 213-227.

Aase K. K. (1987): Stochastic Control of Geometric Processes. *J. Appl. Prob.* 24, 97-104.

Aase K. K. (1988): Admissible Investment Strategies in Continuous Trading. *Stochastic Processes and their Applications* 30, 291-301.

Ansel, J. P. and C. Stricker (1992): Lois de martingale, densités et décomposition de Föllmer Schweizer. *Ann. Inst. H. Poincaré* 28, 375-392.

Arrow, K. J. (1976): *Essays in the Theory of Risk-Bearing*. Amsterdam: North-Holland Publishing Company.

Bielecki, T. R. and S. R. Pliska (1999): Risk-Sensitive Dynamic Asset Management. *Appl. Math. Optim.* 39, 337-360.

Bielecki, T. R. and S. R. Pliska (2000): A Risk Sensitive Intertemporal CAPM, With Application to Fixed Income Management. *Workingpaper*.

Bismut, J. M. (1973): Conjugate convex functions in optimal stochastic control, *J. Math. Anal. Appl.* 44, 384-404.

Bismut, J. M. (1975): Growth and optimal intertemporal allocations of risk, *J. Economic Theory* 10, 239-287.

Breiman, L. (1960): The Capital Growth Critirion and Continuous-Time Models. *Naval Research Logistics Quarterly* 7, 647-651.
(Reprinted in Ziemba and Vickson (1975)).

Delbaen, F. and W. Schachermayer (1994): A general version of the fundamental theorem of asset pricing. *Math. Ann.* 300, 463-520.

Delbaen, F. and W. Schachermayer (1996): Attainable claims with p'th moments, *Ann. Inst. Henri Poincaré* 32(6), 743-763.

Delbaen, F., P. Monat, W. Schachermayer, M. Schweizer and C. Stricker (1997): Weighted norm inequalities and hedging in incomplete markets, *Finance and Stochastics* 1, 181-227.

Foldes, L. (1991): Optimal Sure Portfolio Plans. *Mathematical Finance* 1(2), 15-55.

Föllmer, H. and M. Schweizer (1990): Hedging of continuous claims under incomplete information. In M. Davis and R. Elliot (eds.), *Applied Stochastic Analysis*. pp. 389-414. London and New York: Gordon and Breach.

Geman, H., N. El Karoui and J. Rochet (1995): Changes of Numéraire, Changes of Probability Measure and Option Pricing, *J. Appl. Prob.* 32, 443-458.

Goll, T. and J. Kallsen (2000): Optimal portfolios for logarithmic utility. *Stoch. Proc. Appl.* 89, 31-48.

Gourieroux, C., J. P. Laurent and H. Pham (1998): Mean-Variance Hedging and Numéraire. *Mathematical Finance* 8(3), 179-200.

Grandits, P. and L. Krawczyk (1998): Closedness of some spaces of stochastic integrals. In J. Azéma, M. Émery, M. Ledoux and M. Yor

(eds.), *Séminaire de Probabilités XXXII*, 73-85. Berlin, New York: Springer.

Harrison, J. and R. Pliska (1981): Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stoch. Proc. Appl.* 11, 215-260.

He, H. and N. D. Pearson (1991): Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite dimensional case. *Mathematical Finance* 1, 1-10.

He, H. and N. D. Pearson (1991): Consumption and portfolio policies with incomplete markets and short-sale constraints: the infinite dimensional case. *Journal of Economic Theory* 54, 259-304.

Jacod, J. (1979): *Calcul Stochastique et Problèmes de Martingales*. Berlin New York: Springer-Verlag.

Jacod, J. and A. N. Shiryaev (1987): *Limit Theorems for Stochastic Processes*. New York: Springer-Verlag.

Karatzas, I., J. P. Lehoczky, S. E. Shreve and G.-L. Xu (1991): Martingale and Duality Methods for Utility Maximization in an Incomplete Market. *SIAM Journal of Control and Optimization* 29 (3), 702-730.

Karatzas, I. and S. E. Shreve (1999): *Methods of Mathematical Finance*. New York: Springer-Verlag.

Kelly, J. L. (1956): A new interpretation of the information rate. *Bell System Technical Journal* 35, 917-926.

Kramkov, D. and W. Schachermayer (1999): The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets. *Annals of Applied Probability* 9(3), 904-950.

Laurent, J. P. and H. Pham (1999): Dynamic programming and mean-variance hedging. *Finance and Stochastics* 3, 83-110.

Leland, H. E. (1972): On Turnpike Portfolios. In *Mathematical Methods in Investment and Finance*. Szegö, G. P. and K. Shell, eds. Amsterdam: North-Holland Publishing Company. p. 24-33.

Leitner, J. (2000): Mean-Variance Efficiency and Intertemporal Price for Risk. Discussion Paper Series, CoFE, Nr. 00/35.

Markowitz, H. M. (1952): Portfolio Selection, *Journal of Finance* 7, 77-91.

Markowitz, H. M. (1987): *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. Oxford: Basil Blackwell.

Merton, R. C. (1990): *Continuous-Time Finance*. Oxford: Basil Blackwell.

Mossin, J. (1968): Optimal Multiperiod Portfolio Policies. *The Journal of Business*, 215-229

Pliska S. R. (1986): A Stochastic Calculus Model of Continuous Trading: Optimal Portfolios. *Mathematics of Operations Research* 11(2), 371-382.

Pratt, J. W. (1964): Risk aversion in the small and in the large, *Econometrica* 32, 122-136.

Pham, H., T. Rheinländer and M. Schweizer (1998): Mean-variance hedging for continuous processes: New proofs and examples. *Finance and Stochastics* 2, 173-198.

Rockafellar, R. T. (1970): *Convex Analysis*. Princeton: Princeton University Press.

Schachermayer, W. (2000): Optimal Investment in Incomplete Markets when Wealth may become negative. *Workingpaper*.

Sharpe, W. F. (1964): Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance* 19, 425-442.

Sharpe, W. F. (2000): *Portfolio Theory and Capital Markets*. New York: McGraw-Hill.

Yong, J. and X. Y. Zhou (1999): *Stochastic controls: Hamiltonian systems and HJB equations*. Berlin, New York: Springer-Verlag.

Ziemba, W. T. and R. G. Vickson (1975): *Stochastic Optimization Models in Finance*. New York: Academic Press.