Kohlmann, Michael; Shanjian, Tang

Working Paper

Optimal Control of Linear Stochastic Systems with Singular Costs, and the Mean-Variance Hedging Problem with Stochastic Market Conditions

CoFE Discussion Paper, No. 00/13

Provided in Cooperation with:
University of Konstanz, Center of Finance and Econometrics (CoFE)

Suggested Citation: Kohlmann, Michael; Shanjian, Tang (2000) : Optimal Control of Linear Stochastic Systems with Singular Costs, and the Mean-Variance Hedging Problem with Stochastic Market Conditions, CoFE Discussion Paper, No. 00/13, University of Konstanz, Center of Finance and Econometrics (CoFE), Konstanz,
https://nbn-resolving.de/urn:nbn:de:bsz:352-opus-4963

This Version is available at:
https://hdl.handle.net/10419/85170

Optimal Control of Linear Stochastic Systems with Singular Costs, and the Mean-Variance Hedging Problem with Stochastic Market Conditions

Michael Kohlmann*
Department of Mathematics and Statistics
University of Konstanz, D-78457, Konstanz, Germany

Shanjian Tang*, †
Department of Mathematics
and the Laboratory of Mathematics
for Nonlinear Sciences at Fudan University,
Fudan University, Shanghai 200433, China

May 11, 2000

Abstract

The optimal control problem is considered for linear stochastic systems with a singular cost. A new uniformly convex structure is formulated, and its consequences on the existence and uniqueness of optimal controls and on the uniform convexity of the value function are proved. In particular, the singular quadratic cost case with random coefficients is discussed and the existence and uniqueness results on the associated *nonlinear singular backward stochastic Riccati differential equations* are obtained under our structure conditions, which generalize Bismut-Peng's existence and uniqueness on *nonlinear regular backward stochastic Riccati equations* to *nonlinear singular backward stochastic Riccati equations*. Finally, applications are given to the mean-variance hedging problem with random market conditions, and an explicit characterization for the optimal hedging portfolio is given in terms of the adapted solution of the associated backward stochastic Riccati differential equation.

Key words: singular optimal stochastic control, linear quadratic stochastic control with random coefficients, nonlinear singular backward stochastic Riccati differential equation, existence and uniqueness, mean-variance hedging

*Both authors gratefully acknowledge the support by the Center of Finance and Econometrics, University of Konstanz.

†Supported by a Research Fellowship from the Alexander von Humboldt Foundation and by the National Natural Science Foundation of China under Grant No. 79790130.

AMS Subject Classification. 49N10, 91B28, 93C05

Abbreviated title: Optimal Stochastic Control with a Singular Cost

1 Introduction

In this paper, we consider the optimal control problem of the following linear stochastic system

$$\begin{cases} dX(t) &= [A(t)X(t) + B(t)u(t)]\,dt + \sum_{i=1}^{d}[C_i(t)X(t) + D_i(t)u(t)]\,dw_i(t), \\ X(0) &= x, \quad u(t) \in R^m \end{cases} \tag{1}$$

under the following singular cost

$$J(u;0,x) = EM(X(T)) + E\int_0^T \widetilde{M}(s,X(s))\,ds. \tag{2}$$

Here, $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed complete probability space on which is defined a standard $\mathcal{F}_t$-adapted d-dimensional Brownian motion $w(t) \equiv (w_1(t), \cdots, w_d(t))^*$. Assume that $\mathcal{F}_t$ is the completion, by the totality $\mathcal{N}$ of all null sets of $\mathcal{F}$, of the natural filtration $\{\mathcal{F}_t^w\}$ generated by w. We assume that $M(x)$ is $\mathcal{F}_T$-measurable and uniformly convex in x, and $\widetilde{M}(t,x)$ is $\mathcal{F}_t$-measurable and convex in x. Denote by $\{\mathcal{F}_t^2, 0 \leq t \leq T\}$ the P-augmented natural filtration generated by the $(d-d_0)$-dimensional Brownian motion $(w_{d_0+1}, \ldots, w_d)$. Assume that all the coefficients A, B, C_i, D_i are $\mathcal{F}_t^2$-progressively measurable bounded matrix-valued processes, defined on $\Omega \times [0,T]$, of dimensions $n\times n, n\times m, n\times n, n\times m$ respectively. $X(t)$ stands for the state of the system at time t and $u(\cdot)$ the whole control action imposed to the system, which is required to take values in a previously given nonempty closed convex subset U of the m-dimensional Euclidean space R^m and to be adapted to the previously prescribed filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$.

A new feature of our problem is that the cost J is singular, that is it does not explicitly depend on the control variable u while the admissible control values are possibly unbounded in the n-dimensional Euclidean space R^n. A problem is said to be well-posed, if and only if it has unique solution. The well-posedness of the singular stochastic optimal control problem is concerned. On one hand, in the deterministic case, i.e. when $C_i = 0$ and $D_i = 0$, the above singular control problem is not well-posed in general.

Example 1. (no optimal control) A typical example is to consider the case of

$$n = 1, A = B = 1, M(x) = \widetilde{M}(x) = x^*x, U = R.$$

In this example, for $\forall x \neq 0$, the optimal control problem has no solution. To see this, consider the constant feedback law

$$u = -kX$$

whose corresponding state process is given by $X(t) = x\exp(-(k-1)t), 0 \leq t \leq T$. Therefore its value of the cost functional

$$J(-kX;0,x) = x^2\exp(-2(k-1)T) + \frac{x^2}{2(k-1)}(1-\exp(-2(k-1)T))$$

converges to zero as $k \to \infty$. If the optimal control problem has a solution $\widehat{u}$, then one should have $J(\widehat{u}; 0, x) = 0$, and therefore $X(t) \equiv 0$ for $t \in [0, T]$. This contradicts $X(0) = x \neq 0$.

Example 2. (multiple optimal controls) Consider the case

$$n = 1, A = B = 1, M(x) = x^* x, \widetilde{M}(x) = 0, U = R.$$

In this example, for $\forall x \in R$, the number of optimal controls is infinite: all those controls which shift the state of the system to zero at the terminal time T are optimal!

On the other hand, the above singular model is indeed used in financial economics and the relevant financial problems are well-posed. It is used to formulate the mean-variance hedging problem, for example. Actually, the latter observation motivates us to study the above optimal stochastic control problem with a singular cost.

From the above comments, we see that in order to make our problem well-posed it is necessary to make some assumption which excludes the degenerate case. In fact, we assume that for some positive $\varepsilon > 0$

$$\sum_{i=1}^{d} D_i(t)^* D_i(t) \geq \varepsilon I. \tag{3}$$

We shall call it *the nondegeneracy condition.*

The reader will see later that the above nondegeneracy condition is satisfied in the mean-variance hedging problem.

Theoretically, it is also interesting that the above nondegeneracy condition together with the uniform convexity assumption of the cost in state variable implies a uniform convexity of the cost functional both in control processes and in initial states, which leads in a straightforward way to the existence and uniqueness of optimal controls and the uniform convexity of the value function in the state variable x. The proofs given here are simple applications of the theory of **backward stochastic differential equations** (**BSDEs** in short form), developed by Pardoux and Peng [15].

When the cost function is a positive quadratic form of the terminal state, then our problem becomes a singular linear quadratic stochastic control problem, called a singular stochastic LQ problem. It is associated with a *nonlinear singular* **backward stochastic Riccati differential equation** (**BSRDE** in short form) of the following form

$$\begin{cases} dK(t) &= -[A^*K(t) + K(t)A + \sum_{i=1}^{d} C_i^* K(t) C_i + \sum_{i=d_0+1}^{d} (C_i^* L_i(t) + L_i(t) C_i) \\ &\quad -(K(t)B + \sum_{i=1}^{d} C_i^* K(t) D_i + \sum_{i=d_0+1}^{d} L_i(t) D_i) \\ &\quad \times (\sum_{i=1}^{d} D_i^* K(t) D_i)^{-1} (K(t)B + \sum_{i=1}^{d} C_i^* K(t) D_i + \sum_{i=d_0+1}^{d} L_i(t) D_i)^*] \, dt \\ &\quad + \sum_{i=d_0+1}^{d} L_i(t) \, dw_i(t), \qquad 0 \leq t < T, \\ K(T) &= Q. \end{cases} \tag{4}$$

It should be noted that the above differential equation has singularities in that the inverse of the unknown matrix K is involved. For a *nonlinear regular* **BSRDE**, which has the following form

$$\begin{cases} dK(t) &= -[A^*K(t) + K(t)A + \sum_{i=1}^{d} C_i^*K(t)C_i + \sum_{i=d_0+1}^{d} (C_i^*L_i(t) + L_i(t)C_i) \\ &\quad -(K(t)B + \sum_{i=1}^{d} C_i^*K(t)D_i + \sum_{i=d_0+1}^{d} L_i(t)D_i) \\ &\quad \times(N + \sum_{i=1}^{d} D_i^*K(t)D_i)^{-1}(K(t)B + \sum_{i=1}^{d} C_i^*K(t)D_i + \sum_{i=d_0+1}^{d} L_i(t)D_i)^*]\,dt \\ &\quad + \sum_{i=d_0+1}^{d} L_i(t)\,dw_i(t), \qquad 0 \le t < T, \\ K(T) &= Q. \end{cases} \tag{5}$$

with $N > 0$, an existence and uniqueness result was obtained by Wonham [25] for the case of deterministic coefficients (that is $d_0 = d$), by Bismut [2] for the case of random coefficients with the assumption that $C_{d_0+1} = \cdots = C_d = 0$ and $D_{d_0+1} = \cdots = D_d = 0$, and by Peng [18] for the case of random coefficients with the assumption that $D_{d_0+1} = \cdots = D_d = 0$. The literature on the study of the *nonlinear singular* **BSRDE** (4) is restricted to the framework of deterministic coefficients (that is $d_0 = d$, and (4) is a deterministic Riccati differential eqution) and moreover the assumption that $C_1 = \cdots = C_d = 0$ is made. Even in that quite restricted framework, there are only few positive results: the existence and uniqueness result is obtained by Kohlmann and Zhou [12] for the case of $D_i = I, i = 1, \ldots, d$, under the following additional assumption:

$$A(t) + A^*(t) \ge BB^*(t). \tag{6}$$

The arguments in [12] are based on a result of Chen, Li and Zhou [3].

In this paper, we obtain the existence and uniqueness result on *nonlinear singular* **BSRDE** for a general $C := (C_1, \ldots, C_d)$ and those $D := (D_1, \ldots, D_d)$ which satisfy both the nondegeneracy assumption (3) and the condition that

$$D_{d_0+1} = \cdots = D_d = 0. \tag{7}$$

In our framework, the coefficients are allowed to be random, and the assumption (6) and the condition that $C_1 = \cdots = C_d = 0$ in Kohlmann and Zhou [12] are dispensed with. We use a regular approximating method, and the arguments given here are based on our new observation in Theorem 2.2 (the uniform convexity of the value function, see also Lemma 3.2), which *a priori* asserts that the symmetric matrix associated with the value function of the singular optimal stochastic control problem under consideration is uniformly positive. That new observation enables us to pass a limit in the approximating regular **BSRDEs**. In this way the discussion on singular **BSRDEs** becomes unified and straightforward.

As an application of the above results, the mean-variance hedging problem with random market conditions is considered. The mean-variance hedging problem was initially introduced by Föllmer and Sondermann [5], and later widely studied by Duffie and

Richardson [4], Föllmer and Schweizer [6], Schweizer [21, 22, 23], Hipp [9], Monat and Stricker [14], Pham, Rheinländer and Schweizer [20], Gourieroux, Laurent and Pham [7], and Laurent and Pham [13]. All of these works are based on a projection argument. Recently, Kohlmann and Zhou [12] used a natural LQ theory approach to solve the case of deterministic market conditions. In this paper, the case of random market conditions is solved by using the above results, and the optimal hedging portfolio is characterized by the solution of the associated **BSRDE**. The variance-optimal martingale measure is also characterized in terms of the solution of the associated **BSRDE**. Note that the application of a convex duality method and the dynamic programming principle to finance is well-known by now and the reader is referred to Karatzas and Shreve [11].

The rest of the paper is organized as follows. In Section 2, the optimal control problem is studied for linear stochastic systems with a general singular cost. The nondegenerate assumption (3) on the system and the uniformly convex assumption on the cost function are made. The coefficients of the problem are allowed to be random. The results given here contain the exposition of an implicit uniformly convex structure, the existence and uniqueness for optimal controls, and the uniform convexity of the value function in the state variable. As a particular but important case, the case of a quadratic cost, that is the singular stochastic LQ problem is discussed in Section 3. Via a regular approximating approach, the existence and uniqueness is proved for the adapted solution to the associated nonlinear singular **BSRDE** (4), and in terms of this solution the optimal control is expressed as a closed form. For convenience of subsequent application, a detailed solution to the nonhomogenious singular stochastic LQ problem is given. To illustrate the above results, the mean-variance hedging problem with stochastic market conditions is solved in Section 4, and the natural LQ theory approach is connected to the *artificial hedging numeraire* method of Gourieroux et al [7]. Finally some concluding comments are made in Section 5.

2 Optimal Control of Linear Stochastic Systems with Singular Costs

2.1 Formulation of the problem

Throughout this paper $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed complete probability space on which is defined a standard $\mathcal{F}_t$-adapted d-dimensional Brownian motion $w(t) \equiv (w_1(t), \cdots, w_d(t))^*$. Assume that $\mathcal{F}_t$ is the completion, by the totality $\mathcal{N}$ of all null sets of $\mathcal{F}$, of the natural filtration $\{\mathcal{F}_t^w\}$ generated by w. Denote by $\mathcal{L}^2_{\mathcal{F}}(t, T; R^m)$ the set of all R^m-valued, $\mathcal{F}_t$-adapted stochastic processes ψ on $[t, T]$ such that $E\int_t^T |\psi(s)|^2\, ds < \infty$.

Let A, B, C_i, D_i be $\mathcal{F}_t$-progressively measurable bounded matrix-valued processes, defined on $\Omega \times [0, T]$, of dimensions $n \times n, n \times m, n \times n, n \times m$ respectively, and the random variable $M(x)$ be $\mathcal{F}_T$-measurable and bounded for each $x \in R^n$. Let U be a some nonempty closed convex subset of the Euclidean space R^m.

Consider the following linear stochastic control system parameterized by the initial

data $(x,t) \in R^n \times [0,T]$:

$$\begin{cases} dX(s) &= [AX(s)+Bu(s)]\,ds + \sum_{i=1}^{d}[C_iX(s)+D_iu(s)]\,dw_i(s), \quad t<s\le T, \\ X(t) &= x. \end{cases} \tag{8}$$

An admissible control on $[t,T]$ is an $\mathcal{F}_s$-adapted process $\{u_t, t \le s \le T\}$ with values in U, such that

$$E\int_t^T |u(s)|^2\,ds < \infty.$$

Denote by $U_{ad}(t,T)$ the set of admissible controls on $[t,T]$ and Let U_{ad} be $U_{ad}(0,T)$. For given initial data (t,x) and given $u \in U_{ad}$, the above control system (8) has a unique solution X, also denoted by $X^{t,x;u}$ to indicate its dependence on the triple (x,t,u). For given initial data $(x,0)$, the optimal control problem (denoted by $\mathcal{P}_0$) is the following minimization problem:

$$\textbf{Problem } \mathcal{P}_0 \qquad J(u;0,x) := EM(X^{0,x;u}(T)) = \min_{u\in U_{ad}} !. \tag{9}$$

Following the idea of dynamic programming, consider the following associated optimal control problem parameterized by the initial data (t,x):

$$\min_{u\in U_{ad}(t,T)} J(u;t,x) \tag{10}$$

with

$$J(u;t,x) := E^{\mathcal{F}_t^2} M(X^{t,x;u}(T)). \tag{11}$$

Denote it by $\mathcal{P}_0^{t,x}$. The value function is defined as follows:

$$V(t,x) := \min_{u\in U_{ad}(t,T)} J(u;t,x). \tag{12}$$

Observe that, for any fixed x, $\{V(t,x), 0 \le t \le T\}$ is an $\mathcal{F}_t^2$-adapted real valued stochastic process.

We make the following basic hypothesis. Assume that (a) there is some $\varepsilon > 0$ such that

$$\sum_{i=1}^{d} D_i^* D_i(t) \ge \varepsilon I, \quad \forall t \in [0,T], a.s. \tag{13}$$

and that (2) the function $M : \Omega \times R^n \to R$ is uniformly convex in the second argument:

$$M(x_1) + M(x_2) - 2M(\frac{x_1+x_2}{2}) \ge \varepsilon|x_2-x_1|^2, \quad \forall x_1,x_2 \in R^n, a.s. \tag{14}$$

and satisfies

$$|M(x_1) - M(x_2)| \le \varepsilon_0(1+|x_1|+|x_2|)|x_1-x_2|, \quad \forall x_1,x_2 \in R^n, a.s. \tag{15}$$

for some positive constant ε_0.

Further notation. Throughout this paper, the following additional notaion will be used:

M^*	:	the transpose of any vector or matrix M;
$\|M\|$	:	$= \sqrt{\sum_{ij} m_{ij}^2}$ for any vector or matrix $M = (m_{ij})$;
(M_1, M_2)	:	the inner product of the two vectors M_1 and M_2;
R^n	:	the n-dimensional Euclidean space;
$\mathcal{S}^n$	:	the Euclidean space of all $n \times n$ symmetric matrices;
$\mathcal{S}_+^n$	:	the set of all $n \times n$ nonnegative definite matrices;
$C([0,T];H)$	:	the Banach space of H-valued continuous functions on $[0,T]$, endowed with the maximum norm for a given Hilbert space H;
$\mathcal{L}^2_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued $\mathcal{F}_t$-adapted square-integrable stochastic processes f on $[0,T]$, endowed with the norm $(E\int_0^T \|f(t)\|^2\,dt)^{1/2}$ for a given Euclidean space H;
$\mathcal{L}^\infty_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued, $\mathcal{F}_t$-adapted, essentially bounded stochastic processes f on $[0,T]$, endowed with the norm $\operatorname{ess\,sup}_{t,\omega} \|f(t)\|$ for a given Euclidean space H;
$L^2(\Omega,\mathcal{F},P;H)$	:	the Banach space of H-valued norm-square-integrable random variables on the probability space $(\Omega,\mathcal{F},P)$ for a given Banach space H;

and $L^\infty(\Omega,\mathcal{F},P;C([0,T];R^n))$ is the Banach space of $C([0,T];R^n)$-valued, essentially maximum-norm-bounded random variables f on the probability space $(\Omega,\mathcal{F},P)$, endowed with the norm $\operatorname{ess\,sup}_{\omega\in\Omega} \max_{0\le t\le T} |f(t,\omega)|$.

2.2 The backward feature of the problem, and the associated backward stochastic differential equation

Intuitively, it is easy to see that the singular optimal stochastic control problem under consideration has a backward structure: the optimality of the control process (and therefore the state process) is determined only by the terminal value of the corresponding state process, for the value of the cost functional is completely determined by it. Hence if the singular optimal stochastic control problem $\mathcal{P}_0$ is expected to have a unique optimal control, then it is natural to expect that the terminal value $X(T)$ of the controlled state process $X^{0,x;u}(\cdot)$ should determine, in a unique way the whole control process $u(\cdot)$ under being used, and thus determine simultaneously the whole state process $X^{0,x;u}(\cdot)$ (note that the stochastic differential equation (8) has under our setting a unique solution $X^{0,x;u}(\cdot)$ for given initial state x and given control process $u(\cdot)$). This is to say, for each $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, the following **BSDE**

$$\begin{cases} dX(t) &= [AX(t)+Bu(t)]\,dt + \sum_{i=1}^{d}[C_iX(t)+D_iu(t)(t)]\,dw_i(t), \\ X(T) &= \xi \end{cases} \tag{16}$$

should have unique $\mathcal{F}_t$-adapted solution. So, it should not be surprising that the properties of the control problem $\mathcal{P}_0$ are closely related with the properties of the above **BSDE** (16).

We have

Proposition 2.1. (the backward structure) *Assume that the nondegeneracy condition (13) is satisfied. Then for every $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the* **BSDE** *(16) has a unique $\mathcal{F}_t$-adapted solution $(X(\cdot), u(\cdot))$ with $X \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^n) \cap L^2(\Omega, \mathcal{F}_T, P; C([0,T];R^n))$ and $u \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$.*

Proof Set

$$q_i := C_i X + D_i u, \quad i = 1, \ldots, d.$$

In view of the nondegeneracy condition (13), we have

$$u_i = (\sum_{i=1}^{d} D_i^* D_i)^{-1} \sum_{i=1}^{d} D_i^* (q_i - C_i X), \quad i = 1, \ldots, d.$$

Substituting the above equation into the **BSDE** (16), we have

$$\begin{cases} dX(t) &= [\tilde{A}X(t) + \sum_{i=1}^{d} \tilde{B}_i q_i(t)]\, dt + \sum_{i=1}^{d} q_i(t)\, dw_i(t), \\ X(T) &= \xi \end{cases} \tag{17}$$

with

$$\tilde{A} := A - B(\sum_{i=1}^{d} D_i^* D_i)^{-1} \sum_{i=1}^{d} D_i^* C_i, \quad \tilde{B}_i := B(\sum_{i=1}^{d} D_i^* D_i)^{-1} D_i^*, i = 1, \ldots, d.$$

Note that $\tilde{A}$ and $\tilde{B}_i, i = 1, \ldots, d$, are uniformly bounded. Applying the theory of **BSDE** (see Pardoux and Peng [15]), we conclude that the linear **BSDE** (17) has a unique $\mathcal{F}_t$-adapted solution $(X(\cdot), u(\cdot))$ with $X \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^n) \cap L^2(\Omega, \mathcal{F}_T, P; C([0,T];R^m))$ and $q \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$. Then the existence part of Proposition 2.1 follows. The uniqueness part follows from Lemma 2.1 below.

Proposition 2.1 states a backward structure: the terminal condition determines, in a unique way, both the whole state process $X(\cdot)$ (not only the initial state $X(0)$) and the control process $u(\cdot)$. Moreover, we have the following quantitative characterization to this backward structure, which is called *a priori estimate* of the solution of the **BSDE** (16).

Lemma 2.1. (a priori estimate) *Assume that the assumption (13) is satisfied. If $X(\cdot)$ and $u(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$ satisfy (16), then there is $\beta > 0$, such that*

$$\frac{\varepsilon}{2} E^{\mathcal{F}_t^2} \int_t^T |u(s)|^2\, ds + E^{\mathcal{F}_t^2} |X(t)|^2 \leq \exp\left(\beta(T-t)\right) E^{\mathcal{F}_t^2} |X(T)|^2, \quad 0 \leq t \leq T. \tag{18}$$

Proof Using Itô's formula, we have from (16)

$$\begin{aligned}
& E^{\mathcal{F}_t^2}|X(T)|^2 \\
= \; & E^{\mathcal{F}_t^2}|X(r)|^2 + 2E^{\mathcal{F}_t^2}\int_r^T < AX(s)+Bu(s), X(s) > \, ds \\
& + E^{\mathcal{F}_t^2}\int_r^T \sum_{i=1}^d |C_iX(s)+D_iu(s)|^2\, ds \\
= \; & E^{\mathcal{F}_t^2}|X(r)|^2 + 2E^{\mathcal{F}_t^2}\int_r^T < (A+\sum_{i=1}^d C_i^*C_i)X(s), X(s) > \, ds \\
& + 2E^{\mathcal{F}_t^2}\int_r^T < (B+\sum_{i=1}^d C_i^*D_i)u(s), X(s) > \, ds \\
& + E^{\mathcal{F}_t^2}\int_r^T u^*(s)(\sum_{i=1}^d D_i^*D_i)u(s)\, ds \\
\geq \; & E^{\mathcal{F}_t^2}|X(r)|^2 + \frac{\varepsilon}{2}E^{\mathcal{F}_t^2}\int_r^T |u(s)|^2\, ds - \beta E^{\mathcal{F}_t^2}\int_r^T |X(s)|^2\, ds
\end{aligned} \tag{19}$$

for some positive constant β. Write

$$\rho_r := E^{\mathcal{F}_t^2}|X(r)|^2, \quad t \leq r \leq T. \tag{20}$$

Then, the above reads

$$\rho_t + \frac{\varepsilon}{2}E^{\mathcal{F}_t^2}\int_t^T |u(s)|^2\, ds \leq \rho_T + \beta\int_t^T \rho_s\, ds. \tag{21}$$

By Grownwall's inequality, we have

$$\rho_r \leq \exp(\beta(T-r))\rho_T, \tag{22}$$

$$\rho_t + \frac{\varepsilon}{2}E^{\mathcal{F}_t^2}\int_t^T |u(s)|^2\, ds \leq \exp(\beta(T-t))\rho_T. \tag{23}$$

This concludes the proof.

We remark that the **BSDE** was initially introduced by Bismut [1] for the linear case, and was later developed by Pardoux and Peng [15] for the nonlinear case. The general theory of **BSDE** is well-known by now. The classical form is the **BSDE** (16) with $C_i = 0$ and $D_iu = u_i$ $(i = 1, \ldots, d)$. The proof of Proposition 2.1 shows that the general form of **BSDE** (16) can be transformed into the classical form under the nondegeneracy assumption (13). The *a priori estimate* for the classical form of **BSDE** can be found in Pardoux and Peng [16].

Proposition 2.2. (the closedness of the attainable set) *Assume that the non-degeneracy assumption (13) is satisfied. Then, for every $x \in R^n$, the attainable set $\mathcal{R}(0,x;T)$ at time T from the initial point x at time 0, of the linear stochastic system (1), defined by*

$$\mathcal{R}(0,x;T) := \{X^{0,x;u}(T) : \forall u \in U_{ad}\}, \tag{24}$$

is closed in $L^2(\Omega, \mathcal{F}_T, P)$.

Proof Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of points in $\mathcal{R}(0,x;T)$, and strongly convergence to $\xi \in L^2(\Omega, \mathcal{F}_T, P)$. Then, there are admissible controls u_k such that

$$\xi_k = X^{0,x;u_k}(T), \quad k = 1, 2, \dots$$

By Proposition 2.1, we conclude that there exist $u \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$ and $\widehat{x}$ such that

$$\xi = X^{0,\widehat{x};u}(T).$$

We assert that u is admissible. In fact, the pair $(X_k - X, u_k - u)$ satisfies the **BSDE** (16) with the terminal condition being $\xi_k - \xi$. By Lemma 2.1, we have

$$\frac{\varepsilon}{2} E \int_0^T |u_k(s) - u(s)|^2 \, ds + |x - \widehat{x}|^2 \leq \exp(\beta T) E|\xi_k - \xi|^2.$$

By passing to the limit, we obtain that $\widehat{x} = x$ and that u is a limit point of U_{ad}. Hence $u \in U_{ad}$ (noting that the admissible control class U_{ad} is closed in $\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$) and ξ is attainable.

2.3 A new uniformly convex structure

Theorem 2.1. (the uniformly convex structure) *Assume that (13) and (14) are satisfied. Then, the cost functional $J(\cdot;t,\cdot)$ is uniformly convex in admissible control process u and the state variable x. In fact, we have*

$$\begin{aligned} & J(u_1;t,x_1) + J(u_2;t,x_2) - 2J(\frac{u_1+u_2}{2}; t, \frac{x_1+x_2}{2}) \\ \geq\ & \varepsilon \exp(-\beta(T-t))(\frac{\varepsilon}{2} E^{\mathcal{F}_t^2} \int_t^T |u_1(s) - u_2(s)|^2 \, ds + |x_1 - x_2|^2), \\ & \forall u_1, u_2 \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m), \quad x_1, x_2 \in R^n. \end{aligned} \tag{25}$$

The above theorem shows that our assumptions (13) and (14) result in a —- to our best knowledge —- new uniformly convex structure of the cost functional. This structure is implicit in the sense that the cost functional $J(u;t,x), u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m)$ depends on the control u in an implicit way.

Proof Let $X_1 := X^{t,x_1;u_1}$ and $X_2 := X^{t,x_2;u_2}$. Then, by the uniform convexity of M, we obtain

$$\begin{aligned} & J(u_1;t,x_1) + J(u_2;t,x_2) - 2J(\frac{u_1+u_2}{2}; t, \frac{x_1+x_2}{2}) \\ =\ & E^{\mathcal{F}_t^2}[M(X_1(T)) + M(X_2(T)) - 2M(\frac{X_1(T)+X_2(T)}{2})] \\ \geq\ & \varepsilon E^{\mathcal{F}_t^2} |X_1(T) - X_2(T)|^2. \end{aligned} \tag{26}$$

Set

$$\delta u = u_1 - u_2, \quad \delta X = X_1 - X_2. \tag{27}$$

They satisfy the following stochastic differential equation

$$\begin{cases} d\delta X(r) &= [A\delta X(r) + B\delta u(r)]\, dr + \sum_{i=1}^{d} [C_i \delta X(r) + D_i \delta u(r)]\, dw_i(r), \\ \delta X(t) &= x_1 - x_2. \end{cases} \tag{28}$$

Applying Lemma 2.1, we complete the proof.

2.4 Existence and uniqueness of optimal controls

One immediate consequence of the above uniformly convex structure is the following existence and uniqueness result on optimal controls.

Theorem 2.2. (existence and uniqueness) *Assume that the assumptions (13) and (14) are satisfied. Then if*

$$\inf_{u \in U_{ad}(t,T)} J(u;t,x) > -\infty, \tag{29}$$

there is a unique $\widehat{u} \in U_{ad}(t,T)$ *such that*

$$J(\widehat{u};t,x) = \min_{u \in U_{ad}(t,T)} J(u;t,x).$$

Proof We first show the uniqueness assertion. In fact, assume that u_1 and u_2 are two optimal controls, that is

$$J(u_1;t,x) = J(u_2;t,x) = \min_{u \in U_{ad}(t,T)} J(u;t,x).$$

Then, we have by applying Theorem 2.1 that

$$\begin{aligned} 0 &\geq J(u_1;t,x) + J(u_2;t,x) - 2J(\frac{u_1+u_2}{2};t,x) \\ &\geq \frac{\varepsilon^2}{2} \exp\left(-\beta(T-t)\right) E^{\mathcal{F}_t^2} \int_t^T |u_1(s) - u_2(s)|^2\, ds \end{aligned}$$

which gives $u_1(s) = u_2(s), t \leq s \leq T$.

Then we show the existence. In view of (29), we can choose a sequence of admissible controls $\{u_k\}_{k=1}^{\infty}$ such that

$$\lim_{k \to \infty} J(u_k;t,x) = \inf_{u \in U_{ad}(t,T)} J(u;t,x). \tag{30}$$

We assert that $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{L}_{\mathcal{F}}^2(t,T;R^m)$. In fact, from Theorem 2.1, we have

$$\begin{aligned} & J(u_k;t,x) + J(u_l;t,x) - 2J(\tfrac{u_k+u_l}{2};t,x) \\ \geq\ & \frac{\varepsilon^2}{2} \exp\left(-\beta T\right) E^{\mathcal{F}_t^2} \int_t^T |u_k(s) - u_l(s)|^2\, ds \end{aligned} \tag{31}$$

which implies

$$\overline{\lim_{k,l\to\infty}} E^{\mathcal{F}_t^2}\int_t^T |u_k(s)-u_l(s)|^2\,ds = 0. \tag{32}$$

Then, by the closedness of $U_{ad}(t,T)$ in $\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)$, we conclude that there is $u(\cdot)\in U_{ad}(t,T)$ such that u_k converges strongly in $\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)$. As a consequence, $X^{t,x;u_k}(T)$ converges to $X^{t,x;u}(T)$ strongly in $L^2(\Omega,\mathcal{F}_T,P;R^m)$. In view of (15), we obtain

$$J(u;t,x)=\lim_{k\to\infty} J(u_k;t,x)=\inf_{v\in U_{ad}(t,T)} J(v;t,x), \tag{33}$$

and therefore $u(\cdot)$ is optimal.

2.5 The uniform convexity of the value function

The following presents the second consequence of the above uniformly convex structure.

Theorem 2.3. (the uniform convexity of the value function) *Assume that (13) and (14) are satisfied. Assume that $V(t,x)>-\infty$ for $\forall t\in[0,T]$ and $x\in R^n$. Then, the value function V is uniformly convex in the state variable x. In fact, we have*

$$\begin{array}{ll} & V(t,x_1)+V(t,x_2)-2V(t,\dfrac{x_1+x_2}{2}) \\ \geq & \varepsilon\exp\left(-\beta(T-t)\right)|x_1-x_2|^2, \qquad \forall x_1,x_2\in R^n, P-a.s. \end{array} \tag{34}$$

Proof By Theorem 2.2, there are u_1,u_2,v, such that

$$\begin{array}{rll} V(t,x_1) & = J(u_1;t,x_1), & V(t,x_2)=J(u_2;t,x_2), \\ V(t,\dfrac{x_1+x_2}{2}) & = J(v;t,\dfrac{x_1+x_2}{2}). & \end{array} \tag{35}$$

Therefore,

$$\begin{array}{ll} & V(t,x_1)+V(t,x_2)-2V(t,\dfrac{x_1+x_2}{2}) \\ = & J(u_1;t,x_1)+J(u_2;t,x_2)-2J(v;t,\dfrac{x_1+x_2}{2}) \\ \geq & J(u_1;t,x_1)+J(u_2;t,x_2)-2J(\dfrac{u_1+u_2}{2};t,\dfrac{x_1+x_2}{2}) \\ \geq & \varepsilon\exp\left(-\beta(T-t)\right)(\dfrac{\varepsilon}{2}E^{\mathcal{F}_t^2}\displaystyle\int_t^T |u_1-u_2|^2(s)\,ds+|x_1-x_2|^2) \quad \text{by Theorem 2.1} \end{array} \tag{36}$$

which completes the proof.

Further properties of the value function which are not used in this paper will be discussed elsewhere.

3 The Quadratic Case: singular backward stochastic Riccati equation and the optimal feedback law

3.1 Formulation of the stochastic LQ problem and some historical comments

Consider the optimal control problem (denoted by $\mathcal{P}_0$) concerning the linear stochastic system

$$\begin{cases} dX(t) &= [AX(t)+Bu(t)]\,dt+\sum_{i=1}^{d}[C_iX(t)+D_iu(t)]\,dw_i(t), \\ X(0) &= x, \end{cases} \tag{37}$$

the following general quadratic cost

$$\begin{aligned} J(u;0,x) &= E\int_0^T[(N(s)u(s),u(s))+(G(s)X(s),X(s))]\,ds \\ &\quad +E(QX(T),X(T)) \end{aligned} \tag{38}$$

and the admissible control class $U_{ad}=\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$. Assume that all the coefficients A,B,C_i,D_i,N,G are $\mathcal{F}_t^2$-progressively measurable bounded matrix-valued processes, defined on $\Omega\times[0,T]$, of dimensions $n\times n, n\times m, n\times n, n\times m, m\times m, n\times n$ respectively. Also assume that $Q\in L^\infty(\Omega,\mathcal{F}_T^2,P;\mathcal{S}_+^n)$.

The above problem $\mathcal{P}_0$ with random coefficients was initially studied for the case of both state and control independent system noises W^2(that is, $C_i=D_i=0$ for $i=d_0+1,\dots,d$) by Bismut [2]. He derived the associated Riccati differential equation,

$$\begin{cases} dK(t) &= -[A^*K(t)+K(t)A+\sum_{i=1}^{d_0}C_i^*K(t)C_i+G \\ &\quad -(K(t)B+\sum_{i=1}^{d_0}C_i^*K(t)D_i) \\ &\quad \times(N+\sum_{i=1}^{d_0}D_i^*K(t)D_i)^{-1}(K(t)B+\sum_{i=1}^{d_0}C_i^*K(t)D_i)^*]\,dt \\ &\quad +\sum_{i=d_0+1}^{d}L_i(t)\,dw_i(t), \qquad 0\le t<T, \\ K(T) &= Q. \end{cases} \tag{39}$$

and proved the existence and uniqueness of the solution for that "new" (at that time) type of equation (later it is called **BSDE**). The speciality of this case is that the second unknown variable does not appear in the drift term of the Riccati equation. In this way, for the case of $C_i=D_i=0, i=d_0+1,\dots,d$, he showed the well-posedness of the problem $\mathcal{P}_0$ and obtained the feedback form of the optimal control.

Later, Peng [18] addressed the above general case, and formally derived the associ-

ated nonlinear **BSRDE**

$$\begin{cases} dK(t) &= -[A^*K(t)+K(t)A+\sum_{i=1}^{d}C_i^*K(t)C_i+G+\sum_{i=d_0+1}^{d}(C_i^*L_i(t)+L_i(t)C_i) \\ &\quad -(K(t)B+\sum_{i=1}^{d}C_i^*K(t)D_i+\sum_{i=d_0+1}^{d}L_i(t)D_i) \\ &\quad \times(N+\sum_{i=1}^{d}D_i^*K(t)D_i)^{-1}(K(t)B+\sum_{i=1}^{d}C_i^*K(t)D_i+\sum_{i=d_0+1}^{d}L_i(t)D_i)^*]\,dt \\ &\quad +\sum_{i=d_0+1}^{d}L_i(t)\,dw_i(t), \qquad 0\le t<T, \\ K(T) &= Q. \end{cases} \tag{40}$$

However, he only attacked the above **BSRDE** (40) for the case of $D_i = 0, i = d_0+1, \ldots, d$, that is the following equation

$$\begin{cases} dK(t) &= -[A^*K(t)+K(t)A+\sum_{i=1}^{d}C_i^*K(t)C_i+G \\ &\quad +\sum_{i=d_0+1}^{d}(C_i^*L_i(t)+L_i(t)C_i)-(K(t)B+\sum_{i=1}^{d}C_i^*K(t)D_i) \\ &\quad \times(N+\sum_{i=1}^{d_0}D_i^*K(t)D_i)^{-1}(K(t)B+\sum_{i=1}^{d}C_i^*K(t)D_i)^*]\,dt \\ &\quad +\sum_{i=d_0+1}^{d}L_i(t)\,dw_i(t), \qquad 0\le t<T, \\ K(T) &= Q. \end{cases} \tag{41}$$

The special feature of the Riccati equation in this case is that the second unknown variable appears in the drift term in a linear way. Then, for the case of $D_i = 0, i = d_0+1, \ldots, d$, he proved the well-posedness of the above problem P_0 and expressed the optimal control in a feedback form.

Both Bismut [2] and Peng [18] only considered the regular case, that is they assume that

$$N > 0, Q \ge 0, G(t) \ge 0. \tag{42}$$

For the singular case, the study in the literature is restricted to the context of deterministic coefficients with the assumption that $C_1 = \cdots = C_d = 0$: Chen, Li and Zhou [3] gave an equivalent criterion for the existence and uniqueness of the solution, and Kohlmann and Zhou [12] proved the existence and uniqueness of the solution under the additional assumptions:

$$D = I, A^*(t) + A(t) \ge BB^*(t).$$

3.2 Well-posedness of the stochastic LQ control problem

For the singular case, we have the following

Theorem 3.1. *Assume that the nondegeneracy condition (13) is satisfied and the $\mathcal{F}_T^2$-measurable bounded matrix Q is uniformly positive. Then, the problem $\mathcal{P}_0$ is well-posed.*

Proof Under the assumptions of this theorem, it is obvious that

$$V(t,x) > -\infty, \quad \forall (t,x) \in [0,T] \times R^n.$$

So, by Theorem 2.2, there is unique optimal control.

For the regular case, the following theorem can easily be proved with similar arguments as in the proof of Theorem 2.2.

Theorem 3.2. *Assume that N is uniformly positive, and G, Q are semipositive. Then, the problem $\mathcal{P}_0$ is well-posed.*

3.3 Nonlinear singular backward stochastic Riccati differential equation, and the optimal feedback law

Consider the singular case, that is $N(t) = 0, 0 \leq t \leq T$. Assume

$$Q \in L^\infty(\Omega, \mathcal{F}_T^2, P; \mathcal{S}_+^n), \quad Q \geq \varepsilon I \quad \text{for some positive constant } \varepsilon. \tag{43}$$

For simplicity of presentation, also assume that $G(t) = 0, \forall t \in [0,T]$.

Consider the following regular approximation of the original control problem $\mathcal{P}_0$

$$\textbf{Problem } \mathcal{P}_\alpha \qquad \min_{u \in \mathcal{L}_{\mathcal{F}}^2(t,T;R^m)} J_\alpha(u;t,x) \tag{44}$$

with

$$J_\alpha(u;t,x) = J(u;t,x) + \alpha E^{\mathcal{F}_t^2} \int_t^T |u(s)|^2 \, ds, \qquad \alpha > 0. \tag{45}$$

It associates with the following regular **BSRDE**

$$\begin{cases} dK(t) &= -[A^*K(t) + K(t)A + \sum_{i=1}^d C_i^* K(t) C_i \\ &\quad + \sum_{i=d_0+1}^d (C_i^* L_i(t) + L_i(t) C_i) - (K(t)B + \sum_{i=1}^d C_i^* K(t) D_i) \\ &\quad \times (\alpha I + \sum_{i=1}^{d_0} D_i^* K(t) D_i)^{-1} (K(t)B + \sum_{i=1}^d C_i^* K(t) D_i)^*] \, dt \\ &\quad + \sum_{i=d_0+1}^d L_i(t) \, dw_i(t), \qquad 0 \leq t < T, \\ K(T) &= Q. \end{cases} \tag{46}$$

The value function of the problem $\mathcal{P}_\alpha$ is denoted by $V_\alpha(t,x)$. Next, the following result of Peng [18] is borrowed for subsequent citation.

Lemma 3.1. *For $\forall \alpha > 0$, the regular* **BSRDE** *(46) has a unique $\mathcal{F}_t^2$-adapted solution $(K_\alpha(\cdot), L_\alpha(\cdot))$ with $K_\alpha \in \mathcal{L}^\infty_{\mathcal{F}^2}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];\mathcal{S}^n_+))$ and $L_\alpha \in (\mathcal{L}^2_{\mathcal{F}^2}(0,T;\mathcal{S}^n))^{(d-d_0)}$, and the problem $\mathcal{P}_\alpha$ has unique optimal control $\widehat{u}_\alpha$. Moreover, the optimal control $\widehat{u}_\alpha$ has the following closed form*

$$\widehat{u}_\alpha = -(\alpha I + \sum_{i=1}^{d_0} D_i^* K_\alpha(t) D_i)^{-1}[B^* K_\alpha(t) + \sum_{i=1}^{d} D_i^* K_\alpha(t) C_i]\widehat{X} \tag{47}$$

and the value function

$$V_\alpha(t,x) = (K_\alpha(t)x, x). \tag{48}$$

The relationship between the singular problem $\mathcal{P}_0$ and the regular approximating problem $\mathcal{P}_\alpha$ is discussed in the next lemma.

Lemma 3.2. *Assume that the conditions (13) and (43) are satisfied. Then, for fixed $x \in R^n$, $V_\alpha(t,x)$ converges to $V(t,x)$ strongly both in $\mathcal{L}^\infty_{\mathcal{F}^2}(0,T;R)$ and in $L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];R))$.*

Proof Denote by $\widehat{u}$ the optimal control of the original problem, i.e. $V(t,x) = J(\widehat{u};t,x)$. Then,

$$\begin{aligned} V(t,x) &\leq V_\alpha(t,x) \leq J_\alpha(\widehat{u};t,x) \\ &= J(\widehat{u};t,x) + \alpha E^{\mathcal{F}_t^2}\int_t^T |\widehat{u}(s)|^2\, ds \\ &= V(t,x) + \alpha E^{\mathcal{F}_t^2}\int_t^T |\widehat{u}(s)|^2\, ds. \end{aligned} \tag{49}$$

Therefore,

$$|V_\alpha(t,x) - V(t,x)| \leq \alpha E^{\mathcal{F}_t^2}\int_t^T |\widehat{u}(s)|^2\, ds.$$

It is easy to show that there is a constant $\beta_1 > 0$ such that

$$J(0;t,x) \leq |x|^2 \exp\left(\beta_1(T-t)\right). \tag{50}$$

Noting the positivity of Q and Lemma 2.1, we have

$$J(\widehat{u};t,x) \geq \varepsilon E^{\mathcal{F}_t^2}|X^{t,x;\widehat{u}}(T)|^2 \geq \frac{\varepsilon^2}{2}\exp\left(-\beta(T-t)\right)E^{\mathcal{F}_t^2}\int_t^T |\widehat{u}(s)|^2\, ds. \tag{51}$$

Since

$$J(\widehat{u};t,x) = V(t,x) \leq J(0;t,x),$$

we have

$$\frac{\varepsilon^2}{2}\exp\left(-\beta(T-t)\right)E^{\mathcal{F}_t^2}\int_t^T |\widehat{u}(s)|^2\, ds \leq |x|^2\exp\left(\beta_1(T-t)\right). \tag{52}$$

Concluding the above, we have

$$|V_\alpha(t,x) - V(t,x)| \leq 2\alpha\varepsilon^{-2}|x|^2\exp\left((\beta_1+\beta)(T-t)\right).$$

This completes the proof of this lemma.

With Lemma 3.2, the following is easily proved:

Lemma 3.3. *Assume that the conditions (13) and (43) are satisfied. Then, the value function V has a quadratic expression. More precisely, there is an $\mathcal{F}_t^2$-adapted stochastic process $K(\cdot) \in \mathcal{L}^\infty_{\mathcal{F}^2}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];\mathcal{S}^n_+))$ such that*

$$V(t,x) = (K(t)x, x), \quad \forall (t,x) \in [0,T] \times R^n, P-a.s. \tag{53}$$

Moreover, K_α converges to K strongly in the two spaces

$$\mathcal{L}^\infty_{\mathcal{F}^2}(0,T;\mathcal{S}^n_+) \quad \text{and} \quad L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];\mathcal{S}^n_+)).$$

Lemma 3.4. *Assume that the conditions (13) and (43) are satisfied. Then, the matrix-valued stochastic process $K(\cdot)$ in Lemma 3.3 is uniformly positive with respect to $(t,\omega) \in [0,T] \times \Omega$. Hence, $\{L_\alpha\}$ is a Cauchy sequence in $(\mathcal{L}^2_{\mathcal{F}^2}(0,T;\mathcal{S}^n))^{(d-d_0)}$.*

Proof The uniform positivity of $P(\cdot)$ results from the uniform convexity of the value function $V(t,x)$ in the state variable x. Therefore, in view of Lemma 3.3, when $\alpha > 0$ is sufficiently small, K_α is uniformly positive. At this stage, we can use the standard arguments to conclude that $\{L_\alpha\}$ is a Cauchy sequence in $(\mathcal{L}^2_{\mathcal{F}^2}(0,T;\mathcal{S}^n))^{(d-d_0)}$.

Theorem 3.3. *Assume that the conditions (13) and (43) are satisfied. Then, the singular* **BSRDE** *(4) has a unique $\mathcal{F}_t^2$-adapted solution $(K(\cdot), L(\cdot))$ with*

$$K \in \mathcal{L}^\infty_{\mathcal{F}^2}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];\mathcal{S}^n_+)), \quad L \in (\mathcal{L}^2_{\mathcal{F}^2}(0,T;\mathcal{S}^n))^{(d-d_0)},$$

and $K(t,\omega)$ being uniformly positive w.r.t. (t,ω).

Proof Let L be the limit in $(\mathcal{L}^2_{\mathcal{F}^2}(0,T;\mathcal{S}^n))^{(d-d_0)}$ of the Cauchy sequence $\{L_\alpha\}$. By Lemma 3.4, the $K \in \mathcal{L}^\infty_{\mathcal{F}^2}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];\mathcal{S}^n_+))$ specified in Lemma 3.3 is uniformly positive. Therefore, it is meaningful to take limit in the approximating regular **BSRDEs** (46) by letting $\alpha \to 0$. As a result, $(K(\cdot), L(\cdot))$ is shown to be an $\mathcal{F}_t^2$-adapted solution to the singular **BSRDE** (4).

Next, we show the uniqueness assertion. Assume that $(\widetilde{K}, \widetilde{L})$ is another solution to the singular **BSRDE** (4) with

$$\widetilde{K} \in \mathcal{L}^\infty_{\mathcal{F}^2}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T^2, P; C([0,T];\mathcal{S}^n_+)), \quad \widetilde{L} \in (\mathcal{L}^2_{\mathcal{F}^2}(0,T;\mathcal{S}^n))^{(d-d_0)},$$

and $\widetilde{K}(t,\omega)$ being uniformly positive w.r.t. (t,ω). Our aim is to show that $K(t) = \widetilde{K}(t), \forall t \in [0,T]$, and $L(t) = \widetilde{L}(t), a.s.a.e.t \in [0,T]$. For this purpose, define $F(S,\Gamma,L) : \mathcal{S}^n \times R^{m\times n} \times (\mathcal{S}^n)^{(d-d_0)}$ by

$$\begin{aligned} F(S,\Gamma,L) \;:=\; & (A+B\Gamma)^*S + S(A+B\Gamma) + \sum_{i=1}^{d}(C_i+D_i\Gamma)^*S(C_i+D_i\Gamma) \\ & + \sum_{i=d_0+1}^{d}(C_i^*L_i + L_iC_i), \end{aligned} \tag{54}$$

and $\hat{\Gamma} : \mathcal{S}_+^n \to R^{m\times n}$ by

$$\hat{\Gamma}(S) = -(\sum_{i=1}^{d_0} D_i^* S D_i)^{-1}(B^* S + \sum_{i=1}^{d_0} D_i^* S C_i).$$

Then, we have

$$\begin{cases} -dK(t) &= F(K(t), \hat{\Gamma}(K(t)), L(t))\, dt - \sum\limits_{i=d_0+1}^{d} L_i(t)\, dw_i(t) \\ K(T) &= Q \end{cases} \tag{55}$$

and

$$\begin{cases} -d\widetilde{K}(t) &= F(\widetilde{K}(t), \hat{\Gamma}(\widetilde{K}(t)), \tilde{L}(t))\, dt - \sum\limits_{i=d_0+1}^{d} \tilde{L}_i(t)\, dw_i(t) \\ \widetilde{K}(T) &= Q. \end{cases} \tag{56}$$

Set

$$\delta K(t) = K(t) - \widetilde{K}(t), \quad \delta L_i(t) = L_i(t) - \tilde{L}_i(t), i = d_0+1, \ldots, d, \quad t \in [0, T].$$

They satisfy the following

$$\begin{cases} -d\delta K(t) &= [F(\delta K(t), \hat{\Gamma}(K(t)), \delta L(t)) + \tilde{Q}(t)]\, dt - \sum\limits_{i=d_0+1}^{d} \delta L_i(t)\, dw_i(t), \\ \delta K(T) &= 0 \end{cases} \tag{57}$$

where

$$\begin{aligned} \tilde{Q}(t) &:= F(\widetilde{K}(t), \hat{\Gamma}(K(t)), \tilde{L}(t)) - F(\widetilde{K}(t), \hat{\Gamma}(\widetilde{K}(t)), \tilde{L}(t)) \\ &= (\hat{\Gamma}(K(t)) - \hat{\Gamma}(\widetilde{K}(t)))^* (\sum_{i=1}^{d_0} D_i^* \widetilde{K}(t) D_i)(\hat{\Gamma}(K(t)) - \hat{\Gamma}(\widetilde{K}(t))) \\ &\in \mathcal{S}_+^n. \end{aligned} \tag{58}$$

For given (t, x), let $Y(\cdot)$ be the solution of

$$\begin{cases} dY(s) &= (A + B\hat{\Gamma}(K(s)))Y(s)\, ds + \sum\limits_{i=1}^{d}(C_i + D_i\hat{\Gamma}(K(s)))Y(s)\, dw_i(s), \\ & \qquad t < s \le T, \\ Y(t) &= x. \end{cases} \tag{59}$$

Then,

$$(\delta K(t)x, x) = E^{\mathcal{F}_t^2} \int_t^T (\tilde{Q}(s)Y(s), Y(s))\, ds \ge 0.$$

So, $\delta K(t) \ge 0$ and $K(t) \ge \widetilde{K}(t)$. Identically, $\widetilde{K}(t) \ge K(t)$. This proves that $K(t) = \widetilde{K}(t), \forall t \in [0, T]$. Then it is straightforward to prove that $L = \tilde{L}$.

From the above theorem, the next theorem can be derived in a straightforward way. For the detailed proof, the reader is referred to the proof of Theorem 3.5 below.

Theorem 3.4. *Assume that the conditions (13) and (43) are satisfied. Let (K, L) be the unique $\mathcal{F}_t^2$-adapted solution to the singular* **BSRDE** *(4). Then, the optimal control $\widehat{u}$ of the singular stochastic LQ problem $\mathcal{P}_0$ has the following feedback law*

$$\widehat{u} = -(\sum_{i=1}^{d_0} D_i^* K(t) D_i)^{-1}(B^* K(t) + \sum_{i=1}^{d_0} D_i^* K(t) C_i)\widehat{X}, \tag{60}$$

and the value function $V(t,x), (t,x) \in [0,T]\times R^n$ is a quadratic form given by the following formula

$$V(t,x) = (K(t)x, x), \quad \forall (t,x) \in [0,T] \times R^n, a.s. \tag{61}$$

3.4 The nonhomogenious singular stochastic LQ problem

Assume that

$$\begin{array}{ll} Q \in L^\infty(\Omega, \mathcal{F}_T^2, P; \mathcal{S}_+^n), & Q \geq \varepsilon I \quad \text{for some positive constant } \varepsilon; \\ f, g \in \mathcal{L}_{\mathcal{F}}^2(0,T;R^n), & \xi \in L^2(\Omega, \mathcal{F}_T, P). \end{array} \tag{62}$$

Consider the following optimal control problem (denoted by $\widetilde{\mathcal{P}}_0$):

$$\min_{u \in \mathcal{L}_{\mathcal{F}}^2(0,T;R^m)} EM(X^{0,x;u}(T)) \tag{63}$$

with

$$M(x) = (Q(x-\xi), (x-\xi)) \tag{64}$$

and $X^{t,x;u}(\cdot)$ solving the linear stochastic system

$$\left\{ \begin{array}{rcl} dX(s) &=& [AX(s) + Bu(s) + f(s)]\, dt \\ && + \sum_{i=1}^{d} [C_i X(s) + D_i u(s) + g_i(s)]\, dw_i(t), \quad t < s \leq T, \\ X(t) &=& x, \quad u \in \mathcal{L}_{\mathcal{F}}^2(t,T;R^m). \end{array} \right. \tag{65}$$

The value function V is defined as

$$V(t,x) := \min_{u \in \mathcal{L}_{\mathcal{F}}^2(t,T;R^m)} E^{\mathcal{F}_t} M(X^{t,x;u}(T)), \quad (t,x) \in [0,T] \times R^n. \tag{66}$$

Note that ξ is $\mathcal{F}_T$-measurable and therefore the expectation in (66) is conditioned on $\mathcal{F}_t$ rather than $\mathcal{F}_t^2$. Let (ψ, ϕ) be the $\mathcal{F}_t$-adapted solution of the following **BSDE**

$$\left\{ \begin{array}{rcl} d\psi(t) &=& -\{[A + B\widehat{\Gamma}(K(t))]^* \psi + \sum_{i=1}^{d_0} [C_i + D_i \widehat{\Gamma}(K(t))]^* (\phi_i - K(t) g_i) \\ && + \sum_{i=d_0+1}^{d} C_i^* (\phi_i - K(t) g_i) - K(t) f - \sum_{i=d_0+1}^{d} L_i g_i \}\, dt + \sum_{i=1}^{d} \phi_i\, dw_i(t), \\ \psi(T) &=& Q\xi \end{array} \right. \tag{67}$$

where (K, L) is the unique $\mathcal{F}_t^2$-adapted solution of the singular **BSRDE** (4). With Theorem 3.3, the following can be verified by a pure completion of squares.

Theorem 3.5. *Assume that (13) and (43) are satisfied. Let (K, L) be the unique $\mathcal{F}_t^2$-adapted solution of the singular* **BSRDE** *(4). Then, the optimal control $\widehat{u}$ for the nonhomogenious singular stochastic LQ problem $\widetilde{\mathcal{P}}_0$ exists uniquely and has the following feedback law*

$$\begin{aligned} \widehat{u} &= -(\sum_{i=1}^{d_0} D_i^* K(t) D_i)^{-1}[(B^* K(t) + \sum_{i=1}^{d_0} D_i^* K(t) C_i)\widehat{X} \\ &\quad -B^*\psi(t) + \sum_{i=1}^{d_0} D_i^*(K(t)g_i - \phi_i)]. \end{aligned} \tag{68}$$

The value function $V(t,x), (t,x) \in [0,T] \times R^n$ has the following explicit formula

$$V(t,x) = (K(t)x, x) - 2(\psi(t), x) + V^0(t), \quad (t,x) \in [0,T] \times R^n \tag{69}$$

with

$$\begin{aligned} V^0(t) := \; & E^{\mathcal{F}_t}(Q\xi, \xi) - 2E^{\mathcal{F}_t}\int_t^T (\psi(s), f(s))\, ds \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d}[(Kg_i(s), g_i(s)) - 2(\phi_i(s), g_i(s))]\, ds \\ & - E^{\mathcal{F}_t}\int_t^T ((\sum_{i=1}^{d_0} D_i^* K D_i)u^0, u^0)\, ds. \end{aligned} \tag{70}$$

and

$$u^0(s) := (\sum_{i=1}^{d_0} D_i^* K(s) D_i)^{-1}[B^*\psi(s) + \sum_{i=1}^{d_0} D_i^*(\phi_i(s) - K(s)g_i(s))], \quad t \le s \le T. \tag{71}$$

Proof Set

$$\widetilde{u} = u - \widehat{\Gamma}(K(t))X, \widehat{A} = A + B\widehat{\Gamma}(K(t)), \widehat{C}_i = C_i + D_i\widehat{\Gamma}(K(t)), i = 1, \ldots, d. \tag{72}$$

Then the system (65) reads

$$\begin{cases} dX(s) &= [\widehat{A}X(s) + B\widetilde{u}(s) + f(s)]\, dt \\ & \quad + \sum_{i=1}^{d}[\widehat{C}_i X(s) + D_i\widetilde{u}(s) + g_i(s)]\, dw_i(t), \quad t < s \le T, \\ X(t) &= x, \quad u \in \mathcal{L}_{\mathcal{F}}^2(t, T; R^m). \end{cases} \tag{73}$$

Applying Itô's formula, we have the equation for $\mathcal{X}(t) =: XX^*(t)$:

$$\begin{cases} d\mathcal{X} &= [\widehat{A}\mathcal{X} + \mathcal{X}\widehat{A}^* + X(B\widetilde{u}(s) + f(s))^* + (B\widetilde{u}(s) + f(s))X^*]\, dt \\ & \quad + \sum_{i=1}^{d}[\widehat{C}_i\mathcal{X}C_i^* + \widehat{C}_i X(D_i\widetilde{u}(s) + g_i(s))^* + (D_i\widetilde{u}(s) + g_i(s))X^*\widehat{C}_i^* \\ & \quad + (D_i\widetilde{u}(s) + g_i(s))(D_i\widetilde{u}(s) + g_i(s))^*]\, dt \\ & \quad + \sum_{i=1}^{d}[\widehat{C}_i\mathcal{X} + \mathcal{X}\widehat{C}_i^* + X(D_i\widetilde{u}(s) + g_i(s))^* + (D_i\widetilde{u}(s) + g_i(s))X^*]\, dw_i(t), \\ & \quad t < s \le T, \\ \mathcal{X}(t) &= xx^*, \quad u \in \mathcal{L}_{\mathcal{F}}^2(t, T; R^m). \end{cases} \tag{74}$$

Note that the singular **BSRDE** (4) can be rewritten as

$$\begin{cases} -dK(t) &= F(K(t),\widehat{\Gamma}(K(t)),L(t))\,dt - \sum_{i=d_0+1}^{d} L_i(t)\,dw_i(t) \\ K(T) &= Q \end{cases} \tag{75}$$

where F is defined by (54). So, application of Itô's formula gives

$$\begin{aligned} & E^{\mathcal{F}_t}(QX(T),X(T)) = E^{\mathcal{F}_t}\mathrm{tr}(Q\mathcal{X}(T)) \\ = \;& (K(t)X(t),X(t)) + 2E^{\mathcal{F}_t}\int_t^T (K(s)(B\widetilde{u}(s)+f(s)),X(s))\,ds \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d} 2(K(s)(D_i\widetilde{u}(s)+g_i(s)),\widehat{C}_iX(s))\,ds \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d} (K(s)(D_i\widetilde{u}(s)+g_i(s)),D_i\widetilde{u}(s)+g_i(s))\,ds \\ & + 2E^{\mathcal{F}_t}\int_t^T \sum_{i=d_0+1}^{d} (L_i(s)(D_i\widetilde{u}(s)+g_i(s)),X(s))\,ds, \end{aligned}$$

and

$$\begin{aligned} & E^{\mathcal{F}_t}(Q\xi,X(T)) = E^{\mathcal{F}_t}(\psi(T),X(T)) \\ = \;& (\psi(t),X(t)) + E^{\mathcal{F}_t}\int_t^T (\psi(s),B\widetilde{u}(s)+f(s))\,ds \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d} (\phi_i(s),D_i\widetilde{u}(s)+g_i(s))\,ds \\ & + E^{\mathcal{F}_t}\int_t^T (\sum_{i=1}^{d} \widehat{C}_i^*K(s)g_i(s) + K(s)f(s) + \sum_{i=d_0+1}^{d} L_i(s)g_i(s),X(s))\,ds. \end{aligned}$$

Combining the last two equations, we get

$$\begin{aligned} & E^{\mathcal{F}_t}M(X(T)) = E^{\mathcal{F}_t}(QX(T),X(T)) - 2E^{\mathcal{F}_t}(QX(T),\xi) + E^{\mathcal{F}_t}(Q\xi,\xi) \\ = \;& (KX(t),X(t)) - 2(\psi(t),X(t)) + E^{\mathcal{F}_t}(Q\xi,\xi) \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d} (K(s)(D_i\widetilde{u}(s)+g_i(s)),D_i\widetilde{u}(s)+g_i(s))\,ds \\ & - 2E^{\mathcal{F}_t}\int_t^T (\psi(s),B\widetilde{u}(s)+f(s))\,ds - 2E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d} (\phi_i(s),D_i\widetilde{u}(s)+g_i(s))\,ds \\ = \;& (Kx,x) - 2(\psi(t),x) + E^{\mathcal{F}_t}(Q\xi,\xi) \\ & - 2E^{\mathcal{F}_t}\int_t^T (\psi(s),f(s))\,ds + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d} [(Kg_i(s),g_i(s)) - 2(\phi_i(s),g_i(s))]\,ds \\ & + E^{\mathcal{F}_t}\int_t^T ((\sum_{i=1}^{d_0} D_i^*KD_i)(\widetilde{u}-u^0),\widetilde{u}-u^0)\,ds - E^{\mathcal{F}_t}\int_t^T ((\sum_{i=1}^{d_0} D_i^*KD_i)u^0,u^0)\,ds. \end{aligned}$$

This completes the proof.

4 The Mean-Variance Hedging Problem

In this section, we consider the mean-variance hedging problem when asset prices follow Itô's processes in an incomplete market framework. The market conditions are allowed

to be random, but are assumed to be uniformly bounded which implies that there is an equivalent martingale measure. It will be shown that the mean-variance hedging problem in finance of this context is a special case of the linear quadratic optimal stochastic control problem discussed in the preceding section, and therefore can be solved completely, by using the above results.

4.1 The financial market model

Consider the financial market in which there are $n+1$ primitive assets: one nonrisky asset (the bond) of price process

$$S_0(t) = \exp\left(\int_0^t r(s)\,ds\right), \quad 0 \le t \le T, \tag{76}$$

and n risky assets (the stocks)

$$dS(t) = \mathrm{diag}(S(t))(\mu(t)\,dt + \sigma(t)\,dW^1(t)), \quad 0 \le t \le T. \tag{77}$$

Here $W^1 = (w_1, \ldots, w_n)^*$ is a n-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{F}_t, 0 \le t \le T\}$ is the P-augmentation of the filtration generated by a d-dimensional Brownian motion $W = (w_1, \ldots, w_n, w_{n+1}, \ldots, w_d)^*$ with $d \ge n$. Denote by $W^2 = (w_{n+1}, \ldots, w_d)^*$ the $(d-n)$-dimensional Brownian motion. Assume that the instantaneous interest rate r, the n-dimensional appreciate vector process μ and the volatility matrix $n \times n$ process σ are progressively measurable with respect to $\{\mathcal{F}_t^2, 0 \le t \le T\}$. For simplicity of exposing the main ideas, assume that they are uniformly bounded and there exists a positive constant ε such that

$$\sigma\sigma^*(t) \ge \varepsilon I, \quad 0 \le t \le T, a.s. \tag{78}$$

The risk premium process is given by

$$\lambda(t) = \sigma^{-1}(t)(\mu(t) - r(t)e_n), \quad 0 \le t \le T \tag{79}$$

where $e_n = (1, \ldots, 1)^* \in R^n$.

4.2 Formulation of the problem

For any $x \in R$ and $\pi \in \mathcal{L}_{\mathcal{F}}^2(0, T; R^n)$, define the self-financed wealth process X with initial capital x and with quantity π invested in the risky asset S by

$$\begin{cases} dX(t) &= (rX(t) + (\mu - re_n, \pi))\,dt + \pi^*\sigma\,dW^1(t), \quad 0 < t \le T, \\ X(0) &= x, \quad \pi \in \mathcal{L}_{\mathcal{F}}^2(0, T; R^n). \end{cases} \tag{80}$$

Given a random variable $\xi \in L^2(\Omega, \mathcal{F}, P)$, consider the quadratic optimal control problem:

$$\textbf{Problem } \mathcal{P}_{0,x}(\xi) \qquad \min_{\pi \in \mathcal{L}_{\mathcal{F}}^2(0,T;R^n)} E(X^{0,x;\pi}(T) - \xi)^2 \tag{81}$$

where $X^{0,x;\pi}$ is the solution to the wealth equation (80). The associated value function is denoted by $V(t,x), (t,x) \in [0,T] \times R$. The minimum point of $V(t,x)$ over $x \in R$ for given time t is defined to be the approximate price for the contingent claim ξ at time t.

The problem $\mathcal{P}_{0,x}(\xi)$ is the so-called mean-variance hedging problem in financial economics. It is the one-dimensional case of the singular stochastic LQ problem $\widetilde{\mathcal{P}}_0$. In the next subsection, Theorem 3.5 will be used to give a complete solution to the mean-variance hedging problem $\mathcal{P}_{0,x}(\xi)$.

4.3 A general case of random market conditions: a complete solution

In the case of the mean-variance hedging problem, we have

$$\begin{array}{rclrclrcl} A(t) & = & r(t), & B(t) & = & (b - re_n)^*(t), & C_i(t) & = & 0, \\ D_i(t) & = & \sigma_i^*, & u(t) & = & \pi(t), & & & \\ Q & = & I, & d_0 & = & n, & \displaystyle\sum_{i=1}^{d_0} D_i^* D_i & = & \sum_{i=1}^n \sigma_i \sigma_i^* = \sigma\sigma^* \end{array}$$

where σ_i is the i-th column of the volatility matrix σ. The associated Riccati equation is a linear **BSDE**:

$$\left\{ \begin{array}{rcl} dK & = & -(2r - |\lambda|^2)K\,dt + \displaystyle\sum_{i=n+1}^{d} L_i\,dw_i(t), \quad 0 \le t < T \\ K(T) & = & 1. \end{array} \right. \tag{82}$$

Let (ψ, ϕ) is the $\mathcal{F}_t$-adapted solution of the following **BSDE**

$$\left\{ \begin{array}{rcl} d\psi & = & -\{(r - |\lambda|^2)\psi - \displaystyle\sum_{i=1}^{n} [\sigma_i^*(\sigma\sigma^*)^{-1}(\mu - re_n)]\phi_i\}\,dt + \sum_{i=1}^{d} \phi_i\,dw_i(t), \\ & = & -\{(r - |\lambda|^2)\psi - \displaystyle\sum_{i=1}^{n} \lambda_i\phi_i\}\,dt + \sum_{i=1}^{d} \phi_i\,dw_i(t), \\ \psi(T) & = & \xi \end{array} \right. \tag{83}$$

An immediate application of Theorem 3.5 provides an explicit formula for the optimal hedging portfolio:

$$\begin{array}{rcl} \pi & = & -(\displaystyle\sum_{i=1}^{n} \sigma_i\sigma_i^*)^{-1}\{(b - re_n)[X - K^{-1}(t)\psi(t)] - \sum_{i=1}^{n} \sigma_i\phi_i K^{-1}(t)\} \\ & = & -(\sigma\sigma^*)^{-1}\{(b - re_n)[X - K^{-1}(t)\psi(t)] - \sigma(\phi_1(t), \ldots, \phi_n(t))^* K^{-1}(t)\} \\ & = & -(\sigma^*)^{-1}\lambda[X - K^{-1}(t)\psi(t)] + (\sigma^*)^{-1}(\phi_1(t), \ldots, \phi_n(t))^* K^{-1}(t) \end{array} \tag{84}$$

where (K, L) is the $\mathcal{F}_t^2$-adapted solution to the Riccati equation (82). The value function V is also given by

$$V(t,x) = K(t)x^2 - 2\psi(t)x + E^{\mathcal{F}_t}\xi^2 - E^{\mathcal{F}_t}\int_t^T K^{-1}(s)|\lambda\psi + \phi|^2(s)\,ds \tag{85}$$

where $\phi := (\phi_1, \ldots, \phi_n)^*$. So, the approximate price $p(t)$ at time t for the contingent claim ξ is given by

$$p(t) = K^{-1}(t)\psi(t). \tag{86}$$

The above solution need not introduce the additional concepts of the so-called *hedging numeraire* and *variance-optimal martingale measure,* and therefore is simpler than that of Gourieroux et al [7], and Laurent and Pham [13]. To be connected to the latter, the optimal hedging portfolio (84) is rewritten as

$$\pi = -(\sigma^*)^{-1}\lambda[X - \widetilde{\psi}(t)] + (\sigma^*)^{-1}(\widetilde{\phi}_1(t), \ldots, \widetilde{\phi}_n(t))^*. \tag{87}$$

Here, the pair $(\widetilde{\psi}, \widetilde{\phi})$ is defined as

$$\begin{array}{rcll} \widetilde{\psi}(t) & = & \psi K^{-1}(t), & \\ \widetilde{\phi}_i(t) & = & \phi_i K^{-1}(t), & i = 1, \ldots, n, \\ \widetilde{\phi}_i(t) & = & \phi_i K^{-1}(t) - L_i\psi(t)K^{-2}(t), & i = n+1, \ldots, d, \end{array} \tag{88}$$

and solves the following **BSDE**:

$$\left\{ \begin{array}{rcl} d\widetilde{\psi} & = & \{r\widetilde{\psi} + \sum_{i=1}^{d} \widetilde{\lambda}_i\widetilde{\phi}_i\}\, dt + \sum_{i=1}^{d} \widetilde{\phi}_i\, dw_i(t), \quad 0 \le t < T, \\ \widetilde{\psi}(T) & = & \xi \end{array} \right. \tag{89}$$

with

$$\begin{array}{rcll} \widetilde{\lambda}_i(t) & := & \lambda_i(t), & \forall t \in [0,T], \quad i = 1, \ldots, n, \\ \widetilde{\lambda}_i(t) & := & -K^{-1}(t)L_i(t), & \forall t \in [0,T], \quad i = n+1, \ldots, d. \end{array} \tag{90}$$

The process $\widetilde{\psi}$ is just the *approximate price process*, and the **BSDE** (89) is the *approximate pricing equation.*

Note that the optimal hedging portfolio (84) consists of the following two parts:

$$\pi^1(t) := -(\sigma^*)^{-1}\lambda X(t), \quad 0 \le t \le T \tag{91}$$

and

$$\pi^0(t) := (\sigma^*)^{-1}[\lambda\widetilde{\psi}(t) + (\widetilde{\phi}_1(t), \ldots, \widetilde{\phi}_n(t))^*], \quad 0 \le t \le T \tag{92}$$

and satisfies

$$\pi(t) = \pi^1(t) + \pi^0(t), \quad 0 \le t \le T. \tag{93}$$

The first part π^1 is the optimal solution of the *homogenious* mean-variance hedging problem $\mathcal{P}_{0,x}(0)$ (that is the case of $\xi = 0$ for the problem$\mathcal{P}_{0,x}(\xi)$). The corresponding optimal wealth process $X^{0,1;\pi^1}$ is the solution to the following *optimal closed system*

$$\left\{ \begin{array}{rcl} dX(t) & = & X(t)[(r - |\lambda|^2)\, dt - \lambda^*\, dW^1(t)], \quad 0 < t \le T, \\ X(0) & = & 1, \end{array} \right. \tag{94}$$

and is just the *hedging numeraire*. So, the *hedging numeraire* is just *the state (wealth) transition process* of the optimal closed system (94) from time 0, or it is just *the fundamental solution* of the optimal closed system (94).

To understand the quantity $(\widetilde{\lambda}_{n+1},\ldots,\widetilde{\lambda}_d)^*$, consider the **BSDE** satisfied by $(\mathcal{K},\mathcal{L})$

$$\left\{\begin{array}{rcl} d\mathcal{K} &=& \{(2r-|\lambda|^2)\mathcal{K}+\sum\limits_{i=n+1}^{d}\mathcal{L}_i^2\mathcal{K}^{-1}\}\,dt+\sum\limits_{i=n+1}^{d}\mathcal{L}_i\,dw_i(t),\quad 0\le t<T \\ \mathcal{K}(T) &=& 1 \end{array}\right. \tag{95}$$

with $\mathcal{K}:=K^{-1}$ and $\mathcal{L}_i:=-L_iK^{-2}$. It is the singular **BSRDE** for the following singular stochastic LQ problem (denoted by $\mathcal{P}^*_{0,x}$):

$$\textbf{Problem } \mathcal{P}^*_{0,x} \qquad \min_{\theta\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^{d-n})} E|\mathcal{X}^{0,x;\theta}(T)|^2 \tag{96}$$

where $\mathcal{X}^{0,x;\theta}$ is the solution to the following stochastic differential equation

$$\left\{\begin{array}{rcl} d\mathcal{X} &=& \mathcal{X}[-r\,dt-\lambda^*\,dW^1(t)]+\theta^*\,dW^2(t),\quad 0\le t\le T, \\ \mathcal{X}(0) &=& x,\quad \theta\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^{d-n}). \end{array}\right. \tag{97}$$

Its optimal control $\widehat{\theta}=(\widehat{\theta}_{n+1},\cdots,\widehat{\theta}_d)^*$ has the following feedback form

$$\widehat{\theta}_i=-\mathcal{K}^{-1}\mathcal{L}_i(t)\mathcal{X}=-\widetilde{\lambda}_i(t)\mathcal{X},\quad i=n+1,\ldots,d. \tag{98}$$

The problem $\mathcal{P}^*_{0,1}$ is just the so-called *dual problem* of the problem $\mathcal{P}_{0,1}(0)$ in [7, 13], and so the variance-optimal martingale measure is P_* defined as

$$dP_*:=\exp\Big\{-\sum_{i=1}^{d}\int_0^T\widetilde{\lambda}_i(t)\,dw_i(t)-\frac{1}{2}\int_0^T(\sum_{i=1}^{d}\widetilde{\lambda}_i^2)\,dt\Big\}\,dP. \tag{99}$$

P_* is an equivalent martingale measure.

Note that $\widetilde{\psi}$ has the following explicit formula:

$$\widetilde{\psi}(t)=E_*^{\mathcal{F}_t}\xi\exp\,(-\int_t^T r(s)\,ds),\quad 0\le t\le T. \tag{100}$$

Here, the notation $E_*^{\mathcal{F}_t}$ stands for the expectation operator conditioning on the σ-algebra $\mathcal{F}_t$ with respect to the probability P_*. The discounted $\widetilde{\phi}$ is just the integrand of the stochastic-integral-representation of the P^*-martingale $\{E_*^{\mathcal{F}_t}\xi\exp\,(-\int_0^T r(s)\,ds), 0\le t\le T\}$ (w.r.t. the P^*-martingale $W+\int_0\widetilde{\lambda}\,dt$ with $\widetilde{\lambda}:=(\lambda^*,\widetilde{\lambda}_{n+1},\ldots,\widetilde{\lambda}_d)^*$).

4.4 The case of Markovian market conditions

Assume the following Markovian structure for the randomness of the market conditions:

$$r(t,\omega):=r(t,Y_t),\quad \mu(t,\omega):=\mu(t,Y_t),\quad \sigma(t,\omega):=\sigma(t,Y_t) \tag{101}$$

with $\{Y_t, 0\le t\le T\}$ defined by the stochastic differential equation

$$\left\{\begin{array}{rcl} dY_t &=& \eta(t,Y_t)\,dt+\gamma(t,Y_t)\,dW^2(t),\quad 0\le t\le T, \\ Y_0 &=& y\in R^{d-n}. \end{array}\right. \tag{102}$$

In this case, the risk premium process $\{\lambda(t,\omega), 0 \le t \le T\}$ reads

$$\lambda(t,\omega) = \sigma^{-1}(t,Y_t)[\mu(t,Y_t) - r(t,Y_t)e_n], \quad 0 \le t \le T. \tag{103}$$

This context includes the stochastic volatility models usually studied in the literature (Hull and White [10], Stein and Stein [24], Heston [8]).

Under the above assumption, the Riccati equation (82) and the stochastic differential equation (102) constitute *a forward-backward stochastic differential equation.* Then, it is straightforward in the literature that the solution to the Riccati equation (82) can be characterized by the parabolic partial differential equation:

$$\begin{cases} Z_t + (\eta(t,y), Z_y) + \frac{1}{2}\text{tr}\ (\gamma\gamma^*(t,y)Z_{yy}) + (2r - |\lambda|^2)(t,y)Z = 0, \\ \qquad\qquad y \in R^{d-n}, 0 \le t < T, \\ Z(T,y) = 1, \quad y \in R^{d-n} \end{cases} \tag{104}$$

through the relation

$$K(t) = Z(t,Y_t), \qquad L_i(t) = Z_y(t,Y_t)\gamma_i(t,Y_t). \tag{105}$$

The reader is referred to Peng [19], Pardoux and Peng [16], and Pardoux and Tang [17] for details.

5 Concluding Comments

In this paper, a new framework for the stochastic LQ problem is developed where the cost is singular and the coefficients are allowed to be random. The nondegeneracy condition and the uniform positivity of the terminal state weighting matrix Q are assumed. The existence and uniqueness of the solution to the associated nonlinear singular **BSRDE** are proved, and in terms of it the optimal control is expressed as a feedback form. These results can be used to solve the mean-variance hedging problem with random market conditions under some mild conditions. However, our framework has obvious limitations: (a) our model is put into an Itô process setting rather than a semimartingale setting, (b) the uniformly bounded condition on the coefficients is assumed, (c) the randomness of the coefficients is not general enough. Due to these limitations, the general mean-variance hedging problem could not be treated at present with a LQ theory approach.

Acknowledgement The second author would like to thank the hospitality of Department of Mathematics and Statistics, and the Center of Finance and Econometrics, Universität Konstanz, Germany.

References

[1] Bismut, J. M., *Conjugate convex functions in optimal stochastic control,* J. Math. Anal. Appl., 44 (1973), 384–404

[2] Bismut, J. M., *Linear quadratic optimal stochastic control with random coefficients,* SIAM J. Control Optim., 14 (1976), 419–444

[3] Chen, S., Li, X. and Zhou, X., *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim. **36 (1998)**, 1685-1702

[4] Duffie, D. and Richardson, H. R., *Mean-variance hedging in continuous time,* Ann. Appl. Prob., 1 (1991), 1-15

[5] Föllmer, H. and Sonderman, D., *Hedging of non-redundant contingent claims.* In: Mas-Colell, A., Hildebrand, W. (eds.) Contributions to Mathematical Economics. Amsterdam: North Holland 1986, pp. 205-223

[6] Föllmer, H. and Schweizer, M., *Hedging of contingent claims under incomplete information.* In: Davis, M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis. (Stochastics Monographs vol. 5) London-New York: Gordon & Breach, 1991, pp. 389-414

[7] Gourieroux, C., Laurent, J. P. and Pham, H., *Mean-variance hedging and numéraire,* Mathematical Finance, 8 (1998), 179-200.

[8] Heston, S., *A closed-form solution for option with stochastic volatility with applications to bond and currency options.* Rev. Financial Studies, 6 (1993), 327–343

[9] Hipp, C., *Hedging general claims,* Proceedings of the 3rd AFIR colloqium, Rome, Vol. 2, pp. 603-613 (1993)

[10] Hull, J. and White, A., *The pricing of options on assets with stochastic volatilities.* J. Finance, 42 (1987), 281-300

[11] Karatzas, I. and Shreve, S. E., *Methods of Mathematical Finance,* Springer, New York, Berlin, Heidelberg, 1999

[12] Kohlmann, M. and Zhou, X., *Backward stochastic differential equations and stochastic controls: a new perspective,* preprint.

[13] Laurent, J. P. and Pham, H., *Dynamic programming and mean-variance hedging,* Finance and Stochastics, 3 (1999), 83-110

[14] Monat, P. and Stricker, C., *Föllmer-Schweizer decomposition and mean-variance hedging of general claims,* Ann. Probab. 23, 605–628

[15] Pardoux, E. and Peng, S., *Adapted solution of backward stochastic equation,* Systems Control Lett., 14 (1990), 55–61

[16] Pardoux, E. and Peng, S., *Backward stochastic differential equations and quasi-linear parabolic partial differential equations,* in: Rozovskii, B. L., Sowers, R. S. (eds.) Stochastic Partial Differential Equations and Their Applications, Lecture Notes in Control and Information Science 176, Springer, Berlin, Heidelberg, New York 1992, pp. 200–217

[17] Pardoux, E. and Tang, S., *Forward-backward stochastic differential equations with application to quasi-linear partial differential equations of second-order*, Probability Theory and Related Fields, 114 (1999), 123-150

[18] Peng, S., *Stochastic Hamilton-Jacobi-Bellman equations*, SIAM J. Control and Optimization **30 (1992)** , 284-304

[19] Peng, S., *Probabilistic interpretation for systems of semilinear parabolic PDEs,* Stochastics & Stochastic Reports, 37 (1991), 61–74

[20] Pham, H., Rheinländer, T., and Schweizer, M., *Mean-variance hedging for continuous processes: New proofs and examples,* Finance and Stochastics, 2 (1998), 173–198

[21] Schweizer, M., *Mean-variance hedging for general claims,* Ann. Appl. Prob., 2 (1992), 171-179

[22] Schweizer, M., *Approaching random variables by stochastic integrals,* Ann. Probab., 22 (1994), 1536–1575

[23] Schweizer, M., *Approximation pricing and the variance-optimal martingale measure,* Ann. Probab., 24 (1996), 206–236

[24] Stein, E. M. and Stein, J. C., *Stock price distributions with stochastic volatility: an analytic approach,* Rev. Financial Studies, 4 (1991), 727–752

[25] Wonham, W. M., *On a matrix Riccati equation of stochastic control,* SIAM J. Control Optim., 6 (1968), 312–326