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Working Paper

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CoFE Discussion Paper, No. 99/12

Provided in Cooperation with:
University of Konstanz, Center of Finance and Econometrics (CoFE)

Suggested Citation: Adam-Müller, Axel F. A. (1999) : Hedging Price Risk When Real Wealth Matters, CoFE Discussion Paper, No. 99/12, University of Konstanz, Center of Finance and Econometrics (CoFE), Konstanz, https://nbn-resolving.de/urn:nbn:de:bsz:352-opus-3214

This Version is available at:
https://hdl.handle.net/10419/85167

Hedging Price Risk When Real Wealth Matters *

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This version: May 25, 1999

Abstract

This paper analyzes optimal hedging of a tradable risk (e.g. price risk or exchange rate risk) with forward contracts in the presence of untradable inflation risk. Utility is defined over real wealth. Optimal forward positions are derived relative to a given initial exposure in the tradable risk. A nominally unbiased forward market usually implies a non-zero real risk premium and hence some risk taking. If untradable inflation risk is a monotone function of the tradable risk plus noise, cross hedging and speculating on the real risk premium are conflicting objectives; the level of relative risk aversion determines which objective is dominant in a nominally unbiased forward market.

JEL classification: D81, G11, D11

Keywords: hedging, forward market, untradable inflation risk, relative risk aversion

*I would like to thank Marc De Ceuster, Günter Franke, Roland Jeske and Michael Kohlmann for helpful comments and discussions. Remaining errors are mine, of course.

1 Introduction

Consider a risk-averse individual with a riskless and a risky asset. The risky asset exposes the individual to nominal price risk. If, for example, the risky asset is a position in a foreign currency, nominal price risk takes the form of nominal exchange rate risk. When there is a forward market for the risky asset, price risk can be traded. Suppose that the individual maximizes expected utility of nominal wealth. Then, given an unbiased forward market, the individual will completely eliminate price risk as is well-known.

This paper addresses the following question: How is the optimal forward position affected by the existence of a second, multiplicative risk that cannot be traded? This risk can be inflation risk, for example. The question is motivated by the fact that individuals are not primarily interested in nominal wealth per se but instead in consumption or real wealth. Nominal wealth is related to consumption by the prices of consumption goods. For simplicity, assume there is one composite consumption good. If its price is deterministic, the probability distribution of nominal wealth equals that of consumption. If, in contrast, the price of the consumption good is random, the individual is exposed to inflation risk. The purpose of this paper is to analyze the effect of untradable inflation risk on the optimal forward position in a tradable risk. Thus, the paper contributes to the work on optimal decisions in incomplete financial markets.

Inflation risk is assumed to be untradable.[1] In the presence of untradable inflation risk, the individual will generally take into account the joint stochastic behavior of the rate of inflation and the tradable risk when deciding on the optimal forward position in the tradable risk. For example, if the individual's nominal wealth is positively correlated with inflation, he expects to be richer in nominal terms when consumption goods are expensive

[1]However, in Canada, Israel, the U.K. and some developing countries, bonds linked to a price index have been issued. In the U.S., such bonds have been available since February 1997, in France since September 1998.

and poorer when consumption goods are cheap. Consequently, consumption varies less than in the absence of inflation risk. Intuition suggests that this reduces the optimal forward position. However, it will be shown that this is not necessarily correct, depending on the joint distribution of the two risks.

Inflation risk is incorporated in the model by defining the individual's utility function over real wealth. This is in contrast to previous hedging models where utility is defined over nominal wealth.[2] It will be shown in a two-date model that the optimal forward position is determined by (1) the joint distribution of the two risks, (2) the real risk premium in the forward market and (3) the level of relative risk aversion (RRA). If untradable inflation risk is a monotone function of the tradable risk plus a noise term, the results are as follows: Full hedging of the tradable risk is not optimal if the forward market is unbiased. Firstly, correlation implies a non-zero real risk premium if the nominal risk premium in the forward market is zero. Thus, the individual speculates on the real risk premium; a speculative position in the forward market will be optimal. Secondly, correlation allows for cross hedging. The forward market will be used to cross hedge the untradable risk; a cross hedging position will be optimal. As will be shown, the speculative position and the cross hedging position have opposite signs. The level of RRA determines whether speculation or cross hedging is dominant in a nominally unbiased forward market. For stochastically independent risks, cross hedging is impossible. Thus, the untradable risk only affects the extent of speculation. Under logarithmic utility, multiplicative untradable risk is ignored.

The classical hedging problem in which there is only a tradable risk has been analyzed by Danthine (1978), Holthausen (1979) and others. More recent papers analyze the effects of additional risks that are untradable. Benninga et al. (1985) and Adam-Müller (1997) consider the effects of a

[2]Utility defined over real wealth has already been applied to other problems, for example to portfolio problems under inflation risk (see, e.g., Biger, 1975).

second risk that is multiplicatively combined with only the tradable risk; Adam-Müller (1993), Briys et al. (1993) and Franke et al. (1998) consider an independent, additive background risk in initial wealth. Cross hedging is discussed by Anderson and Danthine (1981), Broll et al. (1995), Broll and Wahl (1996) and others.

The first to analyze optimal hedging in the presence of untradable inflation risk were Briys and Schlesinger (1993). They use a state-dependent preference model to analyze the optimal forward position under the assumption that there are only two realizations of the inflation rate.[3] They show that starting from deterministic inflation, untradable inflation risk does not affect the sign of the open position in the tradable risk if marginal utility is state-independent. For state-dependent marginal utility, clear-cut results can only be derived under restricted conditions. In contrast to Briys and Schlesinger (1993), this paper uses state-independent preferences but allows for any probability distribution of the inflation rate.

This paper is organized as follows. Section 2 describes the analytical framework. Sections 3 to 5 analyze the optimal forward position under different sets of assumptions. Section 3 briefly addresses the logarithmic utility case without restricting the joint probability distribution of the two risks. Section 4 assumes stochastic independence but allows for any risk-averse utility function. In Section 5, both assumptions are relaxed. A numerical example is provided in Section 6. Section 7 concludes. All proofs are given in the Appendix.

[3]Briys and Schlesinger (1993) assign a utility function to each of the two possible price vectors for consumption goods. These state-dependent utility functions are defined over nominal wealth as in the classical hedging model.

2 The model

Consider a two-date model where decisions are made at date 0 and uncertainty is resolved at date 1. The individual's initial endowment consists of q units of an asset whose price $\tilde{p}$ is risky, $q > 0$ and $p > 0$ almost surely[4]. The $\tilde{p}$-risk is called *nominal price risk*. In addition, the individual is endowed with a deterministic amount of nominal wealth due at date 1, denoted a, $a \geq 0$. There exists a competitive forward market for the risky asset. Hence, the $\tilde{p}$-risk is tradable. f is the date 0 forward price for delivery of one unit of the risky asset at date 1. a, f, and q are exogeneously given. F is the quantity of the risky asset sold forward. Nominal wealth $\tilde{n}$ at date 1 is given by

$$\tilde{n} = a + \tilde{p}\, q + F(f - \tilde{p})\ . \tag{1}$$

Nominal wealth will be completely consumed at date 1. Therefore, the individual is not interested in nominal but in real wealth. Real wealth is the product of nominal wealth and the purchasing power index $\tilde{z}$, $z \geq 0$ almost surely. The individual consumes a competitively traded composite consumption good with nominal price $1/\tilde{z}$ at date 1. Without loss of generality, we assume $E\tilde{z} = 1$ such that $\tilde{z}$ represents unexpected inflation. The randomness of $\tilde{z}$ is called *inflation risk*. We assume that inflation risk cannot be traded. Thus, $\tilde{z}$ is a multiplicative background risk that applies to the individual's entire nominal wealth. The individual is jointly exposed to nominal price risk and inflation risk. The joint distribution of $\tilde{p}$ and $\tilde{z}$ is known to the individual. Real wealth $\tilde{r}$ at date 1 is given by

$$\tilde{r} = \tilde{z}\,\tilde{n} = \tilde{z}\,\Big(a + f\, q + (F - q)(f - \tilde{p})\Big)\ . \tag{2}$$

[4]Random variables have a tilde, their realizations do not.

At date 0, the individual decides on F. He is assumed to maximize a von Neumann-Morgenstern utility function U defined over real wealth $\tilde{r}$ which is at least twice continuously differentiable and strictly concave. Thus, the individual is risk-averse[5]. The optimization problem is

$$\max_{F} \; E\Big[U(\tilde{r})\Big] \tag{3}$$

subject to equation (2). Since the maximand is strictly concave in F, the optimal value F^{opt} is the unique solution of the first-order condition

$$E\Big[U'(\tilde{r})\tilde{z}(f-\tilde{p})\Big] = 0 \; . \tag{4}$$

We assume $U'(r) \to \infty$ for $r \to 0$ in order to preclude insolvency or starvation. Notice that $r \to 0$ is equivalent to $n \to 0$ since $z > 0$. Thus, we always have $n > 0$.

Equation (2) indicates that the decision problem is equivalent to choosing a nominal speculative position $(F - q)$ given a deterministic nominal endowment of $(a + fq)$. Under full hedging, defined as $F = q$, all $\tilde{p}$-risk is eliminated and nominal wealth is risk-free.

It is useful to repeat some commonly used definitions concerning the forward market and the forward position. The nominal risk premium in the forward market is $E[f - \tilde{p}]$. If it is zero [not zero], the forward market is said to be nominally unbiased [nominally biased]. If the forward price is smaller [higher] than the spot price expected for date 1, the forward market is said to exhibit backwardation [contango]. The real risk premium is $E[\tilde{z}(f - \tilde{p})]$. The forward market is said to be unbiased [biased] in real terms if $E[\tilde{z}(f - \tilde{p})] =$

[5]Hanoch (1977) has shown that risk aversion with respect to wealth and risk aversion with respect to consumption are equivalent. The utility function considered here shares two important properties with the indirect utility function which is defined over income and prices. First, it is homogeneous of degree zero in the price of the consumption good and nominal wealth. Second, it is decreasing in the price of the consumption good.

$[\neq]\, 0$. Finally, underhedging [overhedging] is defined by selling less [more] forward than the endowment, $F < [>]\, q$.

3 Optimal forward positions under logarithmic utility

It is well-known that inflation risk does not matter under logarithmic utility (see, e.g., Adler and Dumas, 1983) since the objective function becomes $E[\ln \tilde{r}] = E[\ln \tilde{z}] + E[\ln \tilde{n}]$ for any joint distribution of $\tilde{p}$ and $\tilde{z}$. Thus, the optimal forward position is the same as in the absence of inflation risk: Underhedging [full hedging] [overhedging] is optimal if and only if the forward market exhibits backwardation [nominal unbiasedness] [contango] (see, e.g., Holthausen, 1979).

It is counterintuitive that the optimal forward position is independent from inflation risk even if price and inflation risk are strongly correlated and inflation variability is high. Therefore, logarithmic utility will be replaced by more general preferences in the following.

4 Optimal forward positions under independent inflation risk

This section deals with the case of independent risks. The analysis applies to all risk-averse utility functions. Under independence, the real risk premium and the nominal risk premium have the same sign since $E[\tilde{z}(f-\tilde{p})] = E\tilde{z}\; E[f-\tilde{p}]$. As Theorem 1 shows, a non-zero risk premium implies a speculative position although any nominally speculative position, $F^{opt} \neq q$, increases expected exposure to inflation risk.

Theorem 1: Suppose $\tilde{p}$ and $\tilde{z}$ are independent. Underhedging [full hedging] [overhedging] is optimal if and only if the forward market is characterized by backwardation [unbiasedness] [contango].

Proof: All proofs are given in the Appendix.

Under independence, forward contracts only allow for hedging the tradable risk. The relation between the optimal forward position F^{opt} and the initial exposure q only depends on the relation between the forward price and the expected spot price. Cross hedging inflation risk is impossible since there is no systematic relation between $\tilde{p}$ and $\tilde{z}$. If the real risk premium is zero, the individual attempts to minimize real risk. Given stochastic independence, this implies a full hedge in the tradable risk. In a nominally biased forward market, the real risk premium differs from zero. Thus, a speculative position is optimal.

The optimal forward position can be decomposed as $F^{opt} = F^n + F^s$. F^n is a pure hedging component which completely eliminates nominal price risk from nominal wealth. Thus, we always have $F^n = q$. The speculative component F^s has the same sign as the nominal and the real risk premium.

Theorem 1 illustrates the well-known result that any risk averter takes a risky position when there is a non-zero risk premium (Arrow, 1965, p. 39). This result still holds for the maximization of expected utility of real wealth if both risks are independent. However, it is important to note that the optimal level F^{opt} is affected by inflation risk whenever $F^{opt} \neq q$.

By definition, inflation risk applies *multiplicatively* to the entire nominal wealth. If independent untradable inflation risk is replaced by an *additive* background risk that is both independent and untradable, the qualitative statement of Theorem 1 still holds (Adam-Müller, 1993; Briys et al., 1993). But if inflation risk is replaced by another independent and untradable multiplicative risk that only applies to $\tilde{p}q$, Theorem 1 no longer holds. In that case, the forward position also depends on the individual's prudence in the sense of Kimball (1990) as shown by Benninga et al. (1985) and Adam-Müller (1997) in the context of exchange rate risk.

5 Optimal forward positions when price risk and inflation risk are dependent

This section deals with stochastically dependent risks. If, for example, the risky asset is a position in a foreign currency, relative purchasing power parity implies a systematic relation between nominal exchange rate risk and inflation risk. The type of tradable risk plays a particularly important role when determining the relevant joint distribution function. In general, any kind of stochastic dependence between the tradable risk and untradable inflation risk is possible. Throughout this section, we concentrate on the following type of stochastic dependence: $\tilde{z}$ is a monotone function of $\tilde{p}$ plus a noise term. More formally, we assume $\tilde{z} = h(\tilde{p}) + \tilde{\varepsilon}$ where $E[\tilde{\varepsilon}] = 0$ and $\tilde{p}$ is conditionally independent of $\tilde{\varepsilon}$, i.e., $E[\tilde{p}|\varepsilon] = E[\tilde{p}]\,\forall\varepsilon$. $h(p)$ is a deterministic differentiable function which monotonically increases or decreases.[6] Hence $cov(\tilde{p}, \tilde{z}) = cov(\tilde{p}, h(\tilde{p})) > [<]\, 0$ if $h'(p) > [<]\, 0$.

The special case where $h(p)$ is a linear function has been applied to describe the joint behavior of spot and futures prices (see, e.g., Benninga et al., 1983, Briys et al., 1993); recently, Broll and Wahl (1996) applied it to the problem of cross hedging exchange rate risk. However, inflation risk may require other specifications of $h(p)$. For example, in an economy with general inflation risk $\tilde{z}$ is linearly related to the *inverse* of $\tilde{p}$.

Under these assumptions, we have two effects. The first is speculation, the second is cross hedging. Let us look at speculation first. The real risk premium is $E[\tilde{z}(f - \tilde{p})] = E\tilde{z}E[f - \tilde{p}] - cov(\tilde{p}, h(\tilde{p}))$. This shows that a nominally unbiased forward market is biased in real terms. Any position in a nominally unbiased forward market changes expected real wealth while leaving expected nominal wealth unchanged. The existence of a real risk premium provides an incentive for speculation. Hence, there will be a speculative position. Second, correlation between inflation risk and nominal price

[6] For $h'(p) = 0\,\forall p$, $\tilde{p}$ and $\tilde{z}$ are independent and Section 4 applies.

risk allows the individual to cross hedge untradable inflation risk. Thus, there will be a cross hedging position as well.

Given these two effects, Theorem 2 characterizes the optimal forward position relative to the initial exposure given a nominally unbiased forward market.

Theorem 2: Suppose $\tilde{z} = h(\tilde{p}) + \tilde{\varepsilon}$. *Suppose further that the forward market is nominally unbiased.*

a) Underhedging is optimal if $h'(p) < [>]\, 0\, \forall p$ *and relative risk aversion is above [below] one for all possible levels of real wealth.*

b) Overhedging is optimal if $h'(p) < [>]\, 0\, \forall p$ *and relative risk aversion is below [above] one for all possible levels of real wealth.*

For interpretation, we break the optimal forward position down into three components as $F^{opt} = F^n + F^s + F^c$. As in Section 4, F^n denotes the hedging component aimed at directly hedging the tradable risk without taking the untradable risk and risk premia into account. Thus, we again have $F^n = q$ which eliminates all tradable risk from nominal wealth.

F^s is a speculative position arising from the existence of a real risk premium.[7] The real risk premium is $-cov(\tilde{p}, h(\tilde{p}))$. The individual speculates by assuming the position F^s. For negatively correlated $\tilde{p}$ and $\tilde{z}$, the real risk premium is positive. Thus, expected real wealth can be raised by taking a positive speculative position $F^s > 0$. For positively correlated $\tilde{p}$ and $\tilde{z}$, $F^s < 0$.

Since correlation between $\tilde{p}$ and $\tilde{z}$ allows one to indirectly hedge the untradable risk, there is a cross hedging component, denoted by F^c. Cross hedging reduces real wealth risk by buying nominal wealth for states with low purchasing power against nominal wealth in states with high purchasing power as compared to the case of independent risks. Thus, for negatively

[7] In the decomposition of Briys and Schlesinger (1993) the speculative component is different from zero if and only if the forward market is nominally biased. Here, the real risk premium matters.

correlated $\tilde{p}$ and $\tilde{z}$, the cross hedging component F^c is negative since this provides higher nominal wealth in states with low purchasing power. Analogously, we have $F^c > 0$ for $cov(\tilde{p}, \tilde{z}) > 0$. This is in line with a result from Broll and Wahl (1996) derived under the assumption of additively combined exchange rate risks: Negative [positive] correlation implies a negative [positive] cross hedging position.

So far, we have demonstrated that F^s and F^c have opposite signs. As Theorem 2 shows, the size of F^s relative to F^c is determined by the level of RRA[8]. For RRA below one, the former dominates the latter, $|F^s| > |F^c|$. Hence, $F^{opt} - F^n = F^{opt} - q = F^s + F^c$ is positive [negative] if $h'(p) <[>] 0$. Loosely speaking, the motivation to speculate on the real risk premium is stronger than the motivation to cross hedge untradable inflation risk if risk aversion is small. Increasing expected real wealth is more attractive than reducing the variability of real wealth via cross hedging. Conversely, the cross hedging component dominates the speculative component if risk aversion is high. Thus, we have $|F^s| < |F^c|$ for RRA above one. (As Section 3 shows for RRA = 1, $F^s = -F^c$ in a nominally unbiased forward market.)

It is important to note that the components of the optimal forward position are not independent. For example, the cross hedging component F^c not only deals with inflation risk arising from the initial position in the risky asset and the nominally risk-free part of the endowment but also with inflation risk arising from expected nominal wealth associated with a speculative position.

The following Corollary extends the results of Theorem 2 by allowing the forward market to be nominally biased. This increases the motivation to speculate.

[8]RRA measures the elasticity of marginal utility with respect to wealth. For RRA below [above] one marginal utility is inelastic [elastic].

Corollary: Suppose $\tilde{z} = h(\tilde{p}) + \tilde{\varepsilon}$.

a) Suppose the forward market is characterized by backwardation. Underhedging is optimal if $h'(p) < [>]\, 0\, \forall p$ *and relative risk aversion is above [below] one for all possible levels of real wealth.*

b) Suppose the forward market is characterized by contango. Overhedging is optimal if $h'(p) < [>]\, 0\, \forall p$ *and relative risk aversion is below [above] one for all possible levels of real wealth.*

The real risk premium is given by $E[\tilde{z}(f-\tilde{p})] = E\tilde{z}E[f-\tilde{p}] - cov(\tilde{p}, h(\tilde{p}))$. Theorem 2 describes the optimal forward position for $E\tilde{z}E[f-\tilde{p}] = 0$. In the Corollary, we enlarge the real risk premium by allowing for a non-zero nominal risk premium while holding its sign constant. Hence, the speculative component grows but leaves the sign of $(F^{opt} - q)$ unchanged as compared to a nominally unbiased forward market.

As Theorem 2 and the Corollary show, the optimal forward position F^{opt} relative to the initial exposure q crucially depends on whether RRA is above one or not. Numerous attempts have been made to empirically investigate the level of constant RRA (CRRA). Evidence is mixed but tends to suggest that CRRA is above one.[9] However, the following example considers CRRA above one as well as CRRA below one.

6 An example

The purpose of this example is to illustrate Theorem 2. It is based on the same assumptions except for the (slightly) stronger assumption of CRRA given by power utility $\breve{U}(r) = r^{\gamma}/\gamma$. Risk aversion implies $\gamma < 1$; CRRA equals $(1-\gamma)$, relative prudence amounts to $(2-\gamma) > 0$. We assume $\tilde{z} = \alpha + \beta/\tilde{p} + \tilde{\varepsilon}$.

Three distributions of real wealth are compared: The first distribution refers to full hedging, $F = q$, and will be used as a benchmark. The sec-

[9] Barsky (1989) provides a short overview.

Table 1: Optimal forward positions in a nominally unbiased forward market

$\hat{\varepsilon}$	$\rho(\tilde{p},\tilde{z})$	problem	$(F-q)$	$E\tilde{r}$	$\sigma(\tilde{r})$	$\min(\tilde{r})$	$\max(\tilde{r})$
0.00	-1.00	full hedging	0.00	1050.00	168.00	882.00	1218.00
		CRRA 1.5	-112.86	1040.97	111.57	929.40	1152.54
		CRRA 0.5	336.00	1076.88	336.00	740.88	1412.88
0.05	-0.95	full hedging	0.00	1050.00	176.01	829.50	1270.50
		CRRA 1.5	-113.30	1040.94	123.14	874.26	1201.95
		CRRA 0.5	336.43	1076.92	340.39	696.61	1474.04
0.10	-0.85	full hedging	0.00	1050.00	198.11	777.00	1323.00
		CRRA 1.5	-114.65	1040.83	152.67	819.42	1250.77
		CRRA 0.5	337.74	1077.02	353.26	652.04	1535.78
0.20	-0.62	full hedging	0.00	1050.00	268.93	672.00	1428.00
		CRRA 1.5	-120.26	1040.38	236.39	710.48	1346.22
		CRRA 0.5	343.12	1077.45	400.72	562.20	1661.32
0.30	-0.47	full hedging	0.00	1050.00	357.00	567.00	1533.00
		CRRA 1.5	-130.60	1039.55	331.90	602.26	1437.66
		CRRA 0.5	352.73	1078.22	469.69	471.76	1790.49
0.40	-0.37	full hedging	0.00	1050.00	452.35	462.00	1638.00
		CRRA 1.5	-147.55	1038.20	431.45	494.46	1522.91
		CRRA 0.5	367.72	1079.42	552.82	381.10	1924.82

ond is the distribution of real wealth given an optimal forward position and CRRA = 1.5, the third is optimal if CRRA = 0.5. The initial position is $a = 1000$ and $q = 50$. There are four possible states at date 1 with equal probability. For $\tilde{\varepsilon}$ as well as for $\tilde{p}$, two realizations are possible. In states 1 and 2, $p = 0.5$; in states 3 and 4, $p = 1.5$. In states 1 and 3, $\varepsilon = \hat{\varepsilon}$, in states 2 and 4, $\varepsilon = -\hat{\varepsilon}$. Thus, $E\tilde{p} = 1$ and $E\tilde{\varepsilon} = 0$. Nominal unbiasedness requires $f = 1$. An increase in $\hat{\varepsilon}$ is a mean preserving spread for $\tilde{\varepsilon}$ and, thus, for $\tilde{z}$. We set $\beta = 0.24$. $\alpha = 0.68$ ensures $E\tilde{z} = 1$.

The first column of Table 1 shows $\hat{\varepsilon}$. A higher $\hat{\varepsilon}$ implies a higher $\tilde{\varepsilon}$-risk. Hence, three wealth distributions are compared for six different scenarios.

The marginal distribution of $\tilde{p}$ is always the same. The correlation coefficient $\rho(\tilde{p}, \tilde{z})$ is given in the second column. The third column denotes the underlying assumptions about full hedging or optimal hedging assuming CRRA = 1.5 or CRRA = 0.5, respectively. The position relative to the initial endowment is given in the fourth column. In the remaining columns, each distribution of real wealth is characterized by its mean $E\tilde{r}$, its standard deviation $\sigma(\tilde{r})$ and the smallest and highest realization of real wealth, $\min(\tilde{r})$ and $\max(\tilde{r})$.

In a nominally unbiased forward market, $E\tilde{n}$ is a constant. The real risk premium, given by $E[\tilde{z}(f - \tilde{p})] = -\beta cov(\tilde{p}, 1/\tilde{p}) = 0.08$, is independent of $\hat{\varepsilon}$. Since it is positive, expected real wealth increases in F. The smaller $|F - q|$ is, the closer expected wealth will be to $E\tilde{z}E\tilde{n} = 1050$. For an underhedging [overhedging] position, expected real wealth is below [above] $E\tilde{z}E\tilde{n}$. Table 1 mirrors Theorem 2 in that underhedging is optimal for CRRA = 1.5 and overhedging is optimal for CRRA = 0.5.[10]

In the first scenario where $\hat{\varepsilon}$ is zero, tradable and untradable risks are perfectly negatively correlated. This is the case of general inflation. For an underhedging position of $(F - q) = -336$, real wealth risk is a constant. If cross hedging were the only objective, this position would be optimal. However, completely eliminating risk is suboptimal since there is a positive real risk premium. Table 1 shows that the extent of speculation significantly differs between the two levels of CRRA.

In contrast to what might have been expected, the optimal open position increases as $\tilde{\varepsilon}$ becomes more volatile. To see why, consider CRRA = 1.5 where cross hedging is the dominant objective. Since correlation between $\tilde{p}$ and $\tilde{z}$ decreases as $\hat{\varepsilon}$ increases, cross hedging becomes less effective. Hence the individual is less prepared to bear additional risks by taking a speculative position. Consequently, $(F - q) < 0$ is reduced. The reduction of F provides additional real wealth in states where real wealth is very low. Therefore,

[10] It follows directly from Section 3 that full hedging is optimal for CRRA = 1.

this behavior is well in line with the notion of prudence[11]. Now, consider CRRA = 0.5. The fact that cross hedging is less effective implies that even less value is set on this already minor objective. Speculation plays an even more dominant role despite the fact that the real risk premium is constant. $(F - q) > 0$ is further increased.

A detailed analysis of the last three columns of Table 1 is left to the reader.

7 Conclusion

This paper incorporates untradable inflation risk into the problem of hedging a tradable risk, e.g., price risk, using forward contracts. Inflation risk applies *multiplicatively* to the entire nominal wealth. Since the decision maker maximizes expected utility of real wealth, untradable inflation risk affects his behavior in the forward market except for logarithmic utility. In the case of stochastically independent price and inflation risk, the nominal and the real risk premium are of the same sign. Hence the latter only has a quantitative effect on the optimal forward position: well-known results on optimal hedging remain valid. In the case of stochastically dependent price and inflation risk, results significantly differ. If untradable inflation risk is a monotone function of the tradable risk plus a noise term, the level of relative risk aversion plays a crucial role for the optimal forward position. This becomes clear when the optimal forward position is broken down into a pure hedging component, a component arising from speculation on the real risk premium and a cross hedging component. The pure hedging component always mirrors the initial endowment and eliminates all tradable risk from nominal wealth. Speculation on the real risk premium and cross hedging are conflicting goals. For

[11] In the case of additive independent untradable risk, prudence implies taking less tradable risk the higher the untradable risk. The example shows that a prudent individual facing *multiplicatively* combined *dependent* risks may also raise the open position in the tradable risk if the variability of the untradable risk grows. However, in both cases the individual protects himself against very low realization of wealth.

relative risk aversion less than one, the motivation to speculate is stronger when the nominal risk premium is zero. For relative risk aversion above one, cross hedging considerations dominate the optimal forward position in a nominally unbiased forward market. In both cases, full hedging is not optimal if the forward market is nominally unbiased since there is a real risk premium.

Appendix

The proofs make use of the uniqueness of the optimal solution. In order to the characterize optimal forward position relative to the initial exposure q, we analyze the deviation from the first-order condition evaluated at full hedging, $F = q$. The sign of the deviation allows us to determine the sign of $F^{opt} - q$.

Define $U'(zb)z = g(z)$ and $a + fF = b > 0$. $E[g(\tilde{z})]$ is always positive since $U'(\cdot)$, $z > 0$. At $F = q$, real wealth simplifies to $\tilde{r} = \tilde{z}b$. Evaluating the first-order condition at $F = q$ gives

$$E\Big[g(\tilde{z})(f - \tilde{p})\Big] = E\Big[g(\tilde{z})\Big]\, E[f - \tilde{p}] - cov\Big(\tilde{p}, g(\tilde{z})\Big)\,. \tag{5}$$

Under independence, the covariance term in (5) vanishes. This implies $sgn\, E[g(\tilde{z})(f - \tilde{p})] = sgn\, E[f - \tilde{p}]$. Since the maximand is strictly concave in F, $sgn\, E[g(\tilde{z})(f - \tilde{p})] > [=]\,[<]\,0$ at $F = q$ is equivalent to $F^{opt} > [=]\,[<]\,q$. This proves Theorem 1.

The proof of Theorem 2 is based on the fact that only the randomness of $\tilde{p}$ is relevant to the sign of the covariance in (5). Assuming $\tilde{z} = h(\tilde{p}) + \tilde{\varepsilon}$, the covariance term in (5) can be signed as follows: RRA $> [<]\, 1\, \forall z$ implies $g'(z) < [>]\, 0\, \forall z$. We have

$$\begin{aligned}
\frac{dg(z(p))}{dp} &= g'(z)h'(p) \\
&= \left[U''(zb)zb + U'(zb)\right] h'(p) \\
&= \left[1 - \mathrm{RRA}(zb)\right]\ U'(zb)\ h'(p) \quad \forall \varepsilon\,.
\end{aligned} \tag{6}$$

$\mathrm{RRA} < [>]\, 1$ combined with $h'(p) > 0\, \forall p$ implies $dg(z(p))/dp > [<]\, 0\, \forall \varepsilon$. By conditional independence of $\tilde{p}$ from $\tilde{\varepsilon}$, we get $cov(\tilde{p}, g(\tilde{z})) > [<]\, 0$. Combined with $h'(p) < 0$, the reverse inequalities are implied.

Finally, sign (5) by signing the product on the RHS via backwardation, unbiasedness and contango and by signing the covariance term via RRA and the sign of $h'(p)$. Theorem 2 and the Corollary directly follow from the strict concavity of the problem.

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