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A Minimal Test for Convex Games and the Shapley Value

Mark Voorneveld¹ Sofia Grahn²

Abstract: A set of necessary and sufficient conditions for convexity of a transferable utility game in terms of its decomposition into unanimity games is shown to be minimal: none of the conditions is redundant. The result is used to provide an axiomatization of the Shapley value on the set of convex games.

Keywords: convex game, unanimity game, Shapley value, axiomatization.

JEL Classification: C71.

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1 Introduction

A game with transferable utility, or a (TU) game for ease of notation, is a tuple (N, v) with $N = \{1, \dots, n\}$ a finite set of $n \geq 2$ players and $v : 2^N \rightarrow \mathbb{R}$ a function that assigns to each coalition $S \subseteq N$ of players a value $v(S) \in \mathbb{R}$ with $v(\emptyset) = 0$. Let G^N denote the set of games with player set N and $\mathcal{C} = \{S \mid S \subseteq N, S \neq \emptyset\}$ the collection of all nonempty coalitions. Shapley (1953) proves that the unanimity games $\{(N, u_T) \in G^N \mid T \in \mathcal{C}\}$ form a basis of the vector space G^N , where (N, u_T) is defined for each $S \subseteq N$ as follows:

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for each game $(N, v) \in G^N$ there exist unique coefficients $(\alpha_T)_{T \in \mathcal{C}}$ such that $v = \sum_{T \in \mathcal{C}} \alpha_T u_T$. Many different classes of games, like airport games (Littlechild and Owen, 1973) and sequencing games (Curiel et al., 1989), can be characterized through restrictions on these coefficients. A game is convex (Shapley, 1971) if

$$\forall S, T \subseteq N : v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \quad (1)$$

or, equivalently, if

$$\forall S, T \subseteq N, \forall i \in N : \text{if } S \subseteq T \subseteq N \setminus \{i\}, \text{ then } v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S). \quad (2)$$

Convex games have nice properties (Shapley, 1971, Ichiishi, 1981): the nonempty core is the unique stable set and coincides with the Weber set (the convex hull of the marginal vectors). Hence, the Shapley value is the barycenter of the core. Moreover, several practical classes of games, like bankruptcy games (O'Neill, 1982, Aumann and Maschler, 1985) and sequencing games (Curiel et al., 1989), turn out to be convex.

The purpose of this paper is threefold: (i) to provide a number of necessary and sufficient conditions on the unanimity coefficients α_T for the game to be convex, (ii) to show that this number is minimal: it is really necessary to check all these conditions, in the sense that it is possible to construct a non-convex game violating an arbitrarily chosen condition, but nevertheless satisfying all remaining conditions, (iii) to provide an axiomatization of the Shapley value on the class of convex games in the spirit of Young (1985).

2 Testing convexity

A large number of convexity conditions in (1) or (2) is redundant. Our next theorem provides a smaller number of such conditions, which are shown to be minimal in the sense that none of them is implied by the others.

Theorem 2.1 The following three conditions are equivalent:

- (a) The game $(N, v) \in G^N$ is convex;
(b) For all $i, j \in N, i \neq j$, and each $S \subseteq N \setminus \{i, j\}$:

$$v(S \cup \{i, j\}) - v(S \cup \{j\}) \geq v(S \cup \{i\}) - v(S). \quad (3)$$

- (c) For all $i, j \in N, i \neq j$, and each $S \subseteq N \setminus \{i, j\}$:

$$\sum_{R \subseteq S} \alpha_{R \cup \{i, j\}} \geq 0. \quad (4)$$

Proof. (a) \Rightarrow (b): In (2), write $T = S \cup \{j\}$ to obtain (3).

(b) \Rightarrow (a): Let $S, T \subseteq N$ and $i \in N$ be such that $S \subseteq T \subseteq N \setminus \{i\}$. We prove that inequality (2) holds. If $T = S$, the inequality is trivial, so assume that $T \neq S$ and write $T \setminus S = \{i_1, \dots, i_m\}$. Repeated application of (3) yields

$$\begin{aligned} v(T \cup \{i\}) - v(T) &= v(S \cup \{i_1, \dots, i_m\} \cup \{i\}) - v(S \cup \{i_1, \dots, i_m\}) \\ &\geq v(S \cup \{i_1, \dots, i_{m-1}\} \cup \{i\}) - v(S \cup \{i_1, \dots, i_{m-1}\}) \\ &\dots \\ &\geq v(S \cup \{i_1\} \cup \{i\}) - v(S \cup \{i_1\}) \\ &\geq v(S \cup \{i\}) - v(S). \end{aligned}$$

(b) \Leftrightarrow (c): Write $v = \sum_T \alpha_T u_T$. Let $i, j \in N, i \neq j$, and $S \subseteq N \setminus \{i, j\}$. Then

$$\begin{aligned} (3) &\Leftrightarrow [v(S \cup \{i, j\}) - v(S \cup \{j\})] - [v(S \cup \{i\}) - v(S)] \geq 0 \\ &\Leftrightarrow \left[\sum_{T: T \subseteq S \cup \{i, j\}} \alpha_T - \sum_{T: T \subseteq S \cup \{j\}} \alpha_T \right] - \left[\sum_{T: T \subseteq S \cup \{i\}} \alpha_T - \sum_{T: T \subseteq S} \alpha_T \right] \geq 0 \\ &\Leftrightarrow \sum_{R: R \subseteq S \cup \{j\}} \alpha_{R \cup \{i\}} - \sum_{R: R \subseteq S} \alpha_{R \cup \{i\}} \geq 0 \\ &\Leftrightarrow \sum_{R: R \subseteq S} \alpha_{R \cup \{i, j\}} \geq 0, \end{aligned}$$

There are $\binom{n}{2}$ ways to choose two players $i, j \in N, i \neq j$, and 2^{n-2} ways to choose a coalition $S \subseteq N \setminus \{i, j\}$, yielding $2^{n-2} \binom{n}{2}$ conditions in (4). To show that — as opposed to (1) or (2) — none of these conditions is redundant, let $i, j \in N, i \neq j$, and $S \subseteq N \setminus \{i, j\}$. We construct a game that violates exactly the convexity condition corresponding with (S, i, j) , while still satisfying all other conditions. Consider the game $(N, v) \in G^N, v = \sum_{T \in \mathcal{C}} \alpha_T u_T$ with for each $T \in \mathcal{C}$:

$$\alpha_T = \begin{cases} -1 & \text{if } T = S \cup \{i, j\}, \\ 1 & \text{if } T = S \cup \{i\} \text{ or } T = S \cup \{j\} \text{ or } R \not\subseteq S \cup \{i, j\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{R \subseteq S} \alpha_{R \cup \{i, j\}} = -1 < 0$, so the condition for (S, i, j) is indeed violated. Let $k, \ell \in N, k \neq \ell$, and $T \subseteq N \setminus \{k, \ell\}$ such that $T \neq S$ or $\{i, j\} \neq \{k, \ell\}$ and consider the condition for the combination (T, k, ℓ) :

$$\sum_{R \subseteq T} \alpha_{R \cup \{k, \ell\}} \geq 0. \quad (5)$$

If $\alpha_{S \cup \{i, j\}}$ does not appear in (5), then the sum is over nonnegative terms, hence nonnegative.

If $\alpha_{S \cup \{i, j\}}$ does appear in (5), discern two cases:

- (a) If $T \neq S$ and $\{i, j\} = \{k, \ell\}$, the fact that $\alpha_{S \cup \{i, j\}}$ appears in (5) implies that $S \cup \{i, j\} = S \cup \{k, \ell\} \subseteq T \cup \{k, \ell\}$, so $S \subseteq T$. Choose $m \in T \setminus S$, which is possible since $T \neq S$. Then $\alpha_{\{m\} \cup \{k, \ell\}} = 1$ appears in (5), compensating for $\alpha_{S \cup \{i, j\}} = -1$ and consequently yielding a nonnegative outcome.
- (b) If $\{i, j\} \neq \{k, \ell\}$, assume without loss of generality that $i \notin \{k, \ell\}$. Since $\alpha_{S \cup \{i, j\}} = -1$ appears in (5), also $\alpha_{S \cup \{j\}} = 1$ appears in (5), compensating the negative number and hence yielding a nonnegative outcome.

Conclude that the condition for (S, i, j) is the unique condition that is violated.

A brief remark on the complexity of our test: $2^{n-2} \binom{n}{2} = \frac{2^n n(n-1)}{8}$ conditions seems quite a lot, until one realizes that the game itself is defined on its 2^n coalitions. In other words, the input size is not the number of players n , but the number of coalitions $x = 2^n$. Thus, testing $\frac{2^n n(n-1)}{8}$ conditions is only of complexity $\mathcal{O}(x(\log x)^2)$.

3 The Shapley value of convex games

As mentioned before, the Shapley value φ , which assigns to each game $(N, v) \in G^N$ the average of its marginal vectors, i.e.,

$$\forall i \in N : \quad \varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)],$$

is of particular appeal in convex games: in such a game, the core is equal to the convex hull of the marginal vectors (the Weber set) and, consequently, the Shapley value is the barycenter of the core.

Young (1985) axiomatizes the Shapley value by replacing the original additivity and dummy axioms of Shapley (1953) with a monotonicity condition. He provides his axiomatization on two different classes of games: the class G^N of all n -player games and the class of n -player superadditive games. See Timmer et al. (2000) for a characterization on a third class of games. The purpose of this section is to axiomatize the Shapley value on the set of convex games in the same spirit as Young (1985).

Let $C^N \subset G^N$ denote the set of convex games. A solution concept on C^N is a function ψ that assigns to each game $(N, v) \in C^N$ a vector $\psi(v) \in \mathbb{R}^n$, specifying a payoff $\psi_i(v)$ to each player $i \in N$.

Theorem 3.1 The Shapley value φ is the unique solution concept ψ on C^N satisfying:

efficiency: For all $(N, v) \in C^N : \sum_{i \in N} \psi_i(v) = v(N)$;

symmetry: For all $(N, v) \in C^N$ and all $i, j \in N$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) = \psi_j(v)$;

strong monotonicity: For all $(N, v), (N, w) \in C^N$ and all $i \in N$, if $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, then $\psi_i(v) \geq \psi_i(w)$.

Proof. The Shapley value φ satisfies the axioms. Conversely, assume that the solution concept ψ on C^N also satisfies the axioms. For each $(N, v) \in C^N$ there are unique numbers $(\alpha_T(v))_{T \in \mathcal{C}}$ such that $v = \sum_{T \in \mathcal{C}} \alpha_T(v) u_T$. For each $t \in \{1, \dots, n\}$, define $\beta_t(v) = \max_{T \in \mathcal{C}, |T|=t} \alpha_T(v)$. For each $T \in \mathcal{C}$, define $\gamma_T(v) = \beta_{|T|}(v) - \alpha_T(v) \geq 0$. The proof proceeds by induction on $k(v) = |\{T \in \mathcal{C} \mid \gamma_T(v) > 0\}|$.

If $k(v) = 0$, then $\alpha_T(v) = \beta_{|T|}(v)$ for all T , so $v = \sum_{T \in \mathcal{C}} \beta_{|T|}(v)$ is a symmetric game. Symmetry and efficiency imply that $\varphi_i(v) = \psi_i(v) = v(N)/n$ for all $i \in N$.

Let $k \in \mathbb{N}$. Assume that φ and ψ coincide on all games $(N, v) \in C^N$ with $k(v) \leq k - 1$ and consider a game $(N, v) \in C^N$ with $k(v) = k$. Take $D = \cap_{T \in \mathcal{C}, \gamma_T(v) > 0} T$. Then $\gamma_N(v) = 0$ implies $D \neq N$. Let $i \in N \setminus D$ and define the auxiliary game

$$\begin{aligned} w^i &= v + \sum_{T \in \mathcal{C}, i \notin T} \gamma_T(v) u_T \\ &= \sum_{T \in \mathcal{C}, i \in T} \alpha_T(v) u_T + \sum_{T \in \mathcal{C}, i \notin T} \beta_{|T|}(v) u_T. \end{aligned} \quad (6)$$

So for the unanimity coefficients $\alpha_T(w^i)$ of (N, w^i) we have

$$\alpha_T(w^i) = \begin{cases} \alpha_T(v) & \text{if } i \in T, \\ \beta_{|T|}(v) & \text{if } i \notin T. \end{cases} \quad (7)$$

(N, w^i) is convex: (N, v) is convex and the unanimity coefficients for coalitions T with $i \notin T$ are increased from $\alpha_T(v)$ to $\beta_{|T|}(v)$, so if the conditions (4) hold for v , they definitely hold for w^i . Moreover, (7) implies that for each $t \in \{1, \dots, n\}$: $\beta_t(w^i) = \max_{T \in \mathcal{C}, |T|=t} \alpha_T(w^i) = \beta_t(v)$ and for each coalition T ,

$$\gamma_T(w^i) = \beta_{|T|}(w^i) - \alpha_T(w^i) = \beta_{|T|}(v) - \alpha_T(w^i) = \begin{cases} \gamma_T(v) & \text{if } i \in T, \\ 0 & \text{if } i \notin T. \end{cases}$$

Conclude that

$$\begin{aligned} k(w^i) &= |\{T \in \mathcal{C} \mid \gamma_T(w^i) > 0\}| \\ &= |\{T \in \mathcal{C} \mid \gamma_T(v) > 0, i \in T\}| \\ &< |\{T \in \mathcal{C} \mid \gamma_T(v) > 0\}| \\ &= k(v), \end{aligned}$$

where the inequality follows from the fact that $i \in N \setminus D$, so $\gamma_T(v) > 0$ and $i \notin T$ for some coalition $T \in \mathcal{C}$. Since $k(w^i) < k(v) = k$, induction implies that

$$\psi(w^i) = \varphi(w^i). \quad (8)$$

For every coalition $S \in \mathcal{C}$ with $i \notin S$, (6) implies

$$\begin{aligned} w^i(S \cup \{i\}) - w(S) &= \sum_{T \in \mathcal{C}, i \in T} \alpha_T [u_T(S \cup \{i\}) - u_T(S)] + \sum_{T \in \mathcal{C}, i \notin T} \beta_{|T|} [u_T(S \cup \{i\}) - u_T(S)] \\ &= \sum_{T \in \mathcal{C}, i \in T} \alpha_T [u_T(S \cup \{i\}) - u_T(S)] + \sum_{T \in \mathcal{C}, i \notin T} \alpha_T [u_T(S \cup \{i\}) - u_T(S)] \\ &= v(S \cup \{i\}) - v(S), \end{aligned}$$

where the second equality follows from the fact that $u_T(S \cup \{i\}) - u_T(S) = 0$ whenever $i \notin T$. Strong monotonicity implies that $\varphi_i(w^i) = \varphi_i(v)$ and $\psi_i(w^i) = \psi_i(v)$. Together with (8), this implies

$$\forall i \in N \setminus D : \quad \varphi_i(v) = \psi_i(v). \quad (9)$$

Next, let $i, j \in D$ and $S \subseteq N \setminus \{i, j\}$. Then

$$\begin{aligned} v(S \cup \{i\}) &= \sum_{T \in \mathcal{C}} \beta_{|T|}(v) u_T(S \cup \{i\}) - \sum_{T \in \mathcal{C}} \gamma_T(v) u_T(S \cup \{i\}) \\ &= \sum_{T \in \mathcal{C}} \beta_{|T|}(v) u_T(S \cup \{j\}) - \sum_{T \in \mathcal{C}} \gamma_T(v) u_T(S \cup \{j\}) \\ &= v(S \cup \{j\}), \end{aligned}$$

where the first and third equality follow from $v = \sum_{T \in \mathcal{C}} \alpha_T(v) u_T$ and the definitions of $\beta_{|T|}$ and γ_T and the second equality follows from symmetry of the game $\sum_{T \in \mathcal{C}} \beta_{|T|} u_T$ and the fact that $\gamma_T(v) > 0$ together with $i, j \in D$ imply that $u_T(S \cup \{i\}) = u_T(S \cup \{j\}) = 0$. Hence any two players $i, j \in D$ are symmetric: $\varphi_i(v) = \varphi_j(v)$ and $\psi_i(v) = \psi_j(v)$. Together with efficiency and (9), this implies that $\varphi_i(v) = \psi_i(v)$ also if $i \in D$, finishing our proof. 2

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