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Prospect Theory and Higher Moments


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Prospect Theory and Higher Moments

Martin Ågren
PROSPECT THEORY AND HIGHER MOMENTS

MARTIN ÅGREN

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Prospect Theory and Higher Moments

Martin Ågren*

October, 2006

Abstract

The paper relates cumulative prospect theory to the moments of returns distributions, e.g. skewness and kurtosis, assuming returns are normal inverse Gaussian distributed. The normal inverse Gaussian distribution parametrizes the first- to forth-order moments, making the investigation straightforward. Cumulative prospect theory utility is found to be positively related to the skewness. However, the relation is negative when probability weighting is set aside. This shows that cumulative prospect theory investors display a preference for skewness through the probability weighting function. Furthermore, the investor’s utility is inverse hump-shape related to the kurtosis. Consequences for portfolio choice issues are studied. The findings, among others, suggest that optimal cumulative prospect theory portfolios are not mean-variance efficient under the normal inverse Gaussian distribution.

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1 Introduction

Behavioral finance has emerged as an alternative approach to financial economics largely because of the difficulties of the traditional theory. The most acclaimed behavioral model of individual decision-making under risk is prospect theory. Kahneman and Tversky (1979) demonstrate a number of individual violations of neoclassical expected utility based on experimental evidence, and in spirit of these violations they propose prospect theory as a more realistic model. Although successful in many applications, the original version has its drawbacks. For one, utility can be derived from gambles of only two outcomes, and, for the other, the attractive property of first-order stochastic dominance does not hold. As a resolution, Tversky and Kahneman (1992) introduce cumulative prospect theory (CPT), where utility is derived from gambles of any number of outcomes, and first-order stochastic dominance holds.

CPT is perhaps the most complete summary of the experimental evidence on attitude to risk. Under CPT, investors derive utility by using a specific value function, and by weighting probabilities subjectively. The latter feature transforms the outcome distribution so that small probabilities are over-weighted, which magnifies the tails of the distribution, and moderate to large ones are under-weighted. The value function differs from standard concave utility functions, e.g. power utility, in three main respects. First, utility is derived from changes in wealth relative to a reference point, as opposed to final levels of wealth. Second, the value function is concave over gains, implying risk aversion, but convex over losses, reflecting a risk-seeking behavior in that domain. Third, losses loom larger than gains do, causing for a kink in the value function at the reference point. This last property, referred to as loss aversion, implies a high sensitivity for small changes in wealth. In contrast, standard concave utility functions display local risk-neutrality.\footnote{Loss aversion is related to the concept of first-order risk aversion. See Epstein and Zin (1990).}

There exists a number of applications of prospect theory and its modified version CPT in financial economics research. Shefrin and Statman (1985) apply prospect theory to help explain the disposition effect, which concerns the disposition of individual investors to sell winning stocks too early, and hold on to losers for too long.\footnote{Recently, however, Barberis and Xiong (2006) argue that prospect theory predicts the opposite of the disposition effect.} A plausible clarification of the ambiguous endowment effect, which refers to the individual tendency to value something more heavily once owned, is put forward by Kahneman, Knetsch, and Thaler (1990). The most celebrated finance application of CPT, and in particular loss aversion, is presented by Benartzi and Thaler (1995) however. Stocks are perceived as more risky among loss-averse investors if
evaluated frequently, since losses occur with greater probability over shorter time horizons. Benartzi and Thaler (1995) show that "myopic loss aversion" can explain the historical magnitude of equity premium over bonds if evaluated yearly. Consequently, they propose a behavioral explanation to the infamous equity premium puzzle of Mehra and Prescott (1985). Barberis, Huang, and Santos (2001) generalize Benartzi and Thaler's (1995) single-period model in a multi-period general equilibrium context. They argue that loss aversion alone does not produce a large enough equity premium, and incorporate an investor sensitivity for prior outcomes as a resolution, causing for a time-varying loss aversion.

The current paper relates to the literature on CPT portfolio choice. Levy, De Giorgi, and Hens (2003) are first to show that CPT efficient portfolios are, in fact, also mean-variance efficient, provided that returns are normally distributed. They, also, prove that the standard two-period capital asset pricing model (CAPM) is consistent with CPT. The mean-variance optimization algorithm can thus be employed when constructing CPT efficient portfolios, something that is quite remarkable since CPT stands in such sharp contrast to the assumptions of mean-variance analysis, namely expected utility maximization and global risk aversion. Levy and Levy (2004) and Barberis and Huang (2005) present analogous proofs to solidify the result. However, the normality assumption weakens the general understanding of CPT in portfolio choice issues. Financial returns distributions are often skewed and fat-tailed, which are characteristics that the normal distribution cannot model since it is fully determined by the mean and the variance. A natural question that comes to mind is how CPT utility is related to the higher-order moments, e.g., skewness and kurtosis. Since loss aversion implies an asymmetric preference over gains and losses, and probability weighting magnifies the tails of the returns distribution, the question is relevant. Furthermore, are CPT portfolios mean-variance efficient under more general distributional assumptions than normality? These issues are addressed in the current paper.

Few previous studies within the literature of optimal asset allocation consider higher-order moments. Kraus and Litzenberger (1976) present an unconditional three-moment CAPM, and find that investors with standard concave utility functions like skewness. This result is in line with Arditti (1967), who shows that most standard concave utility functions, e.g., logarithmic and power utility imply a preference for skewness, since they fulfill the condition of non-increasing absolute risk aversion. Harvey and Siddique (2000) expand the conditional CAPM to include

3The puzzle concerns the inability to explain the historical magnitude of U.S. equity premium within a standard consumption-based general equilibrium model at reasonable parameter values.
coskewness with the market, which helps to explain the cross-section of equity returns. Furthermore, Ägren (2006) presents a technical extension to the work of Benartzi and Thaler (1995) by incorporating conditional heteroskedasticity in returns, in contrast to the original temporal independence assumption. The results show that overall longer evaluation periods are needed under conditional heteroskedasticity when considering both U.S. and Swedish data. Consequently, Ägren (2006) argues that prospect theory utility is sensitive to the distributional assumption of returns, especially concerning the skewness.

Assuming returns are normal inverse Gaussian (NIG) distributed, this paper addresses the implications of higher moments for CPT portfolio choice. The NIG distribution, presented by Barndorff-Nielsen (1997), is a four parameter distribution with the desirable property of parameter-dependent higher-order moments. Eriksson, Forsberg, and Ghysels (2005) present a useful transformation of the NIG distribution’s parameters so that its probability density can be expressed as a function of the first four cumulants. Cumulants are a set of distributional descriptive constants just like moments are. The first and second cumulants equal the respective first and second central moments, i.e. the mean and the variance, while skewness and kurtosis are simple normalizations of the third and forth cumulants, respectively. The transformed alternative parameterization makes a straightforward link between utility and cumulants possible so that the effects of a change in one specific distributional characteristic, such as skewness, can be analyzed in isolation, i.e., without affecting the other ones. This makes the NIG distribution highly suitable for the current investigation.

The paper considers a risky portfolio with NIG distributed return in a single-period framework. There are two main objectives. First, an analysis of portfolio utility in relation to the return’s distributional characteristics is conducted, where three kinds of investor preferences are considered: CPT, CPT without investor probability weighting, which is referred to as expected loss aversion (ELA), and expected power (EP) utility. In this way, the implications of probability weighting and loss aversion can be separated, and compared with traditional utility theory. Second, the CPT portfolio choice is examined by optimizing the allocation to a risky and a relatively risk-free asset. Both analyses involve model-parameter calibrations to empirical estimates.

I show that investor utility is positively related to the portfolio’s mean, and negatively related to its variance, irrespective of the preference scheme. Intuitively,
the result for the variance is somewhat surprising considering that CPT investors are risk-seeking over losses. Loss aversion dominates however, implying a preference for low-variance portfolios. Furthermore, the relation between utility and skewness is negative when ELA preferences are considered, but turns positive when probability weighting is introduced, i.e., when CPT preferences are embraced in full. This shows that CPT investors display a preference for skewness through the probability weighting function. Essentially, loss aversion makes the ELA investor sensitive to the probability of small losses, while CPT investors, over-weighting the probability of extreme outcomes, care more about the probability of large losses. While CPT investors prefer lottery-type gambles with positively skewed outcomes as they might receive a large gain, the ELA investor is averse to such gambles since they incur a small but almost sure loss.

Utility and kurtosis are positively related under ELA, but inverse hump-shape related under CPT. The relation is difficult to explain, and is quite sensitive to the level of loss aversion and degree of probability weighting. The extent to which the investor suffers from loss aversion in relation to her degree of probability weighting determines the relation between CPT utility and kurtosis.

What implications do these results have for the optimal asset allocation? To answer this question, the CPT portfolio choice problem is analyzed under the NIG distributional assumption. Related research includes Aït-Sahalia and Brandt (2001), who study the optimal set of predictive variables for portfolio choice over different preference schemes, among them CPT, and Berkelaar, Kouwenberg, and Post (2004), who analyze the optimal investment strategy of CPT investors when assuming general Ito processes for asset prices. The two studies do not consider probability weighting however, but analyze what I refer to as ELA preferences. Neither do they consider the effects of skewness and kurtosis on the portfolio choice.

Consistent with Aït-Sahalia and Brandt (2001) and Berkelaar et al. (2004), I find strong horizon effects in the investor’s asset allocation. The portion of stocks progresses heavily as the horizon increases. Moreover, the results suggest that CPT optimal portfolios are not mean-variance efficient under the NIG assumption, with the investor typically placing a relatively larger weight on stocks when higher moments are taken into account.

The rest of the paper is outlined as follows: Section 2 introduces CPT, and explains how to derive CPT utility under a distributional assumption. Section 3 presents the NIG distribution in general, as well as in a more useful alternative form. Section 4 analyzes investor utility as a function of the portfolio’s mean, variance, skewness, and kurtosis. Section 5 turns to the optimal portfolio choice of CPT investors. Section 6 concludes.
2 Cumulative Prospect Theory

Tversky and Kahneman (1992) present two cornerstone functions for CPT utility: a value function over outcomes, \( v(\cdot) \), and a weighting function over cumulative probabilities, \( w(\cdot) \). The CPT utility of a gamble \( G \) with stochastic return \( X \) is defined as

\[
U(G) \equiv E^w [v(X)],
\]

where \( E^w \[ \cdot \] \) is the unconditional expectations operator under subjective probability weighting, indicated by \( w \), and \( v(X) \) is the value function.

2.1 Value Function

The value function derives utility from gains and losses, and not from final wealth as traditional utility functions do. Tversky and Kahneman (1992) suggest the following functional form:

\[
v(x) = \begin{cases} 
(x - \bar{x})^\gamma & \text{if } x \geq \bar{x} \\
-\lambda(\bar{x} - x)^\gamma & \text{if } x < \bar{x}
\end{cases},
\]

where outcomes \( x \) are separated into gains and losses with respect to a reference point \( \bar{x} \), which is thought of as a sure alternative to the risky gamble.\(^6\)

The value function in (2) exhibits loss aversion when \( \lambda > 1 \), which is motivated by the experimental finding that individual investors are more sensitive to losses than to gains. Although its expected value is positive, a fifty-fifty bet of winning $200 or losing $100 is generally rejected, since a loss of $100 is perceived as more painful than a gain of $200 is enjoyable. Moreover, (2) allows for risk aversion over gains but risk-seeking over losses when \( \gamma < 1 \). Consider a gamble with a fifty percent chance of winning $100 or nothing to the alternative of receiving $50 for sure. Most individuals would prefer the sure gain to the risky gamble since they are risk-averse over gains. They prefer the expected value to the gamble. In comparison, consider a gamble with a fifty percent probability of losing $100 or nothing. When choosing between this gamble and the alternative of giving up $50 for sure, experimental evidence shows that individuals generally prefer to take on the gamble. They are risk-seeking over losses, and, hence, favor the gamble to its expected value.

Figure 1 illustrates the value function for a few parameter-value combinations and with a zero reference return. Loss aversion causes the value function to be kinked at the reference point, reflecting a dramatic change in marginal utility. With \( \gamma < 1 \), the value function becomes concave over gains and convex over losses. Tversky and

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\(^6\)When considering the gamble of investing in a portfolio of risky assets, a common assumption is to let the average return on a risk-free asset represent the investor’s reference return.
The figure illustrates the cumulative prospect theory value function over returns (%) for a few parameter-value combinations, and with a zero reference return. Tversky and Kahneman (1992) suggest $\hat{\lambda} = 2.25$ and $\hat{\gamma} = 0.88$.

Kahneman (1992) conduct individual experiments, and estimate the value function’s parameters to $\hat{\lambda} = 2.25$ and $\hat{\gamma} = 0.88$.

### 2.2 Probability Weighting

The probability weighting function $w(\cdot)$ applies to cumulative probabilities. Essentially, it over-weights small probabilities so that the tails of the distribution are magnified. This feature of CPT stems from experimental evidence showing that individuals perceive extreme events as more likely to occur than they really are.$^7$ Furthermore, moderate to large probabilities are under-weighted, which reflects the pessimism individuals might feel toward a relatively sure outcome. Tversky and Kahneman (1992) propose the following function:

$$w(P) = \frac{P^\tau}{(P^\tau + (1 - P)^\tau)^{1/\tau}}, \quad (3)$$

where $P$ is the objective cumulative probability, and $\tau \in (0, 1]$ is a function parameter.$^8$ In (3), cumulative probabilities are weighted non-linearly to the extent determined by $\tau$. Since it is cumulative probabilities that are weighted and not

$^7$For instance, why do people buy lottery tickets?

$^8$Other functional forms of probability weighting have been proposed. See, e.g., Prelec (1998).
the actual ones, CPT is consistent with first-order stochastic dominance.\(^9\) Moreover, probability weighting should not be associated with a change of probability measure, since the weighted probabilities, in fact, need not sum up to one.\(^{10}\)

**Figure 2: The Weighting Function**

Figure 2 illustrates the cumulative prospect theory weighting function for a few parameter values. Tversky and Kahneman (1992) suggest \(\hat{\tau} = 0.65\).

Figure 2 illustrates \(w(P)\) for a few values of \(\tau\). When \(\tau = 1\), the function collapses so that \(w(P) = P\), and the CPT investor treats probabilities linearly. A value of \(\tau < 1\) introduces probability weighting, and the lower the value the more prominent the weighting becomes. Tversky and Kahneman (1992) suggest \(\hat{\tau} = 0.65\) by way of individual experiments.\(^{11}\)

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\(^9\)The original version of prospect theory weights actual probabilities, and, therefore, lacks the property of first-order stochastic dominance.

\(^{10}\)For this reason, Kahneman and Tversky (1979) refer to the weighted probabilities as decision weights.

\(^{11}\)Actually, Tversky and Kahneman (1992) estimate \(\tau\) to 0.61 in the gains domain, and 0.69 in the loss domain. For simplicity, I approximate the value of \(\tau\) with the average of these two estimates.
2.3 Incorporating a Distributional Assumption

Consider a risky portfolio, $G$, with a stochastic return, $X$, that is continuously distributed. CPT utility, defined in (1), is then derived as

$$U(G; \theta) \equiv U(\theta)$$

$$= - \int_{-\infty}^{\infty} v(x) dw(1 - F(x)) + \int_{-\infty}^{\infty} v(x) dw(F(x))$$

$$= \int_{\bar{x}}^{\infty} v(x) w'(1 - F(x)) f(x) dx + \int_{-\infty}^{\bar{x}} v(x) w'(F(x)) f(x) dx,$$

where $f(\cdot)$ is the probability density function of $X$, $F(\cdot)$ is the corresponding cumulative distribution function, $v(\cdot)$ is the value function in (2), $w(\cdot)$ is the weighting function in (3), and $\theta$ is a vector of parameters. Tversky and Kahneman (1992) consider gambles with discrete outcomes for which CPT utility is expressed differently. Similar to Barberis and Huang (2005), the expression presented here is adjusted to allow for continuous probability distributions.

Notice, in (4), that the weighting function applies differently in the domain of gains and in the domain of losses. Moreover, utility is expressed using both the Riemann-Stieltjes integral as well as the Riemann integral. Although the former expression is, perhaps, easier to relate to Tversky and Kahneman’s (1992) discrete representation, the latter one is attractive for computational reasons.

Expression (4) holds under any continuous distributional assumption for $X$. In this paper, I assume $X$ is NIG distributed.

3 Normal Inverse Gaussian Distribution

The NIG distribution is introduced by Barndorff-Nielsen (1997) in an application to stochastic volatility modeling. It is a mixture of the normal distribution and the inverse Gaussian (IG) distribution.\textsuperscript{12} Formally, if a normally distributed variable $X$ has its variance drawn from the IG distribution, i.e.,

$$X|Z = z \sim N(\mu, z),$$

where

$$Z \sim IG \left( \delta, \sqrt{\alpha^2 - \beta^2} \right),$$

\textsuperscript{12}The IG distribution is defined over the interval $[0, \infty)$. The name stems from the fact that the cumulant generating function of an IG distributed variable is the inverse of the cumulant generating function of a normally (Gaussian) distributed variable.
then $X$ is NIG distributed with parameters $\alpha$, $\beta$, $\mu$, and $\delta$. Since I apply a result of Eriksson et al. (2005), their standard parametrization is used: $\bar{\alpha} = \delta \alpha$ and $\bar{\beta} = \delta \beta$.

The $NIG(\bar{\alpha}, \bar{\beta}, \mu, \delta)$ probability density function is given by

$$f_{NIG}(x; \bar{\alpha}, \bar{\beta}, \mu, \delta) = \frac{\bar{\alpha}}{\pi \delta} \exp \left( \sqrt{\bar{\alpha}^2 - \bar{\beta}^2} - \frac{\bar{\beta} \mu}{\delta} \right) K_1 \left[ \frac{\bar{\alpha} \sqrt{1 + \left( \frac{x - \mu}{\delta} \right)^2}}{\sqrt{1 + \left( \frac{x - \mu}{\delta} \right)^2}} \right] \exp \left( \frac{\bar{\beta}}{\delta} x \right),$$

where $x \in \mathbb{R}$, $\bar{\alpha} > 0$, $0 < \bar{\beta} < \bar{\alpha}$, $\delta > 0$, $\mu \in \mathbb{R}$, and $K_1$ is the modified Bessel function of third order with index 1. The mean, the variance, the skewness, and the kurtosis of $X \sim NIG(\bar{\alpha}, \bar{\beta}, \mu, \delta)$ are given by

$$E[X] = \mu + \frac{\bar{\beta} \delta}{\sqrt{\bar{\alpha}^2 - \bar{\beta}^2}} \quad (6)$$

$$V[X] = \frac{\delta^2 \bar{\alpha}^2}{(\bar{\alpha}^2 - \bar{\beta}^2)^{3/2}} \quad (7)$$

$$S[X] = \frac{3\bar{\beta}}{\bar{\alpha}(\bar{\alpha}^2 - \bar{\beta}^2)^{1/4}} \quad (8)$$

and

$$K[X] = \frac{12\bar{\beta}^2 + 3\bar{\alpha}^2}{\bar{\alpha}^2 \sqrt{\bar{\alpha}^2 - \bar{\beta}^2}} \quad (9)$$

respectively.

While the normal distribution has zero skewness and a kurtosis equal to three, equations (8) and (9) show that a NIG distributed variable has parameter-dependent skewness and kurtosis. Explicitly, the parameters of the NIG density can be interpreted as follows: $\bar{\alpha}$ and $\bar{\beta}$ are shape parameters with $\bar{\beta}$ expressing the skewness of the distribution, and, when $\bar{\beta} = 0$, $\bar{\alpha}$ determining the amount of excess kurtosis.13

The parameter $\mu$ is a location parameter, and $\delta$ is a scale parameter.14

To illustrate the NIG distribution’s ability to capture the characteristics of financial returns distributions, consider the real six-month returns of the S&P 500 composite index from Ibbotson Associates. Table 1 reports on summary statistics. Over the sample period of January 1926 to December 2003, the returns average at 5.56 percent, with a standard deviation of 15.13 percent. The skewness and kurtosis equal 1.07 and 9.21, respectively, indicating that the data is non-normally distributed. Indeed, the Jarque-Bera test of normality is highly significant. Figure 3 illustrates both the empirical stock returns distribution (panel A), as well as two

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13 Excess kurtosis refers to the amount of kurtosis that exceeds that of the normal distribution.
14 To read more on the NIG distribution and its use in stochastic volatility modeling, see, e.g., Andersson (2001) and Forsberg (2002).
Table 1: Summary Statistics for Financial Returns

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P500, nominal returns</th>
<th>S&amp;P 500, real returns</th>
<th>U.S. 30-day bill, real returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>One-month horizon</td>
<td>Horizon (months)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Mean (%)</td>
<td>0.99</td>
<td>0.90</td>
<td>5.56</td>
</tr>
<tr>
<td>Max. (%)</td>
<td>42.56</td>
<td>53.64</td>
<td>113.31</td>
</tr>
<tr>
<td>Min. (%)</td>
<td>-29.73</td>
<td>-25.48</td>
<td>-44.03</td>
</tr>
<tr>
<td>Std. dev. (%)</td>
<td>5.62</td>
<td>5.85</td>
<td>15.13</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.39</td>
<td>1.62</td>
<td>1.07</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>12.45</td>
<td>19.80</td>
<td>9.21</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.18</td>
<td>0.15</td>
<td>0.37</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>3488</td>
<td>11357</td>
<td>1663</td>
</tr>
<tr>
<td></td>
<td>(&lt;0.001)</td>
<td>(&lt;0.001)</td>
<td>(&lt;0.001)</td>
</tr>
<tr>
<td>No. obs.</td>
<td>936</td>
<td>936</td>
<td>931</td>
</tr>
</tbody>
</table>

The table reports on summary statistics for continuously compounded returns on the S&P 500 composite index and a U.S. 30-day Treasury bill, provided by Ibbotson Associates. The time period stretches from January 1926 to December 2003. One-month, six-month and twelve-month horizons are considered. Jarque-Bera is a test over skewness and kurtosis under the null of normality, where skewness equals zero and kurtosis is equal to three. p-values are in parentheses.
approximations, where both NIG and normality are assumed (panel B). Notice that
the NIG distribution captures both the skewness and the kurtosis of the empirical
distribution.

**Figure 3: Empirical and Approximate Distributions**

![Empirical and Approximate Distributions]

The figure illustrates the empirical distribution of S&P 500 real six-month returns (%) from January 1926 to December 2003 (panel A), together with two approximate distributions, namely the normal and normal inverse Gaussian distributions (panel B).

### 3.1 An Alternative Parameterization

Analyzing the relationship between CPT utility and the distributional characteristics of the portfolio’s return is complicated given its standard parameterization. Although the mean (6) and the variance (7) are quite easily altered by varying $\mu$ and $\delta$, respectively, it seems difficult to change, e.g., the distributional skewness without affecting another central moment. It would be preferable to parameterize the NIG distribution as a function of its mean, variance, skewness and kurtosis directly, instead of indirectly via the standard parameters $\alpha$, $\beta$, $\mu$, and $\delta$. Such an alternative parameterization would imply that an individual moment’s influence on utility can be analyzed in isolation, i.e., without affecting the other moments.

Eriksson et al. (2005) show that if the first four cumulants of $X$ exist, and fulfill a regularity condition, the NIG density can be expressed as a function of these first four cumulants. Cumulants are a set of descriptive constants of a distribution just like moments are, and, in some instances, they are more useful than moments.\(^\text{15}\)

\(^{15}\)To read more on moments and cumulants, see chapter 3 of Kendall and Stuart (1963).
The result of Eriksson et al. (2005) is very useful since the first and the second cumulants equal the mean and the variance, respectively, and the skewness and the kurtosis are simple normalizations of the third and forth cumulants.

Specifically, if we let $\kappa_1, \kappa_2, \kappa_3,$ and $\kappa_4$ denote the first four cumulants of the probability distribution of a stochastic variable $X$, the mean, the variance, the skewness, and the kurtosis of $X$ are given by

\begin{align*}
E[X] &= \kappa_1, \\
V[X] &= \kappa_2, \\
S[X] &= \frac{\kappa_3}{\kappa_2^{3/2}}, \\
K[X] &= \frac{\kappa_4}{\kappa_2^{2}} + 3,
\end{align*}

and respectively. Using (6)-(9) and (10)-(13), Eriksson et al. (2005) show that the NIG parameters $\bar{\alpha}, \bar{\beta}, \mu,$ and $\delta$ can be expressed as functions of the first four cumulants $\kappa_1, \kappa_2, \kappa_3,$ and $\kappa_4$. The following parameter transformations are presented:

\begin{align*}
\bar{\alpha} &= 3 \frac{4/\rho + 1}{\sqrt{1 - \rho^{-1}} \kappa_4}, \\
\bar{\beta} &= 3 \frac{\text{signum}(\kappa_3)}{\sqrt{\rho}} \frac{4/\rho + 1}{\sqrt{1 - \rho^{-1}} \kappa_4}, \\
\mu &= \kappa_1 - \frac{\text{signum}(\kappa_3)}{\sqrt{\rho}} \sqrt{(12/\rho + 3) \frac{\kappa_3^2}{\kappa_4}}, \\
\delta &= \sqrt{3(4/\rho + 1)(1 - \rho^{-1}) \frac{\kappa_3^2}{\kappa_4}},
\end{align*}

where $\rho = 3\kappa_4\kappa_2\kappa_3^{-2} - 4$.$^{16}$ The transformation is valid under the regularity condition $\rho > 1$.

Equations (5) and (14)-(17) imply an alternative parametrization of the NIG density, denoted $\tilde{f}_{NIG}$, which is a direct function of the first four cumulants, i.e., $\tilde{f}_{NIG} = \tilde{f}_{NIG}(x; \{\kappa_i\}_{i=1}^4)$. Using this alternative NIG density, one can approximate an empirical distribution by estimating its first four cumulants, $\{\kappa_i\}_{i=1}^4$, instead of estimating the standard NIG parameters $\bar{\alpha}, \bar{\beta}, \mu$ and $\delta$. More importantly, a single distributional characteristic can be altered without affecting the other ones, making the study of CPT utility in relation to a specific moment possible.

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$^{16}$The function signum($x$) equals the sign of $x$. 
4 Utility in Relation to Distributional Characteristics

This section presents an analysis of single-period portfolio utility in relation to the portfolio return’s distributional characteristics. Three kinds of investor preferences are considered, namely CPT, CPT without probability weighting, i.e., ELA, and EP utility preferences. The first two cases are considered in order to separate the effects of the value and weighting functions. EP utility preferences are considered to compare CPT with traditional utility theory.

4.1 Investor Utility with NIG Distributed Returns

Consider a single-period portfolio with NIG distributed stochastic return. Following (4), CPT utility is derived as

$$U(\theta) = \int_{-\infty}^{\bar{x}} (x - \bar{x})^\gamma w(1 - \bar{F}_{NIG}(x; \{\kappa_i\}_{i=1}^4)) \bar{f}_{NIG}(x; \{\kappa_i\}_{i=1}^4) dx$$
$$-\lambda \int_{-\infty}^{\bar{x}} (\bar{x} - x)^\gamma w(\bar{F}_{NIG}(x; \{\kappa_i\}_{i=1}^4)) \bar{f}_{NIG}(x; \{\kappa_i\}_{i=1}^4) dx,$$

(18)

where \(\bar{f}_{NIG}\) is the alternative NIG density function, \(\bar{F}_{NIG}\) is the corresponding cumulative distribution function, and \(w(\cdot)\) is the probability weighting function (3).\(^{17}\)

Utility parameters are gathered in \(\theta = (\gamma, \lambda, \tau, \bar{x}, \{\kappa_i\}_{i=1}^4)'\), where \(\gamma\) reflects risk aversion over gains and risk-seeking over losses, \(\lambda\) measures loss aversion, \(\tau\) determines the degree of probability weighting, \(\bar{x}\) is the reference return that separates gains from losses, and \(\{\kappa_i\}_{i=1}^4\) are the first four cumulants of the portfolio’s returns distribution.

Consider the case when \(\tau = 1\) in (18). The weighting function in (3) then collapses so that objective probabilities are considered, and utility becomes

$$U(\theta)|_{\tau=1} = \int_{-\infty}^{\bar{x}} (x - \bar{x})^\gamma \bar{f}_{NIG}(x; \{\kappa_i\}_{i=1}^4) dx$$
$$-\lambda \int_{-\infty}^{\bar{x}} (\bar{x} - x)^\gamma \bar{f}_{NIG}(x; \{\kappa_i\}_{i=1}^4) dx,$$

(19)

which is referred to as ELA utility.

EP utility under a NIG assumption is derived similarly to ELA utility, however using a different utility function. Replacing the value function in (19) by the constant

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\(^{17}\)To my knowledge, there is actually no closed form expression of the NIG c.d.f. It is, however, easily derived numerically using \(\bar{F}_{NIG}(x) = \int_{-\infty}^{x} \bar{f}_{NIG}(t) dt\).
relative risk aversion (CRRA) power utility function \( v(w) = \frac{w^{1-\eta}}{1-\eta} \), where \( w = 1 + \frac{x}{100} \) (\( x \) in percent) is final wealth, EP utility is formalized as

\[
V(\psi) = \int_{-\infty}^{\infty} \frac{(1 + \frac{x}{100})^{1-\eta}}{1-\eta} f_{NIG}(x; \{\kappa_i\}_{i=1}^{4}) dx, \tag{20}
\]

where \( \eta \) is the parameter of constant relative risk aversion, and \( \psi = (\eta, \{\kappa_i\}_{i=1}^{4})' \) is a parameter vector.

4.2 Analysis Procedure

Utility is analyzed in relation to the portfolio return’s distributional characteristics through the following procedure:

1. Consider one of the utility functions (18), (19) and (20), and calibrate its parameters using experimental or empirical estimates.

2. Vary a cumulant value of choice and register the variation in derived utility. Recall that a change in \( \kappa_i \) affects either the mean, the variance, the skewness, or the kurtosis, according to (10)-(13).\(^{18}\)

3. Illustrate utility as a function of the analyzed distributional characteristic graphically.

4. Carry out steps 2 and 3 for the other cumulants.

5. Carry out steps 2-4 for the other utility functions.

The procedure involves a calibration of the parameters in its first step. I use the Tversky and Kahneman (1992) estimates of \( \hat{\lambda} = 2.25, \hat{\gamma} = 0.88, \) and \( \hat{\tau} = 0.65 \), for CPT utility. The weighting function parameter is set to one when ELA utility is considered, implying objective probabilities. The parameter of relative risk aversion of EP utility is set to \( \eta = 3 \), which is reasonable.\(^{19}\) The first four cumulants, \( \{\kappa_i\}_{i=1}^{4} \), are estimated using the historical monthly nominal returns of the S&P 500 composite index. Table 1 presents summary statistics. The mean and the standard deviation equal 0.99 and 5.62 percent, respectively, while the skewness is 0.39 and the kurtosis equals to 12.45. Moreover, the investor’s reference return for CPT

\(^{18}\)Changing \( \kappa_2 \) alters the variance of the distribution, as (11) shows, but the measures of the skewness in (12) and the kurtosis in (13) are also affected. The latter changes are only matters of normalization however, and are not of concern. Specifically, the actual distributional skewness is not affected by \( \kappa_2 \), only its normalized measure.

\(^{19}\)See, e.g., Mehra and Prescott (1985).
and ELA utility, $\bar{x}$, is set to the risk-free nominal interest rate measured by the average return on a U.S. 30-day Treasury bill, which equals 0.31 percent.\(^2\) Hence, the gamble of investing in a single-period stock portfolio is considered, with the reference investment being a risk-free bill.

The derivation of utility in the second step involves numerical integration or quadrature. The Matlab programming function \textit{quad} is applied.

### 4.3 Results

Figures 4-7 illustrate ELA and EP utility as functions of the mean, the variance, the skewness, and the kurtosis, respectively, in panels A. Analogous functions for CPT utility are presented graphically in panels A of (8)-(11). To help clarify the distributional variations, panel B of each figure displays the two outermost distributions of analysis. For example, since changes in the mean vary within the interval of 0.5 percent to 1.4 percent, as panel A of figure 4 shows, panel B gives plots of two distributions with respective means equal to 0.5 percent and 1.4 percent, all other things equal.

**Figure 4:** Expected Utility in Relation to Mean

The figure plots expected loss-averse (ELA) and expected power (EP) utility as functions of the distributional mean of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B). In panel A, ELA (EP) utility is measured on the left (right) axis.

\(^2\)The average T-bills return is from Ibbotson Associates.
The figure plots expected loss-averse (ELA) and expected power (EP) utility as functions of the distributional variance of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B). In panel A, ELA (EP) utility is measured on the left (right) axis.

4.3.1 ELA and EP Utility

Figures 4 and 5 (panels A) show that ELA and EP utility are both positively related to the mean of the underlying returns distribution, and negatively related to its variance. The intuition for ELA preferences is that a higher mean decreases the probability of a loss, increasing utility, while a higher variance spreads the distribution and, hence, increases the probability of a loss, which decreases utility.

Illustrations of ELA and EP utility as functions of the skewness and the kurtosis are presented in figures 6 and 7, respectively. The figures also show the two outermost distributions, where the skewness equals either -2 or 2 (figure 6), and the kurtosis is either 3 or 20 (figure 7). A slightly hump-shaped relation between ELA utility and the skewness is shown. At reasonable levels of the skewness for stock returns, say greater than -1, utility falls as the skewness rises.\textsuperscript{21} Intuitively, when the skewness increases, the left tail of the distribution attenuates while the right tail fattens, but the center mass moves in the opposite direction to preserve the mean. Although the effect on the tails of the distribution increases ELA utility, since the probability of large losses is reduced, the adjustment of the center mass has a negative effect, since small losses become more probable. ELA utility falls

\textsuperscript{21}In table 1, the stock market returns skewness is greater than -1 overall.
Figure 6: Expected Utility in Relation to Skewness

The figure plots expected loss-averse (ELA) and expected power (EP) utility as functions of the distributional skewness of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B). In panel A, ELA (EP) utility is measured on the left (right) axis.

Figure 7: Expected Utility in Relation to Kurtosis

The figure plots expected loss-averse (ELA) and expected power (EP) utility as functions of the distributional kurtosis of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B). In panel A, ELA (EP) utility is measured on the left (right) axis.
when the distributional skewness increases since loss aversion induces an investor sensitivity to small losses.

Figure 7 presents ELA and EP utility plotted against a kurtosis between 3 and 20. The graph for ELA utility in panel A is clearly positively sloped, meaning that ELA utility increases with kurtosis. A plot of the two outermost examined distributions, found in panel B, helps in understanding this result. When kurtosis increases, the distributional tail masses thicken but the center mass becomes more peaked and concentrated around the mean. Although extreme negative returns become more likely, the effect is not large enough to offset the implications of a fall in the probability of small losses. Again, it is the effect on the probability of small losses that is decisive for the outcome. ELA utility rises since the probability of small losses decreases.

The results for ELA utility contrast to EP utility, which rises with the skewness, and falls when the kurtosis increases. The former result is expected following Arditti (1967), who shows that most standard concave utility functions, e.g., logarithmic and power utility imply a preference for skewness, since they fulfill the condition of non-increasing absolute risk aversion. The latter result, however, is new to the literature as far as the author is aware of. Intuitively, EP utility maximizers are most sensitive to the probability of larger outcomes since they do not exhibit first-order risk aversion. Thus, EP utility falls as the kurtosis increases.

4.3.2 CPT Utility

Let us now turn to CPT preferences, and include probability weighting in the analysis. Figures 8 and 9 illustrate CPT utility as respective functions of the mean and the variance. The graphs are similar to the ones for ELA utility; high-mean and low-variance portfolios are preferred by CPT investors too. However, the results for the skewness and the kurtosis change dramatically. Compared with ELA preferences, figures 10 and 11 show that utility now rises with the skewness, and is inverse hump-shape related to the kurtosis. Probability weighting causes small (cumulative) probabilities to be over-weighted so that the tails of the returns distribution are magnified. Hence, with a change in the skewness or the kurtosis, the effects on the probability tail-masses, i.e. the probability of extreme outcomes, is of greater importance. CPT utility rises since an increasing skewness attenuates the left tail. Of course, the probability of small losses still increases with a larger skewness, but the over-weighting of small probabilities dominates this effect.

The relation to the kurtosis is more complicated to explain. The inverse hump-shaped function in figure 11 makes it unclear which aspect of the distributional
Figure 8: CPT Utility in Relation to Mean

The figure plots cumulative prospect theory (CPT) utility as a function of the distributional mean of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B).

Figure 9: CPT Utility in Relation to Variance

The figure plots cumulative prospect theory (CPT) utility as a function of the distributional variance of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B).
The figure plots cumulative prospect theory (CPT) utility as a function of the distributional variance of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B).

The figure plots cumulative prospect theory (CPT) utility as a function of the distributional kurtosis of a single-period risky investment with the other distributional characteristics held constant (panel A), and the two outermost analyzed distributions (panel B).
change, following an increase in the kurtosis, that is most important for CPT utility. A larger kurtosis accentuates the tails, which raises the probability of large losses, while making the distribution more pointy, decreasing the probability of small losses. Since the first effect has bad implications for utility, and the second has good ones, the inverse hump-shaped function is likely the result of a balance between the two effects at the specified preference-parameter values, i.e., the ones provided by Tversky and Kahneman (1992).

4.4 Sensitivity Analysis

So far, the analysis has assumed Tversky and Kahneman’s (1992) estimates of the value and weighting functions’ parameters, i.e., \((\lambda, \gamma, \tau) = (2.25, 0.88, 0.65)\), but with \(\tau = 1\) for ELA utility. Are the obtained results sensitive to changes in these estimates? The question is analyzed by fixing the distributional parameters, i.e., the first four cumulants at their empirical estimates, and by varying the CPT preference parameters.

Parameter-value variations do not have any drastic effects on the results for the mean or the variance. Utility is negatively related to the variance so long as the investor is loss-averse, i.e., \(\lambda > 1\). This is the case despite a heavy degree of investor risk-seeking over losses, measured by \(\gamma\). A preference for high-variance portfolios appears when \(\lambda = 1\) however. Indeed, if the investor is risk-neutral with \((\lambda, \gamma) = (1, 1)\), she only has concern for a large return, irrespective of the level of risk, and the probability of large returns increases with a higher variance.

Not so surprising, the results for the skewness and the kurtosis turn out to be quite parameter sensitive, especially to the weighting function parameter \(\tau\). Recall that the investor’s preference for skewness and kurtosis changes quite dramatically when introducing probability weighting. Figure 6 shows a negative relation between ELA utility and the skewness when the skewness is greater than -1, while in figure 10, where probability weighting is considered, a clear positive relation is presented. What degree of probability weighting is sufficient to achieve this positive relation? Experimenting with different values, a \(\tau\) of 0.90 turns out to be adequate. In fact, the CPT investor has a preference for skewness so long as \(\tau \leq 0.90\), regardless of the level of loss aversion or degree of risk aversion/risk-seeking. Probability weighting is clearly the driving source of the CPT preference for skewed portfolios.

The positive relation between ELA utility and the kurtosis, previously explained to be driven by loss aversion, is presented in figure 7. When probability weighting is introduced, figure 11 presents an inverse hump-shaped relation however. Varying the parameter values, it is quite obvious that the level of loss aversion and the
degree of probability weighting have counteracting effects on utility. When $\lambda > 1$ and $(\gamma, \tau) = (1, 1)$, i.e. the investor suffers from "pure" loss aversion and weights probabilities linearly, utility is positively related to the kurtosis. The loss-averse investor’s sensitivity to the probability of small losses causes this result. On the contrary, when $\tau < 1$, $\gamma = 1$, and $\lambda > 1$ but close to one, i.e. the investor is mildly loss-averse and distorts probabilities, utility is negatively related to the kurtosis, which concerns the probability of large losses and the investor’s probability overweighting of such. In the general case of $\lambda > 1$ and $\tau < 1$, the interplay between the level of loss aversion and the degree of probability weighting implies an inverse hump-shaped relation, where the relation is first negative at low values of the kurtosis, but turns positive at larger ones. With $(\lambda, \gamma) = (2.25, 0.88)$, the relation to the kurtosis is positive and monotonic when $0.75 < \tau \leq 1$, but inverse hump-shaped related when $\tau \leq 0.75$.

5 Optimal Portfolio Choice with NIG Distributed Returns

This section turns to the single-period portfolio choice of CPT investors. Aït-Sahalia and Brandt (2001) and Berkelaar, Kouwenberg, and Post (2004) conduct similar studies, however without investigating the effects of higher-order moments on optimal asset allocation. Neither do the two studies consider probability weighting, but focus on loss aversion and the ELA investor’s behavior. Having found that probability weighting is a crucial ingredient of CPT when returns are non-normally distributed, a complete study of CPT portfolio choice includes this property.

The optimal allocation to a risky asset and a relatively risk-free asset is examined under the assumption of a NIG distributed portfolio return. To examine the effects of skewness and kurtosis on the portfolio choice, the normality assumption is also considered in comparison to the NIG. I study the investment strategies of both the ELA investor, who applies objective probabilities, and the complete CPT investor, who weights probabilities subjectively.

5.1 Data Set

The risky and the relatively risk-free assets are represented by continuously compounded real returns of the S&P 500 composite index and a U.S. 30-day Treasury bill, respectively. Real and not nominal returns are used in the analysis, since real returns are more kind to NIG approximations; the regularity condition does not hold
for nominal returns, while real returns cause no problem.\textsuperscript{22} Investment horizons of one, six, and twelve months are considered, where a moving window is used when calculating the lower frequency data.

Summary statistics of the data across all frequencies are reported on in table 1. Over the sample period of January 1926 to December 2003, the monthly real aggregate stock return has averaged at 0.90 percent, compared with the real bill return of 0.06 percent. The empirical monthly standard deviations of the two assets are 5.85 and 0.52 percent. The mean returns increase at longer horizons, but so do the standard deviations, naturally. Yearly returns average at 11.50 and 0.77 percent and have standard deviations of 24.29 and 4.06 percent for the stock and bill assets, respectively.

Over the one-, six-, and twelve-month horizons the skewness of the empirical stock returns distributions are 1.62, 1.07, and 2.17, respectively, and the respective kurtosis are 19.80, 9.21, and 20.71. Hence, neither the skewness nor the kurtosis is monotonically increasing or decreasing as the horizon increases. All data series, including the ones for real bill returns, deviate from normality to such an extent that the Jarque-Bera test statistics are significant throughout.

5.2 Portfolio Choice Problem

Formally, the portfolio choice problem is stated as

\[
\max_{q_s,q_{tb}} E^w[v(X)] = \int_{\mathbb{R}} (x - \bar{x})^\gamma u'(1 - F(x; \xi)) f(x; \xi) dx \\
- \lambda \int_{-\infty}^{\bar{x}} (\bar{x} - x)^\gamma u'(F(x; \xi)) f(x; \xi) dx,
\]

subject to

\[
X = q_s X_s + q_{tb} X_{tb},
\]

and

\[
q_s + q_{tb} = 1, \\
q_s, q_{tb} \in [0, 1],
\]

where \(q_s\) (\(q_{tb}\)) denotes the weight of stocks (bills), \(X\) is the composed portfolio’s stochastic return, \(f(x; \xi)\) is the probability density function of \(X\), \(F(x; \xi)\) is the corresponding cumulative distribution function, \(\xi\) is a vector of distributional para-

\textsuperscript{22}Nominal Treasury bills have empirical returns distributions that are far from "bell-shaped", resulting in cumulant estimates that do not fulfill the NIG regularity condition.
meters, and $X_s$ ($X_b$) is the stochastic return on the stock (bill) asset. The portfolio’s return is assumed either NIG or normally distributed. The constraints (23) imply that short selling is not allowed.\textsuperscript{23}

### 5.3 Results

Table 2 reports on the optimal portfolio weights of stocks and bills of an ELA investor with loss aversion parameter $\lambda$ equal to 1, 2.25, or 3, and risk aversion/risk-seeking parameter $\gamma$ equal to 0.6, 0.88, or 1. Panel A presents the results under the NIG assumption, and panel B under normality. The sharpe ratio, i.e. the mean divided by the standard deviation of the optimal portfolio composition, is also provided.

The results show that an investor who does not value losses any more than she does gains, i.e. $\lambda = 1$, allocates one hundred percent to stocks over all horizons, irrespective of the degree of risk aversion/risk-seeking and whether NIG or normality is assumed. Loss aversion is the investor’s main source of aversion to risk, and without it she is practically risk-neutral.

Consistent with previous studies such as Aït-Sahalia and Brandt (2001), the investor’s portfolio choice displays large horizon effects. Larger weights are placed on stocks as the horizon increases. Under the NIG assumption, an ELA investor with $(\lambda, \gamma) = (2.25, 0.88)$ increases her allocation to stocks from 5.2 to one hundred percent when the investment horizon rises from one to six months. This is quite a dramatic increase.\textsuperscript{24} With a higher loss aversion of $\lambda = 3$, the allocations to risky stocks over the horizons are also very progressive; 3.7 percent at the one-month, 33 percent at the six-months, and one hundred percent at the yearly horizon. Benartzi and Thaler (1995) explain that loss-averse investors perceive stocks as less risky at longer horizons, since losses occur with smaller probability.\textsuperscript{25} On the contrary, Merton (1969) and Samuelson (1969) show that the portfolio choice under traditional expected utility preferences are horizon independent, so long as returns are i.i.d.\textsuperscript{26}

The ELA investor allocates to a fairly similar portfolio under normality as she does under the NIG assumption, as panel B shows. Previously, it was found that the ELA investor cares about the probability mass surrounding the reference return, particularly the probability of small losses. Similar weights are obtained under the

\textsuperscript{23}The optimization problem (21) is solved by using the Matlab constrained minimization routine \textit{fmincon}.

\textsuperscript{24}The weight on stocks is one hundred percent at the yearly horizon as well.

\textsuperscript{25}The stock return’s probability mass moves further away from the reference return as the horizon increases.

\textsuperscript{26}Barberis (2000) shows that this result breaks down if returns are somehow predictable, e.g., mean-reverting.
Table 2: Single-Period Portfolio Choice of ELA Investors

<table>
<thead>
<tr>
<th>Panel A: NIG Assumption</th>
<th>One-Month Horizon</th>
<th>Six-Month Horizon</th>
<th>Twelve-Month Horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$q_s$</td>
<td>$q_{tb}$</td>
<td>$S$</td>
</tr>
<tr>
<td>$\lambda = 1$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>$\gamma = 0.6$</td>
<td>1</td>
<td>0</td>
<td>0.154</td>
</tr>
<tr>
<td>$\gamma = 0.88$</td>
<td>1</td>
<td>0</td>
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</tr>
<tr>
<td>$\gamma = 1$</td>
<td>1</td>
<td>0</td>
<td>0.154</td>
</tr>
</tbody>
</table>

| $\lambda = 2.25$       |      |          |        |      |          |        |      |          |        |
| $\gamma = 0.6$         | 0.042| 0.958    | 0.166  | 1    | 0        | 0.368  | 1    | 0        | 0.473  |
| $\gamma = 0.88$        | 0.052| 0.948    | 0.173  | 1    | 0        | 0.368  | 1    | 0        | 0.473  |
| $\gamma = 1$           | 0.061| 0.939    | 0.178  | 1    | 0        | 0.368  | 1    | 0        | 0.473  |

| $\lambda = 3$          |      |          |        |      |          |        |      |          |        |
| $\gamma = 0.6$         | 0.035| 0.965    | 0.160  | 0.213| 0.787    | 0.387  | 1    | 0        | 0.473  |
| $\gamma = 0.88$        | 0.037| 0.963    | 0.162  | 0.333| 0.667    | 0.390  | 1    | 0        | 0.473  |
| $\gamma = 1$           | 0.038| 0.962    | 0.163  | 0.445| 0.555    | 0.385  | 1    | 0        | 0.473  |

<table>
<thead>
<tr>
<th>Panel B: Normality Assumption</th>
<th>One-Month Horizon</th>
<th>Six-Month Horizon</th>
<th>Twelve-Month Horizon</th>
</tr>
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<tbody>
<tr>
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<td>$q_s$</td>
<td>$q_{tb}$</td>
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<tr>
<td>$\lambda = 1$</td>
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<tr>
<td>$\gamma = 0.6$</td>
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<td>$\gamma = 0.88$</td>
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<tr>
<td>$\gamma = 1$</td>
<td>1</td>
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<td>0.154</td>
</tr>
</tbody>
</table>

| $\lambda = 2.25$             |      |          |        |      |          |        |      |          |        |
| $\gamma = 0.6$               | 0.064| 0.938    | 0.178  | 0.553| 0.447    | 0.381  | 1    | 0        | 0.473  |
| $\gamma = 0.88$              | 0.051| 0.949    | 0.172  | 1    | 0        | 0.368  | 1    | 0        | 0.473  |
| $\gamma = 1$                 | 0.047| 0.953    | 0.169  | 1    | 0        | 0.368  | 1    | 0        | 0.473  |

| $\lambda = 3$                |      |          |        |      |          |        |      |          |        |
| $\gamma = 0.6$               | 0.048| 0.952    | 0.170  | 0.232| 0.768    | 0.389  | 0.502| 0.498    | 0.488  |
| $\gamma = 0.88$              | 0.037| 0.963    | 0.162  | 0.209| 0.791    | 0.386  | 1    | 0        | 0.473  |
| $\gamma = 1$                 | 0.034| 0.966    | 0.159  | 0.205| 0.795    | 0.386  | 1    | 0        | 0.473  |

The table shows optimal portfolio weights of stocks ($q_s$) and Treasury bills ($q_{tb}$) of an expected loss-averse investor with single-period objective:

$$\max_{q_s,q_{tb}} E[v(X)],$$

where $E[\cdot]$ is the expectations operator,

$$v(x) = \begin{cases} 
(x - \bar{x})^\gamma & \text{if } x \geq 0 \\
-\lambda(\bar{x} - x)^\gamma & \text{if } x < 0 
\end{cases},$$

and $\bar{x}$ is the average return on Treasury bills. The portfolio return $X$ is assumed either NIG distributed (panel A) or normally distributed (panel B). The investor horizon is either one, six, or twelve months. $S$ is the Sharpe ratio. Restrictions $q_s, q_{tb} \in [0,1]$ and $q_s + q_{tb} = 1$ are imposed in the optimization.
NIG and normality assumptions since higher moments primarily affect the distribu-
tional tails.

Table 3 reports on the optimal asset allocation to stocks and bills of a CPT
investor with probability weighting parameter $\tau = 0.65$, and varying value function
parameters. The investor weights probabilities so that the portfolio’s distribution is
subjectively transformed, magnifying its tails. Panel A presents the results under
the NIG distributional return assumption, and panel B under normality. First,
compared with the results of table 2, the horizon effects are still present, which
does not come as a surprise. Second, the results at the monthly horizon resemble
the corresponding ones obtained without probability weighting, where only a minor
portion of stocks is chosen. Whether a NIG or a normality assumption is applied
does not seem to matter here either. Essentially, the stock returns’ variance is too
dominating at the monthly horizon for them to be attractive.

There are quite striking differ-
cences between tables 2 and 3 at the longer horizons
however. Consider the optimal weights under the NIG assumption in panel A,
with an investment horizon of six months. Instead of investing fully in stocks, the
CPT investor with Tversky and Kahneman (1992) estimates of the value function
parameters places 45 percent in stocks and 55 percent in bills. The intuition is that
the probability weighting investor perceives stocks as more risky, since the left tail
is magnified. Although stocks are positively skewed at the six-month horizon, which
is a positive for CPT utility, they are not skewed enough to offset the fear of a large
loss, which is enhanced by the large stock distributional kurtosis. Thus it seems
that kurtosis has a negative effect on CPT utility in this case.

Under the normal distribution, Levy et al. (2003), among others, show that CPT
is consistent with mean-variance efficiency. Hence, the optimal portfolios presented
in panel B of table 3 are mean-variance efficient. Are the CPT portfolios obtained
under the NIG assumption (panel A) mean-variance efficient too? Considering the
large differences in optimal weights shown in panels A and B, this does not seem to be
the case. For instance, at the yearly horizon, the CPT investor with $(\lambda, \gamma) = (3, 0.88)$
chooses to allocate 45.7 percent in stocks under the NIG assumption, but only 18.2
percent under normality. Such a disparity between optimal allocations indicates that
the there are other aspects of the distribution besides the mean and the variance
that are important to the CPT investor. Plausibly, the positive skewness of the
yearly stock returns distribution makes the CPT investor want to deviate from the
mean-variance portfolio, and choose a portfolio composition with a larger weight of
stocks.

Consider the Sharpe ratios of table 3. Since the optimal portfolios of panel B are
obtained under the normal distribution, which is fully characterized by the mean
### Table 3: Single-Period Portfolio Choice of CPT Investors

#### Panel A: NIG Assumption

<table>
<thead>
<tr>
<th></th>
<th>One-Month Horizon</th>
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<td>$\lambda = 1$</td>
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</tr>
<tr>
<td>$\gamma = 0.6$</td>
<td>0.048</td>
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#### Panel B: Normality Assumption

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The table shows optimal portfolio weights of stocks ($q_s$) and Treasury bills ($q_{tb}$) of a cumulative prospect theory investor with single-period objective:

$$\max_{q_s, q_{tb}} E^{w}[v(X)],$$

where $E^{w}[\cdot]$ is the expectations operator under probability weighting,

$$v(x) = \begin{cases} (x - \bar{x})^\gamma & \text{if } x \geq 0, \\ -\lambda(\bar{x} - x)^\gamma & \text{if } x < 0, \end{cases}$$

and $\bar{x}$ is the average return on Treasury bills. The probability weighting parameter is set to $\tau = 0.65$. The portfolio return $X$ is assumed either NIG distributed (panel A) or normally distributed (panel B). The investor horizon is either one, six, or twelve months. $S$ is the Sharpe ratio. Restrictions $q_s, q_{tb} \in [0, 1]$ and $q_s + q_{tb} = 1$ are imposed.
and the standard deviation, it is fair to believe that these portfolios have the largest attainable Sharpe ratio. However, the fact that the portfolios are mean-variance efficient undermines this reasoning. Mean-to-variance efficiency sets and mean-to-standard deviation efficiency sets are not equivalent. This, plausibly, explains why some optimal portfolios of panel A, obtained under the NIG distribution, have larger Sharpe ratios than the corresponding portfolios obtained under normality. For instance, at the monthly horizon with \((\lambda, \gamma) = (2.25, 1)\), the Sharpe ratio is 0.155 under normality, but 0.181 under the NIG assumption.

6 Conclusions

The paper examines the CPT utility of a NIG distributed portfolio return in a single-period context. The NIG assumption allows for a straightforward approach to analyzing utility in relation to the return’s distributional characteristics mean, variance, skewness, and kurtosis. Moreover, the optimal portfolio choice is analyzed, paying special interest to the implications of higher moments and probability weighting, which have received little attention in the previous literature. The main findings can be summarized as follows: First, CPT investors prefer high-mean and low-variance portfolios, since such portfolios imply smaller loss-probabilities. Second, skewness typically has a negative impact on utility when probability weighting is not considered. Once probabilities are subjectively transformed however, a clear preference for skewness appears. This shows that CPT investors display a preference for skewness through the probability weighting function. Third, utility is positively related to kurtosis when the investor treats probabilities objectively, but inverse hump-shape related when introducing probability weighting. The latter result is quite sensitive to the level of loss aversion in relation to the degree of probability weighting.

What implications do these results have for the portfolio choice? To answer this question, the CPT optimal asset allocation is analyzed under the NIG distributional assumption. Consistent with the previous literature, CPT investors are progressive in their allocation to stocks over the investment horizon. While the optimal portfolio might only consist of a small portion of stocks at the monthly horizon, the CPT investor with Tversky and Kahneman (1992) parameter estimates will prefer an all-stocks portfolio at the yearly horizon. Furthermore, the optimal portfolio composition may differ quite dramatically when higher-order moments are accounted for. Specifically, CPT portfolios do not seem to be mean-variance efficient under the NIG assumption, and they typically consist of a relatively larger
portion of stocks. Probability weighting causes this result. Since higher moments
are important to the CPT investor, the main priority is not mean-variance efficiency
but a more complicated preference-scheme including all first four moments.
References


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