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Numerical Calculation of an Asymmetric Supply Function Equilibrium with Capacity Constraints

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Numerical Calculation of an Asymmetric Supply Function Equilibrium with Capacity Constraints[1]

Pär Holmberg[2]

March 29, 2005

Abstract

Producers submit committed supply functions to a procurement auction, e.g. an electricity auction, before the uncertain demand has been realized. In the Supply Function Equilibrium (SFE), every firm chooses the bid maximizing his expected profit given the bids of the competitors. In case of asymmetric producers with general cost functions, previous work has shown that it is very difficult to find valid SFE. This paper presents a new numerical procedure that can solve the problem. It comprises numerical integration and an optimization algorithm that searches an end-condition. The procedure is illustrated by an example with three asymmetric firms.

Keywords: supply function equilibrium, uniform-price auction, numerical integration, oligopoly, asymmetry, capacity constraint, wholesale electricity market

JEL codes: C61, D43, D44, L11, L13, L94

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1. INTRODUCTION

The Supply Function Equilibrium (SFE) was introduced by Klemperer & Meyer in 1989 [8]. The equilibrium concept assumes that producers submit bids simultaneously in a one-shot game. In the non-cooperative Nash Equilibrium, each producer commits to the supply function that maximizes his expected profit given the bids of the competitors and the properties of the uncertain demand. In 1992, Bolle [2] and Green & Newbery [5] observed that the set-up has many similarities with the organization of most electricity markets. Since then, the equilibrium is often used when modeling bidding behavior in electric power auctions. There are a few SFE papers with other applications. The model can be applied to any uniform price auction where valuations/costs are certain and common knowledge, quantity discreteness is negligible and demand is uncertain.

Klemperer & Meyer show that all smooth SFE are characterized by a differential equation, which in this paper is called the KM first-order condition. In the general case, there is a continuum of possible SFE that fulfill this first-order condition [8]. However, with capacity constraints one can often drastically reduce the set of SFE candidates [4]; at least for inelastic demand. The set of SFE can be further reduced by allowing extreme demand outcomes, i.e. that there is a positive probability—it can be arbitrarily small—that the capacity constraints of all firms — but possibly the largest — bind. Then one can show analytically that there is a unique equilibrium in some specific cases. One case is symmetric producers with strictly convex cost functions [6]. The other is for producers with identical constant marginal costs and asymmetric capacities [7]. A reservation price, i.e. a price cap $\overline{p}$, is needed to limit the equilibrium price. Inelastic demand, a reservation price and the possibility of extreme demand outcomes, are all realistic assumptions for electric power markets, and especially so for balancing markets [6].

But in reality, firms typically have both non-constant marginal costs and asymmetric production capacities. In this general case, the KM first-order conditions—one for each firm—constitute a system of non-autonomous ordinary differential equations. To analytically solve this system is very difficult and probably impossible. Baldick & Hogan [1] calculate approximate asymmetric SFE by numerically integrating the system of ordinary differential equations. They note that it is generally very difficult to find solutions that do not violate the requirement that supply functions must be non-decreasing[3]. The three exceptions are:

[3] Non-decreasing supply functions are required by most electricity auctions. Further, profit maximizing firms would, under normal conditions, not submit decreasing supply functions [8].

symmetric firms with identical cost functions, cases with affine solutions —i.e. affine marginal costs and no capacity constraints— and when there are small variations in the demand.

In this paper, I suggest, a new numerical algorithm to find a valid SFE. It is intended for $N \geq 2$ asymmetric firms and cost functions more general than the three special cases mentioned by Baldick & Hogan. The equilibrium consists of piece-wise smooth supply functions and is inspired by the unique equilibrium previously derived for asymmetric producers with constant marginal costs [7]. Some of the analytically derived properties are conjectured to be valid also for increasing marginal costs. These properties are: Large firms have more market power and have larger mark-ups for any percentage of the capacity. Hence, capacity constraints of smaller firms bind earlier. Let p_i be the price at which the capacity constraint of firm i starts to bind. Arrange the producers according to size, starting with the smallest firm. The capacity constraint of the second largest firm starts to bind at the price cap. Thus $C'(0) < p_1 < \ldots < p_{N-1} = \overline{p},$ where $C()$ is the aggregate cost function. The largest producer offers its remaining capacity ΔS_N with a perfectly elastic supply at the price cap. All firms offer their first unit of power at the lowest marginal cost, as if under Bertrand competition, which is in agreement with general results for uniform price auctions [9].

To ensure an equilibrium with the conjectured properties, the following two assumptions are made. First, the larger of any two firms has weakly larger marginal cost for any percentage of the capacity[4]. Second, all firms has the same marginal cost at zero supply[5]. The constants $\Delta S_N, p_1, p_2, \ldots p_{N-2}$ are unknown, so far. Given these constants, the terminal conditions of the system of KM first-order conditions are known and the supply functions of all firms can be solved by numerical integration. The numerical integration starts at the price cap and proceeds in the direction of decreasing prices. The integration is terminated as soon as any supply function violates the requirements, a supply function must be non-decreasing and non-negative. The function $\Gamma(p_1, \ldots p_{N-2}, \Delta S_N)$ is equal to the terminated price. In theory, all considered SFE candidates should fulfill $\Gamma(p_1 \ldots p_{N-2}, \Delta S_N) = C'(0)$. In practice, however, one has to be somewhat forgiving due to numerical errors. If there is a unique equilibrium, it can be found by an optimization algorithm minimizing Γ.

[4] It might be enough to assume that the larger of any two firms has a weakly higher marginal cost for the last unit. It should be possible to numerically calculate asymmetric SFE for even more general cost functions. However, then adjustments of the conjecture might be needed, i.e. the order in which the capacity constraints bind.

[5] It should be possible to numerically calculate asymmetric SFE also when firms have different marginal costs at zero demand, but then firms will offer their first units of power at different prices.

Section 2 introduces the notation and assumptions used in the analysis of this paper. The KM first-order conditions of the conjectured SFE are presented in Section 3. In Section 4, the numerical algorithm is applied to an example with three firms. The algorithm returns one solution that approximately fulfills the first-order condition and the non-decreasing requirement. It is graphically verified that no firm will find it profitable to deviate from the equilibrium candidate. The accepted production of the equilibrium is inefficient, because mark-ups are asymmetric. The paper is concluded in Section 5.

2. NOTATION AND ASSUMPTIONS

Except for firms' capacities and costs, the notation and market assumptions are the same as in previous papers by Holmberg [6,7]. There are N asymmetric producers. The bid of each firm i consists of a piece-wise smooth — i.e. piece-wise twice continuously differentiable — non-decreasing and left-hand continuous supply function $S_i(p)$. In most electricity auctions, supply functions are required to be non-decreasing. The aggregate supply of the competitors of firm i is denoted $S_{-i}(p)$ and the total supply is denoted $S(p)$.

Let $\bar{\varepsilon}_i$ be the capacity constraint of producer i. Without loss of generality, we can order firms according to their capacity, i.e. $\bar{\varepsilon}_1 < \bar{\varepsilon}_2 < \ldots < \bar{\varepsilon}_N$. The total capacity is designated by $\bar{\varepsilon}$, i.e. $\bar{\varepsilon} = \sum_{i=1}^{N} \bar{\varepsilon}_i$. Let p_i denote the price, at which firm i chooses to offer his last unit, i.e. $S_i(p_i) = \bar{\varepsilon}_i$.

Denote the inelastic demand by ε and its probability density function by $f(\varepsilon)$. I assume that demand is always non-negative[6]. The density function is continuously differentiable and has a convex support set that includes zero demand. To get a unique equilibrium, extreme demand outcomes are allowed for, i.e. ε such that $\varepsilon > S(\bar{p})$ occur with a positive probability[7]. In equilibrium this implies that the capacity constraints of all firms, but possibly the largest one, bind with a positive probability. The reservation price $\bar{p}$ ensures that the demand is zero above the price cap. Accordingly, the market price equals the price cap when $\varepsilon > \bar{\varepsilon}$, if there are such demand outcomes.

[6] As in [6], it is straightforward to extend the analysis to negative demand, which is relevant for balancing markets.

[7] Note that $S(\bar{p})$ does not include ΔS_N, as supply functions are left-hand continuous.

All firms have increasing, strictly convex and twice continuously differentiable cost functions. Among other things this implies increasing marginal costs. Denote the aggregated cost function of all firms by *C(S).* For the cost functions of the individual firms, it is assumed that $C_i'(S_i) \geq C_j'(S_j)$, if $\frac{S_i}{\varepsilon_i} = \frac{S_j}{\varepsilon_j}$ and *i*>*j*. Further, $C_i'(0) = C_j'(0)$. These assumptions are made to ensure an equilibrium with the conjectured properties. For more general cost functions, adjustments of the conjecture might be necessary.

The residual demand of an arbitrary producer *i* is denoted by $q_i(p,\varepsilon)$. As long as the supply functions of his competitors are not perfectly elastic at *p,* his residual demand is:

$$q_i(\varepsilon, p) = \varepsilon - S_{-i}(p). \qquad (1)$$

3. THE CONJECTURED SFE

For symmetric producers [6] and producers with asymmetric capacities and identical constant marginal costs [7], it has been shown that there is a unique equilibrium with the following properties:

- All producers offer their first units of power at the price $C'(0)$.
- All supply functions are twice continuously differentiable, except at points where the capacity constraint of some producer starts to bind.
- There are no supply functions with perfectly elastic segments below the price cap and only the largest firm *N* can have a perfectly elastic segment at the price cap. This implies that all supply functions $S_i(p)$ are continuous below the price cap.
- Firms with non-binding capacity constraints do not have supply functions with inelastic segments.
- Below the price cap, all supply functions with non-binding capacity constraints fulfill the KM first-order condition.
- $C'(0) < p_1 < p_2 < \ldots < p_{N-1} = \overline{p}$.

It is conjectured that these properties are true also for the asymmetric firms studied in this paper[8]. The conjecture is the basis of the numerical algorithm developed below.

[8] It is likely that they can be proven by means of the analytical tools used in previous work [1,6,7].

3.1. Necessary conditions

Assuming that competitors do not have perfectly elastic supply functions below the price cap, the residual demand of an arbitrary producer *i* is given by (1). Hence, for given demand and price, the profit of producer *i* is:

$$\pi_i(\varepsilon, p) = \underbrace{[\varepsilon - S_{-i}(p)]}_{S_i} p - C_i[\varepsilon - S_{-i}(p)], \text{ if } S_i \le \bar{\varepsilon}_i \text{ and } p < \bar{p}. \quad (2)$$

In the traditional SFE literature, see e.g. [8], the KM first-order condition is derived by simply differentiating (2) with respect to *p*.

$$S_i(p) - S_{-i}'(p)[p - C_i'(S_i(p))] = 0. \quad (3)$$

Below the price cap, all supply functions with non-binding capacity constraints fulfill the KM first-order condition. This implies that all SFE candidates are given by *N*-1 systems of differential equations. The first system has *N* differential equations and is valid for the price interval $(C'(0), p_1)$. The second system has *N*-1 differential equations and is valid for the price interval (p_1, p_2) and so on. The continuity assumption links the end-conditions of the systems of differential equations. Including the end-conditions, the *N*-1 systems of differential equations are:

$$\begin{array}{lll}
p \in (C'(0), p_1) & \begin{cases} S_1(p) - S_{-1}'[p - C_1'(S_1(p))] = 0 \\ \vdots \\ S_N(p) - S_{-N}'[p - C_N'(S_N(p))] = 0 \end{cases} & \begin{cases} S_1(p_1) = \bar{\varepsilon}_1 \\ \vdots \\ S_N(p_{1-}) = S_N(p_{1+}) \end{cases} \\
\vdots & \vdots & \vdots \\
p \in (p_{N-3}, p_{N-2}) & \begin{cases} S_{N-2}(p) - S_{-(N-2)}'[p - C_{N-2}'(\cdot)] = 0 \\ S_{N-1}(p) - S_{-(N-1)}'[p - C_{N-1}'(\cdot)] = 0 \\ S_N(p) - S_{-N}'[p - C_N'(S_N(p))] = 0 \end{cases} & \begin{cases} S_{N-2}(p_{N-2}) = \bar{\varepsilon}_{N-2} \\ S_{N-1}(p_{N-2-}) = S_{N-1}(p_{N-2+}) \\ S_N(p_{N-2-}) = S_N(p_{N-2+}) \end{cases} \\
p \in (p_{N-2}, \bar{p}) & \begin{cases} S_{N-1}(p) - S_{-(N-1)}'[p - C_{N-1}'(\cdot)] = 0 \\ S_N(p) - S_{-N}'[p - C_N'(S_N(p))] = 0 \end{cases} & \begin{cases} S_{N-1}(\bar{p}) = \bar{\varepsilon}_{N-1} \\ S_N(\bar{p}) = \bar{\varepsilon}_N - \Delta S_N \end{cases}
\end{array} \quad (4)$$

Given a set of values $\{p_1, p_2, \ldots p_{N-2}, \Delta S_N\}$, the *N*-1 systems of differential equations can be solved backwards. One must start with the price interval $p \in (p_{N-2}, \bar{p})$, for which all the end-conditions are known, i.e. $S_{N-1}(\bar{p}) = \bar{\varepsilon}_{N-1}$ and $S_N(\bar{p}) = \bar{\varepsilon}_N - \Delta S_N$. Thus $S_N(p)$ and $S_{N-1}(p)$ can be calculated for $p \in (p_{N-2}, \bar{p})$. This solution can then be used to determine the end-conditions for the price interval $p \in (p_{N-3}, p_{N-2})$. After solving the system of differential

equations associated with this price interval, one can proceed with the interval $p \in (p_{N-4}, p_{N-3})$ and so on.

The integration of the systems of ODE starts at the price cap and proceeds in the direction of decreasing prices. It terminates as soon as any supply function violates the non-decreasing and non-negative constraints. The function $\Gamma(p_1, p_2, \ldots p_{N-2}, \Delta S_N)$ returns the terminated price. According to the conjecture, all producers will in equilibrium offer their first unit of power at $C'(0)$. Thus theoretically all SFE candidates must fulfill $\Gamma[p_1, p_2, \ldots p_{N-2}, \Delta S_N] = C'(0)$.

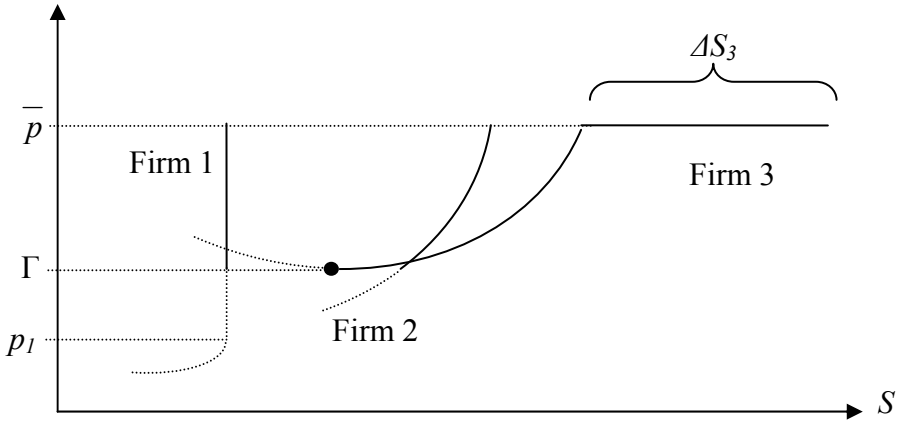

Fig. 1. *The integration starts at the price cap proceeds in the direction of decreasing prices and is terminated as soon as any supply function becomes invalid. Γ is defined by the terminated price.*

3.2. A sufficient condition

The first-order condition and $\Gamma[p_1, p_2, \ldots p_{N-2}, \Delta S_N] = C'(0)$ are necessary conditions for SFE, but not sufficient. An extremum with valid supply functions is guaranteed, but one cannot be sure that it is a globally best response for a producer to follow the SFE candidate, even if the competitors follow strategies implied by the candidate. A sufficiently strong second-order condition is that the market price of the candidate globally maximizes $\pi_i(\varepsilon, p)$ for every ε, given that the competitors follow the equilibrium candidate

3.3. The numerical algorithm

For asymmetric producers with general cost functions, it is very difficult or even impossible to calculate SFE analytically. Nevertheless, the system of differential equations in (4) can be solved by numerical integration, given $\{p_1, p_2, \ldots p_{N-2}, \Delta S_N\}$. By griding the space and by optimization algorithms, values $\{p_1, p_2, \ldots p_{N-2}, \Delta S_N\}$ that (nearly) fulfill $\Gamma = C'(0)$ can be found. Considering numerical errors, one can in practice not rule out SFE candidates that almost fulfill $\Gamma[p_1, p_2, \ldots p_{N-2}, \Delta S_N] = C'(0)$. The second-order condition can be checked graphically or numerically.

4. AN EXAMPLE WITH THREE ASYMMETRIC FIRMS

The numerical procedure to find valid SFE is illustrated by an example with three firms. Their production capacities are: $\bar{\varepsilon}_1 = \frac{\bar{\varepsilon}}{7}$, $\bar{\varepsilon}_2 = \frac{2\bar{\varepsilon}}{7}$ and $\bar{\varepsilon}_3 = \frac{4\bar{\varepsilon}}{7}$. Further, the marginal cost function of all firms is linear $C_i' = c\left(1 + \frac{S_i}{\bar{\varepsilon}_i}\right)$ up to the capacity constraint. Assume further that the price cap is $\bar{p} = 4c$.

4.1. Necessary conditions

The KM first-order conditions of the SFE candidates corresponding to (4) are given by the following set of 2 systems of differential equations:

$$p \in (c, p_1) \quad \begin{cases} S_1(p) - S'_{-1}(p)\left[p - c\left(1 + \frac{7S_1}{\bar{\varepsilon}}\right)\right] = 0 \\ S_2(p) - S'_{-2}(p)\left[p - c\left(1 + \frac{7S_2}{2\bar{\varepsilon}}\right)\right] = 0 \\ S_3(p) - S'_{-3}(p)\left[p - c\left(1 + \frac{7S_3}{4\bar{\varepsilon}}\right)\right] = 0 \end{cases} \quad \begin{cases} S_1(p_1) = \frac{\bar{\varepsilon}}{7} \\ S_2(p_{1-}) = S_2(p_{1+}) \\ S_3(p_{1-}) = S_3(p_{1+}) \end{cases}$$

$$p \in (p_1, 4c) \quad \begin{cases} S_2(p) - S'_{-2}(p)\left[p - c\left(1 + \frac{7S_2}{2\bar{\varepsilon}}\right)\right] = 0 \\ S_3(p) - S'_{-3}(p)\left[p - c\left(1 + \frac{7S_3}{4\bar{\varepsilon}}\right)\right] = 0 \end{cases} \quad \begin{cases} S_2(4c) = \frac{2\bar{\varepsilon}}{7} \\ S_3(4c) = \frac{4\bar{\varepsilon}}{7} - \Delta S_3 \end{cases}$$

The variables can be normalized such that $p = c\tilde{p}$ and $S_i(p) = \bar{\varepsilon}\tilde{S}_i(\tilde{p})$. Then

$$\widetilde{p}\in(1,\widetilde{p}_1)\quad\begin{cases}\widetilde{S}_1(\widetilde{p})-\widetilde{S}'_{-1}(\widetilde{p})\left[\widetilde{p}-\left(1+7\widetilde{S}_1\right)\right]=0\\ \widetilde{S}_2(\widetilde{p})-\widetilde{S}'_{-2}(\widetilde{p})\left[\widetilde{p}-\left(1+\dfrac{7\widetilde{S}_2}{2}\right)\right]=0\\ \widetilde{S}_3(\widetilde{p})-\widetilde{S}'_{-3}(\widetilde{p})\left[\widetilde{p}-\left(1+\dfrac{7\widetilde{S}_3}{4}\right)\right]=0\end{cases}\quad\begin{cases}\widetilde{S}_1(\widetilde{p}_1)=\dfrac{1}{7}\\ \widetilde{S}_2(\widetilde{p}_{1-})=\widetilde{S}_2(\widetilde{p}_{1+})\\ \widetilde{S}_3(\widetilde{p}_{1-})=\widetilde{S}_3(\widetilde{p}_{1+}).\end{cases}$$

$$\widetilde{p}\in(\widetilde{p}_1,4)\quad\begin{cases}\widetilde{S}_2(\widetilde{p})-\widetilde{S}'_{-2}(\widetilde{p})\left[\widetilde{p}-\left(1+\dfrac{7\widetilde{S}_2}{2}\right)\right]=0\\ \widetilde{S}_3(\widetilde{p})-\widetilde{S}'_{-3}(\widetilde{p})\left[\widetilde{p}-\left(1+\dfrac{7\widetilde{S}_3}{4}\right)\right]=0\end{cases}\quad\begin{cases}\widetilde{S}_2(4)=\dfrac{2}{7}\\ \widetilde{S}_3(4)=\dfrac{4}{7}-\Delta\widetilde{S}_3\end{cases}\qquad(5)$$

Given a set of values $\{\widetilde{p}_1,\Delta\widetilde{S}_3\}$, the system representing the price interval $(\widetilde{p}_1,4)$ can be solved by numerical integration[9]. This solution gives end-conditions for the system of differential equations valid for $\widetilde{p}\in(1,\widetilde{p}_1)$, which can be solved in its turn.

The next step is to check whether the calculated supply functions violate the non-decreasing and non-negative requirements. The function $\widetilde{\Gamma}(\widetilde{p}_1,\Delta\widetilde{S}_3)$ returns the first price (starting from the price cap), for which any supply function violates any of the requirements. Most parameter values generate $\widetilde{\Gamma}(\widetilde{p}_1,\Delta\widetilde{S}_3)>1$ and are accordingly not SFE.

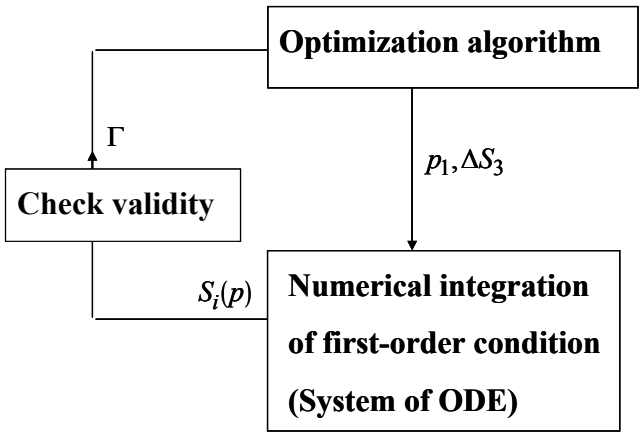

Fig. 2. *The numerical procedure to find valid SFE.*

An example of a parameter set that generates a non-valid SFE is $\widetilde{p}_1=4$ and $\Delta\widetilde{S}_3=0$. This is the boundary condition, if one sees the price cap as a public signal that will coordinate the bids[10]. This assumption has been suggested by Baldick & Hogan, as it gives a unique SFE for

[9] See Appendix for details of the numerical integration.
[10] All supply functions are smooth up to the price cap, where all capacity constraints bind.

symmetric producers [1]. They observe, however, that for asymmetric producers the public signal assumption often leads to invalid SFE as in Fig. 3. In this case, the supply functions violate several of the requirements, as supply functions should be both non-negative and non-decreasing.

To get an idea of the parameter space for which $\tilde{\Gamma}\left(\tilde{p}_1, \Delta\tilde{S}_3\right) = 1$, $\tilde{\Gamma}$ is calculated for a grid with 400x400 points in the space $\left(\tilde{p}_1, \Delta\tilde{S}_3\right) \in [1,4] \times \left[0, \frac{4}{7}\right]$. The result is presented as a contour plot in Fig. 4.

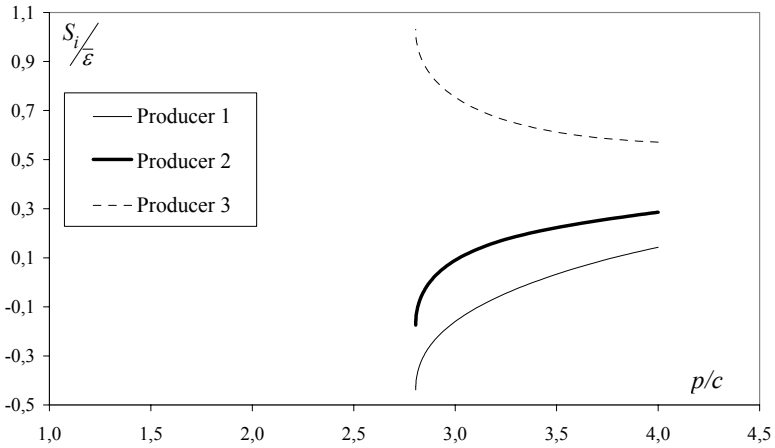

Fig. 3. *The parameter set* $\tilde{p}_1 = 4$ *and* $\Delta\tilde{S}_3 = 0$ *generates invalid supply functions. This is the boundary condition when one sees the price cap as a public signal that coordinates the bids.*

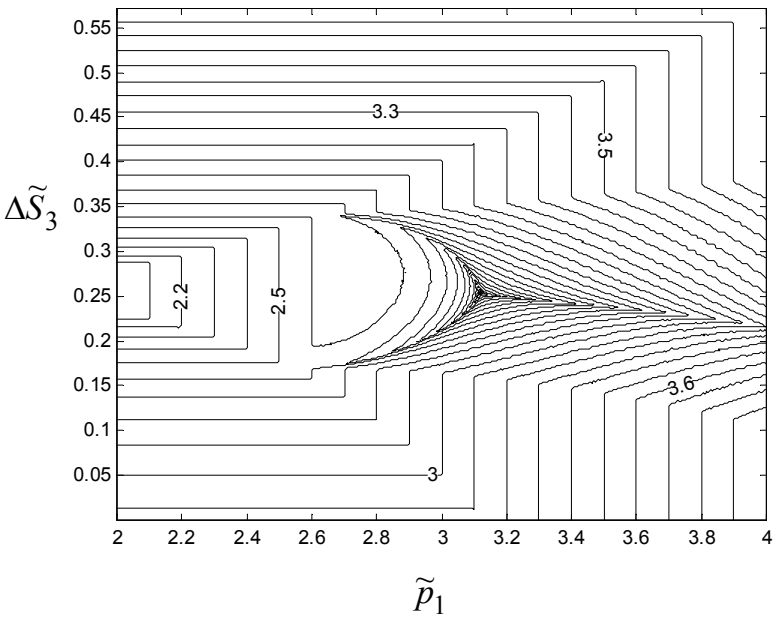

Fig. 4. *Contour plot of* $\tilde{\Gamma}\left(\tilde{p}_1, \Delta\tilde{S}_3\right)$. *There is a minimum around* $\tilde{p}_1 \approx 3$ *and* $\Delta\tilde{S}_3 \approx 0.3$.

The contour plot in Fig. 4 indicates that $\widetilde{\Gamma}\left(\widetilde{p}_1, \Delta\widetilde{S}_3\right)$ has a minimum around $\widetilde{p}_1 \approx 3$ and $\Delta\widetilde{S}_3 \approx 0.3$. By means of an optimization algorithm[11], the estimated minimum of $\widetilde{\Gamma}\left(\widetilde{p}_1, \Delta\widetilde{S}_3\right)$ is located to $\widetilde{p}_1 \approx 3.117$ and $\Delta\widetilde{S}_3 \approx 0.2541,$ and the min-value is roughly 1.005[12].

$\widetilde{\Gamma} \approx 1.005$ is very close to, but still above $\widetilde{\Gamma} = 1,$ which is necessary for the conjectured SFE. The difference may, however, be explained by the numerical sensitivity of the solution. If there is a unique SFE, which one would intuitively expect from previous SFE studies [6,7], then there is a unique set of $\left\{\widetilde{p}_1, \Delta\widetilde{S}_3\right\}$ that gives valid supply functions. Thus there is a unique valid triple of trajectories associated with this set that fulfills the systems of KM first-order conditions in (5). The slightest deviation from this triple, due to a small numerical error, will lead to invalid supply functions, and $\widetilde{\Gamma}\left[\widetilde{p}_1, \Delta\widetilde{S}_3\right] > 1.$ Thus $\widetilde{\Gamma} \approx 1.005$ does, strictly speaking, neither rule out nor secure a SFE.

The calculated supply functions for the set $\left\{\widetilde{p}_1 \approx 3.117, \Delta\widetilde{S}_3 \approx 0.2541\right\}$ are plotted in Fig. 5.

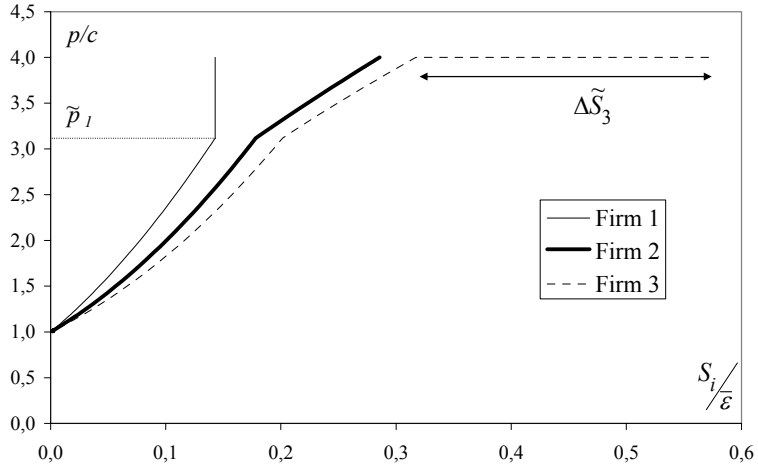

Fig. 5. *The equilibrium candidate*.

As shown in a previous paper, the unique asymmetric equilibrium for constant marginal costs is piece-wise symmetric [7]. Two arbitrary producers have the same supply function, unless the capacity constraint of one of them is binding. With the strictly convex cost functions assumed in this paper, it will be more expensive for a smaller firm to produce a given supply

[11] The fminsearch algorithm, a simplex search method of Matlab, was used in the calculation.
[12] The estimation depends on tolerances used in the numerical integration.

compared to a larger firm. Thus it is expected that producers with more capacity sell more at every price, as in Fig. 5. Still it is apparent that the largest firm uses his market power extensively. More than 40% of the capacity of producer 3 is not offered below the price cap.

Note that firm 2 and 3 have a kink in their supply functions at p_1. A discontinuous increase in the elasticity of the supplies of firm 2 and 3 at this point ensures that the elasticity of their residual demand is continuous. It follows from the KM first-order condition that this is necessary, if the supply functions of firm 2 and 3 are to be continuous at p_1.

4.2. The second-order condition

Does the candidate fulfill the sufficient second-order condition? Denote the supply functions of the SFE candidate in Fig. 5 by $S_i^X(p)$ and denote its market price by $p^X(\varepsilon)$. Given $S_{-i}^X(p)$, does $p^X(\varepsilon)$ globally maximize $\pi_i(\varepsilon, p)$ for every ε? To get an indication, the iso-profit lines of all producers are plotted in Fig. 6-8 together with $p^X(\varepsilon)$. For a local extremum, a vertical line — corresponding to a constant ε — should have a tangency point with the iso-profit line at $p^X(\varepsilon)$. This corresponds to the KM first-order condition and seems to be true for every demand for all producers with non-binding capacity constraints. For such firms, one can deduce from the shape of the iso-profit lines that the profit is globally maximized at $p^X(\varepsilon)$.

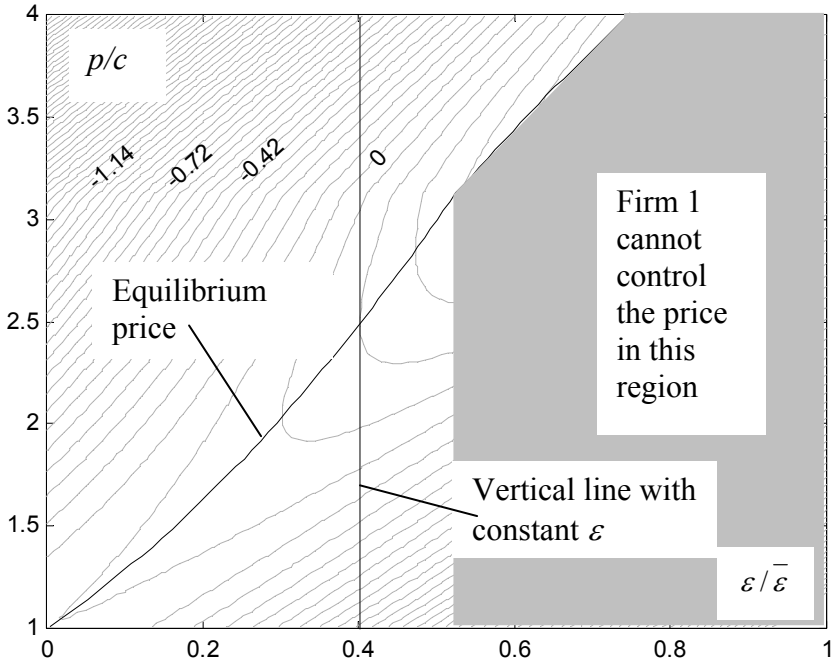

Fig. 6. *Iso-profit lines of firm 1.*

The tangency condition is not necessarily fulfilled in regions where producers cannot control the price due to a binding capacity constraint or a binding price cap. For example, firm 1 cannot, due to his capacity constraint, unilaterally push the price below $p^X(\varepsilon)$ for $\varepsilon > S^X(p_1)$. By increasing his mark-ups, he is still able to increase the market price. However, according to Fig. 6, such deviations decrease his profit. Neither firm 1 nor 2 can control the price for $\varepsilon > S^X(\overline{p})$. Their capacity constraints prevent them from reducing the price and the price cap prevents them from increasing the price. Firm 3 could reduce the price for $\varepsilon > S^X(\overline{p})$, but according to Fig. 8 it would not be profitable[13]. Thus it seems that $p^X(\varepsilon)$ globally maximizes $\pi_i(\varepsilon, p)$ for every ε, if the aggregate supply of the competitors is given by $S_{-i}^X(p)$.

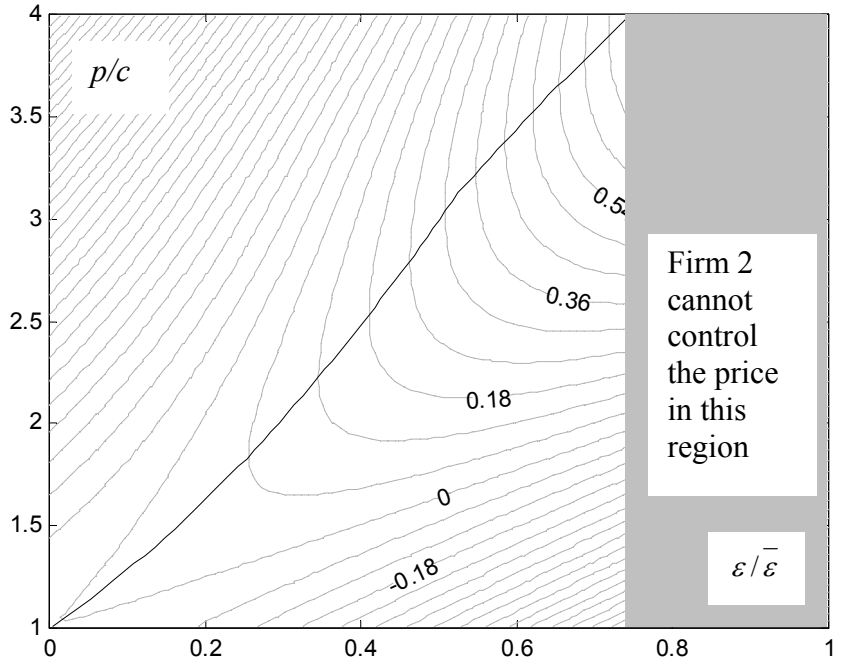

Fig. 7. *Iso-profit lines of firm 2.*

[13] Note that $S^X(\overline{p})$ does not include ΔS_N, as supply functions are left-hand continuous.

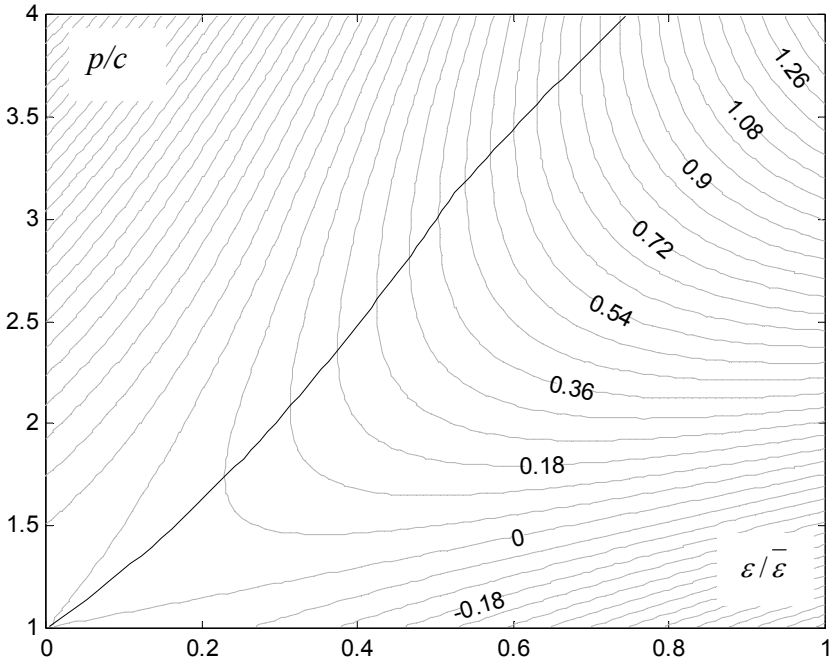

Fig. 8. *Iso-profit lines of firm 3.*

4.3. Welfare loss

With symmetric cost functions, as in [6], or asymmetric capacities and identical constant marginal costs, as in [7], there is no inefficiency, as demand is inelastic and all firms operate at the same marginal cost. But there is a welfare loss, if marginal costs are increasing and large firms have larger mark-ups for every marginal cost, as in Fig. 5. The production is inefficient, as some units with a high marginal cost will be accepted from small firms instead of cheaper production from larger firms. For the example with three firms, the welfare loss is illustrated in Fig. 9.

In the example with three firms, the welfare loss is, relatively speaking, largest for the demand outcome $\varepsilon = S^X(p_1)$, where the capacity constraint of firm 1 starts to bind. For higher demand, production from firm 2 and 3 that is cheaper than the most expensive generator of firm 1 is accepted, and the cost ratio decreases. There is another kink in the ratio, when the capacity constraint of firm 2 binds. The production is optimal when the whole capacity is needed, i.e. $\varepsilon \geq \bar{\varepsilon}$.

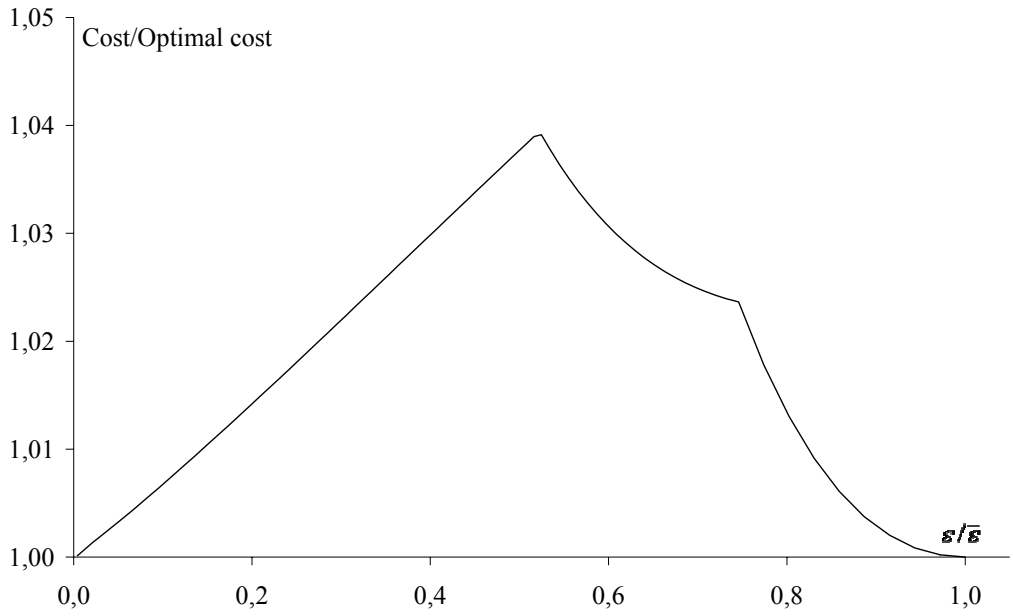

Fig. 9. *The total production cost relative the optimal cost for the example with three firms.*

The problem of inefficient production has lead von der Fehr & Harbord [3] to suggest that electric power markets should consider Vickrey auctions instead of uniform-price auctions. The advantage with the Vickrey auction is that it is optimal for producers to bid their true marginal costs, as they are offered an information rent.

5. CONCLUSIONS

Firms typically have non-constant marginal costs and asymmetric production capacities. In this general case, the first-order conditions of a Supply Function Equilibrium (SFE) constitute a system of non-autonomous ordinary differential equations. Solving such a system analytically is very difficult and probably impossible. Nevertheless, it can be solved by numerical integration. There is one problem, however, electricity auctions normally require non-decreasing supply functions and it has been observed by Baldick & Hogan that numerically calculated asymmetric SFE:s tend to violate this restriction [1]. The three exceptions are: symmetric firms with identical cost functions, cases with affine solutions —i.e. affine marginal costs and no capacity constraints— and when there are small variations in the demand.

In this paper a numerical procedure is suggested that can solve the problem of invalid asymmetric SFE. It is conjectured that the general asymmetric SFE has properties similar to those found in the case of constant marginal costs, which has been analyzed in [7]. The capacity constraints of small firms bind at lower prices compared to firms with a higher

capacity. The capacity constraint of the second largest firm starts to bind at the price cap. The largest firm has a perfectly elastic supply ΔS_N at the price cap. Except for the two largest firms, the prices p_i at which the capacity constraints of firms bind are unknown constants, and so is ΔS_N. The first-order conditions of this assumed equilibrium yield N-1 systems of non-autonomous ordinary differential equations. Given $\{p_1, p_2, \ldots p_{N-2}, \Delta S_N\}$, the set of systems can be solved by means of numerical integration. One starts at the price cap and proceeds in the direction of a decreasing p. When any of the supply functions violates the restrictions—a supply function is increasing and non-negative—the integration is terminated. The function $\Gamma(p_1, \ldots p_{N-2}, \Delta S_N)$ returns the price at which the integration terminates. For a valid SFE candidate, Γ must, in theory, return the marginal cost of the cheapest unit. Based on the results for asymmetric producers with constant marginal costs one would intuitively expect a unique SFE. Then the equilibrium can be found by an optimization algorithm minimizing Γ.

The procedure for finding asymmetric SFE candidates is illustrated by an example with three firms and linear marginal costs. Contour plots of Γ indicate that it has a unique minimum just above the marginal cost of the cheapest unit. This is expected as a numerical error would force the probably unique triple of SFE trajectories slightly off their track. Further, numerically calculated iso-profit lines indicate that no producer will find it profitable to unilaterally deviate from the SFE candidate. Thus the second-order condition seems to be fulfilled.

At the price, for which the capacity constraint of the smallest firm starts to bind, the elasticity of the supply of the two larger firms will increase discontinuously. This ensures that the elasticity of the residual demand of the two firms is continuous at p. Thus in equilibrium, all firms, but the smallest, will have kinks in their supply functions below their capacity constraint.

The numerical procedure could be generalized to any increasing and convex cost function and, with enough computer power, any number of firms. The procedure is more likely to generate valid SFE with the conjectured properties, if the larger of any two firms has weakly larger marginal cost for any percentage of the capacity and if all firms have the same marginal cost at zero supply. With adjustments in the conjectured properties, e.g. the order in which firms' capacities bind, it should be possible to apply the method to even more general cost functions.

For asymmetric firms with increasing marginal costs, asymmetric mark-ups imply inefficient production. The reason is that large firms have more market power. For every

marginal cost, small firms have lower mark-ups compared to large firms. Hence, some costly generators of small firms will be accepted instead of cheaper production from larger firms.

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APPENDIX

The numerical integration is performed in Matlab. It has been observed by Newbery that the coupled differential equations associated with SFE are stiff and highly sensitive to the starting point chosen for the numerical integration [10]. The example studied in this paper has the same problem. Thus a robust solver is used, the ode15s of Matlab with the backward differentiation option.

When using numerical integration algorithms, it is often necessary to rewrite the system of differential equations on the form $x'(t) = f(x)$. This transformation is illustrated for the system of differential equations below. The first-order condition is:

$$\begin{cases} S_1(p) - S'_{-1}(p)\left[p - C_1'(S_1(p))\right] = 0 \\ \vdots \\ S_N(p) - S'_{-N}(p)\left[p - C_N'(S_N(p))\right] = 0 \end{cases}$$

The system can be rewritten on the following form:

$$\begin{cases} \dfrac{S_1(p)}{p - C_1'(S_1(p))} = S'_{-1}(p) \\ \vdots \\ \dfrac{S_N(p)}{p - C_N'(S_N(p))} = S'_{-N}(p) \end{cases} \tag{6}$$

Summing over all equalities yields:

$$\sum_{j=1}^{N} \frac{S_j(p)}{p - C_j'(S_j(p))} = (N-1)S'(p).$$

As $S_{-i}'(p) = S'(p) - S_i'(p)$, the system in (6) can now be rewritten:

$$\begin{cases} S_1'(p) = \dfrac{1}{N-1}\displaystyle\sum_{j=1}^{N} \frac{S_j(p)}{p - C_j'(S_j(p))} - \frac{S_1(p)}{p - C_1'(S_1(p))} \\ \vdots \\ S_N'(p) = \dfrac{1}{N-1}\displaystyle\sum_{j=1}^{N} \frac{S_j(p)}{p - C_j'(S_j(p))} - \frac{S_N(p)}{p - C_N'(S_N(p))}, \end{cases}$$

which has the form $x'(t) = f(x)$. A more general expression, which also considers elastic demand, has been derived by Baldick & Hogan [1].

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