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# Inequality Generating Processes and Measurement of the Matthew Effect

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## INEQUALITY GENERATING PROCESSES AND MEASUREMENT OF THE MATTHEW EFFECT

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# Inequality Generating Processes and Measurement of the Matthew Effect<sup>1</sup>

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**Abstract:** The first aim of this paper is to clarify the differences and relationships between cumulative advantage/disadvantage and the Matthew effect. Its second aim, which is also its main contribution, is not only to present a new measure of the Matthew effect, but also to show how to estimate this effect from data and how to make statistical inference. We argue that one should utilize the positivity of the natural logarithm of the largest generalized eigenvalue for a non-linear dynamic process as evidence when claiming that the Matthew effect is present in the dynamic process that generates individuals' socio-economic life-courses. Thus, our measure of the Matthew effect focuses on the dynamic process that generates socio-economic inequality and not on the outcome of this process.

**Keywords:** Cumulative advantage, cumulative disadvantage, dynamic process, inequality, inter-individual change, intra-individual change, life-course, Matthew effect and socio-economic status.

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## 1 Introduction

The extent of, and the mechanisms behind, inequality in society are problems of fundamental concern in the social sciences, especially whenever inequality disrupts social harmony. As Jasso and Kotz (2008) explain in their review of methods of measuring inequality, there are two sorts of inequality in the literature: (i) within-group inequality, or inequality between persons; and (ii) between-group inequality. Examples of within-group inequality are income and wealth inequalities (see, e.g., Champernowne and Cowell, 1998, Gottschalk and Smeeding, 1997, and Karoly and Burtless, 1995), whereas black-white and male-female earnings gaps are examples of between-group inequality (see, e.g., Blau and Kahn, 2000, Darity and Mason, 1998, and Harkness, 1996). The most sought-after goal in this context is the development of a proper understanding of inequality-generating mechanisms because this would allow correct action to be taken to ameliorate further social disharmony (see, e.g., Neckerman and Torche, 2007).

During the past few decades, there has been increased interest in the development of inequality over time. This is exemplified by a comparison of the economic well-being of the elderly with that of those in their working years. The finding that spurred this research is the fact that, contrary to widespread belief, economic well-being among the elderly is not distributed more equally than among those who work (see Radner, 1987). For instance, adopting a longitudinal perspective and using U.S. data, Crystal and Waehrer (1996) show that within-cohort inequality increases later in life (see also selected contributions in Crystal and Shea, 2003). The authors therefore emphasize the importance of adopting a life-course perspective in an effort to cultivate an appropriate understanding of such a finding (see also O’Rand and Henretta, 1999).

A popular model in life-course research that has achieved widespread acceptance in the literature is the cumulative advantage/disadvantage model proposed by Crystal and Shea (1990). This model focuses on how inequality can be magnified over the life-course: “[T]hose who are initially advantaged [...] are more likely to receive a good education, leading to good jobs, leading to better health and better pension coverage, leading to higher savings and better postretirement benefit income” (see p. 437 in Crystal and Shea, 1990). In other words, it is not just economic well-being that drives the process leading to an increase in within-cohort inequality; other areas of life, such as education and health, also have an impact (see, e.g., Dupre, 2008, Ferraro and Kelley-Moore, 2003, Lynch, 2003, Ross and Wu, 1996, and Willson et al., 2007).

A concept that is closely related to that of cumulative advantage is the Matthew effect. This term is derived from the Gospel according to Matthew, in which Jesus says: “*For unto every one that hath shall be given, and he shall have abundance: but from him that hath not shall be taken away even that which he hath*” (see Matthew 25:29, King James Version). Even though the Matthew effect is referenced in life-course research (see Dannefer, 1987, for an early example), it is a much more common concept in the sociology of science because of Merton’s (1968, 1988) observation that better known scientists tend to get more credit than less well-known scientists for similar achievements. Consequently, better known scientists attract more resources at the cost of lesser-known scientists, which results in a further widening of the gap between better known and less known scientists’ resources and a widening of the gap between their achievements as well.

Clearly, cumulative advantage and the Matthew effect are fruitful concepts in life-course research. However, as DiPrete and Eirich (2006) point out in their review of different cumulative advantage processes and how they may lead to inequality, if further progress in research is to be made, there is a need for more explicit attention to methodological issues in the application of different concepts in the

cumulative advantage literature. The first aim of this paper is to clarify the differences and relationships between cumulative advantage/disadvantage and the Matthew effect. Our second aim, which is also our main contribution, is not only to present a new measure of the Matthew effect, but also to show how to estimate this effect from data and how to make statistical inference.

Specifically, we argue in this paper that a proper measure of the Matthew effect will focus on the *mechanism* or *dynamic process* that generates socio-economic inequality and not on the outcome of this process. In other words, to better understand the inequality-generating dynamic process, we should measure the Matthew effect and not, so to speak, the *effect* of the Matthew effect. It can therefore be argued that it might be better to rename the Matthew effect the Matthew mechanism. In any event, our measure of the Matthew effect coincides with the Mertonian understanding of this concept (referred to here as the Matthew doctrine; see, e.g., Merton, 1968, 1988):

“Taken out of its spiritual context and placed in a wholly secular context, the Matthew doctrine would seem to hold that the posited process must result in a boundlessly growing inequality of wealth, however wealth is construed in any sphere of human activity. Conceived of as a locally ongoing process and not as a single event, the practice of giving unto everyone that hath much while taking from everyone that hath little will lead to the rich getting forever richer while the poor become poorer. Increasingly absolute and not only relative deprivation would be the continuing order of the day. But as we know, things are not as simple as all that. After all, the extrapolation of local exponentials is notoriously misleading” (see pp. 609-610 in Merton, 1988).

R.K. Merton also writes the following regarding the Matthew effect:

“[T]he Matthew effect is the accruing of large increments of peer recognition to scientists of great repute for particular contributions in contrast to the minimizing or withholding of such recognition for scientists who have not yet made their mark. The biblical parable generates a corresponding sociological parable” (see p. 609 in Merton, 1988).

Three things are worth noting in these quotations. First, the Matthew effect or Matthew mechanism is a process that results in inequality. Second, this process is ongoing and therefore dynamic. Third, R.K. Merton is correct in claiming that such a dynamic process may lead to boundlessly increasing inequality. However, there are dynamic processes that are bounded in the sense that no one becomes infinitely rich, but that, at the same time, are still “*capable of magnifying small differences over time*” (see p. 272 in DiPrete and Eirich, 2006). Specifically, a dynamic process characterized by the Matthew mechanism must be a non-linear process to have these properties.

It is quite natural to assume that one must know the actual form of the non-linear dynamic process that generates individuals’ socio-economic life-courses to be able to estimate the Matthew effect. However, this is not true. Instead, using the celebrated embedding theorem by Takens (1981), it becomes possible to reconstruct the dynamics using only a scalar time series and thereafter to estimate the generalized eigenvalues of the reconstructed process. As a result, positivity of the natural logarithm of the largest generalized eigenvalue indicates that the Matthew effect is present in the dynamic process that generates individuals’ socio-economic life-courses. Furthermore, because asymptotic theory is available for statistical inference, a scalar time series of a socio-economic variable is sufficient to conclude whether the Matthew effect is present in the mechanism that generates socio-economic inequality.

The rest of this paper is organized as follows. In Section 2, we present our new measure of the Matthew effect, and in Section 3, we show how to estimate the Matthew effect from data and how to make

statistical inference. In Section 4, we discuss the value added by our measure of the Matthew effect and also outline directions for future research, especially with regard to the theoretical modeling of socio-economic processes that lead to increased inequality over time. Section 5 concludes the paper.

## 2 Measuring the Matthew effect

This section consists of three parts. In Sections 2.1 and 2.2, we outline the relationships between socio-economic status, inequality in socio-economic status and how it changes over time, cumulative advantage and disadvantage, intra- and inter-individual changes in socio-economic status and the Matthew effect. As will be shown, there are not always clear-cut relationships between all of these terms. In Section 2.3, we argue that one should focus on the *dynamic process* that generates socio-economic inequality when measuring the Matthew effect and not on individuals' socio-economic life-courses *per se*. Specifically, we argue that one should utilize the positivity of the natural logarithm of the largest generalized eigenvalue for a non-linear dynamic process as evidence when claiming that the Matthew effect is present in the dynamic process that generates individuals' socio-economic life-courses.

### 2.1 Adam and Eve

Consider a population with a large number of individuals, where two of them are named Adam and Eve, and assume that the socio-economic status of Adam is fully described by the state variable, or  $n$ -tuple,  $S_t^{Adam} \in \mathbb{R}^n$  at time  $t$ .<sup>4</sup> In other words, the socio-economic status of Adam is described with the help of  $n$  variables. These may include education level, health status, income level, and several other socio-economic variables:

$$(1) \quad S_t^{Adam} = [education_t^{Adam}, health_t^{Adam}, income_t^{Adam}, other\ variables].$$

As will become clear below (in Section 2.3), it is not necessary for us to know exactly what these other socio-economic variables are in Adam's case. Nevertheless, if we assume that Adam and Eve are cohabiting, it is reasonable to think that Eve's income also affects Adam's socio-economic status:

$$(2) \quad S_t^{Adam} = [education_t^{Adam}, health_t^{Adam}, income_t^{Adam}, income_t^{Eve}, other\ variables].$$

The mechanism might be that Eve's income means that Adam no longer has to work as hard (compared to when he was alone) to achieve a certain standard of living. Moreover, it might also be the case that Adam does not want to work such long hours any longer (as he did when he was alone); instead, he might like to spend time with Eve. It is also reasonable to think that Eve's socio-economic status at time  $t$ ,  $S_t^{Eve} \in \mathbb{R}^n$ , is affected in a similar way:

$$(3) \quad S_t^{Eve} = [education_t^{Eve}, health_t^{Eve}, income_t^{Eve}, income_t^{Adam}, other\ variables].$$

The ordering of the variables in the  $n$ -tuples is usually important. However, because the exact ordering of the variables does not matter for the measure of the Matthew effect that we will present below (in Section 2.3), we do not spend time here elaborating on this issue. Let us instead continue on to answer the following questions: What about the socio-economic inequality between Adam and Eve? Is there some natural measure of this inequality and how it changes over time?

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<sup>4</sup> Below (in Section 2.2), we introduce Cain and Abel to our story and subsequently all the other individuals in the population.

Let us first define the distance between the socio-economic statuses of Adam and Eve. Naturally, at time  $t = 0$ , the distance  $d \in \mathbb{R}$  between Adam's and Eve's socio-economic statuses is<sup>5</sup>

$$(4) \quad d_0^{Adam,Eve} \equiv \|S_0^{Adam} - S_0^{Eve}\|.$$

Let us thereafter calculate the distance between Adam's and Eve's socio-economic statuses at times  $t = 1, t = 2$ , etc., up to and including time  $t = t_{max}$ , which gives us the time series  $\{d_t^{Adam,Eve}\}_{t=0}^{t_{max}}$ .

If the trend in the aforementioned time series is upward-sloping, the trajectories of Adam's and Eve's socio-economic statuses diverge over time, which means that we have an inter-individual divergence of trajectories. This pattern in the time-paths of the trajectories is also what we typically interpret as the Matthew effect. This effect, however, can be described in a more sophisticated way if we anticipate how it will be defined below (in Section 2.3): an inter-individual divergence of the trajectories of individuals' socio-economic statuses is a sign that the mechanism that generates the trajectories is characterized by the Matthew effect. Be aware that the trajectories of the individuals' socio-economic statuses can also be referred to as the individuals' socio-economic life-courses.

However, if the trend in the time series  $\{d_t^{Adam,Eve}\}_{t=0}^{t_{max}}$  is downward-sloping, the trajectories of Adam's and Eve's socio-economic statuses converge over time, which means that we have an inter-individual convergence of trajectories. See Figures 1 and 2 for an illustration of these two cases.

[Figures 1 and 2 about here.]

If we start with the Matthew effect, the reasons why we observe a divergence of the trajectories for Adam's and Eve's socio-economic statuses are threefold: (i) Adam experiences cumulative advantage because the time series of values  $v \in \mathbb{R}$  of his socio-economic status,  $\{v_t^{Adam}\}_{t=0}^{t_{max}}$ , where

$$(5) \quad v_t^{Adam} \equiv d_t^{Adam,0} = \|S_t^{Adam} - [0, \dots, 0]\|,$$

is upward-sloping, whereas Eve's time series of values for her socio-economic status,  $\{v_t^{Eve}\}_{t=0}^{t_{max}}$ , is downward-sloping, which means that she experiences cumulative disadvantage (or that Adam experiences cumulative disadvantage and Eve experiences cumulative advantage). (ii) Both of them experience cumulative advantage, but Adam's (Eve's) time series of values of his (her) socio-economic status is more strongly upward-sloping than Eve's (Adam's) time series of values. Finally, (iii) both of them experience cumulative disadvantage, but Adam's (Eve's) time series of values for his (her) socio-economic status is more strongly downward-sloping than Eve's (Adam's) time series of values.<sup>6</sup> See Figures 3, 4 and 5 for an illustration of these three cases.

[Figures 3, 4 and 5 about here.]

In all cases, we have assumed that the distance between Adam's and Eve's socio-economic statuses at time  $t = 0$  is short (i.e.,  $d_0^{Adam,Eve}$  is small) because a Matthew effect process should be capable of magnifying small differences over time. In other words, if Adam's and Eve's trajectories for their so-

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<sup>5</sup> Notice that to be able to calculate the distance in (4), we have to know all the values of the  $n$  variables that define Adam's socio-economic status as well as all the values of the  $n$  variables that define Eve's socio-economic status. However, as will become clear below (in Section 2.3), this is no longer necessary when calculating the Matthew effect.

<sup>6</sup> A time series of values of socio-economic status that is flat is a special case of either cumulative advantage or cumulative disadvantage.



cio-economic statuses diverge, but the trajectories were never close from the beginning, it is not really the Matthew effect we are dealing with. Why is this conceptually important? Essentially, we would like to understand why two individuals who initially enjoyed similar socio-economic statuses may end up in very different positions in life. Thus, it is not equally important to understand why two individuals who have never enjoyed similar socio-economic statuses attain different positions in life.

The reasons why we observe a convergence of the trajectories for Adam's and Eve's socio-economic statuses are also threefold: (i) Adam (Eve) experiences cumulative advantage, whereas Eve (Adam) experiences cumulative disadvantage; (ii) both of them experience cumulative advantage, but Adam's (Eve's) time series of values of his (her) socio-economic status is more strongly upward-sloping than Eve's (Adam's) time series of values; and (iii) both of them experience cumulative disadvantage, but Adam's (Eve's) time series of values for his (her) socio-economic status is more strongly downward-sloping than Eve's (Adam's) time series of values. Thus, we have the same cases as above when we observed the Matthew effect. The key here is, of course, whether Adam or Eve had the better socio-economic status at time  $t = 0$  (i.e., if  $v_0^{Adam} > v_0^{Eve}$  or if  $v_0^{Adam} < v_0^{Eve}$ ).

The simple point we would like to make here is that there is no one-to-one correspondence between, on the one hand, intra-individual changes in socio-economic status (which is the experience of either cumulative advantage or cumulative disadvantage) and, on the other hand, inter-individual convergence or divergence of the trajectories of individuals' socio-economic statuses. There is also an obvious explanation for the lack of such a clear-cut relationship: the *intra*-individual change in socio-economic status is a *micro*-level phenomenon, whereas the *inter*-individual change in socio-economic status is a *macro*-level phenomenon, which has consequences for the theoretical modeling of socio-economic processes that are able to produce the Matthew effect (to be discussed below in Section 4.2).

It should be clear from the aforementioned that we interpret cumulative advantage as an absolute improvement in socio-economic status and not, as it is sometimes interpreted in the literature, as a relative improvement in socio-economic status. The reason is, again, that we view cumulative advantage as an intra-individual or micro-level phenomenon, whereas the Matthew effect is viewed as an inter-individual or macro-level phenomenon.

*Let us sum up the discussion so far. As we can see above when looking at the trajectories for Adam's and Eve's socio-economic statuses, it is easy to determine whether the Matthew effect is present by looking at the time series  $\{d_t^{Adam,Eve}\}_{t=0}^{t_{max}}$  that shows how the distance between the socio-economic statuses of Adam and Eve evolves over time. Specifically, when the trend in this time series is upward-sloping, the Matthew effect is present, whereas this is not the case when the trend in the time series is downward-sloping.*

## 2.2 Adam, Eve, Cain and Abel

What happens if we introduce Cain and Abel to our story? How do we define the Matthew effect in this case? One route *could* be to compare the trajectories of individuals' socio-economic statuses within each pair of trajectories in the population simply by looking at the trends in the following six time series:  $\{d_t^{Adam,Eve}\}_{t=0}^{t_{max}}$ ,  $\{d_t^{Adam,Cain}\}_{t=0}^{t_{max}}$ ,  $\{d_t^{Adam,Abel}\}_{t=0}^{t_{max}}$ ,  $\{d_t^{Eve,Cain}\}_{t=0}^{t_{max}}$ ,  $\{d_t^{Eve,Abel}\}_{t=0}^{t_{max}}$  and  $\{d_t^{Cain,Abel}\}_{t=0}^{t_{max}}$ . Specifically, if the trends in all of these time series are upward-sloping, we can certainly call this the Matthew effect because all of the trajectories diverge from each other. However, quite obviously, such a definition of the Matthew effect would be too restrictive.

Think, for example, of the situation in which all of the time series above are upward-sloping except for the downward-sloping time series  $\{d_t^{Cain,Abel}\}_{t=0}^{t_{max}}$ . The reason could be that Cain kills Abel (at time  $t = t_0$ ), with the consequences that Abel's socio-economic status abruptly drops to an  $n$ -tuple with zeros,  $S_{t_0}^{Abel} = [0, \dots, 0]$ , and that Cain's socio-economic status also drops quite quickly. Moreover, the explanation for why their socio-economic life-courses converge is that Abel's initial socio-economic status was higher than Cain's (because God accepted Abel's sacrifice but rejected Cain's sacrifice).

While this example is simplistic, it still illustrates the fact that we do not observe the Matthew effect if we adopt the definition that *all* individuals' socio-economic life-courses must diverge from each other to have such an effect. This is not satisfactory. A better definition of the Matthew effect could instead be that after taking the average of the slopes of all of the time series showing how the distance between the socio-economic statuses of two individuals evolves over time, this average-slope should be positive if the Matthew effect is in play. However, even though this definition represents an improvement over the former definition of the Matthew effect, it still suffers from two drawbacks.

The first drawback relates to what DiPrete and Eirich (2006) argue is a well-defined Matthew effect process (or a cumulative advantage process, as they call it). Namely, such a process should be capable of magnifying small differences over time. This means that we should restrict our attention to the pairs of trajectories of individuals' socio-economic statuses that were initially close to each other.

The second drawback is that we are not simply interested in comparing socio-economic life-courses that have been close in status at the *same* time. Instead, we also wish to compare socio-economic life-courses that have been close in status at *different* times. This condition can be written as follows:

$$(6) \quad d_{t_0, t_1}^{Adam, Eve} \equiv \|S_{t_0}^{Adam} - S_{t_1}^{Eve}\| < \varepsilon.$$

That is, Adam had socio-economic status  $S_{t_0}^{Adam}$  at time  $t = t_0$ , whereas Eve had socio-economic status  $S_{t_1}^{Eve}$  at time  $t = t_1$ , where the distance between the socio-economic statuses is short (i.e.,  $\varepsilon$  is small). One can argue that the comparison in (6) should be restricted to individuals who belong to the same age-cohort. This is because two individuals of different ages but similar socio-economic statuses are not really comparable since one of them had longer time to achieve its socio-economic status.

Certainly, we can without problems reformulate the aforementioned definition of the Matthew effect by taking into account the drawbacks of it. At the same time, the exercise of computing and comparing all of the relevant time series to show how the distance between the socio-economic statuses of two individuals evolves over time becomes rather complex. Let us therefore shift our focus from individuals' socio-economic life-courses to the dynamic process that generates these life-courses.

### 2.3 The Matthew mechanism

The dynamic process, or, shall we say, the dynamic system that generates individuals' socio-economic life-courses is denoted by  $f : \mathbb{R}^{n_f} \rightarrow \mathbb{R}^{n_f}$ , and it describes how the state  $\mathcal{S}_t \in \mathbb{R}^{n_f}$  of the dynamic system evolves over time:

$$(7) \quad \mathcal{S}_{t+1} = f(\mathcal{S}_t).$$

How, then, is the state  $\mathcal{S}_t$  of the dynamic system  $f(\cdot)$  defined? Let us first answer this question before we move on to discuss the properties of the dynamic system  $f(\cdot)$  and present our measure of the Matthew effect.

Recall that we described the socio-economic status of Adam with the help of  $n$  variables in the form of an  $n$ -tuple. We also described the socio-economic status of Eve in the form of an  $n$ -tuple. In fact, each individual in the population has a socio-economic status that is described by an  $n$ -tuple. Specifically, each  $n$ -tuple in the population consists of exactly the same  $n$  variables (even though the ordering of the variables in the  $n$ -tuples varies between individuals and some of the variables of an individual's socio-economic status are, so to speak, turned off—e.g., all variables that concern other individuals). The state  $\mathcal{S}_t$  of the dynamic system  $f(\cdot)$  consists of the same set of  $n$  variables, but also includes two other sets of variables. The first set consists of predetermined variables that make it possible to write the dynamic system  $f(\cdot)$  in first-order form, whereas the second set consists of variables that affect individuals' socio-economic statuses without themselves being variables that define an individual's socio-economic status.<sup>7</sup>

Regarding the assumed properties of the dynamic system  $f(\cdot)$ , we impose the necessary regularity conditions on  $f(\cdot)$  to ensure that the Jacobian  $Df(\cdot)$  with partial derivatives exists, and we also assume that  $f(\cdot)$  is bounded in phase space because we do not observe unbounded dynamics in reality. The latter assumption is often overlooked in the somewhat sparse literature on inequality-generating dynamic processes. We will therefore return to this assumption below (in Section 4.2) when we briefly discuss the relationship between empirical facts and theoretical models explaining these facts, especially the relationship between the Matthew effect and the mathematical-theoretical modeling of socio-economic processes that are able to produce this effect.

The dynamic system  $f(\cdot)$  amplifies the difference between the two states  $\mathcal{S}_0$  and  $\mathcal{S}'_0$  of the dynamic system, where the initial distance  $d_0 \equiv \|\mathcal{S}_0 - \mathcal{S}'_0\| < \varepsilon$  between these states is short (i.e.,  $\varepsilon$  is small):

$$(8) \quad \mathcal{S}_j - \mathcal{S}'_j = f^j(\mathcal{S}_0) - f^j(\mathcal{S}'_0) \cong Df^j(\mathcal{S}_0)(\mathcal{S}_0 - \mathcal{S}'_0),$$

where

$$(9) \quad Df^j(\mathcal{S}_0) = Df(\mathcal{S}_{j-1})Df(\mathcal{S}_{j-2}) \cdots Df(\mathcal{S}_0).$$

We would like to learn how the distance  $d_j \equiv \|\mathcal{S}_j - \mathcal{S}'_j\|$  between the two states  $\mathcal{S}_j$  and  $\mathcal{S}'_j$  of the dynamic system  $f(\cdot)$  is amplified when time approaches infinity (i.e.,  $\lim_{j \rightarrow \infty} d_j$ ). Be aware that even though we are speaking about the amplification of the distance between two neighboring states, this distance cannot increase without limits in each of the  $n_f$  directions in phase space because the dynamic system is bounded.

The amplification of the distance  $d_j$  between the two states  $\mathcal{S}_j$  and  $\mathcal{S}'_j$  of the dynamic system  $f(\cdot)$  is quantified in all  $n_f$  directions in phase space, where these directions are pairwise orthogonal. Thus, because the dynamic system  $f(\cdot)$  is not necessarily a linear system (or, more correctly, because it *cannot* be a linear system and at the same time produce the Matthew effect and be bounded in phase space), the  $n_f$  directions can be viewed as generalized eigenvectors. Furthermore, there are genera-

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<sup>7</sup> Let us assume that the amount of physical capital available to Adam at time  $t - 1$  affects his current income and consequently his current socio-economic status. This means that  $[\text{capital}_{t-1}^{Adam}, \text{income}_t^{Adam}] \subset \mathcal{S}_t$ . Due to the presence of a variable dated in the previous time period, the dynamic system governing the evolution of individuals' socio-economic life-courses is in second-order form. However, introducing the pre-determined variable  $\text{capital}_t^{Adam} \equiv \text{capital}_{t-1}^{Adam}$ , one finds instead that  $[\text{capital}_t^{Adam}, \text{income}_t^{Adam}] \subset \mathcal{S}_t$ . Thus, the dynamic system is now in first-order form.

lized eigenvalues associated with these eigenvectors, and it is the largest of these eigenvalues that is utilized in our measure of the Matthew effect.

To better understand what these generalized eigenvectors and eigenvalues are, consider a linear dynamic system. In fact, assume that the dynamic system is an  $n_f$ -dimensional linear autoregression (with no error term, because it would not affect our argument) and disregard for the moment the need to interpret the variables in the autoregression as socio-economic variables:

$$(10) \quad s_{t+1} = \theta_0 s_t + \theta_1 s_{t-1} + \dots + \theta_{n_f-1} s_{t-n_f+1}.$$

By introducing the vector  $\mathcal{S}_t = [s_t, s_{t-1}, \dots, s_{t-n_f+1}]'$  and the matrix  $\mathcal{F} = \begin{bmatrix} \theta_0 & \theta_1 & \dots & \dots & \theta_{n_f-1} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$ ,

the linear autoregression in (10) can be rewritten as

$$(11) \quad \mathcal{S}_{t+1} = \mathcal{F}\mathcal{S}_t.$$

Because the linear system in (11) evolves in  $n_f$ -dimensional phase space, it is possible to quantify how the system contracts and expands in each of the  $n_f$  one-dimensional directions in phase space simply by solving the following eigenvalue problem:

$$(12) \quad \mathcal{F}\mathbf{e}_i = \epsilon_i \mathbf{e}_i,$$

where the eigenvalues  $\{\epsilon_i\}_{i=1}^{n_f}$  describe the time-evolution of the eigenvectors  $\{\mathbf{e}_i\}_{i=1}^{n_f}$  that, by construction, are pairwise orthogonal and therefore form a basis for  $\mathbb{R}^{n_f}$ . However, it is the quantities  $\{\lambda_i \equiv \log_e |\epsilon_i|\}_{i=1}^{n_f}$  that we focus on hereafter; they are the Lyapunov exponents for the system.

Let us return to the dynamic system  $f(\cdot)$  and define the Lyapunov exponents for this non-linear system. The intuition behind the definition of the Lyapunov exponents in the non-linear case is basically the same as in the linear case. However, one difference is that the eigenvectors now form a basis in the tangent space associated with the state  $\mathcal{S}_t$  of the dynamic system  $f(\cdot)$ . Thus, because the state evolves over time as described by the dynamic system  $f(\cdot)$ , we have not only a time series of states— $\mathcal{S}_0, \mathcal{S}_1$ , etc.—but also a series of tangent spaces associated with these states. Think of the tangent space as a linear approximation of the manifold in the neighborhood around the state  $\mathcal{S}_t$ ; a manifold, in turn, as a space in which every point on the manifold has a neighborhood that locally resembles  $\mathbb{R}^{n_f}$ .<sup>8</sup>

As in the linear case outlined above, the eigenvalues describe the time-evolution of the eigenvectors. Moreover, the Lyapunov exponents are the natural logarithms of the eigenvalues. Thus, there is no real difference between the linear and non-linear cases in terms of the eigenvalues and Lyapunov exponents. The only difference is that they are defined by different sets of eigenvectors. Specifically, the Lyapunov exponents  $\{\lambda_i\}_{i=1}^{n_f}$  for the dynamic system  $f(\cdot)$  are defined by the following  $n_f$  limits:

$$(13) \quad \lambda_i \equiv \lim_{j \rightarrow \infty} \frac{\log_e |Df^j(\mathcal{S}_0)|}{j} = \lim_{j \rightarrow \infty} \frac{1}{j} \sum_{k=0}^{j-1} \log_e |Df(\mathcal{S}_k)|,$$

<sup>8</sup> Think, for example, of the manifold as a sphere and the tangent space as a plane that locally approximates the sphere.

where each limit is taken in the direction identified with the corresponding eigenvector in tangent space. Based on an ergodic theorem (known as Oseledec’s multiplicative ergodic theorem), the  $n_f$  limits in (13) exist and are independent of  $\mathcal{S}_0$  almost surely with respect to the measure induced by the time series  $\{\mathcal{S}_t\}_{t=1}^{\infty}$  (see Guckenheimer and Holmes, 1983). The Lyapunov exponents are ranked from the largest to the smallest value:  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n_f}$ .

If the largest Lyapunov exponent is positive (i.e.,  $\lambda_1 > 0$ ), the dynamic system  $f(\cdot)$  has the property of sensitive dependence on initial conditions: any two trajectories describing the time-paths of the state of the dynamic system  $f(\cdot)$  with arbitrarily close, but not identical, initial conditions (i.e.,  $d_0$  is small) will diverge from each other at an exponential rate, even if the trajectories remain within a bounded space. Specifically, if the initial state of the dynamic system  $f(\cdot)$  is  $\mathcal{S}_0$ , we should observe the following time series of states:  $\mathcal{S}_1, \mathcal{S}_2$ , etc. On the other hand, if the initial state of the dynamic system  $f(\cdot)$  is  $\mathcal{S}'_0$ , we should observe the following time series of states:  $\mathcal{S}'_1, \mathcal{S}'_2$ , etc. The point here is that if  $\lambda_1 > 0$ , the two time series of states will diverge from each other. Thus,  $\lambda_1 > 0$  catches what DiPrete and Eirich (2006) argue is essentially a well-defined inequality-generating dynamic process because it is capable of magnifying small differences over time.

*What are the relationships between the state  $\mathcal{S}_t$  of the dynamic system  $f(\cdot)$ ; the socio-economic statuses of, say, Adam and Eve,  $S_t^{Adam}$  and  $S_t^{Eve}$ ; their respective socio-economic life-courses,  $\{S_t^{Adam}\}_{t=0}^{t_{max}}$  and  $\{S_t^{Eve}\}_{t=0}^{t_{max}}$ ; the Lyapunov exponents  $\{\lambda_i\}_{i=1}^{n_f}$  of the dynamic system  $f(\cdot)$ ; and, most importantly, the Matthew effect?*

Keep in mind that the state  $\mathcal{S}_t$  of the dynamic system  $f(\cdot)$  consists of three sets of variables: (i)  $n$  variables that describe the socio-economic statuses of the individuals in the population; (ii) pre-determined variables that make it possible to write the dynamic system  $f(\cdot)$  in first-order form; and (iii) variables that affect the individuals’ socio-economic statuses without themselves being variables that define an individual’s socio-economic status. Let a simple example illustrate what we mean, where the latter two sets of variables have been summarized under the heading *other variables*:

$$(14) \quad \mathcal{S}_t = [\text{health}_t^{Adam}, \text{income}_t^{Adam}, \text{health}_t^{Eve}, \text{income}_t^{Eve}, \text{other variables}_t].$$

Then, there exists a function  $g_{Adam} : \mathbb{R}^{n_f} \rightarrow \mathbb{R}^n$  that projects the  $n_f$  variables that define the state  $\mathcal{S}_t$  of the dynamic system  $f(\cdot)$  onto the  $n$ -tuple that defines Adam’s socio-economic status,  $S_t^{Adam}$ :

$$(15) \quad g_{Adam}(\mathcal{S}_t) = [\text{health}_t^{Adam}, \text{income}_t^{Adam}].$$

Notice therefore that the socio-economic status of an individual—say, Adam—can also be denoted by  $g_{Adam}(\mathcal{S}_t)$  and not only by  $S_t^{Adam}$ . Furthermore, there exist functions that are analogous to the one in (15) and that do a similar task for all other individuals in the population, including Eve.

As we have just learned, the Lyapunov exponents  $\{\lambda_i\}_{i=1}^{n_f}$  quantify how the distance between the two states  $\mathcal{S}_0$  and  $\mathcal{S}'_0$  of the dynamic system  $f(\cdot)$  evolves over time. Thus, it seems that the Lyapunov exponents are not actually measuring what we discussed above regarding socio-economic inequality and how it changes over time since the socio-economic status of an individual is a proper subset of the state of the dynamic system that generates the socio-economic status. However, this is an illusory problem; to see why, rewrite the state  $\mathcal{S}_t$  in (14) as follows:

$$(16) \quad \mathcal{S}_0 = [S_0^{Adam}, S_0^{Eve}, \text{other variables}_0],$$

where we have dated the state at time  $t = 0$ . In principle, the initial state  $\mathcal{S}'_0$ , which is close to the initial state  $\mathcal{S}_0$ , can be constructed in three different ways: (i) as a distortion of Adam's socio-economic status,  $\mathcal{S}_0^{Adam}$ ; (ii) as a distortion of Eve's socio-economic status,  $\mathcal{S}_0^{Eve}$ ; and (iii) as a distortion of the two sets of variables under the heading *other variables* $_0$ , which is *other variables*' $_0$ . The neighboring state  $\mathcal{S}'_0$  can, of course, be constructed as a combination of these three distortions as well, even though it is always possible to construct it as distortion of Adam's or Eve's socio-economic status.

The point we would like to make here is that if we construct a distortion in, say, Eve's socio-economic status, the time series

$$(17) \quad \lim_{j \rightarrow \infty} \{d_j\}_0^j \equiv \{\|\mathcal{S}_j - \mathcal{S}'_j\|\}_0^j,$$

which is our primary point of interest, has an important property that coincides with a property of the time series

$$(18) \quad \lim_{j \rightarrow \infty} \{d_j^{Adam, Eve}\}_0^j \equiv \{\|\mathcal{S}_j^{Adam} - \mathcal{S}_j^{Eve}\|\}_0^j.$$

Specifically, because the time series in (17) and (18) are generated by the *same* dynamic system  $f(\cdot)$ , the Lyapunov exponents  $\{\lambda_i\}_{i=1}^{n_f}$  are the *same* in the two cases.<sup>9</sup> Furthermore, because there is a large number of individuals in the population, there exists some other individual with a socio-economic status that coincides with Eve's distorted socio-economic status—say, Set,  $\mathcal{S}_j^{Set} = \mathcal{S}_j^{Eve}$ . It is therefore safe to claim that if the socio-economic inequality within the population increases over time, the Matthew effect is at work and  $\lambda_1 > 0$ .

*Let us sum up what we have learned so far. If  $\lambda_1 > 0$ , the dynamic system  $f(\cdot)$  that generates individuals' socio-economic life-courses has the property of sensitive dependence on initial conditions: any two trajectories describing the time-paths of the state of the dynamic system  $f(\cdot)$  with arbitrarily close, but not identical, initial conditions will diverge from each other at an exponential rate, even if the trajectories remain within a bounded space. Thus,  $\lambda_1 > 0$  captures what we and other researchers interpret as the Matthew effect (see the discussion in Section 4.1).*

### 3 Estimating the Matthew effect

This section has three parts. In Section 3.1, we describe the dominant approach in the literature on how to estimate the Lyapunov exponents from data, which also gives us an estimate of the Matthew effect. Thereafter, in Section 3.2, we present the asymptotic theory that is available for statistical inference. Finally, in Section 3.3, we give some ideas about why the approach outlined in Section 3.1 works.

#### 3.1 How to estimate?

Because the actual form of the dynamic system  $f(\cdot)$  is not known, it may seem impossible to estimate the Lyapunov exponents of this system. Fortunately, it is possible to reconstruct the dynamics using only a scalar time series and thereafter to estimate the Lyapunov exponents of the reconstructed dy-

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<sup>9</sup> This will become clear below (in Section 3.1) when we show that a scalar time series generated by a dynamic system is sufficient to estimate the Lyapunov exponents for this system.

dynamic system. For this reason, associate the dynamic system  $f(\cdot)$  with the observer function  $g : \mathbb{R}^{n_f} \rightarrow \mathbb{R}$  that generates

$$(19) \quad s_t = g(\mathcal{S}_t) + \varepsilon_t,$$

where  $s_t \in \mathbb{R}$  is a socio-economic variable and  $\varepsilon_t \in \mathbb{R}$  is the measurement error. Thus, (19) means that the scalar time series  $\{s_t\}_{t=0}^{t_{max}}$  is observed.

The  $t_{max} + 1$  observations in the time series  $\{s_t\}_{t=0}^{t_{max}}$  contain information on unobserved state variables that can, as we will soon see, be utilized to define a state in the present time. In fact, let

$$(20) \quad \mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_{n_{\mathcal{T}}-1})'$$

be the reconstructed trajectory, or orbit, that describes how the reconstructed state  $\mathcal{T}_t \in \mathbb{R}^{n_h}$  evolves over time; additionally, let  $n_{\mathcal{T}}$  be the number of states in the reconstructed trajectory. Moreover, the reconstructed state at time  $t$  is

$$(21) \quad \mathcal{T}_t = \{s_t, s_{t+1}, \dots, s_{t+n_h-1}\},$$

in which  $n_h$  is the embedding dimension.<sup>10</sup> Thus,  $\mathcal{T}$  is an  $n_{\mathcal{T}} \times n_h$  matrix, and the constants  $n_{\mathcal{T}}$ ,  $n_h$  and  $t_{max}$  are related as  $n_{\mathcal{T}} = t_{max} - n_h + 2$ .

Then, given some technical conditions, Takens (1981) proved that the function

$$(22) \quad \Phi(\mathcal{S}_t) = \{g(f^0(\mathcal{S}_t)), g(f^1(\mathcal{S}_t)), \dots, g(f^{n_h-1}(\mathcal{S}_t))\},$$

which maps the  $n_f$ -dimensional unobserved state  $\mathcal{S}_t$  onto (and not only into) the  $n_h$ -dimensional reconstructed state  $\mathcal{T}_t$ , is an embedding when  $n_h > 2n_f$ . This means that the function  $\Phi : \mathbb{R}^{n_f} \rightarrow \mathbb{R}^{n_h}$  is a smooth function that performs a one-to-one coordinate transformation and has a smooth inverse. Be aware that the condition  $n_h > 2n_f$  is a sufficient but not a necessary condition to have an embedding (see Sauer et al., 1991).

Here is the reason for our interest in embeddings: a function that is an embedding preserves topological information about the unknown dynamic system  $f(\cdot)$ , such as the Lyapunov exponents. In particular, such a function induces another function  $h : \mathbb{R}^{n_h} \rightarrow \mathbb{R}^{n_h}$  on the reconstructed trajectory,

$$(23) \quad \mathcal{T}_{t+1} = h(\mathcal{T}_t),$$

which is topologically conjugate to the unknown dynamic system  $f(\cdot)$ :

$$(24) \quad h^j = \Phi \circ f^j \circ \Phi^{-1}(\mathcal{T}_t).$$

$h(\cdot)$  is therefore a reconstructed dynamic system that has the same Lyapunov exponents as the unknown dynamic system  $f(\cdot)$ .

To be able to estimate the Lyapunov exponents of the unknown dynamic system  $f(\cdot)$ , one must first estimate the reconstructed dynamic system  $h(\cdot)$ . However, because

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<sup>10</sup> Actually, it is not necessary for the reconstructed state in (21) to consist of consecutive observations in the time series  $\{s_t\}_{t=0}^{t_{max}}$ , but we disregard other types of reconstructions in this paper.

$$(25) \quad h : \begin{pmatrix} s_t \\ s_{t+1} \\ \vdots \\ s_{t+n_h-1} \end{pmatrix} \rightarrow \begin{pmatrix} s_{t+1} \\ s_{t+2} \\ \vdots \\ h'(s_t, s_{t+1}, \dots, s_{t+n_h-1}) \end{pmatrix},$$

the estimation of the reconstructed dynamic system  $h(\cdot)$  reduces to the estimation of  $h'(\cdot)$ :

$$(26) \quad s_{t+n_h} = h'(s_t, s_{t+1}, \dots, s_{t+n_h-1});$$

this is essentially a non-linear autoregression of order  $n_h$  (with no error term). Moreover, because the Jacobian  $Dh(\cdot)$  on the reconstructed state  $\mathcal{T}_t$  is

$$(27) \quad Dh(\mathcal{T}_t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h'}{\partial s_t} & \frac{\partial h'}{\partial s_{t+1}} & \frac{\partial h'}{\partial s_{t+2}} & \dots & \frac{\partial h'}{\partial s_{t+n_h-1}} \end{pmatrix},$$

a feed-forward neural network should be used to estimate the above derivatives and thus to consistently estimate the Lyapunov exponents (see Dechert and Gencay, 1992, and Gencay and Dechert, 1992). This is because Hornik et al. (1990) have shown that a function and its derivatives of any unknown functional form can be approximated arbitrarily accurately by such a network.

Specifically, after having estimated the derivatives in (27) with a feed-forward neural network, we can compute the Jacobian  $Dh(\cdot)$ . Thus, after having repeated this procedure at each point in time along the reconstructed trajectory  $\mathcal{T}$ , we can compute the Lyapunov exponents for the reconstructed dynamic system  $h(\cdot)$ , which are the same as the Lyapunov exponents for the unknown dynamic system  $f(\cdot)$ :

$$(28) \quad \hat{\lambda}_i = \lim_{n_{\mathcal{T}} \rightarrow \infty} \frac{1}{n_{\mathcal{T}}} \sum_{k=0}^{n_{\mathcal{T}}-1} \log_e |Dh(\mathcal{T}_k)|,$$

where each limit is taken in the direction identified with the corresponding eigenvector in tangent space. Of course, our interest is restricted to the estimate of the largest Lyapunov exponent  $\hat{\lambda}_1$ , because the positivity of this exponent means that the Matthew effect is present in the dynamic system that generates individuals' socio-economic life-courses.

### 3.2 How to make inference?

What about statistical inference? How can we test whether the Matthew effect is present in the dynamic system that generates individuals' socio-economic life-courses? Specifically, how can we test the positivity of the largest Lyapunov exponent for the unknown dynamic system that generates these life-courses?

Shintani and Linton (2004) derive the asymptotic distribution of a neural network estimator of the Lyapunov exponents:

$$(29) \quad \sqrt{n_{\mathcal{T}}}(\hat{\lambda}_i - \lambda_i) \Rightarrow \mathbb{N}(0, \mathcal{V}_i),$$

where  $\{\hat{\lambda}_i\}_{i=1}^{n_h}$  is the estimator of the  $i$ :th Lyapunov exponent based on the  $n_{\mathcal{T}}$  reconstructed states on the trajectory  $\mathcal{T}$ , and where  $\{\mathcal{V}_i\}_{i=1}^{n_h}$  is the variance of the  $i$ :th Lyapunov exponent.



To test the hypothesis that the Matthew effect is present in the dynamic system that generates individuals' socio-economic life-courses, we consider the following null and alternative hypotheses:

$$(30) \quad \begin{cases} \mathbf{H}_0 : \lambda_1 \leq 0, \\ \mathbf{H}_1 : \lambda_1 > 0, \end{cases}$$

where our interest is restricted to the sign of the largest Lyapunov exponent. The test statistic is  $\hat{t} = \frac{\hat{\lambda}_1}{\sqrt{\frac{\hat{\mathcal{V}}_1}{n_T}}}$ , where  $\hat{\mathcal{V}}_1$  is a consistent estimator of  $\mathcal{V}_1$  (see Andrews, 1991). Thus, the null hypothesis of there being no Matthew effect is rejected when  $\hat{t} \geq z_\alpha$ , where the significance level is  $Pr[\mathbb{Z} \geq z_\alpha] = \alpha$  and  $\mathbb{Z}$  is the standard normal random variable.<sup>11 12</sup>

### 3.3 Why does it work?

At the heart of the estimation procedure is Takens' (1981) embedding theorem. Specifically, according to the embedding theorem, the reconstructed dynamic system  $h(\cdot)$  has the same Lyapunov exponents as the unknown dynamic system  $f(\cdot)$  when the embedding dimension  $n_h$  of the reconstructed state  $\mathcal{T}_t$  is at least twice as large as the dimension  $n_f$  of the unobserved state  $\mathcal{S}_t$ . Considering its central importance, is there some intuitive explanation for Takens' (1981) embedding theorem?

For the sake of argument, assume that  $\mathbb{U}_1 \subset \mathbb{M}$  and  $\mathbb{U}_2 \subset \mathbb{M}$  are two subspaces or manifolds of dimensions  $n_1$  and  $n_2$ , respectively, where  $\mathbb{M} \in \mathbb{R}^{n_h}$  is an  $n_h$ -dimensional manifold that surrounds the reconstructed dynamics. Generically, the two subspaces intersect in a subspace of dimension  $n_1 + n_2 - n_h$ . Thus, when this expression is negative, there is no intersection of the two subspaces. Therefore, the self-intersection of an  $n_f$ -dimensional manifold fails to occur when  $n_h > 2n_f$ .

A difficulty here is that the dimension  $n_f$  of the unknown dynamics is itself unknown, which means that the required embedding dimension  $n_h$  is also unknown. This problem can, however, be solved indirectly by making use of a generic property of a proper embedding. The dynamics in original phase space must be completely unfolded in reconstructed phase space. Thus, if the embedding dimension is too low, distant states in original phase space will be close states in reconstructed phase space and will therefore be false neighbors.

There are two closely related methods of calculating the required embedding dimension from scalar time series data for a proper embedding: (i) false nearest neighbors and (ii) the saturation of invariants on the reconstructed dynamics such as the Lyapunov exponents. The first method is based on the aforementioned generic property of a proper embedding, which indicates that upon increasing the em-

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<sup>11</sup> Software that can be used to estimate the Lyapunov exponents and thereafter make statistical inference along the lines described in Sections 3.1 and 3.2 is NETLE 4.1. This software was developed by R. Gencay, C.-M. Kuan and T. Liu, and it can be downloaded at <http://tliu.iweb.bsu.edu/research/index.html>.

<sup>12</sup> An alternative to the technique used by Shintani and Linton (2004) is to derive an empirical distribution for the largest Lyapunov exponent via bootstrapping. In fact, a moving blocks technique can be used to test the positivity of the largest Lyapunov exponent, where the moving blocks are the reconstructed states  $\{\mathcal{T}_t\}_{t=0}^{n_T-1}$  along the reconstructed trajectory  $\mathcal{T}$  (see Bask and Gencay, 1998, and Gencay, 1996). However, Ziehmman et al. (1999) demonstrate that a moving blocks technique is not straightforward for use with multiplicative ergodic statistics because the limits in (13) may not always exist. Another alternative to Shintani and Linton's (2004) technique is that of Bask and de Luna (2002), who argue that a parametric model can be fitted to the time series  $\{\mathcal{S}_t\}_{t=0}^{t_{max}}$ , which thereafter is utilized to estimate the largest Lyapunov exponent, where the estimator of this exponent is consistent and asymptotically normal.

bedding dimension, the dynamics is completely unfolded when there are no false neighbors in reconstructed phase space (see Kennel et al., 1992).

The second method utilizes the fact that when the dynamics is completely unfolded, the Lyapunov exponents are independent of the embedding dimension. If, however, the dynamics is not completely unfolded, these invariants depend on the embedding dimension. Therefore, by increasing the embedding dimension, we achieve a proper embedding when the value of an invariant stops changing (see Fernández-Rodríguez et al., 2005, who present a test for the positivity of the largest Lyapunov exponent based on this observation).

## 4 Discussion

This section has two parts. In Section 4.1, we discuss the value added by our measure of the Matthew effect, and in Section 4.2, we outline directions for future research, especially with regard to the theoretical modeling of socio-economic processes that lead to increased inequality over time.

### 4.1 *What is the value added by measuring $\lambda_1$ ?*

What is the value added by using the largest Lyapunov exponent as a measure of the Matthew effect? Is this a better measure of socio-economic inequality than, for example, the Atkinson, Gini and Theil indices (see Cowell, 2009, for a review of these and other inequality measures)? No, this is not a better measure, simply because the largest Lyapunov exponent is not exactly an inequality measure. It is true that the Matthew effect is closely related to inequality, as we can note from the introductory section and the quotations by Merton (1988). However, the Matthew effect is not exactly the same as inequality.

An inequality measure such as the Gini index measures the degree of socio-economic inequality between persons at a certain point in time, whereas the largest Lyapunov exponent measures how the degree of inequality changes over time between persons with neighboring socio-economic statuses. In other words, a positive largest Lyapunov exponent, which indicates the presence of the Matthew effect, can be associated with a low as well as a high Gini index. The largest Lyapunov exponent, as a measure of the Matthew effect, is therefore a complement to inequality measures such as the Atkinson, Gini and Theil indices.

Then, if the largest Lyapunov exponent is not an inequality measure, is it a proper measure of the Matthew effect? Recall how we summarized in the introductory section what Merton (1988) described as typical characteristics of the Matthew effect. First, the Matthew effect is a process that results in inequality. Second, the Matthew effect is an ongoing process and therefore dynamic. Third, the Matthew effect is a non-linear dynamic process because it is bounded but still capable of magnifying small differences over time. Thus, the largest Lyapunov exponent is a proper measure of the Matthew effect, because a positive largest Lyapunov exponent means that any two trajectories describing the time-paths of the state of a dynamic process with arbitrarily close, but not identical, initial conditions will diverge from each other, even though they remain within a bounded space.

However, we should clearly not just use Merton (1968, 1988) as a benchmark for what characterizes a proper measure of the Matthew effect. Let us therefore complement the descriptions of the Matthew effect in Merton (1968, 1988) with those in Dannefer (2003), who reviews the cumulative advantage literature as it has been applied in gerontology, and DiPrete and Eirich (2006), who urge more explicit

attention to methodological issues in the application of different concepts in the cumulative advantage literature.

Let us begin with DiPrete and Eirich (2006) and their interpretation of cumulative advantage. Even though the majority of scholars invoke cumulative advantage to mean a dynamic process of diverging socio-economic life-courses for individuals, DiPrete and Eirich (2006) argue that there are multiple and sometimes conflicting interpretations of this concept. We agree with them. For example, it seems to us from the literature that cumulative advantage and the Matthew effect are occasionally, and in our opinion mistakenly, used interchangeably. We have instead argued that cumulative advantage is an intra-individual micro-level phenomenon, whereas the outcome of a Matthew effect process is an inter-individual macro-level phenomenon.

Specifically, DiPrete and Eirich (2006) identify three characteristics of cumulative advantage (or the Matthew effect, as we call it). First, the Matthew effect is a “*mechanism for inequality across any temporal process [...] in which a favorable relative position becomes a resource that produces further relative gains*” (see p. 271 in DiPrete and Eirich, 2006). Second, the Matthew effect “*becomes part of an explanation for growing inequality when current levels of accumulation have a direct causal relationship on future levels of accumulation*” (see p. 272 in DiPrete and Eirich, 2006). Third, the Matthew effect “*is capable of magnifying small differences over time*” (see p. 272 in DiPrete and Eirich, 2006). Thus, the characteristics listed here are the same as those emphasized by Merton (1968, 1988).

Let us continue with Dannefer (2003) and how he interprets cumulative advantage, which we here interpret as a characteristic of the Matthew effect:

“[C]umulative advantage/disadvantage can be defined as the systematic tendency for interindividual divergence in a given characteristic (e.g., money, health, or status) with the passage of time. Two terms in this definition warrant special attention. ‘Systematic tendency’ indicates that divergence is not a simple extrapolation from the members’ respective positions at the point of origin; it results from the interaction of a complex of forces. ‘Interindividual divergence’ implies that cumulative advantage/disadvantage is not a property of individuals but of populations or other collectivities (such as cohorts), for which an identifiable set of members can be ranked” (see p. S327 in Dannefer, 2003).

Thus, if we use DiPrete and Eirich (2006) as our point of reference for what characterizes the Matthew effect, the largest Lyapunov exponent is once more established as a proper measure of this effect. The same is true if we use Dannefer (2003) as our point of reference, even though he makes use of somewhat different phrases than do DiPrete and Eirich (2006) and Merton (1988) in the quotations above. In any event, it is clear that Dannefer (2003) refers to a non-linear dynamic process when describing cumulative advantage (or the Matthew effect, as we call it).

## **4.2 Where to go from here?**

We see two important directions for future research. The first is to apply the techniques that we have demonstrated herein to data and examine whether the Matthew effect is present in any of the socio-economic processes we observe in the real world. The second is the mathematical-theoretical modeling of socio-economic processes that can be characterized by the Matthew effect.

First, recall that a scalar time series of a socio-economic variable is sufficient to indicate whether the Matthew effect is present in the dynamic process that generates socio-economic inequality. One chal-

lenge, however, is that the majority of contemporary data are not yet good enough because, for, say, monthly data, the length of the time series must span multiple decades to be satisfactory. The reason for this requirement is that the number of observations in the time series must be in the hundreds to have a reliable estimate of the largest Lyapunov exponent. However, the embedding theorem by Takens (1981) does *not* indicate that it is not possible to reconstruct the dynamics with a multivariate time series. Instead, it only shows that a scalar time series is sufficient to reconstruct the dynamics.

Specifically, the dynamics that we reconstruct using a scalar time series of a socio-economic variable is the attractor for a dynamic process. An attractor, in turn, is a set of points in phase space towards which the trajectories of this dynamic process converge. Moreover, an attractor  $\mathcal{A}$  is indecomposable if there is no proper subset of  $\mathcal{A}$  that also is an attractor (see Eckmann and Ruelle, 1985). The key here is that *all* time series that make up the multivariate time series, which we would like to use to reconstruct the dynamics, must come from the *same* indecomposable attractor. Otherwise, it is a pseudo-attractor that is reconstructed, which has little to do with the attractor for the socio-economic process under scrutiny. However, if all time series come from the same indecomposable attractor, monthly data that span a few years should be sufficient to test for the presence of the Matthew effect.

The second route for future research is the mathematical-theoretical modeling of socio-economic processes that can be characterized by the Matthew effect. Specifically, if the Matthew effect is detected in real world data, socio-economic theory must also be able to produce the Matthew effect; otherwise, the theory will not explain an important property that it is supposed to account for. To the best of our knowledge, there is no such socio-economic theory available today. However, the mathematical formulation of such a theory must take the form of a bounded non-linear dynamic system. The reasons for this are twofold. First, we do not observe unbounded dynamics in reality. Second, the largest Lyapunov exponent cannot be positive in a bounded dynamic system if the system is linear.

Finally yet importantly, because we have argued that cumulative advantage is an intra-individual micro-level phenomenon, whereas the outcome of a Matthew effect process is an inter-individual macro-level phenomenon, it is necessary that the socio-economic model explaining the Matthew effect be a heterogeneous agent model. One class of heterogeneous agent models that fits well for this purpose are social network models—or, to be more precise, actor-based models for network dynamics. This is because a given and well-specified actor-based model has its roots in a particular theory, such as a socio-economic theory, and because there is also a toolbox available for estimation and statistical inference using the kind of longitudinal data typically employed in life-course research (see Snijders et al., 2010).

## 5 Conclusions

The contributions of this paper are twofold. Our first aim was to take up the baton passed by DiPrete and Eirich (2006) and clarify the differences and the relationships between cumulative advantage/disadvantage and the Matthew effect. Our second aim, which also was our main contribution, was not only to present a new measure of the Matthew effect, but also to show how to estimate this effect from data and how to make statistical inference. Regarding the Matthew effect, we have argued that a positive value for the largest Lyapunov exponent means that the effect is present in the dynamic process that generates individuals' socio-economic life-courses. Thus, we have presented a new, proper measure of the Matthew effect, which focuses on the dynamic process that generates socio-economic inequality and not on the outcome of this process. It might also therefore be better to rename the Matthew effect the Matthew mechanism.

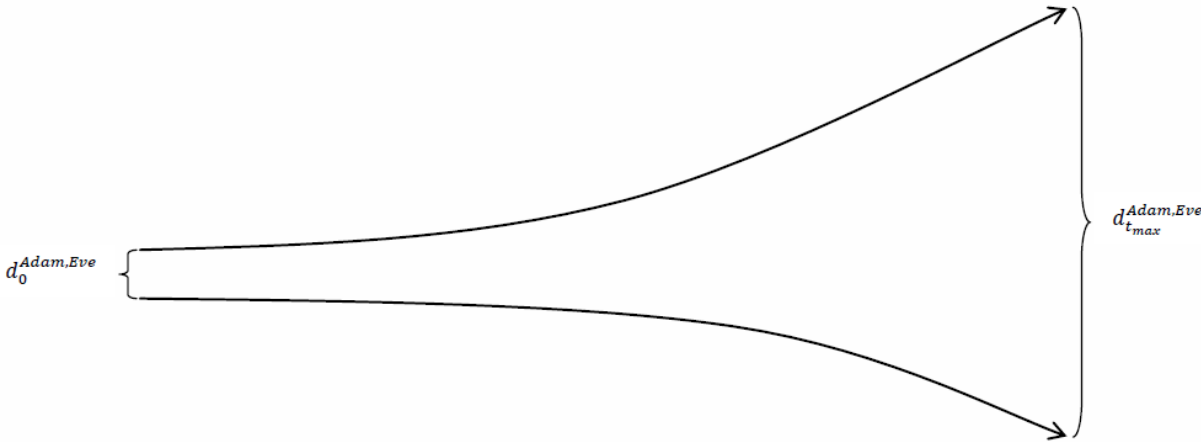
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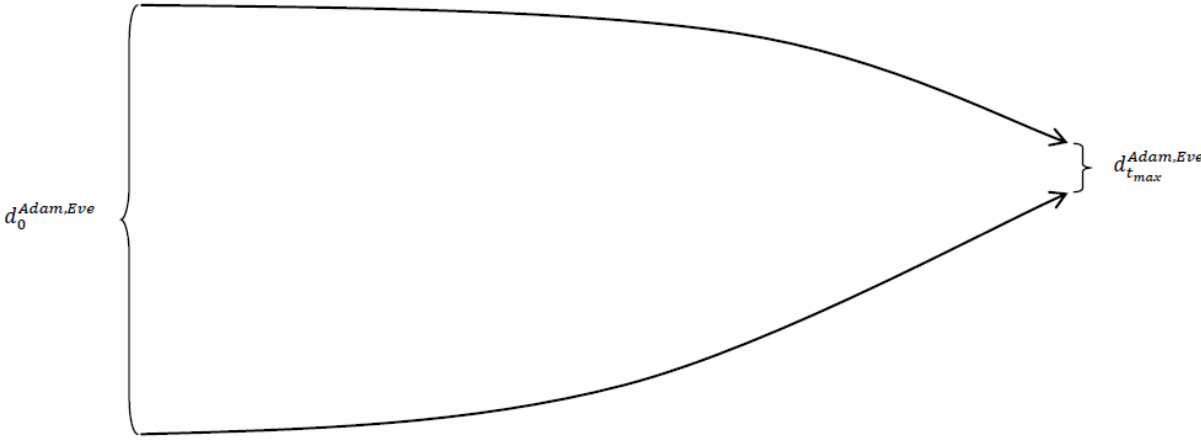
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**Figures**

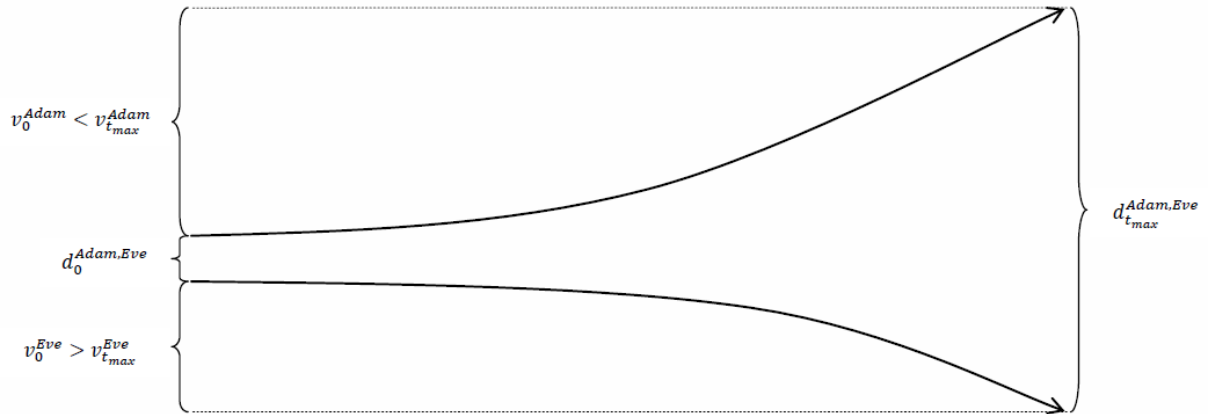


**Figure 1** The distance between Adam’s and Eve’s socio-economic statuses increases over time.

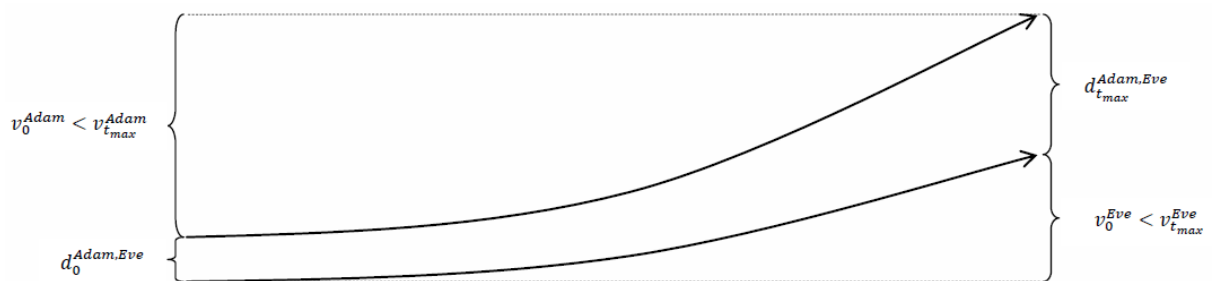


**Figure 2** The distance between Adam’s and Eve’s socio-economic statuses decreases over time.

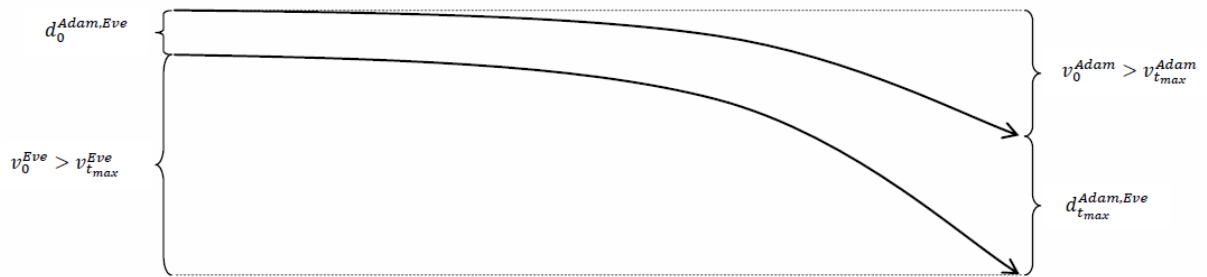




**Figure 3** The distance between Adam's and Eve's socio-economic statuses increases over time when Adam experiences cumulative advantage and Eve experiences cumulative disadvantage.



**Figure 4** The distance between Adam's and Eve's socio-economic statuses increases over time, even when both Adam and Eve experience cumulative advantage.



**Figure 5** The distance between Adam's and Eve's socio-economic statuses increases over time, even when both Adam and Eve experience cumulative disadvantage.

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