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Resources Implies an Unbounded Growth Rate**

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# *Strictly Endogenous Growth with Non-renewable Resources Implies an Unbounded Growth Rate*

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## **Abstract**

Conventional endogenous growth theory relies on the assumption of constant returns to "broad capital". As Solow pointed out, the strength of this assumption is revealed by recognizing that even the slightest touch of increasing returns creates explosive growth: infinite output in finite time! But Solow's observation ignored natural resources. What happens if non-renewable resources enter the "growth engine"? In this case (strictly) endogenous growth requires the technology to be such that there is no upper bound on the sustainable per capita growth rate.

*Keywords:* Endogenous growth; semi-endogenous growth, non-renewable resources, knife-edge.

*JEL Classification:* O4, Q3.

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# 1 Introduction

A recurrent criticism of conventional endogenous growth theory is that it needs the strong assumption of constant returns (at least asymptotically) to the collection of accumulatable factors of production ("broad capital"). Slightly decreasing returns lead to growth petering out unless some exogenous factor, e.g., population, grows. No theoretical reasons have been given why non-diminishing returns should come true, and it seems hard to maintain that the empirical evidence favours the presumption. Moreover, as Solow (1994) reminds us, a kind of "knife-edge" is involved since even the slightest touch of increasing returns generates explosive growth: infinite output in finite time!

The fact that this explosion occurs only a hair's-breadth from the presumed constant returns gives occasion for scepticism towards the realism of conventional endogenous growth theory. But Solow's observation concerns the prototype endogenous growth models which ignores natural resources, in particular non-renewable resources. When such resources enter the "growth engine" in an essential way, endogenous growth in fact *requires* (in the absence of population growth) increasing returns to broad capital, cf. Suzuki (1976) and Groth and Schou (2002).<sup>1</sup>

The present paper shows that this requirement is tantamount to there being no upper bound on the sustainable per capita growth rate. Further, though Solow style explosion is no longer a necessary outcome, it is still within reach. Hence, this framework does not make one feel much easier with the assumption of non-diminishing returns. This assumption still seems too-good-to-be-true.

Our result is based on the one-sector model of Stiglitz (1974a, 1974b), extended to allow for increasing returns at the aggregate level with respect to capital, labour, and the resource; the possibility of increasing returns to capital itself is not excluded. An essential feature is that non-renewable resources are necessary inputs. A number of contributions like Jones and Manuelli (1997), Aghion and Howitt (1998), and Schou (2000) have dealt with the implications of non-renewable resources for endogenous growth. However, in these models natural resources do not appear in the "growth engine" (not even indirectly in the sense of resources being a necessary ingredient in the production of physical capital goods which are then used in the growth-creating sector, e.g., a research sector). This seems unrealistic. After all, most sectors, including educational institutions and research labs, use fossil fuels for heating and transporta-

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<sup>1</sup>We define the growth engine of a model as the set of capital-producing sectors or activities using their own output as an input.

tion purposes, or minerals and oil products for machinery, computers, etc. An early endogenous growth model that indeed does take account of this fact is Suzuki (1976, Section 3). But Suzuki's focus is only on the *sign* of the sustainable growth rate. In contrast, the present paper examines its *size*.

The organization of the paper is as follows. The next section presents the model. In Section 3 the set of "balanced paths" that are attainable from a mere technological point of view is described. Section 4 establishes the unboundedness result. The final section concludes.

## 2 The model

To ensure that the non-renewable resource is necessary for production, but does not *a priori* rule out non-decreasing consumption in the long run, we follow Stiglitz and assume an aggregate production function of Cobb-Douglas form:

$$Y(t) = AK(t)^\alpha N(t)^\beta R(t)^\gamma, \quad A, \alpha > 0, 0 < \beta < 1, 0 < \gamma < 1, \quad (1)$$

where  $Y(t)$  is output,  $K(t)$  is the capital stock,  $N(t)$  is labour input, and  $R(t)$  is input of the non-renewable resource (henceforth, simply called the resource). Stiglitz (1974a, 1974b) and others<sup>2</sup> focus on  $\alpha + \beta + \gamma = 1$ . However, when  $K$  is interpreted as 'broad capital' including technical knowledge and human capital or when positive capital externalities are present, a case can be made for the parameters  $\alpha, \beta$ , and  $\gamma$  summing to some larger value. This is one thing – it is quite another to claim that  $\alpha$  itself is close to one (or larger) as conventional endogenous growth theory does. The empirical foundation seems weak. Here we shall trace out theoretical implications of that strong assumption.

Labour grows at a constant exogenous rate  $n \geq 0$ , i.e.,  $N(t) = N(0)e^{nt}$ ,  $N(0) = N_0 > 0$ . Output is used for consumption and for investment in capital goods so that<sup>3</sup>

$$\dot{K} = Y - C - \delta K, \quad \delta \geq 0, \quad K(0) = K_0 > 0, \quad (2)$$

where  $C$  is total consumption. Capital cannot be "eaten", i.e.,  $C \leq Y$  for all  $t$ . The resource stock  $S$  diminishes with resource extraction:

$$\dot{S} = -R, \quad S(0) = S_0 > 0. \quad (3)$$

Given  $K_0, S_0$ , and  $N_0$ , a path  $(C, Y, K, R, S)_{t=0}^\infty$  is called *feasible* if: (a)  $K$  and  $S$  are continuous functions of  $t$ ; (b)  $C, Y$ , and  $R$  are piecewise continuous functions of  $t$ ;

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<sup>2</sup>E.g., Dasgupta and Heal (1979), and Jones (2002, Ch. 9).

<sup>3</sup>From now on we will suppress the time argument of the variables when not needed for clarity.

(c) the path satisfies (1) for all  $t \geq 0$ , and it satisfies (2) and (3) for all  $t \geq 0$ , except at points of discontinuity of  $C$  and  $R$ ; and (d) the path satisfies the non-negativity constraints  $C, R, K, S \geq 0$  for all  $t \geq 0$ . Condition (3) and the non-negativity constraint on  $S$  imply the restriction

$$\int_0^{\infty} R(t)dt \leq S_0, \quad (4)$$

showing the finite upper bound on cumulative extraction of the resource over the infinite future.

We shall be concerned with feasible paths where  $C, Y, K, R$ , and  $S$  are (strictly) positive for all  $t \geq 0$  ("no collapse"). Such paths are called *interior* feasible paths. Writing  $g_a$  for  $\dot{a}/a$ , by logarithmic differentiation in (1) we get

$$g_Y = \alpha g_K + \beta n + \gamma g_R. \quad (5)$$

A feasible path  $(C, Y, K, R, S)_{t=0}^{\infty}$  is called *efficient* if there does not exist another feasible path  $(\hat{C}, \hat{Y}, \hat{K}, \hat{R}, \hat{S})_{t=0}^{\infty}$  with the same initial condition such that  $\hat{C}(t) \geq C(t)$  for all  $t \geq 0$  and  $\hat{C}(t) > C(t)$  in some open time interval. A feasible path  $(C, Y, K, R, S)_{t=0}^{\infty}$  is called *inefficient* if it is not efficient. Obviously, since  $\partial Y/\partial R > 0$ , an efficient path satisfies the condition

$$\lim_{t \rightarrow \infty} S = 0. \quad (6)$$

Further, as is well-known<sup>4</sup>, an interior efficient path satisfies the *Hotelling Rule*

$$\frac{d(\frac{\partial Y}{\partial R})/dt}{\frac{\partial Y}{\partial R}} = \frac{\partial Y}{\partial K} - \delta, \quad (7)$$

stating that the return ("capital gain") on leaving the marginal unit of the resource in the ground must equal the marginal return on the alternative asset (reproducible capital). Using the Cobb-Douglas specification, this no-arbitrage condition gives

$$g_Y - g_R = \alpha z - \delta, \quad (8)$$

where  $z \equiv Y/K$ . For later use we define  $x \equiv C/K$  and  $u \equiv R/S$ .

### 3 Balanced paths

An interior efficient path  $(C, Y, K, R, S)_{t=0}^{\infty}$  is called a *balanced growth path* (henceforth abbreviated BGP) if  $C, Y$ , and  $K$  change with constant relative rates (some or all of which may be negative). The values taken by the variables along a BGP are marked by \*.

**Lemma 1** *For any BGP the following holds:*

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<sup>4</sup>Dasgupta and Heal, 1979, p. 214 ff.

- (i)  $g_S = g_R = -u = -u^*$ , where  $u^*$  is some positive constant;
- (ii)  $g_C = g_K = g_Y = g_Y^*$ , some constant;
- (iii)  $z$  and  $x$  are positive constants.
- (iv)  $g_Y^*$  and  $u^*$  satisfy  $(1 - \alpha)g_Y^* + \gamma u^* = \beta n$ .

*Proof* See Appendix.

Letting  $c$  denote per capita consumption,  $C/N$ , we have  $g_c = g_C - n$ , and along a BGP  $g_c^* = g_Y^* - n$ .

It is convenient to consider a constant saving ratio, hence we introduce a parameter  $s$ ,  $0 \leq s < 1$ , such that  $Y - C = sY$ . This provides enough structure for a determinate dynamics to arise. And it does not diminish the set of possible BGP's since any BGP has a constant saving rate  $(Y - C)/Y = 1 - x/z$ . Introducing maximization of an intertemporal utility function would restrict the set of possible BGP's. Stiglitz (1974a) and Suzuki (1976) take  $g_c^*$  as a parameter instead of  $s$  which is less convenient since  $g_c^*$  is not a genuine control variable.

Now,  $x = (1 - s)z$ , and  $x > 0$  whenever  $z > 0$ . Condition (2) can be written

$$g_K = sz - \delta. \quad (9)$$

Along a BGP we have

$$g_c^* = sz^* - (n + \delta), \quad (10)$$

from (9) and (ii) of Lemma 1.

To find  $z^*$  and characterize the dynamics with a constant  $s$ , we derive the differential equations of the model. First, with (5) and (9), (8) gives

$$g_Y = \frac{(s - \gamma)\alpha z + \beta n - (\alpha - \gamma)\delta}{1 - \gamma}. \quad (11)$$

Inserting this and (9) into the identity  $\dot{z}/z = g_Y - g_K$  yields

$$\dot{z} = \left[ \frac{(\alpha + \gamma - 1)s - \alpha\gamma}{1 - \gamma} z + \frac{\beta n + (1 - \alpha)\delta}{1 - \gamma} \right] z. \quad (12)$$

Further, by (3),  $\dot{u}/u = g_R - g_S = g_R + u$ , and using (8) and (11) we get

$$\dot{u} = \left[ -\frac{1 - s}{1 - \gamma} \alpha z + u + \frac{\beta n + (1 - \alpha)\delta}{1 - \gamma} \right] u. \quad (13)$$

The dynamics of  $z$  and  $u$  are completely described by the triangular system (12)–(13).

Now, suppose  $(\alpha + \gamma - 1)s \neq \alpha\gamma$ . Then, by Lemma 1, along a BGP the system (12)–(13) is in steady state with

$$z^* = \frac{\beta n + (1 - \alpha)\delta}{\alpha\gamma - (\alpha + \gamma - 1)s} \equiv \bar{z}, \quad (14)$$

$$u^* = -\frac{1-s}{1-\gamma}\alpha\bar{z} - \frac{\beta n + (1-\alpha)\delta}{1-\gamma} = (\alpha - s)\bar{z} \equiv \bar{u}, \quad \text{and} \quad (15)$$

$$x^* = (1-s)\bar{z} \equiv \bar{x}. \quad (16)$$

The *per capita* growth rate is

$$g_c^* = \frac{s(\alpha + \beta - 1)n - (\alpha - s)(n + \delta)\gamma}{\alpha\gamma - (\alpha + \gamma - 1)s} \equiv \bar{g}_c, \quad (17)$$

by (10) and (14).

**Lemma 2** *If and only if  $\alpha \leq 1$ , then  $\alpha + \gamma - 1 \leq \alpha\gamma$ .*

*Proof*  $\alpha > 1 \Leftrightarrow 1 - \gamma < \alpha(1 - \gamma) \Leftrightarrow \alpha\gamma < \alpha + \gamma - 1$ .  $\square$

**Lemma 3** *Let  $0 \leq s < 1$ . A BGP exists only if  $s < \alpha$ .*

*Proof* For a BGP  $z^*$  and  $u^*$  are positive, by definition. Suppose  $(\alpha + \gamma - 1)s \neq \alpha\gamma$ ; then, in view of (15),  $s < \alpha$  is required. Suppose instead  $(\alpha + \gamma - 1)s = \alpha\gamma$ . Then, since  $0 \leq s < 1$ ,  $\alpha + \gamma - 1 > \alpha\gamma$ ; hence,  $\alpha > 1$ , by Lemma 2, implying  $s < \alpha$ .  $\square$

As to the question of stability, notice that though  $z$  and  $u$  are ‘jump variables’, by substituting  $uS$  for  $R$  in the production function (1) we get  $z = AK^{\alpha-1}N^\beta u^\gamma S^{\gamma+\lambda}$ , showing that, given  $K, N$ , and  $S$ , the values of  $z$  and  $u$  are not independent. Hence, when the two eigenvalues of the system (12)–(13) are of opposite sign, we shall say that the system is saddle-point stable. In case the eigenvalues are positive (or have positive real parts) we shall call the system unstable.

**Proposition 1** *Let  $s$  be given such that  $0 \leq s < \min(\alpha, 1)$ . Then:*

(i) *If  $(\alpha + \gamma - 1)s < \alpha\gamma$ , there exists a BGP  $(g_c^*, x^*, z^*, u^*)$ , if and only if*

$$\beta n + (1 - \alpha)\delta > 0. \quad (18)$$

*Further, a BGP has  $(g_c^*, x^*, z^*, u^*) = (\bar{g}_c, \bar{x}, \bar{z}, \bar{u})$  and is saddle-point stable.*

(ii) *If  $(\alpha + \gamma - 1)s > \alpha\gamma$ , there exists a BGP  $(g_c^*, x^*, z^*, u^*)$ , if and only if the inequality in (18) is reversed. Again  $(g_c^*, x^*, z^*, u^*) = (\bar{g}_c, \bar{x}, \bar{z}, \bar{u})$ , but the BGP is unstable.*

- (iii) If  $(\alpha + \gamma - 1)s = \alpha\gamma$ , no BGP exists if  $\beta n + (1 - \alpha)\delta \neq 0$ ; otherwise, there exists a continuum of BGP's, indexed by a constant  $z^* > 0$  such that with this  $z^*$  and  $s = \alpha\gamma/(\alpha + \gamma - 1)$  we have  $g_c^* = sz^* - (n + \delta)$ ,  $u^* = (\alpha - s)z^*$ , and  $x^* = (1 - s)z^*$ .

*Proof* See Appendix.

In view of Lemma 2, a sufficient, but not necessary, condition for case (i) to arise is that  $\alpha \leq 1$ . Case (ii) can arise only if  $\alpha > 1$ . Also the double "knife-edge" case (iii), where the system has hysteresis ("history matters"), requires  $\alpha > 1$ .

## 4 Growth

When is it possible to maintain steady growth?

**Proposition 2** *By appropriate choice of  $s$  in the interval  $0 < s < \min(\alpha, 1)$  there exists a BGP with  $g_c^* > 0$  if and only if*

$$(\alpha + \beta - 1)n > 0 \quad \text{or} \quad \alpha > 1. \quad (19)$$

*Proof* See Appendix.

Thus for the technology to allow steady positive *per capita* growth (with an indispensable resource and without exogenous technical progress), either increasing returns to the capital-*cum*-labour input combined with population growth *or* increasing returns to capital itself is needed. At least one of these conditions is required in order that capital accumulation can offset the effects of the inevitable decline in resource use over time<sup>5</sup>. More astonishing perhaps is:

**Corollary** *Stability of a BGP with positive per capita growth requires population growth.*

*Proof* Consider a BGP. If  $n = 0$ , then, by (19),  $g_c^* > 0$  requires  $\alpha > 1$ , and (18) is invalidated.  $\square$

The result that population growth is required for stability of positive per capita growth holds also in a Ramsey-type optimal growth setting with non-renewable resources entering the growth engine in an essential way (Groth and Schou 2002).

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<sup>5</sup>Of course this presupposes an elasticity of substitution between the resource and the other inputs not larger than one as implied by the Cobb-Douglas specification (1). Historical evidence for the US may indicate otherwise (Nordhaus 1992). In any event, it is difficult to predict the technological substitution possibilities one century ahead, say.

Essentially, Proposition 2 was proved for the case  $\delta = 0$  in Suzuki (1976, Section 3).



Define *strictly endogenous growth* to occur if per capita consumption grows at a constant positive rate in the long run even in the absence of *any* exogenously growing factor.<sup>6</sup> The numerator of (17) gives a hint that if there is strictly endogenous growth ( $\alpha > 1$ ), then  $g_c^*$  can be made arbitrarily high by choosing  $s$  close to the number  $\tilde{s} \equiv \alpha\gamma/(\alpha + \gamma - 1)$ . Indeed we have the slightly stronger result:

**Proposition 3** *Assume that either  $\alpha > 1$  or ( $\alpha = 1$  and  $n > 0$ ). For a given  $g_c^* > -(n + \delta)$  define*

$$\hat{s} \equiv \frac{\alpha\gamma(g_c^* + n + \delta)}{(\alpha + \gamma - 1)(g_c^* + n + \delta + \frac{\beta n + (1-\alpha)\delta}{\alpha + \gamma - 1})}. \quad (20)$$

(i) *If  $\beta n + (1 - \alpha)\delta \geq 0$ , then any  $g_c^* \in (0, \infty)$  in a BGP can be supported by choosing  $s = \hat{s} \in (0, 1)$ .*

(ii) *If  $\beta n + (1 - \alpha)\delta < 0$ , then there exists a number  $g_0$ , possibly positive, such that any  $g_c^* \in (g_0, \infty)$  in a BGP can be supported by choosing  $s = \hat{s} \in (0, 1)$ .*

*Proof* See Appendix.

*Remark 1* Lemma 2 and the formula (17) make it clear that  $\alpha \geq 1$  is necessary for unbounded growth. If  $\alpha < 1$ , then  $\alpha + \gamma - 1 < \alpha\gamma$ , implying, for all  $s \in (0, 1)$ ,  $\alpha\gamma - (\alpha + \gamma - 1)s > \alpha\gamma - (\alpha + \gamma - 1) > 0$ . Hence, the numerator of (17) is bounded away from zero.

*Remark 2* The case considered by Solow (1994) is without non-renewable resources and corresponds to  $\gamma = n = \delta = 0$ . In this case (12) reduces to  $\dot{z} = (\alpha - 1)sz^2$ , from which follows the "explosion" result – infinite output in finite time – whenever  $\alpha > 1$  and  $s > 0$ . With  $\alpha = 1.05$ ,  $s = 0.1$ ,  $z(0) = 1$ , and one year as the time unit (Solow's example), the Big Bang is only 200 years ahead. With "capital" being interpreted as "broad capital", an initial output-capital ratio lower than one might be more realistic. But this line of thought justifies a larger  $s$  as well, and it is the product of  $z(0)$  and  $s$  which matters for the date (from "now") of the Big Bang.

Since, by definition, strictly endogenous growth obtains only when  $g_c^* > 0$  is possible for  $n = 0$ , it follows from Proposition 2 that strictly endogenous growth requires  $\alpha > 1$ . Hence, we have from Proposition 3:

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<sup>6</sup>In contrast, *semi-endogenous* growth is said to occur when sustained per capita growth is driven by some internal mechanism, but requires exogenous growth in *some* variable, typically the labour force. By relying on less demanding parameter values a model featuring semi-endogenous growth may be an attractive alternative to one featuring strictly endogenous growth.

**Corollary** *Strictly endogenous growth (with non-renewable resources entering the growth engine in an essential way) implies that, technologically, any growth rate is sustainable.*

The conclusion is that when non-renewable resources enter the picture in an essential way, assuming strictly endogenous growth is tantamount to assuming that the Land of Cockaigne, though not just round the corner, may not be far away, if the appropriate saving is made. Albeit the one-sector structure of the model makes calibration difficult, let us take as a reference point:  $\alpha = 0.90$ ,  $\beta = 0.50$ ,  $\gamma = 0.02$ ,  $s = 0.28$ ,  $n = 0.01$ , and  $\delta = 0.05$ . Then  $z^* = 0.25$ ,  $u^* = 0.04$ , and  $g_c^* = 0.01$ . If  $\alpha$  is increased to 1.025 and  $\beta$  decreased to 0.375 (leaving the elasticity of scale unchanged at 1.42), then  $z^* = 0.32$ ,  $u^* = 0.07$ , and  $g_c^* = 0.03$ . But if also  $s$  is increased, say to 0.42, then  $z^* = 1.56$ ,  $u^* = 0.27$ , and  $g_c^* = 0.60$ . The adjustment speed of  $(z - z^*)/z$  is, by (12),  $[\beta n + (1 - \alpha)\delta]/(1 - \gamma) = 0.0026$  in this case, implying a half-life equal to 272 years.

Alternatively, if we do not aim at balanced growth, notice that (5) and (9) imply

$$\dot{z} = [(\alpha - 1)sz + \beta n + (1 - \alpha)\delta - \gamma u] z.$$

Hence, whenever  $\alpha > 1$ ,  $\beta n + (1 - \alpha)\delta > 0$ , and  $s > 0$ , infinite output in finite time can be obtained by keeping the extraction rate  $u$  constant at a level not larger than  $[\beta n + (1 - \alpha)\delta]/\gamma$ .

## 5 Concluding remarks

As Solow (1994) pointed out, the strength of the constant-returns-to-capital assumption is revealed by recognizing that even the slightest touch of increasing returns generates explosive growth. This paper has examined what happens if non-renewable resources are necessary inputs in the "growth engine" so that (strictly) endogenous growth *requires* increasing returns to broad capital. Then (a) any positive per capita growth rate is sustainable, and (b) Solow style explosion is still a possibility. Hence, it seems fair to conclude that endogenous growth is still very demanding as to its technology assumption. This underlines Solow's scepticism towards believing in continuing exponential growth in the long run if there is neither increasing population or some irreducible, exogenous element in the process of technical progress.

## 6 Appendix

**Proof of Lemma 1.** Consider a BGP  $(C, Y, K, R, S)_{t=0}^\infty$ . By definition,  $C, Y, K, R, S > 0$  for all  $t \geq 0$ , hence,  $z, x$  and  $u > 0$  for all  $t \geq 0$ . Further,  $g_C, g_Y$ , and  $g_K$  are constant.

(i) By constancy of  $g_Y$  and  $g_K$ , in view of (5),  $g_R$  is a constant. Now,  $R(t) = R(0)e^{g_R t}$ , and in view of  $R(0) > 0$ ,  $g_R < 0$  since otherwise (4) would be violated. Integrating (3) gives  $S(t) = S(0) - R(0) \int_0^t e^{g_R \tau} d\tau = S(0) - \frac{R(0)}{g_R}(e^{g_R t} - 1) \rightarrow S(0) + \frac{R(0)}{g_R}$  for  $t \rightarrow \infty$  since  $g_R < 0$ . Hence, by (6),  $S(0) + \frac{R(0)}{g_R} = 0$ , implying  $S(t) = -\frac{R(0)}{g_R}e^{g_R t} = S(0)e^{g_R t}$ , so that, by (3),  $-u = g_S = g_R < 0$ . The implied constant value of  $u$  is called  $u^*$  and is positive. (ii) and (iii) By constancy of  $g_Y, g_R$ , and  $u$ , by (8) also  $z \equiv Y/K$  is constant, implying  $g_Y = g_K$ . By constancy of  $g_K$  and  $z$ , also  $x$  is constant, in view of (2). Hence,  $g_C = g_K = g_Y \equiv g_Y^*$ . (iv) Insert  $g_R = g_S = -u^*$  and  $g_K = g_Y \equiv g_Y^*$  into (5).  $\square$

**Proof of Proposition 1.** Let  $0 \leq s < \min(\alpha, 1)$ . (i) Assume  $(\alpha + \gamma - 1)s < \alpha\gamma$ . Then, (14) and (17) are valid, and we have  $\bar{z} > 0$ , if and only if (18) holds. Assume (18) holds. Then, by (15), since  $s < \alpha$  we have  $\bar{u} > 0$ , and since  $s < 1$  we have  $\bar{x} > 0$ , in view of (16). Further, the eigenvalues of the triangular system (12)–(13) are  $\frac{(\alpha+\gamma-1)s-\alpha\gamma}{1-\gamma}\bar{z} < 0$  and  $\bar{u} > 0$ ; hence, the system is saddle-point stable. (ii) The proof for the case  $(\alpha + \gamma - 1)s > \alpha\gamma$  is similar, but now  $\frac{(\alpha+\gamma-1)s-\alpha\gamma}{1-\gamma}\bar{z} > 0$ , and therefore a BGP, when it exists, is unstable. (iii) Assume  $(\alpha + \gamma - 1)s = \alpha\gamma$ . Then, by (12) and (13), a BGP exists if and only if  $\beta n + (1 - \alpha)\delta = 0$ ; when this condition holds, any positive constant  $z^*$  is consistent with a BGP with  $g_c^*$  determined by (10) and  $u^*$  determined by (8), (9), and (i) and (ii) of Lemma 1.  $\square$

**Proof of Proposition 2.** "if": First, consider the case  $\alpha > 1$ . Here, by Lemma 2,  $\tilde{s} \equiv \alpha\gamma/(\alpha + \gamma - 1) \in (0, 1)$ . If  $\beta n + (1 - \alpha)\delta > 0$ , choose  $s$  in the open interval  $(0, \tilde{s})$ , but close enough to  $\tilde{s}$  to make  $\bar{z} > (n + \delta)/s$ , i.e.,  $g_c^* > 0$  by (10). In view of (14) and  $\alpha + \gamma > 1$ , this can always be done. If  $\beta n + (1 - \alpha)\delta < 0$ , choose  $s$  in the open interval  $(\tilde{s}, 1)$ , but again close enough to  $\tilde{s}$  to make  $\bar{z} > (n + \delta)/s$ , i.e.,  $g_c^* > 0$ , by (10). Finally, if  $\beta n + (1 - \alpha)\delta = 0$ , choose  $s = \tilde{s}$  and an arbitrary  $z^* > (n + \delta)/\tilde{s}$ . Then  $g_c^* > 0$ , by (iii) of Proposition 1. Now, consider the case  $\alpha \leq 1$ . Assume  $(\alpha + \beta - 1)n > 0$ . Then  $n > 0$ , and therefore  $\beta n + (1 - \alpha)\delta > 0$ . From  $0 < \alpha \leq 1$  follows  $\alpha\gamma \geq \alpha + \gamma - 1$ , by Lemma 2. Choose  $s$  in the open interval  $(0, \alpha)$ ; then  $\alpha\gamma > (\alpha + \gamma - 1)s$ . By choosing  $s$  close enough to  $\alpha$ ,  $g_c^* > 0$ , from (17).

"only if": Consider a BGP. Then, by Lemma 1,  $u^* > 0$ . Hence, by (iv) of Lemma 1,  $(1 - \alpha)(g_c^* + n) < \beta n \Leftrightarrow (1 - \alpha)g_c^* < (\alpha + \beta - 1)n$ . Now, if  $g_c^* > 0$ ,  $\alpha \leq 1$  is seen to imply  $(\alpha + \beta - 1)n > 0$ .  $\square$

**Proof of Proposition 3.** Let  $\alpha > 1$  or  $(\alpha = 1$  and  $n > 0)$ . Define  $\tilde{s} \equiv \alpha\gamma/(\alpha + \gamma - 1)$ . Then  $0 < \tilde{s} \leq 1$ , by Lemma 2. (i) Let  $g_c^*$  be an arbitrary positive number and consider, first, the case  $\beta n + (1 - \alpha)\delta > 0$ . Here, for  $\hat{s}$  given in (20),  $0 < \hat{s} < \alpha\gamma/(\alpha + \gamma - 1) \equiv \tilde{s} \leq 1$ . By (17), to obtain growth at the rate  $g_c^*$ ,

$$s = \frac{\alpha\gamma(g_c^* + n + \delta)}{(\alpha + \gamma - 1)g_c^* + \gamma(n + \delta) + (\alpha + \beta - 1)n} \quad (21)$$

is required. After reordering of the numerator this  $s$  is seen to be identical to  $\hat{s}$  in (20), as was to be shown. Now, consider instead the case  $\beta n + (1 - \alpha)\delta = 0$ . Here, by (20),  $\hat{s} = \tilde{s} > 0$ , and, since  $(\alpha = 1$  and  $n > 0)$  is excluded,  $\alpha > 1$ . Therefore, by Lemma 2,  $\tilde{s} < 1$ . Choose  $s = \hat{s} = \tilde{s}$  and  $z = z^* = (g_c^* + n + \delta)/\tilde{s}$ . Then, by (iii) of Proposition 1, the desired  $g_c^*$  is realized.

(ii) Assume  $\beta n + (1 - \alpha)\delta < 0$ . Then  $\alpha > 1$ , and  $0 < \tilde{s} < 1$ , by Lemma 2. Now, by (20),  $\hat{s} > \tilde{s}$ . Let  $s$  be such that  $\tilde{s} < s < 1$  always. If  $s \rightarrow 1$ , then, by (17),  $g_c^*$  decreases towards

$$\frac{(\alpha + \beta - 1)n - (\alpha - 1)(n + \delta)\gamma}{\alpha\gamma - (\alpha + \gamma - 1)} \equiv g_0, \quad (22)$$

and if  $s \rightarrow \tilde{s}$ , then  $g_c^*$  increases towards  $\infty$ . Hence, any  $g_c^* \in (g_0, \infty)$  can be realized by choosing  $s = \hat{s} \in (0, 1)$ . Since  $\alpha\gamma < \alpha + \gamma - 1$ ,  $g_0$  is a positive number if and only if  $n < (\alpha - 1)\gamma\delta/[\beta + (\alpha - 1)(1 - \gamma)]$ .  $\square$

*Remark* A peculiar feature is that in case  $\beta n + (1 - \alpha)\delta < 0$ , i.e., when existence of a BGP requires, by Proposition 1, that  $s > \tilde{s}$ , then  $g_c^*$  is a *decreasing* function of  $s$ . The explanation is that in this case not only is  $\alpha$  above 1 (implying a growth potential without bound), but the BGP is unstable. It is well-known that when there is instability, comparative statics give "paradoxical" results. To put it differently, in this case, when  $s$  increases, the output-capital ratio required to maintain balanced growth (avoid explosive growth) decreases proportionately *more*, so that  $g_c^*(= sz^* - (n + \delta))$  declines.

## References

- Aghion, Philippe, & Peter Howitt, 1998: *Endogenous Growth Theory*, MIT Press, Cambridge, MA.
- Dasgupta, Partha, & Geoffrey Heal, 1979: *Economic Theory and Exhaustible Resources*. Cambridge University Press, Cambridge.

- Groth, Christian, and Poul Schou, 2002: Can Nonrenewable Resources Alleviate the Knife-edge Character of Endogenous Growth? *Oxford Economic Papers*, 54, 386-411.
- Jones, Charles I., 2002: *Introduction to Economic Growth*, 2. ed., W.W. Norton, New York.
- Jones, Larry E., and Rodolfo E. Manuelli, 1997: The Sources of Growth, *Journal of Economic Dynamics and Control*, 21, 75-114.
- Nordhaus, William, 1992: Lethal Model 2: The Limits to Growth Revisited, *Brookings Papers on Economic Activity*, 2, 1-43.
- Schou, Poul, 2000: Polluting Nonrenewable Resources and Growth. *Environmental and Resource Economics*, 16, 211-27.
- Solow, Robert M., 1994: Perspectives on Growth Theory. *Journal of Economic Perspectives*, 8, 45-54.
- Stiglitz, Joseph, 1974a: Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths. *Review of Economic Studies*, Symposium Issue, 123-137.
- Stiglitz, Joseph, 1974b: Growth with Exhaustible Natural Resources: The Competitive Economy. *Review of Economic Studies*, Symposium Issue, 139-52.
- Suzuki, Hideo, 1976: On the Possibility of Steadily Growing *per capita* Consumption in an Economy with a Wasting and Non-Replenishable Resource. *Review of Economic Studies* 43, 527-35.