

EPRU

**Economic Policy Research Unit
Institute of Economics
University of Copenhagen
Studiestræde 6
DK-1455 Copenhagen K
DENMARK
Tel: (+45) 3532 4411
Fax: (+45) 3532 4444
Web: <http://www.econ.ku.dk/epru/>**

How to Analyze the Investment-Uncertainty Relationship in Real Option Models

Diderik Lund

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How to analyze the investment–uncertainty relationship in real option models?

Diderik Lund*

Department of Economics, University of Oslo

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Abstract

The real options tradition originally predicted a decreasing relationship between uncertainty and investment, through the positive effect of higher uncertainty on the trigger level for revenue relative to costs. An opposing effect on the probability of reaching the level has been identified, yielding a total effect with ambiguous sign. This paper makes three points. The “opposing” effect is not always opposing. Systematic risk cannot generally be assumed to increase with volatility. A probability is not the best measure of investment. The sign of the total effect is again ambiguous. This ambiguity is illustrated, depending on specification of model and parameters.

KEYWORDS: investment, uncertainty, real options, stochastic control

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*Address: P.O.Box 1095, Blindern, NO-0317 Oslo, Norway, phone +47 22855129, fax +47 22855035, e-mail diderik.lund@econ.uio.no, web page <http://folk.uio.no/dilund>. This paper was written while the author was visiting the Economic Policy Research Unit, Institute of Economics, University of Copenhagen. He is grateful for their hospitality.

1 Introduction

The theory of real options (Dixit and Pindyck 1994, Trigeorgis 1996) prescribes optimal investment rules under uncertainty. From these follow predictions for the macroeconomic relationship between measures of uncertainty on one hand and aggregate investment on the other. Originally the relationship was seen as decreasing, see, e.g., Pindyck (1991, p. 1123, p. 1131). More recently this has been challenged by Sarkar (2000, 2003), who derives a theoretical relationship which can be both increasing and decreasing.

This paper discusses the interpretation of Sarkar's results, and takes a closer look at the "opposing effect" he identifies. The idea that uncertainty goes up can have several meanings. Moreover, it is not obvious how to measure the response in aggregate investment. Anyhow, the message that the relationship is not always monotone decreasing, survives.

While Sarkar (2003) introduces a mean-reverting process for the output price, for which there are good arguments, most of the discussion here relates to the geometric Brownian motion (GBM) analysis in Sarkar (2000). The reason is that this is simpler and much more well known, and that most of the arguments relate to both analyses.

2 The model

This follows Sarkar (2000), except that different values for the investment cost will be considered. Firm i has the opportunity to invest K_i , whereby it will start a perpetual revenue stream x_t , where t denotes time. The investment can at most take place once, at any time in the future. While K_i is known and fixed¹, the stream x_t is a GBM with drift,

$$dx_t = \mu x_t dt + \sigma x_t dz_t, \tag{1}$$

where the drift parameter μ and the volatility parameter σ are constants, and dz_t is the increment of a standard Wiener process, z_t . Valuation follows the single-beta version of the ICAPM of Merton (1973), so that the present value of the perpetual revenue stream, if and when the investment is undertaken, is x_t/δ , where $\delta \equiv r + \lambda\rho\sigma - \mu$. Here, r is the

¹For uncertainty in K , see McDonald and Siegel (1986).

riskless interest rate, λ is the market price of risk, and ρ is the correlation between dz_t and the return on the market portfolio, all assumed to be constants.

The optimal time for firm i to invest is the first time x_t reaches a trigger level x_i^* from below. Defining the constant α as

$$\alpha \equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}, \quad (2)$$

the optimal x_i^* is

$$x_i^* = \frac{K_i \alpha \delta}{\alpha - 1}, \quad (3)$$

cf. McDonald (2003, p. 393). It is assumed that the process starts out at $x_0 < x_i^*$. Otherwise it would be optimal to invest immediately.

The probability of x_t reaching the critical x_i^* (at least once) within a time horizon T , is

$$\Pr(\text{invest in } i) = \Phi\left(\frac{\nu T - \ln(x_i^*/x_0)}{\sigma\sqrt{T}}\right) + \left(\frac{x_i^*}{x_0}\right)^{2\nu/\sigma^2} \Phi\left(\frac{-\nu T - \ln(x_i^*/x_0)}{\sigma\sqrt{T}}\right), \quad (4)$$

where $\nu \equiv \mu - \sigma^2/2$, and Φ is the standard normal distribution function, cf. Etheridge (2002, p. 69f).

The probability formula has three elements. Let

$$Q \equiv (x_i^*/x_0)^{2\nu/\sigma^2} \equiv (x_i^*/x_0)^{-1} (x_i^*/x_0)^{2\mu/\sigma^2}, \quad (5)$$

and let the two Φ expressions in (4) be Φ_1 and Φ_2 , respectively. All probabilities to be discussed are conditional on some $x_0 < x_i^*$. The first of these, Φ_1 , is the probability that x_T (at the horizon, T) exceeds x_i^* , cf. equation (20.12) of McDonald (2003). It follows that the other term in (4), now called $Q\Phi_2$, is the probability that x_t exceeds x_i^* during some interval(s) between 0 and T , but returns to a value below x_i^* at T .

Since x_i^* is increasing in σ , the original point of view was that investment is decreasing in σ : Some projects which would have been undertaken in the near future² for a low value of σ , will be postponed when it is realized that a higher σ applies. Metcalf and Hassett (1995) and Sarkar (2000) suggest an opposing effect: The probability of an increase in x_t up to a *given* trigger level x_i^* within some given time horizon, T , is higher, the higher is σ . This supposedly goes in the direction of an increasing relationship between uncertainty

²But see the discussion in section 7 below.

and investment, and Sarkar (2000) indeed demonstrates in a numerical example that the effect of σ on the probability has an ambiguous sign.

3 The probability of exceeding a trigger level

More can be said about the “opposing effect.” This is really the effect of σ on the probability in (4) when x_i^* is held constant (but $\nu = \mu - \sigma^2/2$ is allowed to vary with σ). Holding x_i^* constant simplifies the partial derivative sufficiently to get an idea of its sign. This turns out to be ambiguous, not always positive, as the suggestion of an “opposing” effect seems to be based on.

A higher σ gives a higher probability both for higher and lower outcomes, while reducing the probability for outcomes close to the expected path. The effects at T (on Φ_1) are easiest to grasp. Intuitively one would believe that the effect of σ on the probability that $x_T > x_i^*$ depends on whether $x_i^* > E(x_T)$. To develop this idea, start with the normal distribution and consider the probability of $\ln(x_T)$ exceeding $\ln(x_i^*)$ in a case when $E[\ln(x_T)]$ does not depend on σ (contrary to the present model, which has $E[\ln(x_T)] = \ln(x_0) + T(\mu - \sigma^2/2)$). It can be shown that this probability is increasing in σ if and only if

$$\ln(x_i^*) \geq E[\ln(x_T)] \Leftrightarrow \ln\left(\frac{x_i^*}{x_0}\right) - T\left(\mu - \frac{\sigma^2}{2}\right) \geq 0. \quad (6)$$

But consider now the present model. The partial derivative $\partial\Phi_1/\partial\sigma|_{x_i^*}$ is positive if and only if

$$\ln\left(\frac{x_i^*}{x_0}\right) - T\left(\mu + \frac{\sigma^2}{2}\right) \geq 0. \quad (7)$$

This does not always hold, but depends on parameter values. For given values of the other parameters, the expression can be made negative either by a sufficiently high μ , by a sufficiently high σ , or (except if μ is far below zero) by a sufficiently high T . For the “opposing effect” this means³ that a higher σ can easily lead to a lower probability of exceeding a *given* trigger level, x_i^* .

³So far only the distribution at T has been considered. The effect via $Q\Phi_2$ is also ambiguous, and is considered below.

The condition $x_i^* > E(x_T)$ is equivalent to

$$\ln\left(\frac{x_i^*}{x_0}\right) - T\mu > 0, \quad (8)$$

which is implied by (7). However, there is no implication in the opposite direction, from (8) to (7). We may well have (as long as $\sigma > 0$) that $E(x_T)$ is somewhat less than x_i^* , but at the same time Φ_1 is decreasing⁴ in σ for a given x_i^* . This shows that the reversal of the “opposing” effect should not be ignored (or ascribed to an unreasonably high $E(x_T)$) as far as Φ_1 is concerned. The effect of σ on Φ_1 for a given x_i^* may be negative even when $E(x_T)$ is less than the trigger level.

The differences between the three inequalities, (8), (7), and (6), are due to the facts that the \ln function is non-linear (which explains the difference between (6) and (8)) and that $E[\ln(x_T)]$ depends on σ by assumption (which determines (7)).

What about the other term in (4)? An increase in σ seems to have an ambiguous effect on $Q\Phi_2$. This depends on

$$\left.\frac{\partial(Q\Phi_2)}{\partial\sigma}\right|_{x_i^*} = \frac{Q}{\sigma^2} \left\{ \varphi_2 \cdot \left[\frac{\ln(x_i^*/x_0)}{\sqrt{T}} + \sqrt{T} \left(\mu + \frac{\sigma^2}{2} \right) \right] - 4\Phi_2 \ln\left(\frac{x_i^*}{x_0}\right) \frac{\mu}{\sigma} \right\}, \quad (9)$$

where φ_2 is the normal density corresponding to Φ_2 . No detailed discussion is offered here, whereas the total effects show up in the numerical analysis in section 6.

4 What does it mean to increase uncertainty?

Increased uncertainty in this model is obviously taken to mean a higher σ . There is a conceptual problem in analyzing the response to this, namely whether this increase is seen as a one-time, unexpected increase to a new, constant σ value, or something else. Taken literally the model does not allow for changes in σ . But the usual comparative statics analysis considers a starting situation in which some parameters are given, and makes the experiment of changing one parameter, regarding the others as fixed. Perhaps a model with stochastic volatility and optimal response to this would have been better suited to make predictions about the effects of changing volatility.

⁴This possibility vanishes if $\nu = \mu - \sigma^2$ is held constant when σ is varied, instead of holding μ constant.

But even within the standard comparative statics of the present model, there is the question of which parameters are seen as fixed when σ is increased. Two alternatives to Sarkar's assumptions are given below. The first of these (δ fixed) has a clear economic interpretation, while the second (ν fixed) is more speculative.

4.1 Keeping δ fixed

Equations (2) and (3) above conceal the relationship $\delta = r + \lambda\rho\sigma - \mu$. Sarkar (2000) regards r, λ, ρ , and μ as constants when analyzing changes in σ . This is in line with the comparative statics analysis in Dixit and Pindyck (1994, p. 179). However, McDonald and Siegel (1986) choose another assumption, and point out (their footnote 14) that there may be two interpretations of increased σ : One which is uncorrelated with the market portfolio and one which is not.

One great achievement of financial option theory, starting with Black and Scholes (1973), is that its results do not depend on systematic risk. When there is a rate-of-return shortfall (McDonald and Siegel, 1984), δ , it is possible that systematic risk plays a role through the term $\lambda\rho\sigma$. But consider what happens if uncertainty is increased by addition or multiplication of a random variable which is stochastically independent of the other variables.

Let dz_m be the increment of the market portfolio, while dz is introduced above. For clarity, use now the notation dz_x and σ_x for what have been dz and σ above. For this discussion dz_m and dz_x are treated as any random variables, disregarding their infinitesimal dimension. Their covariance is σ_{xm} , and their correlation is $\rho = \sigma_{xm}/\sigma_x\sigma_m$. Let superscript o denote values at the outset, and superscript a denote values after an increase in uncertainty. Two specifications of this increase will be considered.

First, let ε be a random variable (with strictly positive variance) which is stochastically independent of (dz_m, dz_x) , with $E(\varepsilon) = 0$. Let $dz_x^a \equiv dz_x^o + \varepsilon$. Then $\sigma_{xm}^a = \text{cov}(dz_x^o + \varepsilon, dz_m) = \text{cov}(dz_x^o, dz_m) = \sigma_{xm}^o$, while $\sigma_x^a = \sqrt{\text{var}(dz_x^o + \varepsilon)} > \sqrt{\text{var}(dz_x^o)} = \sigma_x^o$. Thus, if an increase in σ_x takes the form of adding a random variable which is independent like this, then the covariance and the well-known β (from the ICAPM) are unaffected. This means that the correlation is reduced, and that $\lambda\rho\sigma$, and thus δ , are unaffected.

A similar effect follows from multiplicative uncertainty. Let ψ be a random variable (with strictly positive variance) which is stochastically independent of (dz_m, dz_x) , with $E(\psi) = 1$. In this case, redefine $dz_x^a \equiv \psi dz_x^o$. Then $\sigma_{xm}^a = \text{cov}(\psi dz_x^o, dz_m) = \text{cov}(dz_x^o, dz_m) = \sigma_{xm}^o$, while $\sigma_x^a = \sqrt{\text{var}(\psi dz_x^o)} > \sqrt{\text{var}(dz_x^o)} = \sigma_x^o$. Thus, if an increase in σ_x takes the form of multiplication with a random variable which is independent like this, then the covariance is unaffected. Again the correlation is reduced, and $\beta, \lambda\rho\sigma$, and thus δ , are unaffected.

These two examples show that there may be good reasons to consider the kind of increase in σ which does not affect δ : This is what happens if the increase is independent of the other random variables. It should be noted, however, that it is impossible to maintain a fixed covariance if σ_x is reduced to zero. In order for the similar type of independence to work when σ_x is reduced, it is a condition that dz_x , before the reduction, can be written as a sum of two stochastic variables, one of which is independent of the vector of the other and dz_m .

4.2 Keeping ν fixed

Equation (4) conceals the relationship $\nu = \mu - \sigma^2/2$. The parameter μ relates to x_t through $E(x_t) = x_0 e^{\mu t}$. But this is really the expectation of $x_t = x_0 e^{\nu t + \sigma z_t}$, where $\nu \equiv \mu - \sigma^2/2$. Instead of holding μ fixed, one could hold ν , the drift of $\ln x$ constant. The process $y_t \equiv \ln x_t$ is a Brownian motion with drift, $dy_t = \nu dt + \sigma dz_t$. The parameter ν relates to y_t through $E(y_t) = y_0 + \nu t$. To treat ν as a basic parameter of the model instead of μ follows rather naturally from equation (4). The reason is, of course, that the derivation of this probability uses the normal distribution of y . With ν fixed, the dependence on σ in (4) is simplified.

Unfortunately (for the purpose of simplicity), there is not a very good justification why one would want to hold ν fixed instead of μ . After all, x_t is supposed to be the directly observable variable. On the other hand, researchers are well acquainted with $\ln x_t$, as most financial research is done on logarithmic data. If the expected drift is estimated on these, it may be more consistent to keep that parameter fixed.

5 How to measure the effect on investment?

The suggestion in Sarkar (2000) is to measure the effect on investment by the effect on the probability of investment during some time interval. This measure has the advantage of simplicity, i.e., one does not have to specify too many parameters in order to arrive at a number. The disadvantage is that it is not necessarily proportional to what one wants to measure, investment. Figure 1 in Sarkar (2000) shows that the probability is a concave function of σ , first increasing, then decreasing. Whether this carries over to investment is not so clear.

An alternative measure could be based on heterogeneity of investment projects. The simplest extension seems to be to assume that the investment cost varies across projects. The distribution of these costs will obviously be important. For some applications there may be reasons to introduce a particular distribution. For the present discussion a discrete approximation to a uniform distribution will be used. This allows a calculation of expected investment over some time horizon. If the projects are indexed by i , and the costs are K_1, \dots, K_n , then the expected investment over a horizon of length T is

$$E(\text{aggregate investment}) = \sum_{i=1}^n E(\text{inv. in project } i) = \sum_{i=1}^n \Pr(\text{inv. in project } i) K_i. \quad (10)$$

The introduction of K_i gives an additional non-linearity in the expression.

6 Consequences for numerical results

As pointed out by Sarkar (2000), the sign of the effect on investment is ambiguous. While this was true when ρ was assumed fixed, it is still true when δ and/or ν are assumed fixed. Figures 1–5 show the probability of investment within 5 years as functions of the volatility, σ , and reproduce the numerical example in Sarkar (2000) with some modifications. In these calculations⁵ there is only one potential project with $K_1 = 1$. Figures 6–8 show expected investment with a distribution of project costs. The parameter values are identical to those in Sarkar (2000) except where noted. In particular, $r = 0.1$ and $x_0 = 0.1$ are used throughout.

⁵The computer programs used are available at <http://folk.uio.no/dilund/realopt>.

Figure 1 reproduces Sarkar (2000) exactly, with $\mu = 0$, $\rho = 0.7$, $\lambda = 0.4$, which implies that δ increases as σ is increased. The solid curve reproduces (and verifies) Sarkar's Figure 1. The dashed curve shows $\delta/10$, scaled down to fit in the diagram. δ is increasing from 10 percent to more than 26 percent. These are high numbers, and although δ can vary a lot empirically, most observations seem to be below 10 percent, cf. Milonas and Henker (2001) for crude oil and Heaney (2002) for copper, lead and zinc. x_i^* is increasing monotonically from 0.1041 to 0.485 as σ goes from 0.01 to 0.6.

Figure 2 makes only one change compared with Figure 1, by setting $\mu = 0.01$. (The scale on the vertical axis has changed, and δ is now shown unscaled.) This minor increase from $\mu = 0$ changes the probability curve dramatically, and it is now strictly decreasing. At low values of σ , the probability of investing is much higher, cf. the discussion in section 3. x_i^* is increasing from 0.1006 to 0.476.

Figure 3 is based on calculations where again $\mu = 0$, but this time without any effect of μ on x_i^* , as δ is held fixed⁶ at 0.05. ν is not held fixed, but varies according to $\nu = \mu - \sigma^2/2$. ν is thus negative, and increasingly so as σ increases. The effect of keeping δ fixed (and lower) is to increase the probability of investing. x_i^* is increasing from 0.1001 to 0.308.

Figure 4 makes the additional change that ν is now also kept fixed, at zero. This means that $\mu = \nu + \sigma^2/2$ is positive and increasing. This, of course, increases the probability of investing as compared with Figure 3, but the curve is still strictly decreasing as function of σ . x_i^* is increasing exactly as in the calculations underlying Figure 3.

Figure 5 has the same parameter values as Figure 3, except that K_i is now increased from 1 to 2. This gives a dramatically different curve, increasing with an inflection point. Notice that the scale on the vertical axis is different from that of Figure 3. x_i^* is increasing from 0.2002 to 0.616, exactly twice those values relating to Figure 3, cf. equation (3). This makes the probability of reaching the trigger level extremely low for low σ , and low even for higher σ values.

Figures 6, 7, and 8 reproduce the parameter vectors and the solid curves of Figures 1, 3, and 4, respectively. But now two additional potential projects are added, with $K_2 = 2$ and $K_3 = 3$. The dashed curves show expected aggregate investment within 5 years when

⁶For Figures 3–5 and 7–8, λ and ρ are not relevant for the calculations.

only firms 1 and 2 exist, while the dotted curves show expected aggregate investment when all three firms exist. To understand this in more detail, consider Figure 7. The difference between the dashed and the solid curve is the second term in the sum in (4). This is a probability multiplied by $K_2 = 2$. This particular probability is already shown in Figure 5. For $\sigma = 0.6$ it takes the value 0.06, and the difference between these two curves in Figure 7 is thus equal to $0.12 = 2 \cdot 0.06$ at $\sigma = 0.6$.

Since the probabilities are generally increasing functions of σ for higher values of K_i , like in Figure 5, the differences between the curves are increasing in σ , and in Figure 8 the expected aggregate investment becomes non-monotone even if the probability for $K_1 = 1$ is monotone decreasing.

Hopefully a numerical exercise like this can shed some light on the mechanisms relating uncertainty to these measures of investment. It is quite clear that the relationship is neither straightforward to define nor (for many interpretations) strictly decreasing.

7 Discussion

There will, with probability one, be long periods of time during which x_t is less than its previous maximum value. This is particularly pronounced under GBM, even with some positive drift. If all potential projects are known from many years back, there will be long periods without any investment at all in the model presented, cf. Lund (1993). This means that aggregate investment in x_t -yielding projects will be erratic. Perhaps a smoother aggregate investment can be obtained if there are many different revenue processes, not perfectly correlated, or if there is a stream of new potential investment projects.

The unreasonable features of GBM is one reason why Sarkar (2003) and other researchers have considered alternative stochastic processes. This paper will not discuss those in any detail. However, Sarkar (2003), after mentioning in the abstract the two opposing effects of uncertainty on investment, proposes “incorporating a third factor, the effect of mean-reversion on systematic risk.”

While there may well be a link between mean reversion and systematic risk, it is not clear that this influences the relationship between σ and investment. One should again take

care to specify what parameters are kept constant in the comparative statics. Changing σ is in itself not necessarily an argument for a change in systematic risk.

8 Conclusion

In the analysis of the effect of uncertainty on investment, there may or may not be an opposing effect to the traditionally observed negative effect. Furthermore, one must be careful in the definition of a change in uncertainty, even in comparative statics of a real options model based on GBM with constant volatility. It is not obvious that systematic risk is affected. One should also consider carefully what is a meaningful measure of investment in the model. The probability of a particular investment being undertaken within some time horizon is not proportional to expected aggregate investment, which is a possible measure of investment.

The numerical exercise undertaken here clearly show that depending on the specification of the model and its parameters and the measure of investment, the relationship between investment and uncertainty can be either strictly increasing, strictly decreasing, or non-monotone. This confirms the qualitative conclusion of Sarkar (2003).

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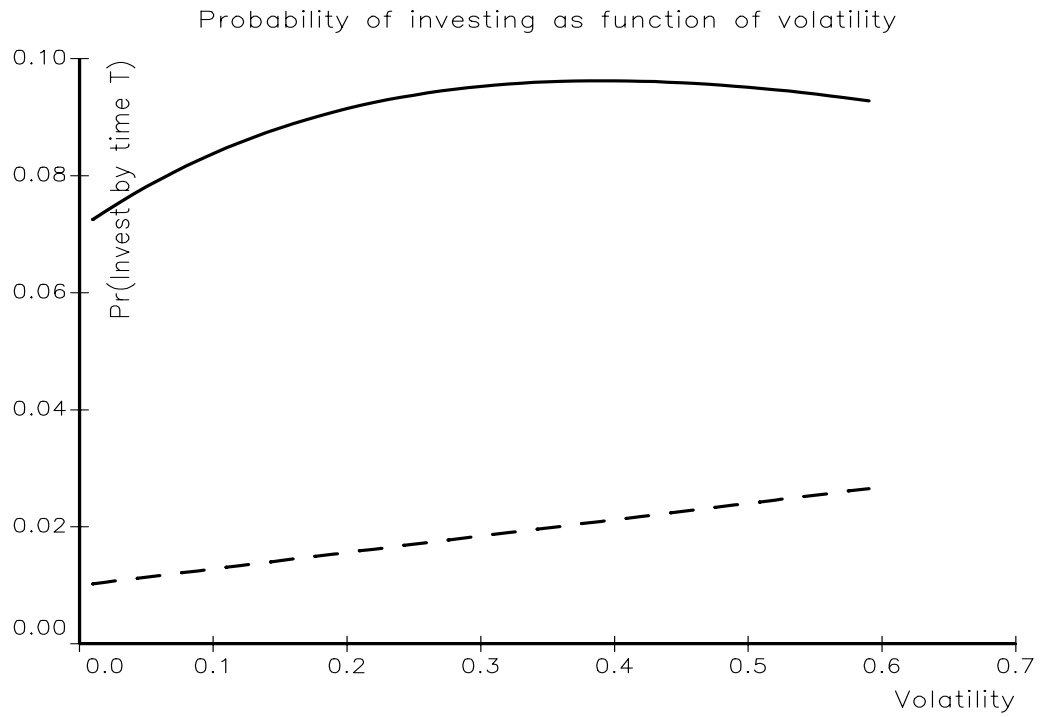


Figure 1: Probability of investment in 5 years; $\mu = 0, r = 0.1, \rho = 0.7, \lambda = 0.4, x_0 = 0.1, K = 1$; dashed curve shows $\delta/10$

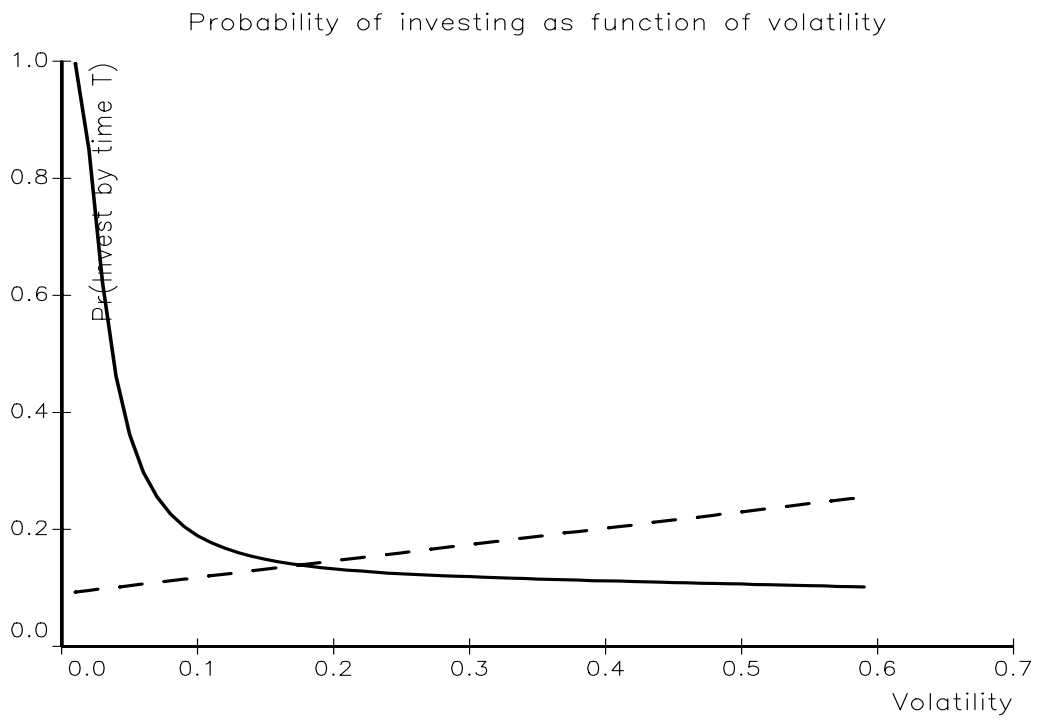


Figure 2: Probability of investment in 5 years; $\mu = 0.01, r = 0.1, \rho = 0.7, \lambda = 0.4, x_0 = 0.1, K = 1$; dashed curve shows δ

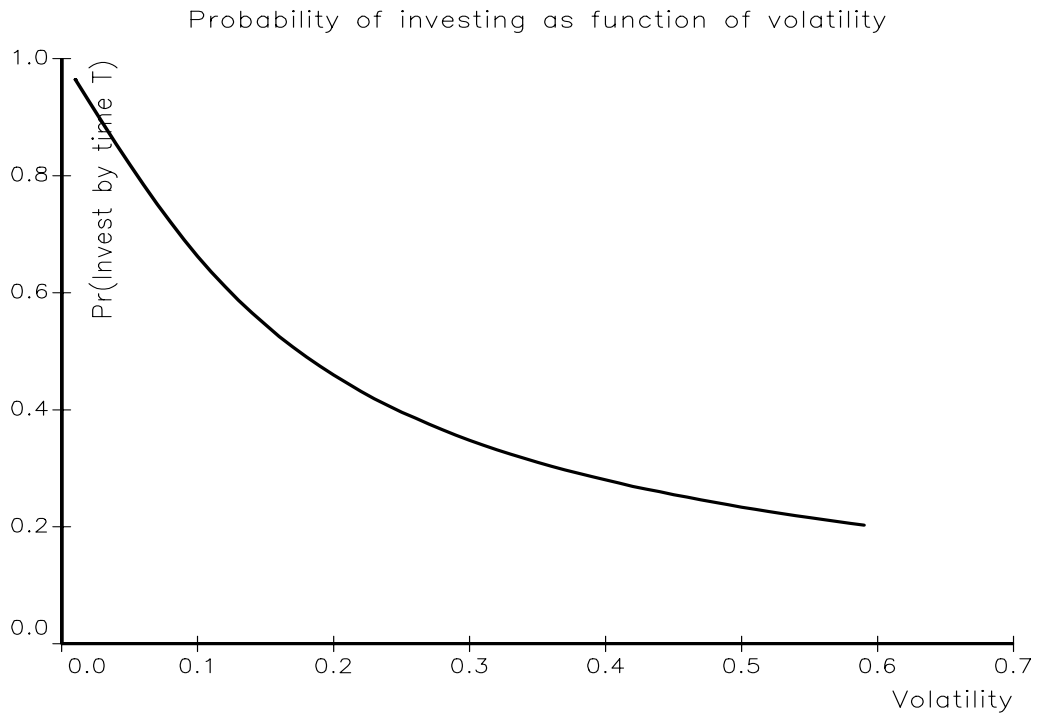


Figure 3: Probability of investment in 5 years; $\mu = 0, r = 0.1, x_0 = 0.1, K = 1; \delta$ fixed at 0.05

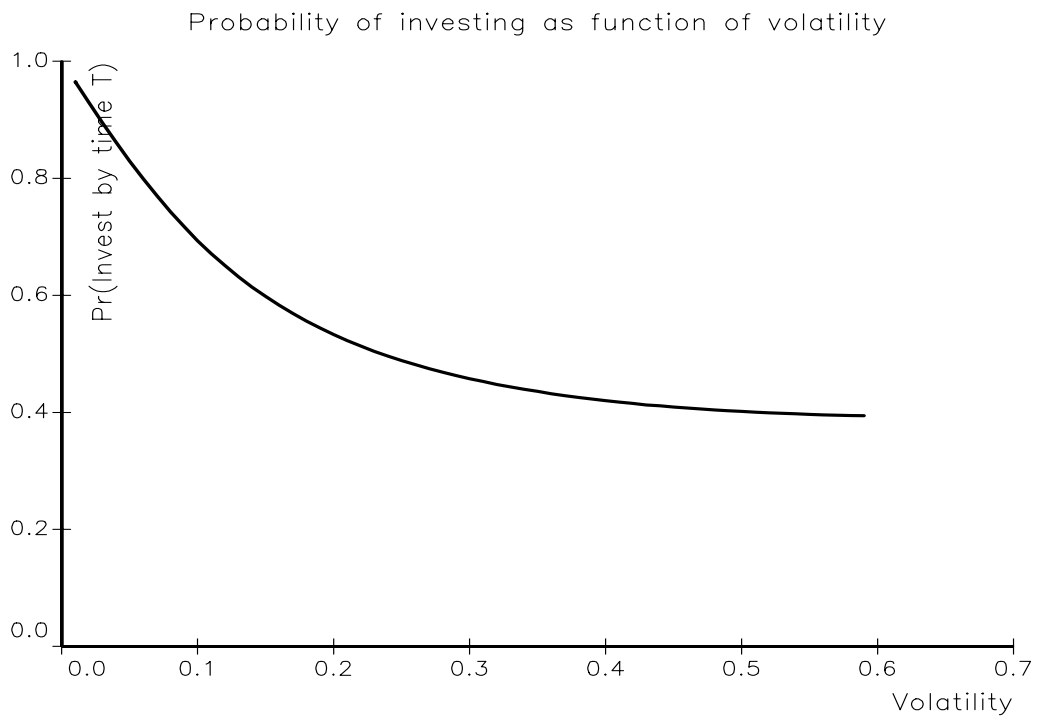


Figure 4: Probability of investment in 5 years; $r = 0.1, x_0 = 0.1, K = 1$; δ fixed at 0.05; ν fixed at 0

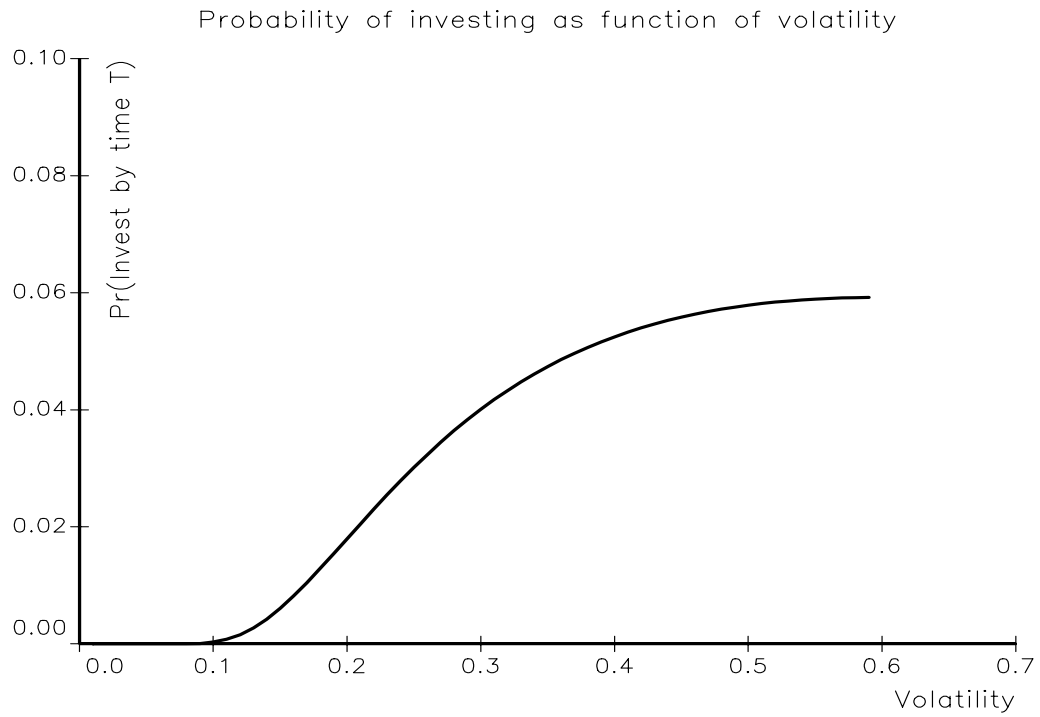


Figure 5: Probability of investment in 5 years; $\mu = 0, r = 0.1, x_0 = 0.1, K = 2; \delta$ fixed at 0.05

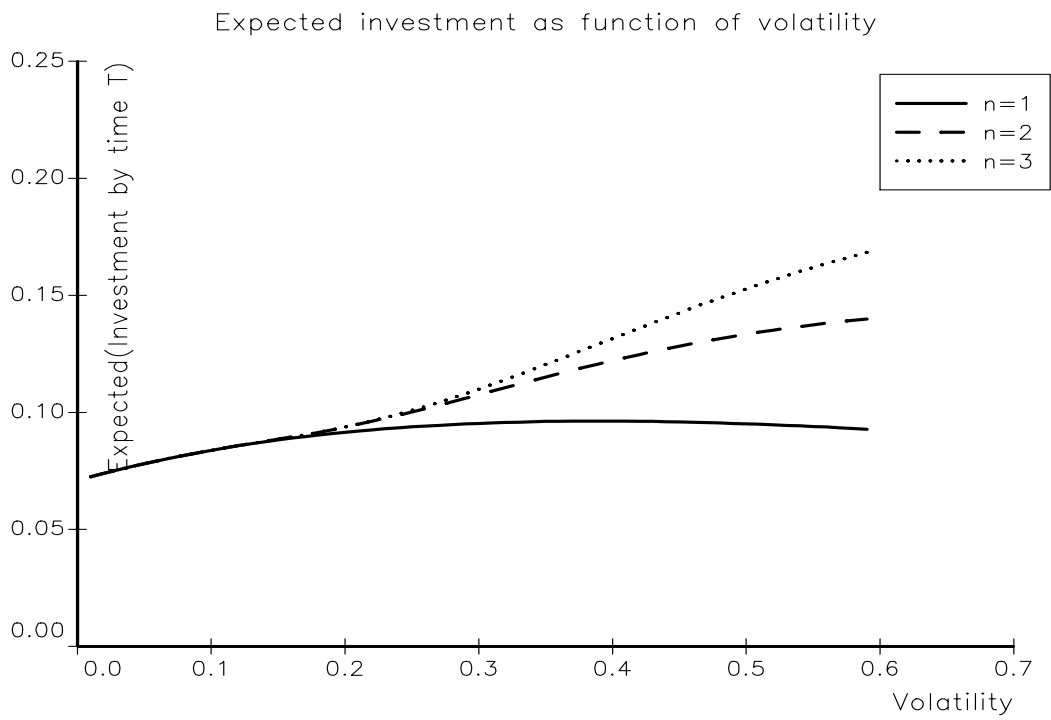


Figure 6: Expected investment in 5 years; $\mu = 0, r = 0.1, \rho = 0.7, \lambda = 0.4, x_0 = 0.1$; curves show 1, 2, and 3 projects (cumulatively), with $K_i = i$

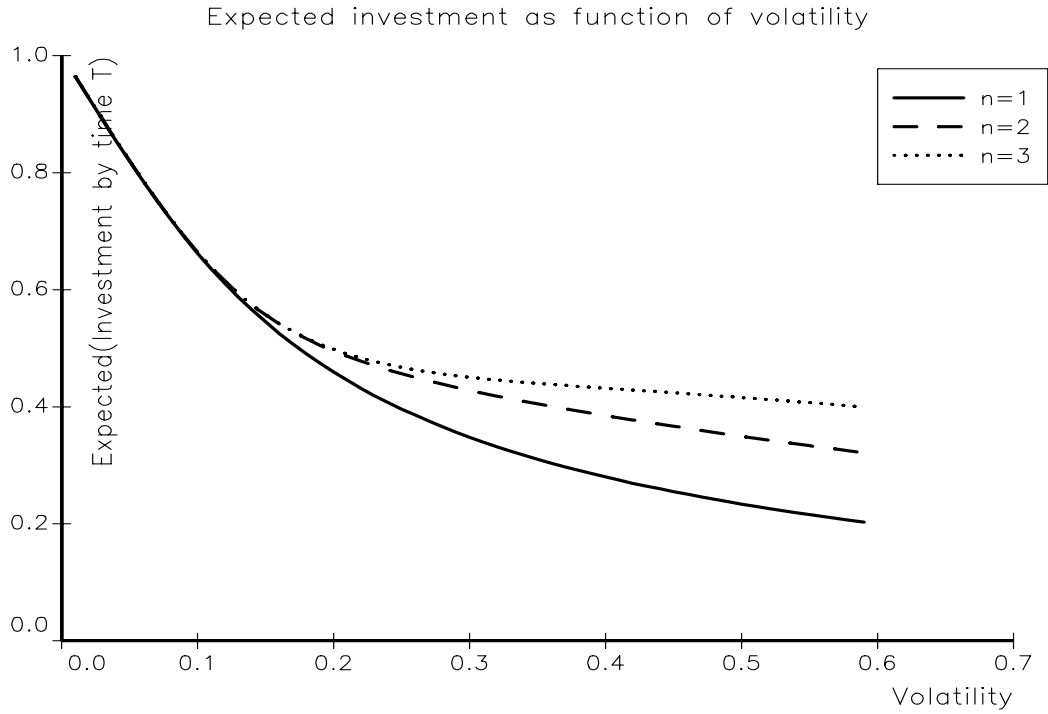


Figure 7: Expected investment in 5 years; $\mu = 0, r = 0.1, x_0 = 0.1$; δ fixed at 0.05; curves show 1, 2, and 3 projects (cumulatively), with $K_i = i$

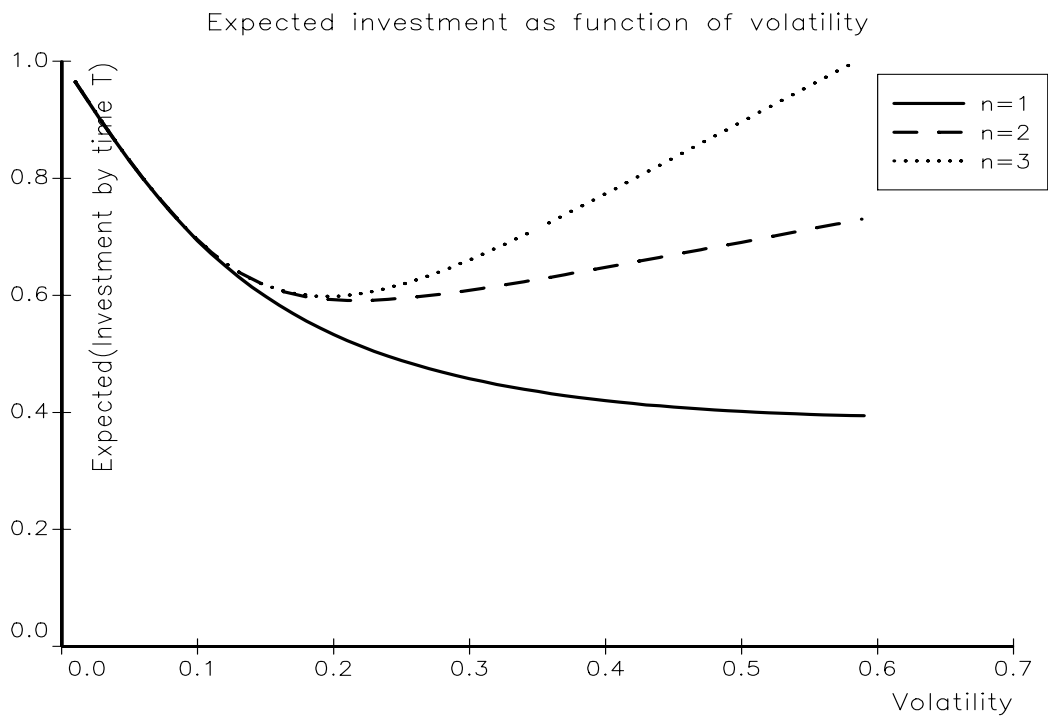


Figure 8: Expected investment in 5 years; $r = 0.1, x_0 = 0.1$; δ fixed at 0.05; ν fixed at 0; curves show 1, 2, and 3 projects (cumulatively), with $K_i = i$