

# How Small Shocks and Heterogeneous Expectations can Create Swings in the Exchange Rate\*

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## Abstract

What can explain the persistent fluctuations observed in non-fixed exchange rates? We use a version of the Kareken-Wallace two-country overlapping generations model to explain this empirical phenomenon. The agents use an adaptive learning rule to forecast expected prices in both countries instead of having perfect foresight as in the original Kareken and Wallace model. There are different but constant speeds of adjustment in the two countries. The constant speed of adjustment combined with a small shock to the money supply in one of the countries creates swings in the exchange rate. This is illustrated in various computational experiments.

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## 1. Introduction

The economics of exchange rates has generated a vast literature over the last decades, where a number of questions have been addressed such as the efficiency of exchange rate markets, the determination of exchange rates, the effectiveness of intervention in foreign exchange markets etc., see e.g. Frenkel and Razin (1992), Frankel (1993) and Taylor (1995) for recent overviews. The empirical literature on exchange rates in general shows the existence of persistent fluctuations in exchange rates, and that these movements are not related to fundamentals such as money supply, GNP, consumption etc., see e.g. Meese and Rogoff (1983), and Dornbusch and Frankel (1988). It has been difficult to explain these observed movements in exchange rates within the existing theoretical framework. In this paper we try to give an explanation of these persistent fluctuations observed in exchange rates.

The literature has been dealing with this issue for some time. Wallace (1979) states that since fiat money is intrinsically useless there is no fundamental value of one currency in terms of the other. The result is that all trade with currencies is purely speculative and the equilibrium exchange rate is indeterminate. Kareken and Wallace (1981) show this indeterminacy within a deterministic two-country overlapping generations model. This is done under the assumption of free trade, no capital restrictions, no intervention from the government in the foreign exchange market, and it is assumed that all agents have perfect foresight. However, an important feature is that the agents are indifferent between holding the different currencies, i.e. an equilibrium (arbitrage) condition requires the rates of return of the two currencies to be equal. The perfect foresight assumption and the equilibrium condition result in a constant exchange rate, and there are no fluctuations in the exchange rate. Hence the perfect foresight equilibria of this model do not allow for the possibility of the exchange rate to fluctuate.

In order to model exchange rate fluctuations, Manuelli and Peck (1990) introduce uncertainty in the endowments and show the existence of volatile exchange rates within a rational expectations framework. Fluctuations in the exchange rate originate from the statistical properties of the endowment processes, and the stochastic equilibria could easily as well be obtained from sunspot processes as Manuelli and Peck explain. However, sunspot processes are exogeneously given in the sense that they do not give an explanation of why and how agents coordinate on a particular sunspot equilibrium. King, Wallace and Weber (1992) show similar results in a model with infinitely-lived agents, constraints on what type of money the agents can hold and non-fundamental uncertainty (sunspot processes).

Exchange rate models usually assume that agents have rational expectations. However, data seems to reject the rational expectations hypothesis on foreign exchange markets, see e.g. Dornbusch (1989). Evans (1986) suggests that the fluctuations in the sterling-dollar rate between 1981-1984 are related to "*disequilibrium expectations*". Goldberg and Frydman (1996) also argue that a potential source for fluctuations and the poor result that standard exchange rate models have in explaining exchange rate fluctuations, is linked to the formation of expectations in foreign exchange markets. Sargent (1993) substitutes the perfect foresight agents with boundedly rational agents within the Kareken and Wallace framework. Agents use a least squares learning rule to update on their savings and portfolio choices. In this case the exchange rate path is determinate, the recursive learning rule converges to a stationary rational expectations equilibrium and the speed of convergence is quite fast. However, the exchange rate path is dependent on the initial conditions, i.e. a different set of initial conditions give a different limit value of the exchange rate, thus it is "*history-dependent*" as Sargent (1993) writes. Since the exchange rate settles down to the steady state value rather fast, fluctuations in the exchange rate are eliminated within a few periods. In Arifovic (1996) the agents use a genetic algorithm to update on their savings and portfolio choices instead of having perfect foresight as in the Kareken and Wallace model. Arifovic shows that the stationary rational expectations equilibrium exchange rate is unstable under the genetic algorithm. The determination of the exchange rate depends on the portfolio choice and the portfolio choice is updated through the genetic algorithm. However, the genetic algorithm introduces uncertainty in the model in a very specific way and may seem as an odd way to update your beliefs regarding savings and portfolio choices. In fact the portfolio choice will actually always fluctuate within Arifovic's setting, but the savings rate converges to a pareto-optimal steady state.

In this paper we follow Sargent and Arifovic by assuming that all agents are boundedly rational. In our set-up the agents use an adaptive learning rule to forecast future prices. The adaptive learning rule is used extensively in other parts of the macroeconomic literature to show convergence towards a rational expectations equilibrium, see e.g. Evans and Honkapohja (1995b) for an overview. We will use this line of approach in the exchange rate model of Kareken and Wallace. The adaptive learning rule has several nice features, it is simple and convergence to a rational expectations equilibrium under learning can be quite fast. We do not discuss the rate of convergence in this paper, but give conditions for the steady state to be stable under learning. We have heterogeneity among agents' expecta-

tions within a class of adaptive learning rules. The stability condition corresponds not surprisingly to the stability condition with homogenous agents. We introduce uncertainty in the model through a small<sup>1</sup> shock to the money supply in one of the two countries. The steady state is dependent on the distribution of the shock, thus we have a "noisy" steady state corresponding to the steady state without noise. The combination of adaptive learning rules and the stochastic shock to the money supply generates persistent exchange rate fluctuations.

The model is compared to the persistent fluctuations that has been observed in for example the Dutch and Belgian exchange rate vis-a-vis the D-Mark. However, we do not 'test' for example the goodness of fit of the model between actual and simulated exchange rate since there is no obvious way to do this. Section 2 describes the model and in section 3 the learning rules are introduced and the conditions for a steady state exchange rate to be stable under learning are found. The simulation of the model and the initial conditions are explained in section 4. In section 5 the simulation results are compared to the exchange rate movements between NLG/DEM and BEF/DEM. Section 6 concludes.

## 2. The Model

Consider a two-country, overlapping generations model. At time  $t$ ,  $t \geq 1$ , there are born  $N$  agents in each country, an agent lives for two periods. There is an old generation at time  $t = 1$  called generation 0. There is one good, the good is the same in both countries and can be traded between the agents in both countries without any restrictions. There are two types of currencies in each period. One for each country. The agents can hold both types of currencies without any restrictions. There is free-trade, no capital restrictions and no intervention in the foreign exchange market.

An agent of generation  $t$ ,  $t \geq 1$ , of country  $i$ ,  $i = 1, 2$  has the endowment  $\omega^1$  of the good at time  $t$  and  $\omega^2$  of the good at time  $t + 1$ . The preferences are given by  $u(c_{i,t}(t), c_{i,t}(t + 1)) = \ln c_{i,t}(t) + \ln c_{i,t}(t + 1)$ , where  $c_{i,t}(t)$  is the consumption at time  $t$  and  $c_{i,t}(t + 1)$  is consumption at time  $t + 1$  of the agent born at time  $t$  in country  $i$ . The agent solves the following maximisation problem :

$$\max_{c_{i,t}(t), c_{i,t}(t+1), m_{i,2}(t), m_{i,2}(t)} \ln c_{i,t}(t) + \ln c_{i,t}(t + 1)$$

*subject to*

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<sup>1</sup>What is meant by a small shock will be defined below.

$$c_{i,t}(t) + \frac{m_{i,1}(t)}{p_1(t)} + \frac{m_{i,2}(t)}{p_2(t)} \leq \omega^1 \quad (2.1)$$

$$c_{i,t}(t+1) \leq \omega^2 + \frac{m_{i,1}(t)}{p_1^e(t+1)} + \frac{m_{i,2}(t)}{p_2^e(t+1)} \quad (2.2)$$

$$c_{i,t}(t) \geq 0, c_{i,t}(t+1) \geq 0, m_{i,1}(t) \geq 0, m_{i,2}(t) \geq 0$$

where  $m_{i,1}(t)$  is the agent's demand for currency 1 at time  $t$ .  $m_{i,2}(t)$  is the agent's nominal holdings of currency 2 acquired at time  $t$ .  $p_1(t)$  is the nominal price of the good in terms of currency 1 at time  $t$ .  $p_2(t)$  is the nominal price of the good in terms of currency 2 at time  $t$ . The expected prices in country 1 and 2 at time  $t+1$  are denoted by  $p_1^e(t+1)$  and  $p_2^e(t+1)$ , where  $p_1^e(t+1)$  is the forecast made by the agent in country 1 and  $p_2^e(t+1)$  is the forecast made by the agent in country 2<sup>2</sup>. An agent's savings,  $s_i(t)$ , in the first period of life, are equal to the sum of real holdings of currency 1,  $m_{i,1}(t)/p_1(t)$  and real holdings of currency 2,  $m_{i,2}(t)/p_2(t)$ .

The exchange rate is defined as  $E(t) = p_1(t)/p_2(t)$ . Kareken and Wallace (1981) assume that agents are indifferent between the two currencies, thus requiring that the *actual* rates of return of the currencies are equal. We assume that the *expected* rates of return should be equal. Let  $R_i^e(t)$  denote the expected gross real rate of return between period  $t$  and  $t+1$  in country  $i$ , then

$$R_1^e(t) = R_2^e(t) \quad (2.3)$$

where

$$R_i^e(t) = \frac{p_i(t)}{p_i^e(t+1)}, \quad i = 1, 2 \quad (2.4)$$

In other words the *expected* real interest rates are equal in the two countries<sup>3</sup>. Rearranging (2.3) using (2.4) gives the following expression

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<sup>2</sup>The agents in country 1 know the forecast in country 2 and vice versa. This might seem odd, but it does not change the results if we assume that agents in country 1 (country 2) make their own expectations about the price in country 2 (country 1).

<sup>3</sup>The assumption is useful in finding the temporary equilibrium conditions below. However if we do not use this assumption this does not make significant changes in the temporary equilibrium conditions.

$$\frac{p_1^e(t+1)}{p_2^e(t+1)} = \frac{p_1(t)}{p_2(t)}$$

which states that the expected exchange rate equals the current actual exchange rate

$$E^e(t+1) = E(t)$$

A consequence of the condition (2.3) is that the expected exchange rate is equal to the actual exchange rate. (2.3) is an important assumption in the model. Kareken and Wallace (1981) assume that all agents have perfect foresight such that  $p_i^e(t+1) = p_i(t+1)$ . In this case the condition is  $R_1(t) = R_2(t)$  such that  $E(t+1) = E^e(t+1)$  and

$$\begin{aligned} E(t+1) &= E(t) \Rightarrow \\ E(t) &= E, \text{ for all } t \geq 0. \end{aligned}$$

Hence the exchange rate is constant, and this arbitrage condition under perfect foresight is exactly what renders the exchange rate indeterminate as shown in the appendix.

## 2.1. Solving the households' maximisation problem

We can rewrite the budget constraints for the consumer using (2.3), such that consumption at time  $t+1$  is a function of consumption at time  $t$ , the initial endowments and the expected real interest rate, thus

$$c_{i,t}(t+1) = \omega^2 + R_i^e(t)\omega^1 - R_i^e(t)c_{i,t}(t), \text{ for } i = 1, 2. \quad (2.5)$$

Given (2.5), the first-order condition for the households' maximisation problem is given by

$$\frac{1}{c_{i,t}(t)} - \frac{R_i^e(t)}{\omega^2 + R_i^e(t)\omega^1 - R_i^e(t)c_{i,t}(t)} = 0 \Rightarrow$$

$$c_i(t) = \frac{1}{2} \left( \omega^1 + \frac{1}{R_i^e(t)} \omega^2 \right), \text{ for } i = 1, 2.$$

The savings for an agent born at time  $t$  of country  $i$  is

$$s_i(t) = \omega^1 - c_{i,t}(t) = \frac{1}{2} \left( \omega^1 - \frac{1}{R_i^e(t)} \omega^2 \right), \quad i = 1, 2 \quad (2.6)$$

We assume  $\omega^1$  is sufficiently large compared to  $\omega^2$  to overcome the problem that savings might be negative.

**Assumption 1.**  $\omega^1 \gg \omega^2$ .

Aggregate real savings in country  $i$  are given by

$$S_i(t) = N s_i(t), \quad i = 1, 2 \quad (2.7)$$

The world total real savings equals

$$S(t) = S_1(t) + S_2(t) = N (s_1(t) + s_2(t))$$

The equilibrium condition  $R_1^e(t) = R_2^e(t)$  determines world savings as

$$S(t) = N \left\{ \omega^1 - \frac{1}{R_1^e(t)} \omega^2 \right\}.$$

Let  $H_i(t)$  denote the money supply in country  $i$ ,  $i = 1, 2$  at time  $t$ . The aggregated real money supply  $M^s$  equals

$$M^s = \frac{H_1(t)}{p_1(t)} + \frac{H_2(t)}{p_2(t)} = \frac{H_1(t)}{p_1(t)} + \frac{H_2(t)}{p_1(t)} E(t)$$

since  $E(t) = p_1(t)/p_2(t)$ . The real money supply is denominated in the price level of country 1, however this could also be done in terms of  $p_2(t)$ .

## 2.2. Equilibrium

In an equilibrium we have that real savings equals real money supply:

$$N \left( \omega^1 - \frac{\omega^2}{R_1^e(t)} \right) = \frac{H_1(t)}{p_1(t)} + \frac{H_2(t)}{p_1(t)} E(t) \Rightarrow$$

$$p_1(t) = \frac{\omega^2}{\omega^1} p_1^e(t+1) + \frac{H_1(t) + H_2(t) E(t)}{N \omega^1}$$

The price level in country 2 can be determined in the same way, where the money supply is denoted in country 2 prices,  $M^s = H_1(t)/(p_2(t)E(t)) + H_2(t)/p_2(t)$ . The temporary equilibrium conditions are given by

$$p_1(t) = \frac{\omega^2}{\omega^1} p_1^e(t+1) + \left( H_1(t) + H_2(t) \frac{p_1^e(t+1)}{p_2^e(t+1)} \right) \frac{1}{N\omega^1}$$

$$p_2(t) = \frac{\omega^2}{\omega^1} p_2^e(t+1) + \left( H_1(t) \frac{p_2^e(t+1)}{p_1^e(t+1)} + H_2(t) \right) \frac{1}{N\omega^1}$$

where we insert  $E(t) = E^e(t+1)$ . In a temporary equilibrium the actual price in country  $i, i = 1, 2$  is a function of the expected prices in both countries and the nominal money supply in both countries:

$$p_1(t) = F(p_1^e(t+1), p_2^e(t+1), H_1(t), H_2(t)) \quad (2.8)$$

$$p_2(t) = G(p_1^e(t+1), p_2^e(t+1), H_1(t), H_2(t)) \quad (2.9)$$

Let the money supply  $\{H_1(t), H_2(t)\}_{t=0}^\infty$  be given by the sequence of beliefs  $\{p_1^e(t), p_2^e(t)\}_{t=0}^\infty$  and define an equilibrium price sequence  $\{p_1(t), p_2(t)\}_{t=0}^\infty$  by (2.8) and (2.9). From the price sequence we get the exchange rate, savings, consumption and welfare.

**Definition 1.** *A perfect foresight equilibrium is a sequence  $\{p_1(t), p_2(t)\}_{t=0}^\infty$  where*

$$\begin{aligned} p_1^e(t+1) &= p_1(t+1) \\ p_2^e(t+1) &= p_2(t+1) \end{aligned}$$

and  $\{p_1(t), p_2(t)\}_{t=0}^\infty$  satisfies (2.8) and (2.9).

If the money supply is constant,  $H_1(t) = H_1$  and  $H_2(t) = H_2$  for all  $t \geq 0$ , the steady state prices  $(\bar{p}_1, \bar{p}_2)$  are given by

$$\bar{p}_1 = F(\bar{p}_1, \bar{p}_2, H_1, H_2) = F(\bar{p}_1, \bar{p}_2). \quad (2.10)$$

$$\bar{p}_2 = G(\bar{p}_1, \bar{p}_2, H_1, H_2) = G(\bar{p}_1, \bar{p}_2). \quad (2.11)$$

The steady state exchange rate is given by  $\bar{e} = \bar{p}_1/\bar{p}_2$ .



### 2.3. Uncertainty in the money supply

We assume there is uncertainty about the money supply in one of the two countries. Formally the money supply follows the following process for country 1 and 2 respectively

$$H_1(t) = H_1, \text{ for all } t \geq 0. \quad (2.12)$$

$$H_2(t) = H_2 + \varepsilon(t), \text{ for all } t \geq 0. \quad (2.13)$$

where  $\varepsilon(t)$  is an i.i.d. shock to the money supply in country 2. It is not necessary to also have uncertainty in country 1's money supply in order to generate fluctuations in the exchange rate. Let the money supply be given by (2.12) and (2.13), then the definition of a steady state depends on the distribution of the shock.

**Definition 2.** *The steady state process  $(\bar{p}_1(\varepsilon), \bar{p}_2(\varepsilon))$  is given by*

$$\begin{aligned} \bar{p}_1(\varepsilon) &= F(\bar{p}_1(\nu), \bar{p}_2(\nu), \varepsilon) \\ \bar{p}_2(\varepsilon) &= G(\bar{p}_1(\nu), \bar{p}_2(\nu), \varepsilon) \end{aligned}$$

where  $\nu$  has the same distribution as  $\varepsilon(t)$ .

Evans and Honkapohja (1995a) show that the "noisy" steady state in definition 2 corresponds to the deterministic steady state  $(\bar{p}_1, \bar{p}_2)$  provided that the range of the shock is not too "large", i.e. suppose  $\varepsilon(t)$  has a bounded support, then the length of the support is sufficiently small.

### 3. The learning rules

The main point of this paper is to track the effects of the shock to the money supply and investigate the case where the agents do not have perfect foresight (or rational expectations), but update their beliefs according to an adaptive learning rule. The learning behaviour is used extensively in the macroeconomic literature to show convergence to a rational expectations equilibrium, see e.g. Bray (1982), Lucas (1986) and Sargent (1993). If the adaptive learning rule converges towards a rational expectations equilibrium the rate of convergence is usually quite fast, see e.g. Bray and Savin (1986) and Marcet and Sargent (1995). In this respect

the adaptive learning rule is quite "sensible". Adaptive learning has also been proposed as a selection device when there are multiple rational expectations equilibria. The learning rule tends to pick a sensible solution, see e.g. Marimon and Sunder (1993) and Evans and Honkapohja (1995c). However, as Woodford (1990) shows, convergence may not always be to a stationary rational expectations equilibrium.

The learning behaviour we use in this paper is known as adaptive or econometric learning. An agent born at time  $t$  in country  $i$  forecast on the price tomorrow in country  $i$ ,  $p_i^e(t+1)$ , according to the adaptive learning rule given by

$$p_i^e(t+1) = p_i^e(t) + a_i(t)(p_i(t-1) - p_i^e(t)), \quad i = 1, 2 \quad (3.1)$$

Notice here that we use  $p_i(t-1)$  instead of  $p_i(t)$ , this is done to avoid simultaneity in determining  $p_i^e(t+1)$ , since  $p_i(t)$  depend on  $p_1^e(t+1)$  and  $p_2^e(t+1)$  according to the temporary equilibrium conditions (2.8) and (2.9). This is not essential for the results, but is done to simplify the stability analysis. This learning rule can be viewed as if the agents in country  $i$  know the economy is in a steady state, but do not know the actual value of the steady state. The agent is trying to estimate this value using a weighted average of past observations of the price in country  $i$ :

$$p_i^e(t+1) = a_i(t)p_i(t-1) + (1-a_i(t))a_i(t-1)p_i(t-2) + (1-a_i(t))(1-a_i(t-1))a_i(t-2)p_i(t-3) + \dots \quad (3.2)$$

where (3.2) is (3.1) rewritten in terms of past prices. The parameter  $a_i(t)$  is the gain parameter known as the speed of adjustment or the rate of convergence. The gain parameter states how much weight the agents place on the "error" term  $p_i(t-1) - p_i^e(t)$ . The rate of convergence is exogenously determined. Evans and Honkapohja (1993) and Marcet and Nicolini (1996) discuss a criteria for choosing an optimal rate of convergence,  $a_i(t)$ , and whether  $a_i(t)$  should be constant or time-variant, i.e. decreasing in  $t$ :

$$\begin{aligned} a_i(t) &= a_i, \text{ for all } t \text{ or} \\ a_i(t) &= \frac{a_i}{t}, \text{ for all } t. \end{aligned}$$

If  $a_i(t)$  is constant the learning rule can track changes better than a decreasing rate of convergence, since a constant rate of convergence place more weight on

recent observations, while a decreasing rate of convergence place more weight on past observations. Below when we introduce noise in the money supply we discuss the choice between a constant or a decreasing gain parameter regarding stability issues. The 'choice' of an optimal rate of convergence is left for future work.

Most types of forecasting rules are ad hoc. However, adaptive learning provides a fairly simple means of modelling expectations in economic theory whilst postulating a mode of behaviour upon the part of the economic agents which seems evidently simple. The belief that agents learn from experience is obviously a more sensible starting point than both 1) the implicit assumption that they are totally devoid of memory, *static expectations*, or 2) that they have knowledge of everything, *perfect foresight*.

Laboratory experiments conducted by Marimon and Sunder (1993) and Evans, Honkapohja and Marimon (1996), suggest that agents use an adaptive learning rule. However, Marimon and Sunder (1993) are inconclusive on what type of adaptive learning rule the agents use. In some experiments conducted by Marimon and Sunder the evidence was for an adaptive learning rule like (3.1), but in other experiments another type of adaptive learning rule was used, for example an extra lag added to (3.1) or perhaps use information based on time  $t - 2$  information instead of information based on time  $t - 1$  as in (3.1). It might be useful to allow for more than one type of learning rule in the model. However this is left for future research.

We should note that agents using the adaptive learning rule are boundedly rational, since they are mis-specified while they are learning, i.e. they are outside a rational expectations equilibrium.

We make the following assumption on the speed of adjustment  $a_1(t)$  and  $a_2(t)$  in the two countries.

$$0 < a_1(t) \leq a_2(t) \leq 1.$$

$$\sum_{t=1}^{\infty} a_i(t) = \infty, i = 1, 2.$$

We allow for the possibility of having different speeds of adjustment in the two countries. This is not essential for the results, as long as we have different initial conditions we still have to keep track of the agents' forecasts in both countries during the learning transition. It should be noted that with constant speeds the

learning rules do not converge. Notice also that the effect from  $p_1^e(t)$  on  $p_2^e(t+1)$  through  $p_2(t-1) = G(p_1^e(t), p_2^e(t))$  is less than the effect from  $p_2^e(t)$  on  $p_1^e(t+1)$  through  $p_1(t-1) = F(p_1^e(t), p_2^e(t))$ . We might interpret this as a combination of a "big" country versus a "small" country, where the small country is more dependent on the "big" country than the opposite. For example, in the case with Belgium vis-a-vis Germany there could be a difference in the speed of adjustment.

### 3.1. The local stability of the steady state under learning

We find the conditions for a steady state to be **locally** stable under learning. Let  $H_1(t) = H_1$ ,  $H_2(t) = H_2$  and  $a_1(t) = a_1$ ,  $a_2(t) = a_2$  for all  $t$ . The dynamic system is given by the temporary equilibrium conditions (2.8), (2.9) and the learning rules (3.1) :

$$p_1^e(t+1) = p_1^e(t) + a_1 (F(p_1^e(t), p_2^e(t)) - p_1^e(t)) \quad (3.3)$$

$$p_2^e(t+1) = p_2^e(t) + a_2 (G(p_1^e(t), p_2^e(t)) - p_2^e(t)) \quad (3.4)$$

and we have a fully recursive system in  $p_1^e(t)$  and  $p_2^e(t)$ . In order to show that the recursive system given by (3.3) and (3.4) converge to a steady state  $(\bar{p}_1, \bar{p}_2)$ , we linearise (3.3) and (3.4) around  $(\bar{p}_1, \bar{p}_2)$  and find the eigenvalues of the Jacobian of (3.3) and (3.4) evaluated at  $(\bar{p}_1, \bar{p}_2)$ . The Jacobian is given by

$$J(\bar{p}_1, \bar{p}_2) = \begin{pmatrix} 1 + a_1(F_1(\bar{p}_1, \bar{p}_2) - 1) & a_1 F_2(\bar{p}_1, \bar{p}_2) \\ a_2 G_1(\bar{p}_1, \bar{p}_2) & 1 + a_2(G_2(\bar{p}_1, \bar{p}_2) - 1) \end{pmatrix}$$

where  $F_1(\bar{p}_1, \bar{p}_2)$  is the partial derivative of  $F$  w.r.t.  $p_1^e(t)$  evaluated at  $(\bar{p}_1, \bar{p}_2)$  etc. If the eigenvalues of  $J(\bar{p}_1, \bar{p}_2)$  are less than 1 in absolute value, the dynamic system (3.3) and (3.4) is locally stable under learning, and converge to the steady state  $(\bar{p}_1, \bar{p}_2)$  for nearby starting values. A condition for the eigenvalues to be less than 1 in absolute value is given by Azariadis (1993):

$$-2 < Tr(J(\bar{p}_1, \bar{p}_2)) < 2 \quad (3.5)$$

$$-1 < Det(J(\bar{p}_1, \bar{p}_2)) < 1 \quad (3.6)$$

where  $Tr(J)$  is the trace of the  $J$  and  $Det(J)$  is the determinant of  $J$ . The conditions (3.5) and (3.6) are reduced to  $F_1(\bar{p}_1, \bar{p}_2) < 1$  and  $G_2(\bar{p}_1, \bar{p}_2) < 1$  as shown in the appendix. We can summarise our result in the following proposition.

**Proposition 1.** *Given assumption 1. If  $F_1(\bar{p}_1, \bar{p}_2) < 1$  and  $G_2(\bar{p}_1, \bar{p}_2) < 1$ , the dynamic system (3.3) and (3.4) converges to a steady state.*

**Proof.** See appendix.

The condition on the partial derivatives in proposition 1,  $F_1(\bar{p}_1, \bar{p}_2) < 1$  and  $G_2(\bar{p}_1, \bar{p}_2) < 1$  corresponds to the case with homogenous agents, see e.g. Evans and Honkapohja (1995a).

In this case the dynamic system (3.3) and (3.4) converges to a steady state  $(\bar{p}_1, \bar{p}_2)$  and the exchange rate  $E(t)$  converges to a steady state exchange rate  $\bar{E} = \bar{p}_1/\bar{p}_2$ . There are no fluctuations in the limit, and if the convergence is "fast" the exchange rate is constant even after a short while.

### 3.2. Where do the fluctuations in the exchange rate originate from?

Let us assume that the money supply is no longer constant, but given by (2.12) and (2.13). The dynamic system (3.3) and (3.4) is now stochastic and depends on the distribution of the shock  $\varepsilon(t)$  (below denoted just by  $\varepsilon$ ):

$$p_1^e(t+1) = p_1^e(t) + a_1 (F(p_1^e(t), p_2^e(t), \varepsilon) - p_1^e(t)) \quad (3.7)$$

$$p_2^e(t+1) = p_2^e(t) + a_2 (G(p_1^e(t), p_2^e(t), \varepsilon) - p_2^e(t)) \quad (3.8)$$

If the shock is sufficiently small and the conditions in Proposition 1 are satisfied, the dynamic system (3.7) and (3.8) fluctuates around the steady state  $(\bar{p}_1, \bar{p}_2)$ . This is a result of the speed of learning  $a_i$  being constant, such that agents produce a noisy forecast even in the limit. The dynamic system does not converge to a steady state in this case. To see this write (3.7) and (3.8) as

$$x(t+1) = x(t) + Q(x(t), \varepsilon) \quad (3.9)$$

where  $x(t) = \begin{pmatrix} p_1^e(t) \\ p_2^e(t) \end{pmatrix}$  and  $Q(x(t), \varepsilon) = \begin{pmatrix} a_1 (F(p_1^e(t), p_2^e(t), \varepsilon) - p_1^e(t)) \\ a_2 (G(p_1^e(t), p_2^e(t), \varepsilon) - p_2^e(t)) \end{pmatrix}$   
then

$$x(t+s) = x(t) + \sum_{k=t}^{t+s} Q(x(k), \varepsilon) \Rightarrow$$

$$x(t+s) - x(t) = \sum_{k=t}^{t+s} Q(x(k), \varepsilon)$$

The sum on the right-hand side is not "weighted" with a factor that is decreasing when  $t$  goes to infinity. This is required to show convergence see. e.g. Honkapohja

(1996). If we instead assume that  $a_i(t) = a_i/t$  then for  $t$  large, the effect from the shock on the expected prices is "eliminated". In this case (3.9) is

$$x(t+1) = x(t) + \frac{1}{t}Q(x(t), \varepsilon) \Rightarrow$$

$$x(t+1) - x(t) = \frac{1}{t}Q(x(t), \varepsilon)$$

and for  $t$  large the right-hand side becomes small, and the shock is "eliminated". However, when  $t$  is decreasing recent observations have less impact compared to past observations. With a constant gain parameter  $a_i$  recent observations have more impact, and the agent can track changes in the exchange rate better than with a decreasing gain parameter. Thus at times with a volatile exchange rate it seems natural to use a constant gain parameter. Here the learning algorithm (3.3) and (3.4) with  $a_i(t) = a_i/t$  converges to  $(\bar{p}_1, \bar{p}_2)$  given the assumption and the requirements on the partial derivatives of  $F$  and  $G$  stated in proposition 1. This follows from a more general proposition in Evans and Honkapohja (1995a) regarding the local convergence of learning algorithms to a steady state. There are still fluctuations in the exchange rate, because the actual prices in country 1 and 2 are affected by the shock through the expected prices  $p_1^e(t+1)$  and  $p_2^e(t+1)$ :

$$E(t) = \frac{p_1(t)}{p_2(t)} = \frac{F(p_1^e(t+1), p_2^e(t+1), \varepsilon)}{G(p_1^e(t+1), p_2^e(t+1), \varepsilon)}$$

We assume in the following that the speed of learning is constant in both countries, such that agents can track recent changes faster at the expense of a "noisy" forecast. The combination of a constant speed of learning in the adaptive learning rule and the monetary shock creates the persistent fluctuations in the exchange rate between the two countries.

#### 4. Simulation results

We choose the following parameter values for the variables in the model.

Parameter :	$a_1$	$a_2$	$\omega^1$	$\omega^2$	$H_1$	$H_2$	$N$
Value	0.48	0.5	10	1	100	120	15

The money supply in country 2 is given by  $H_2(t) = H_2 + \varepsilon(t)$ , where the shock  $\varepsilon(t) = \tau v(t)$  and  $v(t)$  is uniformly distributed over the interval  $[-2, 2]$  with  $\tau = 0.08$ , such that the range of the shock is from  $-0.16$  to  $0.16$ . In both simulations, Germany is Country 2, thus we assume there is a small shock to the German money supply. Country 1 is Belgium and Netherlands, respectively.

The initial conditions for the Belgian/German case is chosen as  $p_1^e(1) = 19.072$  and  $p_2^e(1) = 0.924$  and the steady state is given by  $\bar{p}_1 = 19.93$  and  $\bar{p}_2 = 0.923$  and the steady state exchange rate is  $\bar{e} = 20.62$ . The initial conditions for the expected price levels are chosen to reflect that the exchange rate is fluctuating around a steady state level 20.6-20.7, see figure 1 where the actual BEF/DEM exchange rate is illustrated.

In the Netherlands/Germany simulation we choose the following initial conditions  $p_1^e(1) = 1.74$  and  $p_2^e(1) = 1.547$ , the steady state is given by  $\bar{p}_1 = 1.75$  and  $\bar{p}_2 = 1.547$  with the exchange rate is  $\bar{e} = 1.123$ .

Since the exchange rate paths are dependent on the initial conditions, we choose these carefully in both experiments. The initial conditions are chosen such that it "fits" with the actual values of the exchange rates between Belgium-Germany and Netherlands-Germany. In both simulations the values used for the speed of learning and the money supply in Belgium and Netherlands was the same, we could change these parameters values such that Belgium and Netherlands were different, but that does not change the results.

## 5. An Example: The Dutch and Belgian experience

The model described above can give an explanation of the persistent fluctuations that have been observed for example in the Belgian and Dutch exchange rate over the last years, see figure 1 where the actual BEF/DEM-exchange rate is illustrated and figure 2 where the actual NLG/DEM exchange rate is illustrated. Belgium and Holland have more or less had the same development in fundamentals as in Germany, and it has been certain that both Belgium and Holland would be natural participants in a European Monetary Union. Even though the fundamentals in Germany and Belgium/Holland were just about identical there has still been **persistent** fluctuations in the Dutch and the Belgian exchange rate vis-a-vis the D-Mark. This situation matches our theoretical model where the fluctuations take place around a steady state. Note again that steady state is only locally stable.

Figure 3 and 4 plots the simulated values the theoretical model generates for the BEF/DEM exchange rate and NLG/DEM exchange rate. It is evident that

the model generates persistent fluctuations as observed in the actual exchange rate data for the two countries. The Belgian Franc after the crisis in August 1993 depreciated sharply but returned relatively quickly to its former level again. Since then it has been fluctuating around this level. One can also notice that in the period before the breakdown of the ERM, the exchange rate also exhibited fluctuations within the bands of the ERM.

One interesting aspect of the Netherlands-Germany case is that the fluctuations in the exchange rate are very small, the exchange rate fluctuates between 1.115 and 1.129 as shown by figure 2, however the fluctuations seem to be persistent. The time-series ranges from 1992 till today and covers both a period where ERM was working and now where there is a de facto floating exchange rate regime. But still our theoretical model captures the persistent fluctuations that can be observed in all non-fixed exchange rate regimes. Actually one could conjecture that the regime itself (target zone, free float etc.) in this model is represented by the  $a_i$  in the learning rule. Or in other words the agents use adaptive learning but the speed they adjust with depends on which regime they are in.

## 6. Conclusion

In this paper we have formulated a model that can generate persistent fluctuations in the exchange rate essentially stemming from learning and shocks to the money supply in one of the countries. An important part of generating persistent fluctuations was the fact that expectations were formed via adaptive learning rules and hence under the assumption of bounded rationality. The model used is simple and illustrative and in future research the model could perhaps be extended to include also other parts of the economy e.g. the government sector or labour market considerations.

The theoretical model was also able to capture the development of (at least) two exchange rates. We used two exchange rates close to the D-mark that has had just about the same development in fundamentals in an attempt to isolate the sources of the persistent fluctuations in the exchange rate.

In future work, we will develop the model in an attempt to explain the fluctuations in the USD/DEM exchange rate after 1987. This exchange rate seems to fluctuate between two steady state values, thus there might be a cyclical behaviour in the exchange rate. One way to capture this would be to introduce a government in both countries, that finance a fixed amount of spending by for example by a lump sum tax. In this case the model has two interior steady states.



If the agents use the learning rule combined with a small shock to the money supply the economy will fluctuate between the two steady states depending on the size of the shock.

We will also try to incorporate heterogeneous agents, this could be done in various ways, for example by using different learning rules. The simplest way would be to have one type of learning rule in country 1 and another in country 2, but it seems more reasonable also to have different learning rules within the countries. This is also in line with the experimental evidence.

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## A. Proof of proposition 3.1.

The conditions for the eigenvalues of  $J$  to be less than 1 is given by

$$-2 < Tr(J(\bar{p}_1, \bar{p}_2)) < 2 \quad (\text{A.1})$$

$$-1 < Det(J(\bar{p}_1, \bar{p}_2)) < 1 \quad (\text{A.2})$$

where  $Tr(J)$  is the trace of the  $J$  and  $Det(J)$  is the determinant of  $J$ . Let us begin with (A.1)

$$\begin{aligned} Tr(J) &= 1 + a_1(F_1(\bar{p}_1, \bar{p}_2) - 1) + 1 + a_2(G_2(\bar{p}_1, \bar{p}_2) - 1) \\ &= 2 - (a_1 + a_2) + a_1 F_1(\bar{p}_1, \bar{p}_2) + a_2 G_2(\bar{p}_1, \bar{p}_2) > 0 \end{aligned}$$

since partial derivatives are given by:

$$\begin{aligned} F_1(\bar{p}_1, \bar{p}_2) &= \frac{\omega^2}{\omega^1} + \frac{1}{N\omega^1} \frac{H_2}{\bar{p}_2} > 0. \\ G_2(\bar{p}_1, \bar{p}_2) &= \frac{\omega^2}{\omega^1} + \frac{1}{N\omega^1} \frac{H_1}{\bar{p}_1} > 0. \end{aligned}$$

and  $a_1 < 1$  and  $a_2 < 1$ .

$$Tr(J) = 1 + a_1(F_1(\bar{p}_1, \bar{p}_2) - 1) + a_2(G_2(\bar{p}_1, \bar{p}_2) - 1) < 2 \Rightarrow$$

$$a_1(F_1(\bar{p}_1, \bar{p}_2) - 1) + 1 + a_2(G_2(\bar{p}_1, \bar{p}_2) - 1) < 0 \quad (\text{A.3})$$

We assumed that  $F_1(\bar{p}_1, \bar{p}_2) < 1$  and  $G_2(\bar{p}_1, \bar{p}_2) < 1$ , thus (A.3) is satisfied.

The second condition (A.2) is easy to verify.  $Det(J)$  is given by :

$$\begin{aligned} Det(J) &= [1 + a_1(F_1(\bar{p}_1, \bar{p}_2) - 1)] [1 + a_2(G_2(\bar{p}_1, \bar{p}_2) - 1)] \\ &\quad - a_1 a_2 F_2(\bar{p}_1, \bar{p}_2) G_1(\bar{p}_1, \bar{p}_2) \end{aligned}$$

In order to verify (A.2) we have to look closer at the partial derivatives :

$$\begin{aligned} F_2(\bar{p}_1, \bar{p}_2) &= -\frac{\bar{p}_1}{\bar{p}_2} \frac{H_2}{\bar{p}_2} \frac{1}{N\omega^1} < 0 \\ G_1(\bar{p}_1, \bar{p}_2) &= -\frac{\bar{p}_2}{\bar{p}_1} \frac{H_1}{\bar{p}_1} \frac{1}{N\omega^1} < 0 \end{aligned}$$

thus

$$-a_1 a_2 F_2(\bar{p}_1, \bar{p}_2) G_1(\bar{p}_1, \bar{p}_2) = -a_1 a_2 \frac{H_2}{\bar{p}_2} \frac{1}{N\omega^1} \frac{H_1}{\bar{p}_1} \frac{1}{N\omega^1} < 0$$

Since  $F_1(\bar{p}_1, \bar{p}_2) < 1$  and  $G_2(\bar{p}_1, \bar{p}_2) < 1$ , the first term in  $Det(J)$  is less than 1 and  $Det(J) < 1$ . We only have to show that  $-1 < Det(J)$ . Since  $F_1(\bar{p}_1, \bar{p}_2) < 1$  and  $G_2(\bar{p}_1, \bar{p}_2) < 1$  then  $Det(J) > 1$

$$\frac{1}{N\omega^1} \frac{H_2}{\bar{p}_2} < 1 \text{ and } \frac{1}{N\omega^1} \frac{H_1}{\bar{p}_1} < 1$$

and

$$-1 < -a_1 a_2 \frac{H_2}{\bar{p}_2} \frac{1}{N\omega^1} \frac{H_1}{\bar{p}_1} \frac{1}{N\omega^1}$$

Hence the first term in  $Det(J)$  is positive and  $Det(J) > -1$ .

## B. The indeterminacy proposition.

The indeterminacy proposition formulated in Kareken and Wallace is as follows. We assume agents have perfect foresight. If  $E$  is an exchange rate which sustains a monetary equilibrium where both currencies are valued, then for any other exchange rate  $0 < \tilde{E} < \infty$  there exists a monetary equilibrium where both currencies are valued. In order to see this, choose an exchange rate  $\tilde{E} \neq E$  with the price sequences  $\{\tilde{p}_1(t), \tilde{p}_2(t)\}_t$ , where  $p_i(t) \neq \tilde{p}_i(t)$ , for  $i = 1, 2$  and  $t \geq 0$  such that

$$\frac{H_1(t)}{p_1(t)} + \frac{H_2(t)}{p_2(t)} = \frac{H_1(t)}{\tilde{p}_1(t)} + \frac{H_2(t)}{\tilde{p}_2(t)} \quad (\text{B.1})$$

we can rearrange (B.1) to express  $\tilde{p}_1(t)$  as a function of  $p_1(t)$ ,  $\tilde{E}$  and  $E$

$$\tilde{p}_1(t) = \frac{H_1(t) + \tilde{E}H_2(t)}{H_1(t) + EH_2(t)}p_1(t) \quad (\text{B.2})$$

$$\tilde{p}_2(t) = \tilde{p}_1(t)/\tilde{E} \quad (\text{B.3})$$

If the money supply is constant in both countries,  $H_i(t) = H_i$ ,  $i = 1, 2$ , the price sequences  $\{\tilde{p}_1(t)\}$ ,  $\{\tilde{p}_2(t)\}$  defined by (B.2) and (B.3) give the same real rate of returns as the price sequence  $\{p_1(t), p_2(t)\}_t$  :

$$\begin{aligned} \tilde{R}_i(t) &= \frac{\tilde{p}_i(t)}{\tilde{p}_i(t+1)} \\ &= \frac{(H_1(t) + \tilde{E}H_2(t))p_i(t)}{H_1(t) + EH_2(t)} \frac{H_1(t+1) + EH_2(t+1)}{(H_1(t+1) + \tilde{E}H_2(t+1))p_i(t+1)} \\ &= \frac{p_i(t)}{p_i(t+1)} = R_i(t), \quad i = 1, 2. \end{aligned} \quad (\text{B.4})$$

Since savings, individual and aggregate, depend on the real rate of return according to (2.6) and (2.7), the price sequences  $\{\tilde{p}_1(t)\}$ ,  $\{\tilde{p}_2(t)\}$  give the same savings as the price sequences  $\{p_1(t)\}$ ,  $\{p_2(t)\}$ . This indeterminacy result depends upon the assumption of perfect foresight such that the exchange rate  $E(t) = E$  is constant. We could also allow for growth in the money supply, for example a fixed percentage each period such that

$$H_i(t+1) = (1 + \gamma_i)H_i(t), \quad i = 1, 2$$

If  $\gamma_1 = \gamma_2$ , then (B.4) still holds, but the growth in money supply has to be coordinated between the two countries. Sargent (1993) introduces bonds as well and gives conditions for which the indeterminacy result holds. Kareken and Wallace give conditions for the equilibrium to be determinate, but this involves capital restrictions and trade restrictions.

Figure 1

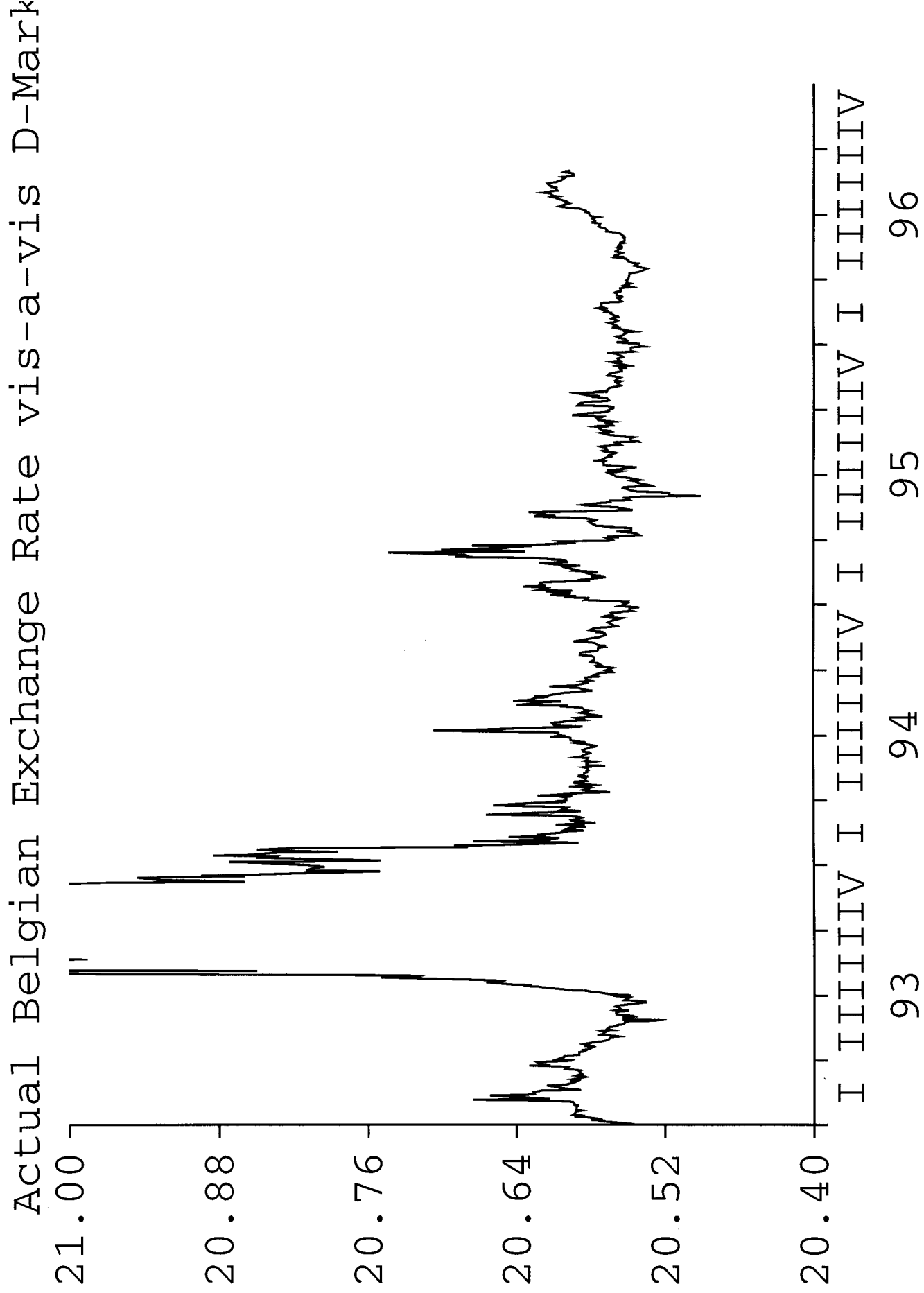
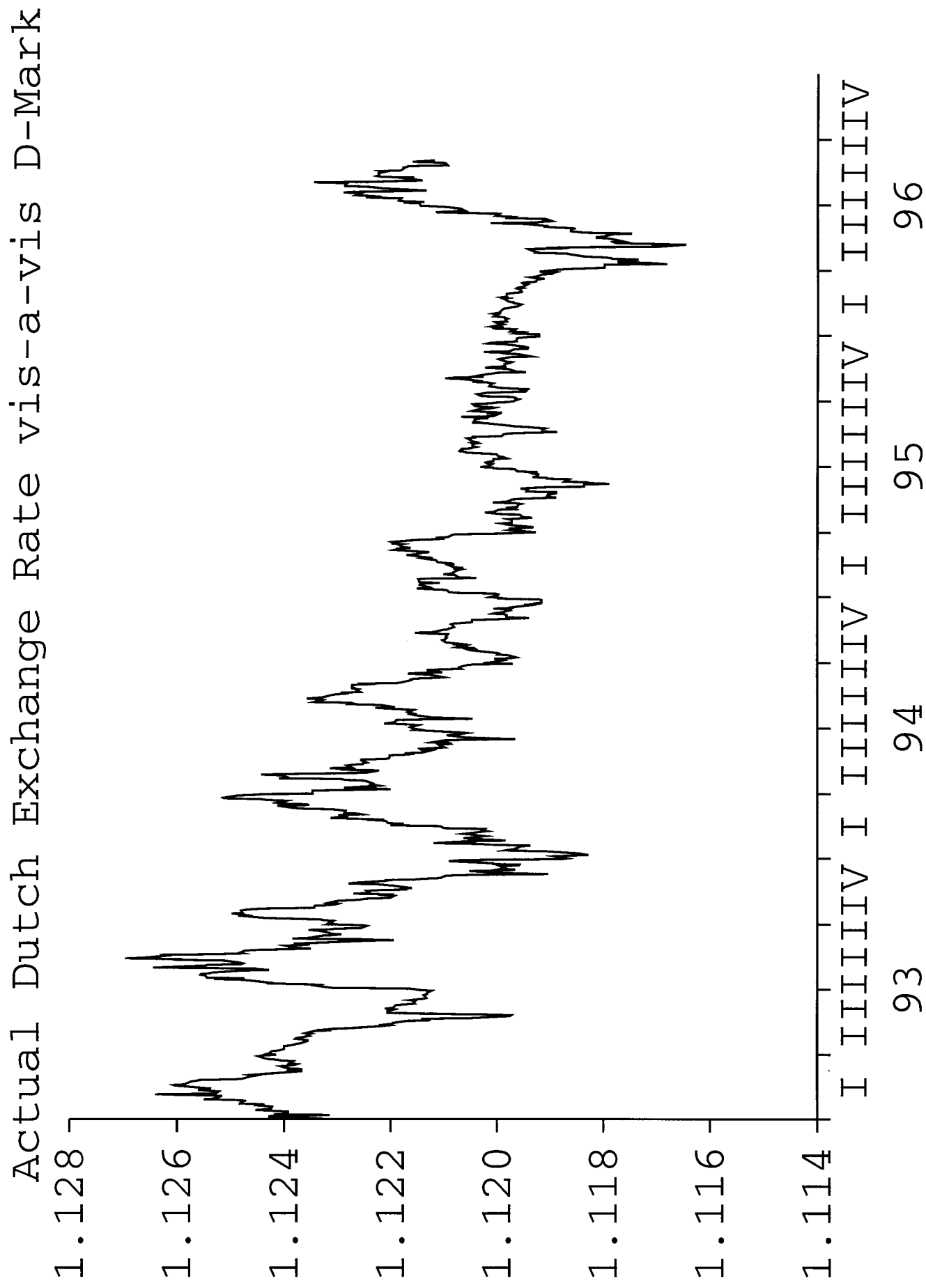


Figure 2





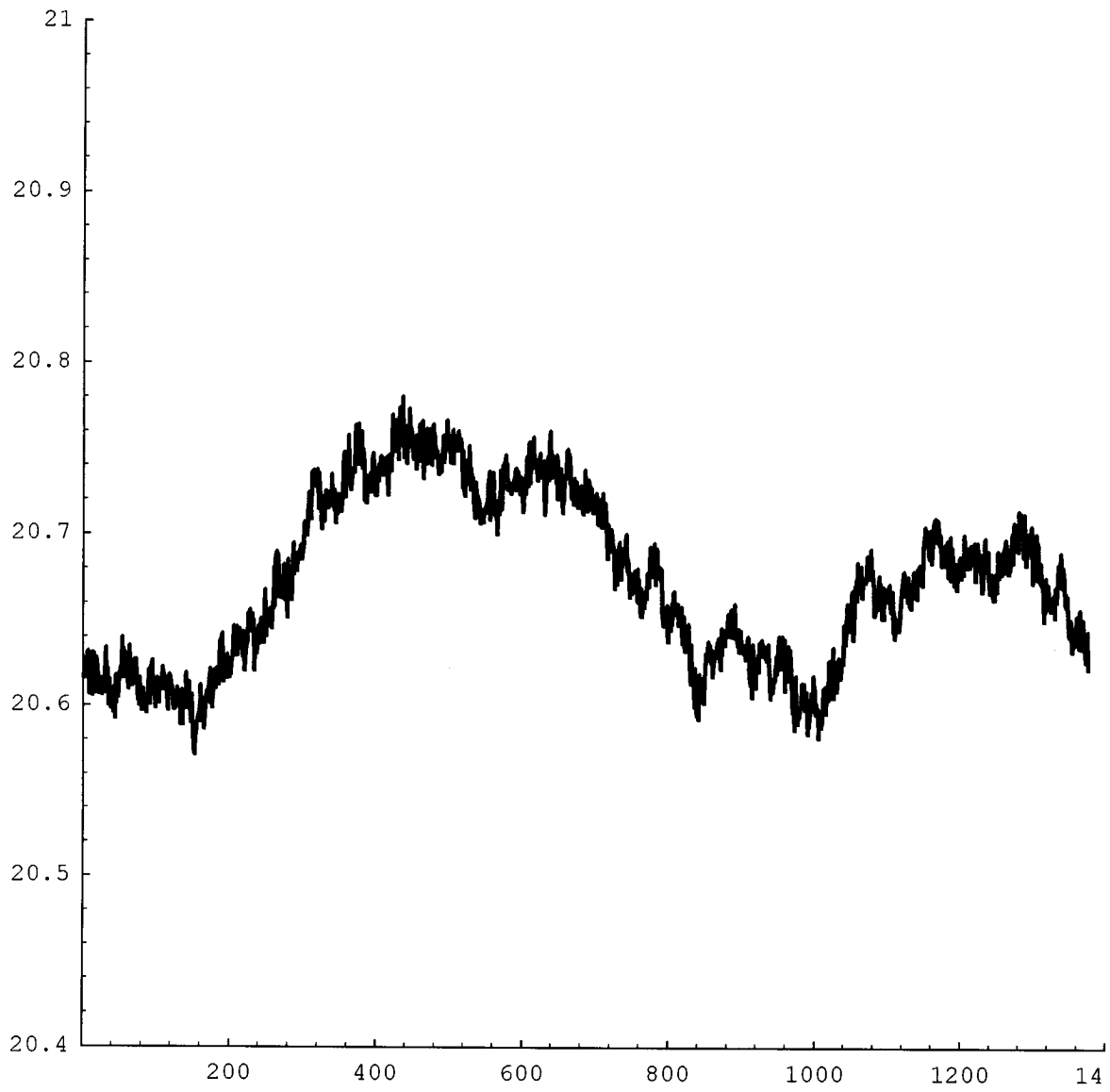
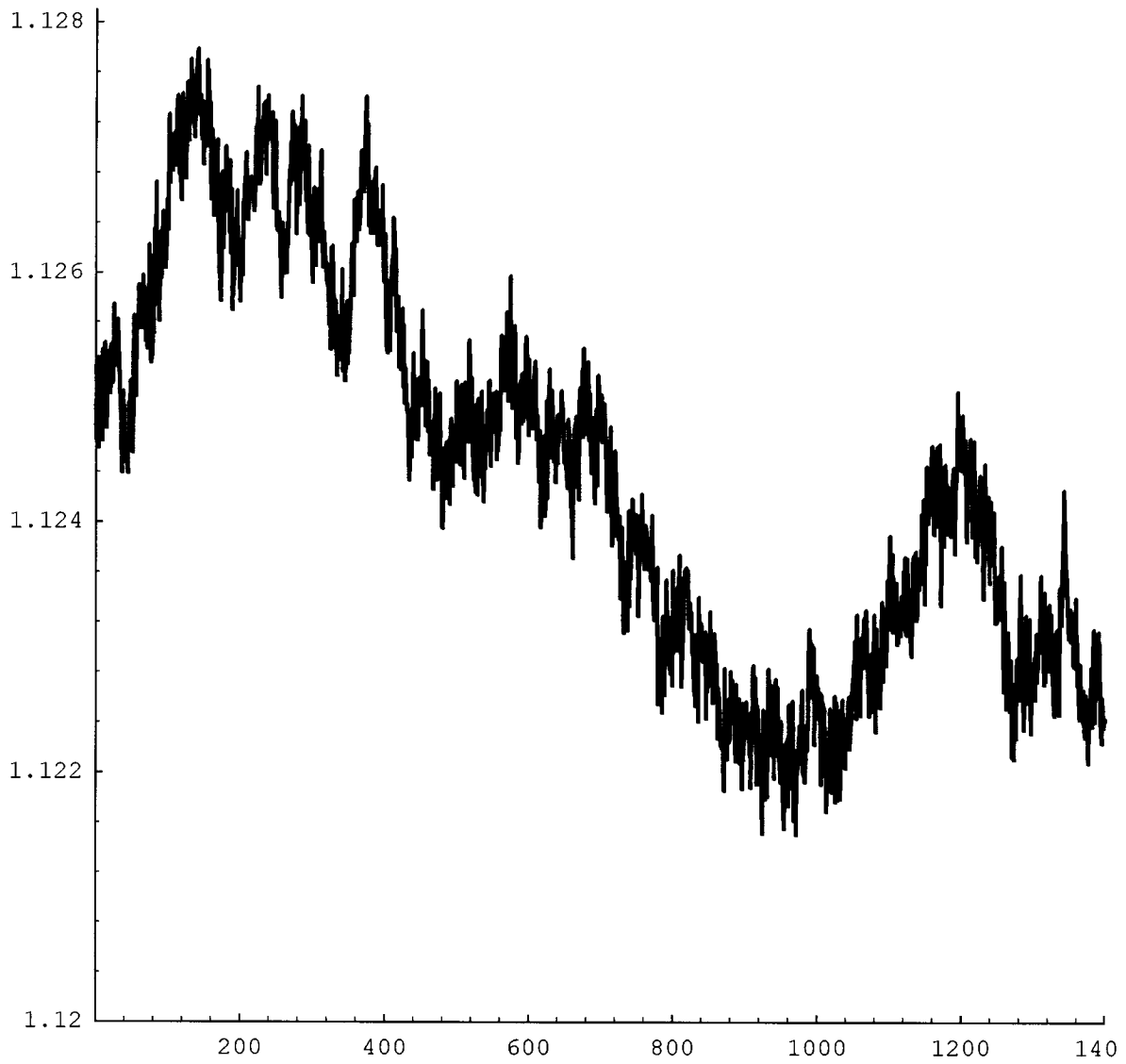


Figure 3: Simulated BEF/DEM  
A simulation with 1400 observations.



**Figure 4: Simulated NLG/DEM**

A simulation with 1400 observations.