

Optimum Taxation and the Allocation of Time*

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Abstract

This paper deals with optimum commodity taxation in Becker's (1965) model of the allocation of time. While the existing public finance literature emphasizes the role of cross elasticities with leisure, I find that the optimal tax system crucially depends on factor shares and elasticities of substitution in household production. In the special case of Leontieff technology, the optimum tax rule depends solely on factor shares and, furthermore, this simple rule maintains the first best allocation. The Becker approach implies, for example, that the social optimum involves a preferential tax treatment of consumer services and possibly even exclusion from the tax base.

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1 Introduction

The theory of optimum commodity taxation was given its modern form by the seminal contributions of Diamond and Mirrlees (1971a,b). Using the emerging duality methods and results in general equilibrium theory, these authors provided a powerful framework to study the properties of optimal tax systems. One of the principal features of the Diamond-Mirrlees framework is a sharp division of consumption and production activities: households consume whereas firms produce. While this assumption has been a very useful abstraction, it is also clear that consumption and production activities are not so easily separated in the real world.

Indeed, as pointed out in the article by Becker (1965), utility-yielding commodities are never bought directly in the market but are rather produced in the household by combining various market-produced goods and services with household time. Becker's theory of the allocation of time has been put to wide use in the analysis of fertility, health, labor supply, transportation and more, but quite surprisingly this framework has not yet had much impact on the theory of taxation. This paper argues that the Becker model is just as powerful a tool in normative tax theory as it has been in many other fields.

The public finance literature devotes a lot of attention to the Ramsey tax problem; that is, how to set commodity tax rates so as to maximize the welfare of a representative household, assuming that the government is subject to an exogenous revenue requirement. The maximization is carried out on the assumption that one good, usually taken to be leisure, cannot be taxed. In essence, this literature provides two fundamental insights. Firstly, goods should be taxed according to their complementarity or substitutability with leisure; the stronger the degree of complementarity, the higher the tax rate. Thus, the structure of optimum taxation crucially depends on compensated cross elasticities with leisure. Secondly, as we have barred ourselves from taxing leisure, the equilibrium inevitably involves a distortion in favor of this commodity. Therefore, the allocation will be second best, not first best.¹

This paper reconsiders these insights in the Becker (1965) model of household production. In this setting there is no such thing as pure leisure but, rather, all

¹Surveys of the literature on optimum commodity taxation can be found in, e.g., Auerbach (1985) and Stern (1990).

utility-yielding commodities take the form of activities that require the input of both market-produced goods and household time. I find that the structure of optimal tax rates is intimately related to household production technology. By contrast, cross elasticities in consumption, so essential in the conventional approach, play a limited role for the optimum tax system. Furthermore, taxation does not necessarily lead to a second best allocation. In the special case of Leontieff technology in household production, the first best allocation is maintained by taxing goods exclusively according to factor shares.

The analysis throws new light on the classic controversy of uniformity versus selectivity in indirect taxation. Two frequently stated arguments in favor of uniform taxation are re-examined. The first argument points to theoretical results showing that, under certain conditions, uniform commodity taxation is the social optimum (see e.g. Sadka, 1977). Employing a Becker approach, the present paper shows that the conditions for uniformity to be the optimum becomes much more special than previously thought. The second argument in favor uniformity concerns our ignorance of the information on which a differentiation of tax rates should be based (see e.g. Deaton, 1981, 1987). The analysis below indicates that the merit of this argument is also weakened once we account for the presence of household production. By imposing additional restrictions on the model household behavior, this approach makes it possible to establish a better intuition for the structure of optimal taxation.

Generally, goods should be taxed according to factor shares and elasticities of substitution in household activities. The more time intensive the activity, the higher the tax rate on market-produced inputs into that activity. And the higher the elasticity of substitution between goods and time in a given activity, the lower the tax rate. The paper takes a closer look on the implications of these theoretical results for the design of real-world tax systems. For example, I find that the optimal tax system involves a preferential tax treatment of consumer services and perhaps even exclusion from the tax base. This policy recommendation is similar in spirit to the numerical results of Piggott and Whalley (1998) who study a VAT base broadening to include services. They find that such a tax reform may involve a welfare loss as production activities shift from the market to the household.

The paper is organized in the following way. Section 2 sets up the model of

household production. Section 3 solves the problem of optimum commodity taxation, while section 4 presents revised versions of the classical propositions of normative tax theory. Section 5 then proceeds to discuss the modeling of household production for the study of taxation. Section 6 investigates some applications of the theoretical model and, finally, section 7 concludes.

2 A model of household production

A representative household derives utility from the consumption of commodities Z_1, Z_2, \dots, Z_n , i.e.

$$U = U(Z_1, Z_2, \dots, Z_n) \quad (1)$$

Following Becker (1965), these commodities are not bought directly in the market but are rather produced in the household by combining market-produced goods and household time. Thus, Z_i is given by

$$Z_i = f_i(X_i, L_i) \quad i = 1, 2, \dots, n \quad (2)$$

where X_i denotes the input of a market-produced good and L_i denotes the input of household time in the production of commodity i . I assume that f has constant returns to scale. The idea behind equation (2) is that market-produced goods are not themselves carriers of utility but are rather combined with household time for transportation, manufacturing and consumption purposes in order to produce some more basic utility-yielding commodities. For example, the television set is combined with time to watch it in order to produce the commodity “watching TV”, and food is combined with time for transportation, cooking and eating so as to produce “dinner”.

This concept of household production is much broader than the understanding of the term in previous papers on taxation with household production by Sandmo (1990), Piggott and Whalley (1998), and Kleven et alia (2000). Inspired by Gronau (1977), the modeling of household production in these papers divides time into three components; work in the market, work in the home, and leisure. A natural distinction between work at home and leisure, suggested by Gronau, is that the former is something you would rather have somebody else do, whereas the latter is

almost impossible to enjoy through a surrogate. From this perspective preparing dinner would perhaps be categorized as work at home, whereas watching TV would be categorized as leisure.²

In one respect the specification in (2) is less general than Becker's (1965) theory of the allocation of time. In the Becker paper X_i is a vector: many different kinds of market-produced goods can go into the production of each home-produced commodity, and the same market-produced good can enter several different household activities. In this paper X_i is a scalar, implying that each type of household production uses only one market produced good, and that each market produced variety enters only one household production activity. This assumption is perhaps not appealing from a descriptive point of view, but it does make for a much simpler analysis, while at the same time keeping the spirit of the results in a more general model.

The optimization problem of households is most conveniently solved by separating production decisions from consumption decisions. First, we solve for the optimal input mix in the production of each commodity, and then we determine optimal consumption of the n different home produced commodities. In the first stage we minimize unit costs in household production. Setting the wage rate equal to one for both producers and consumers (so that time is the untaxed good), this minimization problem can be written in the following way

$$\min \quad P_i \cdot a_{xi} + a_{li} \quad \text{st.} \quad f_i(a_{xi}, a_{li}) = 1 \quad (3)$$

where a_{xi} and a_{li} is the use of market produced goods and household time, respectively, per unit of output in the production of commodity i . The after-tax price of market-produced goods, P_i , is defined as the sum of the producer price, p_i , and a specific tax, t_i . The solution to the above minimization problem is given by factor intensities $a_{xi} = a_{xi}(P_i)$ and $a_{li} = a_{li}(P_i)$, with a_{xi} being a non-increasing function of P_i and a_{li} being a non-decreasing function of P_i . The unit cost function is given by $Q_i(P_i) = P_i \cdot a_{xi}(P_i) + a_{li}(P_i)$ and, by Shephard's Lemma, the first-order derivative of the unit cost function equals

$$\frac{\partial Q_i}{\partial P_i} = \frac{\partial Q_i}{\partial t_i} = a_{xi} \quad i = 1, 2, \dots, n \quad (4)$$

²The literature on household production is surveyed in Gronau (1986).

Thus, a tax on good i raises costs in the household activity using that good, and the increase in costs is exactly equal to the goods intensity.

Let us turn to the second stage of the optimization problem. Consumption decisions must be made in accordance with the budget constraint

$$\sum_{i=1}^n P_i \cdot X_i = L_m \quad (5)$$

where L_m is time devoted to work in the market or, since the wage rate equals one, total market income. In addition, decisions are subject to a time constraint

$$\sum_{i=1}^n L_i + L_m = 1 \quad (6)$$

where the total time available is normalised to one. Combining the budget constraint (5), the time constraint (6) as well as the solution to the cost minimization problem, we get a single overall constraint

$$\sum_{i=1}^n Q_i(P_i) \cdot Z_i = 1 \quad (7)$$

stating that the total cost of consumption, including both the cost of market goods as well as the opportunity cost of time, cannot exceed full income, i.e. the income that would accrue to the household if it devoted all of its time to market work.

The rational households maximize (1) subject to (7). The first order conditions for an optimum are given by

$$\frac{\partial U}{\partial Z_i} - \lambda \cdot Q_i(P_i) = 0 \quad i = 1, 2, \dots, n \quad (8)$$

where λ is the Lagrange multiplier or, equivalently, the marginal utility of income. The solution to the system of equations given in (7) and (8) yields a vector of uncompensated demands $\hat{Z}(Q(P), y)$, where y denotes (hypothetical) non-labor income. The indirect utility function is given by $V(Q(P), y) = U(\hat{Z}(Q(P), y))$. The second stage problem of maximizing (1) subject to (7) is completely conventional and, therefore, all the standard results from consumer theory apply. In particular, we will make use of Roy's Identity, i.e.

$$\frac{\partial V}{\partial Q_i} = -\lambda \hat{Z}_i \quad i = 1, 2, \dots, n \quad (9)$$

and the Slutsky equations

$$\frac{\partial \hat{Z}_i}{\partial Q_j} = \frac{\partial \tilde{Z}_i}{\partial Q_j} - \hat{Z}_j \cdot \frac{\partial \hat{Z}_i}{\partial y} \quad i, j = 1, 2, \dots, n \quad (10)$$

where \tilde{Z}_i denotes the *compensated* demand for commodity i .

How do taxes distort household behavior in this model? It is illuminating to consider the effects of a uniform commodity tax or, equivalently, a proportional income tax. Such a tax leads to two kinds of distortions. First, there will be a substitution from goods to time in the production of each commodity. The magnitude of this effect is determined by elasticities of substitution between goods and time in household production. Second, a uniform commodity tax increases unit costs in all household production activities but, by Shephard's Lemma, costs go up by more in goods intensive types of production. As a result households substitute the consumption of time intensive commodities for the consumption of goods intensive commodities. The size of this second effect is determined, in part, by relative factor intensities. In conclusion, commodity taxes induce a substitution from market time to home time through two channels, and the total magnitude of this distortionary effect is determined by elasticities of substitution and factor intensities in household production. By extension, these parameters will also be crucial for the properties of the optimal tax system.

3 The problem of optimum commodity taxation

This section solves the problem of optimum commodity taxation, assuming an exogenous government revenue requirement, T . We impose the constraint that the required revenue cannot be collected by non-distortionary lump sum taxes. The benevolent government sets commodity taxes so as to maximize the welfare of the representative consumer, subject to the government revenue constraint. Following convention, we assume that the market sector is characterized by a linear technology, implying that producer prices are fixed. In this case the optimization problem can be written as one of maximizing welfare with respect to consumer prices, i.e.

$$\max_{P_1, P_2, \dots, P_n} V(Q(P), y) \quad (11)$$

$$\text{st.} \quad \sum_{i=1}^n (P_i - p_i) \cdot a_{xi}(P_i) \cdot \hat{Z}_i(Q(P), y) = T \quad (12)$$

The first order condition for P_j is given by

$$\begin{aligned} \frac{\partial V}{\partial Q_j} \cdot \frac{\partial Q_j}{\partial P_j} + \mu \cdot \sum_{i=1}^n (P_i - p_i) \cdot a_{xi} \cdot \frac{\partial \hat{Z}_i}{\partial Q_j} \cdot \frac{\partial Q_j}{\partial P_j} \\ + \mu \cdot \left(a_{xj} + (P_j - p_j) \cdot \frac{\partial a_{xj}}{\partial P_j} \right) \cdot \hat{Z}_j = 0 \end{aligned} \quad (13)$$

where μ is the Lagrangian multiplier associated with the government budget constraint. By applying Shephard's Lemma, Roy's Identity and the Slutsky equations, we can rearrange terms so as to obtain

$$\frac{\sum_{i=1}^n a_{xi} \cdot t_i \cdot \frac{\partial \tilde{Z}_j}{\partial Q_i}}{\tilde{Z}_j} + \frac{t_j \cdot \frac{\partial a_{xj}}{\partial P_j}}{a_{xj}} = \theta \quad j = 1, 2, \dots, n \quad (14)$$

where

$$\theta \equiv \frac{\lambda - \mu}{\mu} + \sum_{i=1}^n a_{xi} \cdot t_i \cdot \frac{\partial \hat{Z}_i}{\partial y}$$

The parameter θ is independent of j and can be shown to be negative whenever the revenue requirement, T , is positive. Equation (14) is a Ramsey-type rule. In order to see this note – again by applying Shephard's Lemma – that $a_{xi} \cdot t_i$ is simply equal to the increase in unit costs in the production of commodity i resulting from the imposition of taxes. It is then clear that the first term on the left hand side of equation (14) corresponds to a linear approximation of the relative reduction in the compensated demand for commodity j , that is $\Delta \tilde{Z}_j / \tilde{Z}_j$. Analogously, the second term on the left hand side corresponds to a linear approximation of the relative reduction in the demand for market produced goods per unit of output in the production of commodity j , that is $\Delta a_{xj} / a_{xj}$. Hence, equation (14) states that the sum of substitution effects in consumption, $\Delta \tilde{Z}_j / \tilde{Z}_j$, and production, $\Delta a_{xj} / a_{xj}$, must be the same for all goods. In the simple special case of Leontieff technology the latter term disappears, and optimality then requires that consumption of all household produced commodities is reduced by the same proportion.

The Ramsey rule in equation (14) constitutes the most general representation of the necessary conditions for optimum taxation, and it serves as a useful reminder that the optimality of tax systems is ultimately related to quantities, not prices.

However, the Ramsey rule does not provide any guidance on the structure of tax rates. Nor is it clear from equation (14) what are the consequences of household production for tax policy. To gain a better understanding of these issues, I will now proceed to specialize assumptions further, in order to derive some clear-cut propositions on tax rates. In particular, I will present revised versions of the inverse elasticity rule, the Corlett-Hague rule, and the uniform commodity tax theorem. It is interesting to review these results because they have received a lot of attention in the literature, and because, as it turns out, they highlight the implications of introducing household production in the analysis.

4 A review of some famous propositions

4.1 A revised inverse elasticity rule

In conventional models of optimum taxation, household utility is a function of n market produced goods (which are taxed) and pure leisure (which is not taxed). As is well known this type of framework makes for an extremely simple tax rule in the special case of no cross price effects between taxed goods. In this case the optimal tax rate on a certain good is inversely proportional to the own price elasticity of the compensated demand for that good. This result is one of the oldest in the normative tax literature and it is discussed extensively in Baumol and Bradford (1970) as well as in Sandmo (1976).

In order to derive a result analogous to the inverse elasticity rule, I need to eliminate all cross price effects between taxed goods X_1, X_2, \dots, X_n . In the present framework taxed goods are not related in production, but they are related indirectly through the consumption of home produced commodities Z_1, Z_2, \dots, Z_n . Clearly, one possibility is to assume that there are no cross price effects between Z goods. However, I will follow a slightly different route. In particular, I assume that there exists a commodity, say Z_1 , which is pure leisure so that the goods intensity equals zero. Then I assume that there are no cross price effects between all the other home produced commodities, that is $\partial \tilde{Z}_j / \partial Q_i = 0$ for $i \neq j$ and $i, j = 2, 3, \dots, n$.

Before deriving the optimal tax rates in this simple special case, let me define a couple of key parameters. First, there are the elasticities in consumption and

production

$$\eta_{ji} \equiv \frac{\partial \tilde{Z}_j}{\partial Q_i} \cdot \frac{Q_i}{\tilde{Z}_j}, \quad \sigma_j \equiv -\frac{\partial (a_{xj}/a_{lj})}{\partial P_j} \cdot \frac{P_j}{a_{xj}/a_{lj}} \quad (15)$$

where η_{ji} denotes the elasticity of the compensated demand for commodity j with respect to the price of commodity i , and σ_j denotes the elasticity of substitution between time and goods in the production of commodity j . Furthermore, cost shares in home production are given by

$$\alpha_{lj} \equiv \frac{a_{lj}}{Q_j}, \quad \alpha_{xj} \equiv \frac{P_j \cdot a_{xj}}{Q_j}, \quad \alpha_{lj} + \alpha_{xj} = 1 \quad (16)$$

Using the simplifying assumptions mentioned above as well as the definitions in (15) and (16), we can rewrite the Ramsey rule (14) so as to give

$$\frac{t_j}{P_j} = \frac{\theta}{\eta_{jj} \cdot (1 - \alpha_{lj}) - \sigma_j \cdot \alpha_{lj}} \quad j = 2, 3, \dots, n \quad (17)$$

As η_{jj} is negative and σ_j is defined numerically, the denominator is always negative. If the government collects a positive tax revenue, so that θ is negative, the optimal tax rate on any good j becomes positive. Equation (17) shows that in the absence of household production, so that $\alpha_{lj} = 0$, the model reproduces the classical result that tax rates are proportional to the inverse of the own price elasticity of demand, η_{jj} . The presence of household production introduces two new effects; one effect is related to cost shares, while the other effect is due to substitution between factors. In the special case of Leontieff technology, that is $\sigma_j = 0$, it becomes optimal to impose high tax rates on goods entering time intensive household activities. If technology is not Leontieff, the optimal tax system must also account for the substitutability between factors in different types of household production: the lower the elasticity of substitution, the higher the tax rate.

The revised inverse elasticity rule in equation (17) serves the purpose of pointing out some key parameters in the determination of optimal tax rates, and it also throws some light on the implications of household production for tax policy. However, the derivation rests on a couple of unappealing assumptions; there is no reason to believe that cross price effects between taxed goods are non-existent and, in addition, assuming that pure leisure enters directly into the utility function is at odds with the spirit of the Becker model. In the following section I drop these assumptions and, instead, simplify the analysis by reducing the number of goods.

4.2 A revised Corlett-Hague rule

The structure of optimum commodity taxation in a three-good economy was analyzed in the pioneering article by Corlett and Hague (1953). In a model of two market produced goods (which are taxed) and pure leisure (which is not) these authors found that it is socially optimal to tax goods according to their degree of complementarity with leisure. In particular, the good which is a better complement to leisure should carry a higher rate of tax. In this section we shall see how this result has to be revised in the present context where no such thing as pure leisure exists.

The analogue to the three-good example of Corlett and Hague (op.cit.) is a model with only two utility-yielding commodities (but, of course, there are three goods: X_1 , X_2 , and household time). In this case the optimum tax rule (14) can be written in the following way

$$(\alpha_{x_1} \cdot \eta_{11} - \alpha_{l_1} \cdot \sigma_1) \cdot \frac{t_1}{P_1} + \alpha_{x_2} \cdot \eta_{12} \cdot \frac{t_2}{P_2} = \theta \quad (18)$$

$$\alpha_{x_1} \cdot \eta_{21} \cdot \frac{t_1}{P_1} + (\alpha_{x_2} \cdot \eta_{22} - \alpha_{l_2} \cdot \sigma_2) \cdot \frac{t_2}{P_2} = \theta \quad (19)$$

This is a system of two equations in two unknowns, t_1/P_1 and t_2/P_2 , and we can easily solve it, for example, by using Cramer's rule. We then divide the solution for t_1/P_1 by the solution for t_2/P_2 so as to get an expression of optimal relative tax rates

$$\frac{t_1/P_1}{t_2/P_2} = \frac{\eta_{22} - \eta_{12} - \frac{\alpha_{l_2}}{\alpha_{x_2}} \cdot \sigma_2}{\eta_{11} - \eta_{21} - \frac{\alpha_{l_1}}{\alpha_{x_1}} \cdot \sigma_1} \cdot \frac{\alpha_{x_2}}{\alpha_{x_1}} \quad (20)$$

We can simplify this expression further by exploiting the fact that compensated demands for home produced commodities, \tilde{Z}_1 and \tilde{Z}_2 , are homogeneous of degree zero in prices. This implies, by Eulers Theorem, that

$$\eta_{11} + \eta_{12} = 0 \quad \text{and} \quad \eta_{21} + \eta_{22} = 0 \quad (21)$$

By inserting equation (21) in equation (20) we arrive at a revised version of the Corlett-Hague rule. For pedagogic purposes, let me first look at the simple special case of Leontieff technology, that is $\sigma_1 = \sigma_2 = 0$, and then proceed to look at the case of substitution between factors subsequently. With Leontieff technology

optimum taxation takes the particularly simple form

$$\frac{t_1/P_1}{t_2/P_2} = \frac{1 - \alpha_{l2}}{1 - \alpha_{l1}} = \frac{\alpha_{x2}}{\alpha_{x1}} \quad (22)$$

Instead of taxing goods according to their degree of complementarity with leisure, as suggested by the Corlett-Hague approach, we should tax goods according to factor shares in household production. In particular, it is optimal to levy relatively high tax rates on goods entering time intensive household activities.

Of course, the extreme simplicity of this tax rule depends on the assumption of no substitution between factors in household production. Note, however, that the distortionary effects of taxation in this simple special case are directly comparable to the way taxes distort behavior in the traditional labor-leisure framework. To see this, consider the effects of a uniform commodity tax system. In the model of household production (with Leontieff technology) such a tax system distorts labor supply by leading to a substitution away from goods-intensive consumption towards time-intensive consumption. In the conventional models, a uniform tax system induces households to substitute pure leisure (time intensive consumption) for goods (goods intensive consumption). Clearly, the two effects are similar in spirit.

The point is that the simple statement about optimum taxation in equation (22) does not arise because the assumption of Leontieff technology kills off tax effects that are present in the labor-leisure framework of Corlett and Hague. Instead, the result is an implication of acknowledging that households, in reality, do not derive utility from goods and time separately, but rather from different combinations of goods and time. A simple additional implication of the household production approach is that by taxing goods according to (22) relative unit costs in home production are left unchanged. Thus, there are no tax-induced substitution effects in consumption and, by the Leontieff assumption, there are no substitution effects in production either. In other words, the tax system preserves the first-best allocation.

Another way of deriving the first best result is to insert the tax rule (22) in the budget constraint of households (7). By rearranging terms it can be shown that the tax system corresponds to a proportional tax on the full income of households. As full income is exogenous, such a tax system is of course non-distortionary and the first best allocation is therefore maintained.

Finally, let us drop the simplifying assumption of Leontieff technology and take a look at optimum taxation in the general case. By inserting (21) in (20), we get

$$\frac{t_1/P_1}{t_2/P_2} = \left[\frac{\eta_{11} + \eta_{22} - \frac{\alpha_{i2}}{\alpha_{x2}} \cdot \sigma_2}{\eta_{11} + \eta_{22} - \frac{\alpha_{i1}}{\alpha_{x1}} \cdot \sigma_1} \right] \cdot \frac{\alpha_{x2}}{\alpha_{x1}} \quad (23)$$

The last term is identical to the RHS of equation (22), while the bracketed term is new. Recall that compensated own price elasticities are negative and that elasticities of substitution are defined numerically, so that both numerator and denominator in the bracketed term are negative. Now there is a role for consumption elasticities in the determination of optimum tax rates. However, the magnitude of these parameters, so essential in the labor-leisure model, matter only quantitatively. While factor shares are still important, the optimum tax system also depends on elasticities of substitution between goods and time in household production. *Ceteris Paribus*, commodity inputs into the activity with a higher elasticity of substitution should carry a relatively low rate of taxation. In the special case of perfect substitution in, say, activity 1 we get

$$\lim_{\sigma_1 \rightarrow \infty} \frac{t_1/P_1}{t_2/P_2} = 0 \quad (24)$$

It is socially optimal to exclude good 1 from the tax base. At first glance this special case may seem to be rather extreme and of limited applicability. However, the assumption of perfect substitution is perhaps not entirely off the mark for certain types of services. I will return to this issue in section 6.

4.3 Revised uniform commodity tax theorems

The normative tax literature has devoted quite a lot of attention to identifying necessary and sufficient conditions for the optimality of uniform commodity taxation. Some authors have analyzed this issue in terms of complementarity and substitutability with leisure (Sandmo, 1974 and Sadka, 1977), while other studies focus on the functional form of the utility function (see e.g. Sandmo, 1974; Deaton and Muellbauer, 1980 and Deaton, 1981). According to the first approach, uniform taxation is optimal if and only if all consumer goods have the same compensated elasticities with respect to the wage. A sufficient condition for the equality of these elasticities, according to the second approach, is weak separability of leisure and homotheticity of goods.

These conditions for uniformity to be the optimal solution are clearly special, and there is no reason to believe that they are satisfied in practice. Recent papers argue that taking into account the additional constraints imposed by the existence of household production makes it even more unlikely that uniform taxation is the optimum. Does this conclusion carry over to the present more general model of household production? Can selectivity in indirect taxation be justified by the presence of home production per se? We can state the following uniform commodity tax theorems:

Proposition 1 *The optimum tax system is uniform, i.e. $t_j/P_j = \tau$ for all $j = 1, 2, \dots, n$, if and only if*

$$\sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} + \sigma_j \cdot \alpha_{lj} = -\frac{\theta}{\tau} \quad \text{for all } j = 1, 2, \dots, n. \quad (25)$$

Proof. See appendix A \square

Proposition 2 *The optimum tax system is uniform if and only if such a tax system maximizes labor supply to the market subject to the government budget constraint.*

Proof. See appendix B \square

The labor-leisure model arises as a special case of the present framework if a certain commodity, say Z_1 , has a labor share equal to one, whereas all other commodities have labor shares equal to zero. In this special case, equation (25) becomes $\eta_{j1} = -\theta/\tau$ which is exactly the result of Sadka (1977). In the general case, the left hand side of equation (25) involves both cross price elasticities in consumption as well as factor shares and elasticities of substitution in production. Proposition 1 is easily understood when it is recognized, by proposition 2, that uniform taxation is optimal if and only if such a tax system maximizes labor supply. Intuitively, for a uniform tax system to maximize labor supply, the substitution from market time to home time resulting from a marginal increase in the tax rate on a specific good – starting from uniformity – must be the same for all goods. This is exactly what is stated in equation (25). The first term on the left hand side is the increase in home time resulting from substitution in consumption, whereas the second term is the increase in home time resulting from substitution in production.

A sufficient condition for the optimality of uniform commodity taxation is the equality of all factor shares and elasticities of substitution, that is $\alpha_{li} = \alpha_l$ and $\sigma_i = \sigma$ for all i . For then the left hand side of (25) becomes independent of j :

$$\sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} + \sigma_j \cdot \alpha_{lj} = \alpha_l \cdot \sum_{i=1}^n \eta_{ji} + \sigma \cdot \alpha_l = \sigma \cdot \alpha_l$$

The last equality follows from the homogeneity of degree zero of compensated demands. By contrast, when factor shares and elasticities of substitution do vary across different household activities, the left hand side of (25) will – typically – depend on j , so that uniformity is not the optimum. What do we need, in terms of functional form, to get equality of these parameters? Clearly, an assumption of identical household production functions does *not* do the trick since in this case the labor share as well as the elasticity of substitution in any given activity still depend on the price of goods used in that activity, that is $\alpha_{li} = \alpha_l(P_i)$ and $\sigma_i = \sigma(P_i)$. Since the consumer prices generally vary across goods then so do the α 's and the σ 's. Instead we need the much stronger assumption of identical Cobb-Douglas production functions in all household activities. Of course, it is completely unlikely for this assumption to be satisfied in reality and the result thus indicates just how unlikely it is for uniformity to be the optimum.

5 Becker versus Gronau

The role of household production is to impose additional structure on the model of household behavior, thereby making it possible to derive stronger propositions on the optimal design of tax systems. Of course, the crucial question is what kind of structure should we impose. While the present paper analyses optimum taxation in the Becker (1965) framework, previous papers employ the much more specialized structure of Gronau (1977) or, rather, an adapted version of that model. In order to facilitate a comparison with the Gronau-type specification of previous papers, it is useful to rewrite equations (1) and (2) so as to yield

$$U = \tilde{U}(X_1, X_2, \dots, X_n, L_1, L_2, \dots, L_n) \tag{26}$$

In this formulation the consumer derives utility from n market produced goods as well as n different uses of time. To arrive at the specification of previous papers,

we need to simplify equation (26) in two ways. Firstly, preferences are assumed to be separable in the following way

$$U = \tilde{U}(M(X_1, X_2, \dots, X_i), S(X_{i+1}, X_{i+2}, \dots, X_n), \\ H(L_1, L_2, \dots, L_j), L(L_{j+1}, L_{j+2}, \dots, L_n)) \quad (27)$$

Taxed goods are assumed to be separable into “manufactures” (M) and “services” (S), while time is taken to be separable into “home production” (H) and “leisure” (L). Secondly, home production is assumed to be perfectly substitutable for the market production of services so that utility can be written as

$$U = \tilde{U}(M, S + H, L) \quad (28)$$

This is essentially the specification of Gronau (1977), and it is used to study optimum taxation in studies by Sandmo (1990), Piggott and Whalley (1998), and Kleven et alia (2000). It goes without saying that the assumptions of separability and perfect substitution are quite special. By refraining from such restrictive assumptions, the Becker framework is more general and presumably more realistic than the Gronau model.

In addition to lack of generality, this paper suggests that the Gronau specification does not capture (all) the features of household production that are essential for tax analysis. Firstly, the Becker setup illuminates how optimum commodity taxation is related to factor shares and elasticities of substitution in household production, whereas the Gronau specification is silent about the nature of this relationship. Secondly, the Becker model downplays the role of cross price elasticities in consumption, while the magnitude of these parameters are crucial for the structure of optimum tax rates in the Gronau model, as in the labor-leisure models. Finally, when some commodities are produced exclusively by household time (goods H and L above), taxation will always distort consumption in favor of these commodities and the equilibrium will be second best, not first best. This is not necessarily the case with the Becker framework; in the special case of Leontieff technology it is possible to maintain the first best allocation.

The strength of the Becker approach will become more evident as we turn to applications of the theory in the following section.

6 Applications

6.1 Taxation of consumer services

Previous papers on optimal taxation with household production study the tax treatment of so-called consumer services; i.e. services such as house- and car-repair, cleaning, garden care, housekeeping, child care, cooking and dish-washing. All of these studies adopt the Gronau specification in (28) as their tool of analysis.

The papers by Sandmo (1990) and Kleven et alia (2000) find that the introduction of household production does bias the tax rules towards a relatively lenient taxation of services but, unfortunately, in lack of information on cross elasticities with leisure the results are not clear-cut. Employing a numerical approach, Piggott and Whalley (1998) study a base broadening in indirect taxation to include services; that is the imposition of equal rates starting from a tax system with a zero-rate on services. They argue that the base broadening accompanying the introduction of the Canadian VAT in 1990 was welfare worsening. Other CGE studies by Frederiksen et alia (1995) and Sørensen (1997) reach conclusions for Denmark similar in spirit to those of Piggott and Whalley. But, of course, numerical results always rely on specific functional forms and model calibration.

The above papers take as their point of departure that the home production of consumer services is a perfect substitute for the market production of these services. For example, hiring a domestic help to clean your house is a perfect substitute for doing it yourself. And take-out food is a perfect substitute for a home-cooked meal. In the context of the Becker framework, this implies that the elasticity of substitution between goods and time is very high, perhaps even infinite. If we are willing to accept this basic premise, my analysis leads to quite strong conclusions. As shown in equation (24) it is then optimal to exclude consumer services from the tax base.

Generally speaking, we have seen that consumer services should be taxed leniently if (i) these services are close substitutes for household labor and (ii) the transformation of these services into utility requires little time. It seems quite uncontroversial to assume that market and household labor are very close substitutes in the production of consumer services, indicating a low optimal tax rate on these goods. This conclusion is reinforced once we take into account that these services

enter into activities with low time intensities. For example, it requires very little time to have somebody else clean the house and, likewise, picking up a take-out dinner can be done rather quickly. In fact, the very argument for buying services is often to save time.

Note that if the the optimal tax rate on services is relatively low, but not zero, it may still be optimal to exclude these services from the tax base if there are administrative costs associated with the collection of taxes as in Yitzhaki (1979) and Wilson (1989).

6.2 Taxation of leisure goods

In the literature on optimal commodity taxation a special importance is often attached to leisure goods, i.e. goods used for recreational purposes such as holiday trips, theatre tickets or golf clubs. It seems to be a popular belief among public finance economists that the optimal tax system involves a relatively high rate of tax on leisure goods; see for example Atkinson and Stiglitz (1980, p. 439) or Stern (1990, p. 98). This belief is based on the presumption that leisure *goods* are strongly complementary to leisure *time* and therefore, according to the Corlett-Hague rule, should carry a high rate of tax. But, of course, this is pure speculation as we do not possess hard evidence on the degree of complementarity of different goods and services with leisure.

Again the Becker approach proves useful. In this framework activities are explicitly modelled and we can ask whether market-produced inputs into recreational activities should be taxed at a higher rate than other goods and services? Inspection of the optimum tax rules indicates that this is certainly not always the case. In the case of Leontieff technology taxation is determined solely by factor shares, implying that leisure goods should carry a high rate of tax if and only if forgone earnings constitute a relatively large part of total costs in recreational activities. But, clearly, the goods component of costs is very high in some recreational activities, take for example the luxury cruise in the Mediterranean or the seeing of an opera in the Metropolitan.

Accounting for substitution between factors reinforces the conclusion that optimal taxation do not generally involve heavy taxation of leisure goods. Some recreational activities may very well be characterized by a high elasticity of substitu-

tion between goods and time, implying a low optimal tax rate. For example, people may substitute a cheap, time-intensive mountain hike for the luxury cruise in the Mediterranean. And operagoers may choose to opt for balcony seats rather than the more expensive orchestra seats.

7 Conclusion

A classical insight in the theory of public finance is that the optimum commodity tax system minimizes the substitution away from market work. Thus, the literature has long emphasized a relation between the optimality of tax systems and the allocation of time. From this perspective one may argue that Becker's (1965) theory of the allocation of time is a more natural point of departure than the labor-leisure framework of Diamond and Mirrlees (1971a,b). Indeed, the present paper argues that the Becker framework yields a number of new and interesting results on optimum commodity taxation. These results contribute to the recent, rather policy-oriented, literature on optimum taxation with household production and to the traditional, more theoretical, literature on optimum taxation as it was formulated by Diamond and Mirrlees and many others in the 1970's.

The field of public finance seems to be characterized by a rather strong schism between theory and practice. On the one hand, we have a substantial body of theoretical literature showing that uniformity is unlikely to be the optimum and providing rules for differentiating tax rates among goods and services. On the other hand, most practitioners of public policy advocate uniformity in commodity taxation and they consider any equalization of rates or any base broadening to be a good policy. One of the principal causes for the existence of such a schism is lack of information. Traditional theory suggests that differential rates of tax should be based on cross elasticities with leisure but, in fact, economists are largely ignorant of the magnitude of these parameters. However, the analysis above suggests that household production may alleviate this informational constraint on the practical applicability of economic theory. The introduction of household production imposes additional restrictions on the model of household behavior, thereby helping us to establish a stronger intuition for the likely structure of optimal tax rates.

This paper emphasizes one possible application of the theory as it is argued

that the social optimum involves a lenient taxation of consumer services and perhaps even exclusion of such services from the tax base. Thus, the analysis provides a theoretical underpinning of numerical results by Piggott and Whalley (1998) showing that the Canadian base broadening in 1990 was bad for efficiency. The paper also lends support to claims frequently made in the European policy debate that governments should reduce taxation of services in order to alleviate a distortionary substitution from the formal to the informal economy; see for example Drèze and Sneessens (1994), Lindbeck (1996a,b), and Sørensen (1997). And indeed, in the last decade governments in France and Germany and other European countries have followed this policy prescription by introducing direct subsidies or tax concessions to the consumer service sector.

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A Appendix

A.1 Proof of proposition 1

Suppose that the optimum tax system is uniform. Then there exists a constant τ such that $t_j/P_j = \tau$ for all $j = 1, 2, \dots, n$ solves equation (14), i.e.

$$\sum_{i=1}^n \frac{a_{xi} \cdot P_i}{Q_i} \cdot \frac{\partial \tilde{Z}_j Q_i}{\partial Q_i \tilde{Z}_j} + \frac{\partial a_{xj} P_j}{\partial P_j a_{xj}} = \frac{\theta}{\tau}$$

By using the definitions in equations (15) and (16) this can be rewritten so as to yield

$$\begin{aligned} \sum_{i=1}^n \alpha_{xi} \cdot \eta_{ji} - \sigma_j \cdot \alpha_{lj} &= \sum_{i=1}^n (1 - \alpha_{li}) \cdot \eta_{ji} - \sigma_j \cdot \alpha_{lj} \\ &= \sum_{i=1}^n \eta_{ji} - \sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} - \sigma_j \cdot \alpha_{lj} \\ &= \frac{\theta}{\tau} \end{aligned}$$

Finally, by homogeneity of degree zero of compensated demands, we get

$$\sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} + \sigma_j \cdot \alpha_{lj} = -\frac{\theta}{\tau} \quad (25)$$

This shows that the optimality of uniform taxation implies equation (25).

Conversely, assume that (25) holds. The optimum tax system always obeys the Ramsey rule (14). By definitions (15) and (16), the Ramsey rule can be written as

$$\sum_{i=1}^n \alpha_{xi} \cdot \eta_{ji} \cdot \frac{t_i}{P_i} - \sigma_j \cdot \alpha_{lj} \cdot \frac{t_j}{P_j} = \theta \quad (29)$$

Combining (25) and (29), we get

$$\sum_{i=1}^n \alpha_{xi} \cdot \eta_{ji} \cdot \frac{t_i}{P_i} - \sigma_j \cdot \alpha_{lj} \cdot \frac{t_j}{P_j} = -\sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} \cdot \tau - \sigma_j \cdot \alpha_{lj} \cdot \tau$$

Or, equivalently,

$$\sum_{i=1}^n \alpha_{xi} \cdot \eta_{ji} \cdot \left(\frac{t_i}{P_i} - \tau \right) - \sigma_j \cdot \alpha_{lj} \cdot \left(\frac{t_j}{P_j} - \tau \right) = 0$$

Clearly, $t_j/P_j = \tau$ for all $j = 1, 2, \dots, n$ is a solution to this equation, and thus uniform taxation is optimal. QED.

A.2 Proof of proposition 2

The labor supply function is given by

$$\hat{L}_m = 1 - \sum_{i=1}^n \hat{L}_i = 1 - \sum_{i=1}^n a_{li}(P_i) \cdot \hat{Z}_i(Q(P), y) \quad (30)$$

By differentiating the labor supply function (30) with respect to P_j we get

$$\frac{\partial \hat{L}_m}{\partial P_j} = -\frac{\partial a_{lj}}{\partial P_j} \cdot \hat{Z}_j - \sum_{i=1}^n a_{li} \cdot \frac{\partial \hat{Z}_i}{\partial Q_j} \cdot \frac{\partial Q_j}{\partial P_j} \quad (31)$$

By inserting Shephards Lemma (4), Roy's Identity (9), the Slutsky equation (10) as well as the definitions in (15) and (16) we can rewrite the above expression in the following way

$$\begin{aligned} \frac{\partial \hat{L}_m}{\partial P_j} &= -\frac{\partial a_{lj}}{\partial P_j} \cdot \hat{Z}_j - \frac{\partial Q_j}{\partial P_j} \cdot \sum_{i=1}^n a_{li} \cdot \left(\frac{\partial \hat{Z}_i}{\partial Q_j} - \hat{Z}_j \cdot \frac{\partial \hat{Z}_i}{\partial y} \right) \\ &= -\left[\sigma_j \cdot \alpha_{lj} + \sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} - \sum_{i=1}^n a_{li} \cdot \frac{\partial \hat{Z}_i}{\partial y} \right] \cdot \hat{Z}_j \cdot \frac{\partial Q_j}{\partial P_j} \\ &= \left[\sigma_j \cdot \alpha_{lj} + \sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} - \sum_{i=1}^n a_{li} \cdot \frac{\partial \hat{Z}_i}{\partial y} \right] \cdot \frac{1}{\lambda} \cdot \frac{\partial V}{\partial Q_j} \cdot \frac{\partial Q_j}{\partial P_j} \end{aligned}$$

Since

$$\frac{\partial \hat{L}_m}{\partial y} = -\sum_{i=1}^n a_{li} \cdot \frac{\partial \hat{Z}_i}{\partial y}$$

we can write $\partial \hat{L}_m / \partial P_j$ as

$$\frac{\partial \hat{L}_m}{\partial P_j} = \left[\sum_{i=1}^n \alpha_{li} \cdot \eta_{ji} + \sigma_j \cdot \alpha_{lj} + \frac{\partial \hat{L}_m}{\partial y} \right] \cdot \frac{1}{\lambda} \cdot \frac{\partial V}{\partial P_j} \quad (32)$$

Now suppose that the optimum tax system is uniform. Then, by proposition 1, we know that equation (25) is satisfied. Thus, the first order derivative of the labor supply function is equal to

$$\frac{\partial \hat{L}_m}{\partial P_j} = \left[-\frac{\theta}{\tau} + \frac{\partial \hat{L}_m}{\partial y} \right] \cdot \frac{1}{\lambda} \cdot \frac{\partial V}{\partial P_j} \quad j = 1, 2, \dots, n$$

Thus, the vector $(\partial \hat{L}_m / \partial P_1, \partial \hat{L}_m / \partial P_2, \dots, \partial \hat{L}_m / \partial P_n)$ is a scalar multiple of the vector $(\partial V / \partial P_1, \partial V / \partial P_2, \dots, \partial V / \partial P_n)$. This implies that if the vector (P_1, P_2, \dots, P_n)

maximizes indirect utility, V , subject to the government budget constraint, then this vector also maximizes labor supply, \hat{L}_m , subject to the same constraint.

Conversely, suppose that a uniform tax system maximizes \hat{L}_m subject to the government revenue constraint. Thus, there exists τ such that $t_j/P_j = \tau$ for all j is a solution to the following optimization problem

$$\begin{aligned} & \max_{P_1, P_2, \dots, P_n} \hat{L}_m(P_1, P_2, \dots, P_n) \\ \text{st.} \quad & \sum_{i=1}^n (P_i - p_i) \cdot a_{xi}(P_i) \cdot \hat{Z}_i(Q(P), y) = T \end{aligned}$$

The first order condition for P_j is given by

$$\begin{aligned} & \frac{\partial \hat{L}_m}{\partial P_j} + \nu \cdot \sum_{i=1}^n (P_i - p_i) \cdot a_{xi} \cdot \frac{\partial \hat{Z}_i}{\partial Q_j} \cdot \frac{\partial Q_i}{\partial P_j} \\ & + \nu \cdot \left(a_{xj} + (P_j - p_j) \cdot \frac{\partial a_{xj}}{\partial P_j} \right) \cdot \hat{Z}_j = 0 \end{aligned}$$

and, by assumption, this condition is satisfied for a uniform tax system, i.e.

$$\begin{aligned} & \frac{\partial \hat{L}_m}{\partial P_j} + \nu \cdot \sum_{i=1}^n \tau \cdot a_{xi} \cdot P_i \cdot \frac{\partial \hat{Z}_i}{\partial Q_j} \cdot \frac{\partial Q_i}{\partial P_j} \\ & + \nu \cdot \left(a_{xj} + \tau \cdot \frac{\partial a_{xj}}{\partial P_j} \cdot P_j \right) \cdot \hat{Z}_j = 0 \end{aligned} \tag{33}$$

From the household budget constraint (7) we get

$$\sum_{i=1}^n Q_i(P_i) \cdot \frac{\partial \hat{Z}_i}{\partial Q_j} \cdot \frac{\partial Q_j}{\partial P_j} + \frac{\partial Q_j}{\partial P_j} \cdot \hat{Z}_j = 0 \tag{34}$$

By recalling that $a_{xi} \cdot P_i = Q_i - a_{li}$, we can insert (34) in (33) so as to get

$$\begin{aligned} & \frac{\partial \hat{L}_m}{\partial P_j} - \nu \cdot \tau \cdot \frac{\partial Q_j}{\partial P_j} \cdot \hat{Z}_j - \nu \cdot \tau \cdot \sum_{i=1}^n a_{li} \cdot \frac{\partial \hat{Z}_i}{\partial Q_j} \cdot \frac{\partial Q_j}{\partial P_j} \\ & + \nu \cdot \left(a_{xj} + \tau \cdot \frac{\partial a_{xj}}{\partial P_j} \cdot P_j \right) \cdot \hat{Z}_j = 0 \end{aligned}$$

Now insert (4), (15), (16), and (31) so as to get

$$\frac{\partial \hat{L}_m}{\partial P_j} \cdot (1 + \nu \cdot \tau) + \frac{\partial Q_j}{\partial P_j} \cdot \hat{Z}_j \cdot \nu \cdot (1 - \tau) = 0$$

Finally, by Roy's Identity (9) we can rewrite this expression so as to yield

$$\frac{\partial V}{\partial P_j} = \frac{\lambda \cdot (1 + \nu \cdot \tau)}{\nu \cdot (1 - \tau)} \cdot \frac{\partial \hat{L}_m}{\partial P_j}$$

Thus, the vector $(\partial V/\partial P_1, \partial V/\partial P_2, \dots, \partial V/\partial P_n)$ is a scalar multiple of the vector $(\partial \hat{L}_m/\partial P_1, \partial \hat{L}_m/\partial P_2, \dots, \partial \hat{L}_m/\partial P_n)$. Therefore, if the vector (P_1, P_2, \dots, P_n) maximizes \hat{L}_m subject to the government revenue constraint, then this vector also maximizes indirect utility, V , subject to the same constraint. QED.