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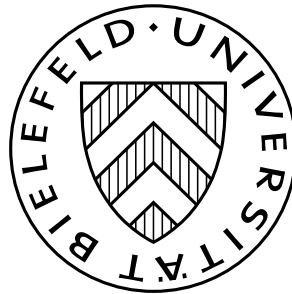
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March 2012

# Coherent Price Systems and Uncertainty-Neutral Valuation

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Patrick Beißner



# Coherent Price Systems and Uncertainty-Neutral Valuation

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March 20, 2012

## Abstract

We consider fundamental questions of arbitrage pricing arising when the uncertainty model is given by a set of possible mutually singular probability measures. With a single probability model, essential equivalence between the absence of arbitrage and the existence of an equivalent martingale measure is a folk theorem, see Harrison and Kreps (1979).

We establish a microeconomic foundation of sublinear price systems and present an extension result. In this context we introduce a prior dependent notion of marketed spaces and viable price systems.

We associate this extension with a canonically altered concept of equivalent symmetric martingale measure sets, in a dynamic trading framework under absence of prior depending arbitrage. We prove the existence of such sets when volatility uncertainty is modeled by a stochastic differential equation, driven by Peng's  $G$ -Brownian motions.

*Key words and phrases:* mutually singular priors, uncertain volatility, sublinear expectation, viability of sublinear price systems, arbitrage, equivalent symmetric martingale measures set (EsMM set), symmetric martingales, Girsanov for  $G$ -Brownian motion

*JEL subject classification:* G13, G14, D46, D52, C62

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# 1 Introduction

In this paper we study a fundamental assumption behind theoretical models in Finance, namely, the assumption of a known single probability measure. Instead, we allow for a set of probability measures  $\mathcal{P}$ , such that we can guarantee awareness of potential model misspecification.<sup>1</sup> We investigate the implications of a related and reasonable arbitrage concept. In this context we suggest a *fair* pricing principle associated with an appropriate martingale concept. The multiple prior setting influences the price system, in terms of the simultaneous control of different null sets. This motivates a worst case pricing theory of possible means.<sup>2</sup>

The pricing of derivatives via arbitrage arguments plays a fundamental role in Finance. Before stating an arbitrage concept, a probability space  $(\Omega, \mathcal{F}, P)$  is fixed such that marketed claims or tradeable assets with trading strategies can be defined. The implicit assumption is that the probabilities are exactly known. The Fundamental Theorem of Asset Pricing (FTAP) then asserts equivalence between the absence of  $P$ -arbitrage in the market model and the existence of a consistent linear price extension so that the market model can price all contingent claims. The equivalent martingale measure is then just an alternative description of this extension via the Riesz representation theorem.

We introduce an uncertainty model described as a set of possibly mutually singular probability measures (or priors). Such models are undominated in the sense that no probability measure exists which controls the zero sets of all other priors in  $\mathcal{P}$ . Our leading motivation is a general form of *volatility uncertainty*. This perspective deviates from models with a term structures of volatilities, i.e. stochastic volatility models, see Heston (1993). We do not formulate the volatility process of a continuous time asset price via another process whose law of motion is exactly known. Instead, the legitimacy of the probability law still depends on an infinite repetition of variable observation, as highlighted by Kolmogoroff (1933). We include this residual uncertainty by giving no concrete model for the stochastics of the volatility process, and instead fix a confidence interval for the volatility variable. We

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<sup>1</sup>The distinction between measurable and unmeasurable uncertainty drawn by Knight (2002) serves in this paper as a starting point for modeling the uncertainty in the economy. Keynes (1937) later argued that single prior models cannot represent irreducible uncertainty.

<sup>2</sup>This was originally noted by de Finetti and Obry (1933).

refer to Avellaneda, Levy, and Paras (1995). Very recent developments in stochastic analysis have established a complete theory in this field, a major objective of which has been the sublinear expectation operator introduced by Peng (2007b).

Unfortunately, a coherent valuation principle changes dramatically when the uncertainty is enlarged by the possibility of different probabilistic scenarios. In order to illustrate this important point, we consider for a moment the uncertainty given by one probability model, i.e.  $\{P\} = \mathcal{P}$ . A (weak) arbitrage refers to a claim  $X$  with zero cost, a  $P$  almost surely positive and with a probably strictly positive payoff. Formally, this can be written as

$$\pi(X) \leq 0, \quad X \geq 0 \text{ } P\text{-a.s.} \quad \text{and } P(X > 0) > 0.$$

The situation changes in the case of an uncertainty model described by a set of mutually singular priors  $\mathcal{P}$ . The second and third condition should be formulated more carefully, because every prior  $P \in \mathcal{P}$  could govern the market. We rewrite the arbitrage condition as

$$\pi(X) \leq 0, \quad X \geq 0 \text{ } P\text{-a.s. for all } P \in \mathcal{P} \quad \text{and } P(X > 0) > 0 \text{ for some } P \in \mathcal{P}.$$

Accepting this new  $\mathcal{P}$ -arbitrage notion one may ask for the structure of the related objects.<sup>3</sup> Suppose we apply the same idea of linear and coherent extensions to the uncertainty model under consideration. Coherence corresponds to strictly positive and continuous functionals on the whole space of claims  $L$  which are consistent with the given market data of marketed claims  $M$ . These claims can be traded frictionless and are priced by a linear functional  $\pi : M \rightarrow \mathbb{R}$ . Hence, the order structure and the underlying topology of  $L$  build the basis of any financial model that asks for coherent pricing. The representation of elements in the topological dual space<sup>4</sup> indicates inconsistencies between positive linear price systems and the concept of  $\mathcal{P}$ -arbitrage. As it is usual, the easy part of establishing an FTAP is deducing an arbitrage free market model from the existence of an equivalent martingale measure  $Q \sim P \in \mathcal{P}$ . When seeking a modified FTAP, the following question (and answer) serves to clarify the issue:

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<sup>3</sup>See Remark 3.14 in Vorbrink (2010) for a discussion of a weaker arbitrage definition and its implication in the  $G$ -framework.

<sup>4</sup>We discuss the precise description in the second part of Introduction and in Section 2.2.

Is the existence of a measure  $Q$  equivalent to some  $P \in \mathcal{P}$  such that prices of all traded assets are  $Q$ -martingales, a sufficient condition to prevent a  $\mathcal{P}$ -arbitrage opportunity?

A short argument gives us a negative answer: Let  $X \in M^0 \subset M$  be a marketed claim with price  $\pi(X) = 0$ . We deduce that  $E^Q[X] = 0$  since  $Q$  is related to a consistent price system. Suppose  $X \in M^0 \cap L_+$  with  $X \geq 0$   $P$ -a.s for every  $P \in \mathcal{P}$  and  $X > 0$   $P'$ -a.s. for some  $P' \in \mathcal{P}$  exists. The point is now,

with  $\{P\} = \mathcal{P}$  we would observe a contradiction since  $Q \sim P$  implies  $E^Q[X] > 0$ . But  $X \in M^0$  may be such that  $P'(X > 0) > 0$  with  $P' \in \mathcal{P}$  is mutually singular to  $Q \sim P$ .

This indicates that our *finer* arbitrage notion is, in general, not consistent with a linear theory of valuation. In other words, a single prior as a pricing measure is not able to contain all the information about the uncertainty.

Since our goal is to suggest a modified framework for a coherent pricing principle, the concept of marketed claim is reformulated by a prior depending notion of possibly marketed spaces  $M_P$ ,  $P \in \mathcal{P}$ . As discussed in Example 3 below, such a step is necessary to deal with the prior dependency of the asset span  $M_P$ . The likeness of marketed spaces depends on the similarity of the involved priors. Here, the possibility of different priors creates uncertainty for a trader who may buy and sell claims which can be achieved frictionless. We associate a linear price system  $\pi_P : M_P \rightarrow \mathbb{R}$  for each marketed space. In this context we posit that coherence is based on *sublinear* price systems,<sup>5</sup> as illustrated this in the following example:

Let the uncertainty model consists of two priors  $\mathcal{P} = \{P, P'\}$ . If  $P$  is the true law, the market model is given by the set of marketed claims  $M_P$  priced by a linear functional  $\pi_P$ . If  $P'$  is the true law, we get  $M_{P'}$  and  $\pi_{P'}$ . For instance, constructing a claim via self-financing strategies implies an equality of portfolio holdings that must be satisfied almost surely only for the particular probability measure. If the trader could choose between the sets  $M_{P'} + M_P$  to create a portfolio, additivity is a natural requirement with the consistency condition  $\pi_{P'} = \pi_P$  on  $M_{P'} \cap M_P$ .<sup>6</sup> However, the trader is neither free to choose a mixture of claims, nor may she choose a scenario. The

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<sup>5</sup>This price system can be seen as an envelope of the price correspondence  $\pi(X) = \{\pi_P(X) : X \in M_P, P \in \mathcal{P}\}$ , as in Clark (1993).

<sup>6</sup>See Heath and Ku (2006) for a discussion.

equality of prices at the intersection is not intuitive, since the different priors create a different price structure in each scenario. We therefore argue, that  $\sup(\pi_{P'}(X), \pi_P(X))$  is a reasonable price for a claim  $X \in M_{P'} \cap M_P$  in our multiple prior framework. This yields to subadditivity. In contrast to the classical law of one price, linearity of the pricing functional is merely true under a fixed prior.<sup>7</sup>

### Outline and results of the paper

We begin with an economic basis for an asset pricing principle. To do so, we introduce an appropriate notion of viability and relate this to the extension of sublinear functionals. Before we give an overview of the results, we describe the primitives of the economy:

The very basic principle of uncertainty is the assumption of different possible future states of the world  $\Omega$ , which is equipped with a  $\sigma$ -algebra  $\mathcal{F}$ . In order to tackle the mutually singular priors, we need some structure in the state space.<sup>8</sup> In the most abstract setting, the states of the world  $\omega \in \Omega$  build a complete separable metric space, also known as a Polish space. The state space contains all realizable path of security prices. For the greater part of the paper, we assume  $\Omega = C([0, T]; \mathbb{R})$ , the Banach space of continuous functions between  $[0, T]$  and  $\mathbb{R}$ , equipped with the supremum norm. In the most general framework, we assume a weakly compact set of priors  $\mathcal{P}$  on the Borel  $\sigma$ -algebra  $\mathcal{F} = \mathcal{B}(\Omega)$ . This encourages us to consider the sublinear expectation operator

$$\mathcal{E}^{\mathcal{P}}(X) = \sup_{P \in \mathcal{P}} E^P[X].$$

In our economy the Banach space of contingent claims  $L^2(\mathcal{P})$  consists of all random variables with a finite variance for all  $P \in \mathcal{P}$ . The primitives of our representative agent economy are given by a preference relation in  $\mathbb{A}(\mathcal{P})$ , the set of convex, continuous, strictly monotone, and rational preferences on  $L^2(\mathcal{P})$ .

The topological dual space, a first candidate for the space of price systems, does not consist of elements which can be represented by a Radon-Nykodym density  $Z$ . Rather, in the present framework, it may be represented by the

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<sup>7</sup>Sublinearity induced by market frictions is conceptionally different. For instance, in Jouini and Kallal (1999) one convex set of marketed claims is equipped with a convex pricing functional, in which case, the possibility of different scenarios is not included.

<sup>8</sup>See Bion-Nadal and Kervarec (2010) for a discussion of different state spaces.

pairs  $(P, Z) \in \mathcal{P} \times L^2(P)$ . With this dual space we introduce our price space of special sublinear price functionals  $\mathfrak{P}(\mathcal{P})$ . Proposition 1 lists the properties of such price functionals and indicates a possible axiomatic approach to the price systems inspired by coherent risk measures.

Sublinear prices can also be motivated by the price systems of partial equilibria, which consists of prior depending linear price functionals  $\pi_P$  restricted to the prior depending marketed spaces  $M_P \subset L^2(P)$ ,  $P \in \mathcal{P}$ . These spaces are joined to a unified marketed space  $\mathbb{M}(\Gamma)$  in terms of an orthonormal basis argument. Here, the sublinearity is already present by a consolidation operation, called  $\Gamma$ , which transforms the given price systems  $\{\pi_P\}_{P \in \mathcal{P}}$  to *one* possibly sublinear system extended to some coherent element in the price space  $\mathfrak{P}(\mathcal{P})$ . Our scenario based viability can then model a preference free equilibrium concept in terms of consolidation of possibilities.

Our first main result, Theorem 1, gives an equivalence between our notion of scenario-based viable price systems, and the extension of sublinear functionals. Our notion of viability, which corresponds to a no trade equilibrium, is then based on sublinear prices so that the price functional acts linearly under a local prior.

In the second part, we consider the dynamic framework on a time interval  $[0, T]$  with an augmented filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ . Its special feature is its reliance on the initial  $\sigma$ -algebra, which does not contain all null sets. Assuming Nutz and Soner (2010), we have that the derivative of the quadratic variation parametrizes the set of priors. The implicit dynamic structure opens the door for a theory of dynamic sublinear expectation based on sublinear conditional expectation operators  $\{\mathcal{E}_t(\cdot)\}_{[0, T]}$ .

With the sublinear conditional expectation, a martingale theory is available which represents a possibilistic model of a fair game against nature. In this fashion, the multiple prior framework allows us to generalize the concept of equivalent martingale measures. Instead of considering *one* probability measure, we suggest that the appropriate concept is a set of priors which is related to the statistical set of priors through a prior depending shift  $Z_P \in L^2(P)$  in the Radon-Nykodym sense: each prior  $P \in \mathcal{P}$  is shifted by a different (state price) density. This creates a new sublinear expectation  $\mathcal{E}^{\mathcal{Q}}$ , generated by a set of priors  $\mathcal{Q}$ . Furthermore we require that the asset price  $S$  under  $\mathcal{E}^{\mathcal{Q}}$  is mean unambiguous, i.e.  $E^{\mathcal{Q}}[S_T] = E^{\mathcal{Q}'}[S_T]$ , for all  $\mathcal{Q}, \mathcal{Q}' \in \mathcal{Q}$ . Such a property is essential for creating a process via a conditional expectation which satisfies the classical martingale representation property, see Appendix B.3. This is true if and only if the martingale is *symmetric*, i.e.  $-S$



is a  $\mathcal{E}^{\mathcal{Q}}$ -martingale as well. This reasoning motivates the modification of the martingale concept, now based on the idea of a fair game under mutually singular uncertainty. The condition that the price process  $S$  is a symmetric martingale motivates qualifying the valuation principle as *uncertainty neutral*.<sup>9</sup>

The principal idea of our modified notion of arbitrage, which we call  $\mathcal{P}$ -arbitrage, and briefly discussed at the beginning of the introduction, was introduced by Vorbrink (2010) for the  $G$ -expectation framework. In Theorem 2 we show that under no  $\mathcal{P}$ -arbitrage there is a one-to-one correspondence between the extensions of Theorem 1 and (special) *equivalent symmetric martingale measure* sets, called EsMM sets. We thus establish an asset pricing theory based on a (discounted) sublinear expectation payoff. Corollary 1 relates EsMM sets to market completeness and to different kinds of arbitrage. Having presented Theorem 1 and Theorem 2, we continue in the same fashion as in the classical literature with a single prior. We consider a special class of asset prices driven by  $G$ -Brownian motion, related to a  $G$ -expectation  $E_G$ .<sup>10</sup> This process is a canonical generalization of the standard Brownian motion, whereas the quadratic variation may move almost arbitrarily in a positive interval. The related  $G$ -heat equation is now a fully nonlinear PDE, see Peng (2007b).

We consider a Black-Scholes like market with uncertain volatilities driven by a  $G$ -Brownian motion  $B^G$ . The stock price  $S$  is modeled as a diffusion

$$dS_t = \mu(t, S_t)d\langle B^G \rangle_t + \sigma(t, S_t)dB_t^G, \quad S_0 = 1, \quad t \in [0, T].$$

This related stochastic calculus comprises a stochastic integral notion, a  $G$ -Itô formula and a martingale representation theorem.<sup>11</sup> In this mutually singular prior setting, the (more evolved) martingale representation property, related to a sublinear conditional expectation, is not equivalent to the completeness of the model, because the volatility uncertainty is encoded in the integrator

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<sup>9</sup>In the finite state case, Dana and Le Van (2010) introduce the notion of a risk adjusted set of priors.

<sup>10</sup> In the mathematical literature, the starting point for consideration is a sublinear expectation space, consisting of the triple  $(\Omega; \mathcal{H}; \mathcal{E})$ , where  $\mathcal{H}$  is a special space of test random variables. If the sublinear expectation space can be represented via the supremum of a set of priors, see Denis, Hu, and Peng (2011), one can take  $(\Omega, \mathcal{B}(\Omega), \mathcal{P})$  as the associated *uncertainty space*. The precise definition of the concept is stated in the Appendix B.

<sup>11</sup>We apply recent results from Xu, Shang, and Zhang (2011), Song (2011), Soner, Touzi, and Zhang (2011b) and Li and Peng (2011).

of the price process. For the density process we introduce an exponential martingale  $\{\mathbf{E}_t\}_{t \in [0, T]}$ <sup>12</sup> for  $G$ -Brownian motion and apply a Girsanov type theorem under the  $G$ -expectation. We observe the following formula

$$\Psi(X) = E_G(\mathbf{E}_T X), \quad X \in L^2(\mathcal{P}),$$

where the valuation still depends on  $\Gamma$  and the interest rate is zero. Example 6 illustrates its usefulness by relating the abstract super-replication, as discussed in Denis and Martini (2006), to an EsMM set.

### Related Literature

We embed the present paper into the existing literature. In Harrison and Kreps (1979) the arbitrage pricing principle provides an economic foundation by relating the notion of equivalent martingale measures with a linear equilibrium price system. Risk neutral pricing, as a precursor, was discovered by Cox and Ross (1976). The idea of arbitrage pricing was introduced by Ross (1976).

The efficient market hypothesis of Fama (1970) introduces information efficiency, a concept closely related to Samuelson (1965), where the notion of a martingale reached neo-classic economics for the first time.<sup>13</sup>

Harrison and Pliska (1981) as well as Kreps (1981) and Yan (1980) continued laying the foundation of arbitrage free pricing. Later, Dalang, Morton, and Willinger (1990), presented a fundamental theorem of asset pricing for finite discrete time. In a general semimartingale framework, the notion of no free lunch with vanishing risk Delbaen and Schachermayer (1994) ensured the existence of an equivalent martingale measure in the given (continuous time) financial market. All these considerations have in common that the uncertainty of the model is given by a single probability measure.

Moving to models with multiple probability measures, pasting martingale measures introduces the intrinsic structure of dynamic convexity, see Riedel (2004) and Delbaen (2006). This type of time consistency is related to recursive equations, see Chen and Epstein (2002), which can result in nonlinear expectation and generates a rational updating principle. Moreover, the backward stochastic differential equations can model drift-uncertainty, a dynamic sublinear expectation, see Peng (1997). However in these models of uncertainty, all priors are related to a reference probability measure, i.e. all priors

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<sup>12</sup>The precise PDE description of the  $G$ -expectation allows to define a universal density. Note that in the more general case we have a prior depending family of densities.

<sup>13</sup> Bachelier (1900) influenced the course of Samuelson's work.

are equivalent or absolutely continuous. Moreover, drift uncertainty does not create a significant change for a valuation principle of contingent claims. The possible insufficiency of equivalent prior models for an imprecise knowledge of the environment motivates the consideration of mutually singular priors as illustrated at the beginning of this introduction. The mathematical discussion of such frameworks can be found for instance in Peng (2007a); Nutz and Soner (2010); Bion-Nadal and Kervarec (2012). Epstein and Ji (2012) provide a discussion in economic terms. Here, the volatility uncertainty is encoded in a non-deterministic quadratic variation of the underlying noise process.

Recalling Gilboa and Schmeidler (1989), this axiomatization of uncertainty aversion represents a non-linear expectation via a worst case analysis. Similarly to risk measures, see Artzner, Delbaen, Eber, and Heath (1999)<sup>14</sup>, the related set of representing priors may be not equivalent to each other. This important change allows an application of financial markets with uncertain volatility. We refer to Denis and Martini (2006) for a pricing principle of claims via a quasi sure stochastic calculus and Avellaneda, Levy, and Paras (1995) for the first intuitive considerations.

Jouini and Kallal (1995) consider a non-linear pricing caused by bid-ask spreads and transaction costs, where the price system is extended to a linear functional. Another classical motivator for nonlinearities is related to superhedging, see Favero, Castagnoli, and Maccheroni (2007). In Araujo, Chateauneuf, and Faro (2012), pricing rules with finitely many state are considered.<sup>15</sup> A price space of sublinear functionals is discussed in Aliprantis and Tourky (2002). We quote the following interpretation of the classical equilibrium concept with linear prices and its meaning (see Aliprantis, Tourky, and Yannelis (2001)):

”A linear price system summarizes the information concerning relative scarcities and at equilibrium approximates the possibly non-linear primitive data of the economy.”

The paper is organized as follows. Section 2 introduces the primitives of the economic model and establishes the connection between our notion of viability and extensions of price systems. Section 3 introduces the security

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<sup>14</sup> Markowitz (1952) postulated the importance of diversification, a fundamental principle in Finance, which corresponds to sublinearity of risk measures.

<sup>15</sup>They establish a characterization of super-replication pricing rules via an identification of the space of frictionless claims.

market model associated with the marketed space. We also discuss the  $G$ -Samuelson model. Section 4 concludes and discusses the results of the paper and list possible extensions. The first part of the appendix presents the details of the model and provides the theorem proofs. In the second part, we discuss mathematical foundations such as the space of price systems and a collection of results of stochastic analysis and  $G$ -expectations.

## 2 Viability and sublinear extensions of prices

We begin by recapping the case where uncertainty is given by an arbitrary probability space  $(\Omega, \mathcal{F}, P)$  as it emphasizes sensible difference with regard to the uncertainty model posit in this paper.

Following, we introduce the uncertainty model as well as the related space of contingent claims. Then we discuss the space of sublinear price functionals. The last subsection is devoted to introducing the economy, where we give an extension result, (see Theorem 1, Section 2.3).

### Background: Classical viability

Let there be two dates  $t = 0, T$ , claims at  $T$  are elements of the classical Hilbert lattice  $L^2(P) = L^2(\Omega, \mathcal{F}, P)$ . Price systems are given by linear and  $L^2(P)$ -continuous<sup>16</sup> functionals. By Riesz representation theorem, elements of the related topological dual can be identified in terms of elements in  $L^2(P)$ . A strictly positive functional  $\Pi : L^2(P) \rightarrow \mathbb{R}$  evaluates a positive random variable  $X$  with  $P(X > 0) > 0$ , such that  $\Pi(X) > 0$ .

A *price system* consists of a (closed) subspace  $M \subset L^2(P)$  and a linear price functional  $\pi : M \rightarrow \mathbb{R}$ . The marketed space consists of contingent claims achievable in a frictionless manner.

$\mathbb{A}(P)$  is the set of rational, convex, strictly monotone and  $L^2(P)$ -continuous preference relations on  $\mathbb{R} \times L^2(P)$ . The consistency condition for economic equilibrium is given by the concept of viability. A price system is *viable* if there exists a preference relation  $\succsim \in \mathbb{A}(P)$  and a bundle  $(\hat{x}, \hat{X}) \in \mathbb{R} \times M$  with

$$(\hat{x}, \hat{X}) \in B(0, 0, \pi, M) \text{ and } (\hat{x}, \hat{X}) \succsim (x, X) \text{ for all } (x, X) \in B(0, 0, \pi, M),$$

where  $B(x, X, \pi, M) = \{(y, Y) \in \mathbb{R} \times M : y + \pi(Y) \leq x + \pi(X)\}$  denotes the budget set. Harrison and Kreps (1979) prove the following fundamental

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<sup>16</sup>The topology is induced by the  $L^2(P)$ -norm.

result:

The price system  $(M, \pi)$  is viable if and only if there is an extension  $\Pi$  of  $\pi$  to  $L^2(P)$  that is strictly positive.

Note that strict positivity implies  $L^2(P)$ -continuity. The proof is achieved by a Hahn-Banach argument and the usage of the properties of  $\succsim$  such that  $\Pi$  creates a linear utility functional and hence a preference relation in  $\mathbb{A}(P)$ .

## 2.1 The uncertainty model and the space of claims

We begin with the underlying uncertainty model. We consider possible scenarios which share neither the same probability measure nor the same zero sets. Therefore it is not possible to assume the existence of a given reference probability measure when the zero sets are not the same. For this reason we need some topological structure in our uncertainty model.

Let  $\Omega$ , the states of the world, be a complete separable metric space equipped with a metric  $d : \Omega \times \Omega \rightarrow \mathbb{R}_+$ ,  $\mathcal{B}(\Omega)$  the Borel  $\sigma$ -algebra of  $\Omega$  and let  $\mathcal{C}_b(\Omega)$  denote the set of all bounded,  $d$ -continuous and  $\mathcal{B}(\Omega)$ -measurable real valued functions.  $\mathcal{M}_1(\Omega)$  defines the set of all probability measures on  $(\Omega, \mathcal{B}(\Omega))$ .

The uncertainty of the model is given by a weakly compact set of probability measure  $\mathcal{P} \subset \mathcal{M}_1(\Omega)$ .<sup>17</sup> In the following example we illustrate a construction for  $\mathcal{P}$ , which we apply in the dynamic setting of Section 3.

**Example 1** *We consider a time interval  $[0, T]$  and the Wiener measure  $P_0$  on the state space of continuous paths starting in zero  $\Omega = \{\omega : \omega \in C([0, T]; \mathbb{R}) : \omega_0 = 0\}$  and the canonical process  $B_t(\omega) = \omega_t$ . Let  $\mathbb{F}^o = \{\mathcal{F}_t^o\}_{t \in [0, T]}$ ,  $\mathcal{F}_t^o = \sigma(B_s, s \in [0, t])$  be the raw filtration of the canonical process  $B$ .*

*The strong formulation of volatility uncertainty is based upon martingale laws with stochastic integrals:*

$$P^\alpha := P_0 \circ (X^\alpha)^{-1}, \quad X_t^\alpha = \int_0^t \alpha_s^{1/2} dB_s,$$

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<sup>17</sup>As shown in Denis, Hu, and Peng (2011), the related capacity  $c(\cdot) = \sup_{P \in \mathcal{P}} P(\cdot)$  is *regular* if and only if the set of priors is relatively compact. Here, regularity refers to a reasonable continuity property. In Appendix B.2, we recall some related notions. Moreover, we give a criterion for the weak compactness of  $\mathcal{P}$  when it is constructed via the quadratic variation and a canonical process.

where the integral is defined  $P_0$  almost surely. The process  $\alpha$  is  $\mathbb{F}^0$ -adapted and has a finite first moment. A set  $\mathcal{D}$  of  $\alpha$ 's build  $\mathcal{P}$  via the associated prior  $P^\alpha$ , such that  $\{P^\alpha : \alpha \in \mathcal{D}\} = \mathcal{P}$  is weakly compact.<sup>18</sup>

We describe the set of contingent claims. Following Huber and Strassen (1973), for each  $\mathcal{B}(\Omega)$ -measurable real function  $X$  such that  $E^P[X]$  exists for every  $P \in \mathcal{P}$ , we define the upper expectation operator  $\mathcal{E}^\mathcal{P}(X) = \sup_{P \in \mathcal{P}} E^P[X]$ .<sup>19</sup> We suggest the following norm for the space of contingent claims, given by the capacity norm  $c_{2,\mathcal{P}}$ , defined on  $\mathcal{C}^b(\Omega)$  by

$$c_{2,\mathcal{P}}(X) = \mathcal{E}^\mathcal{P}(X^2)^{\frac{1}{2}}. \quad (1)$$

Define the closure of  $\mathcal{C}_b(\Omega)$  under  $c_{2,\mathcal{P}}$  norm by  $\mathcal{L}^2(\mathcal{P}) = \mathcal{L}^2(\Omega, \mathcal{B}(\Omega), \mathcal{P})$ .<sup>20</sup> Let  $L^2(\mathcal{P}) = \mathcal{L}^2(\mathcal{P})/\mathcal{N}$  be the quotient space of  $\mathcal{L}^2(\mathcal{P})$  by the  $c_{2,\mathcal{P}}$  null elements  $\mathcal{N}$ . We do not distinguish between classes and their representatives. Two random variables  $X, Y \in L^2(\mathcal{P})$  can be distinguished if there is a prior in  $P \in \mathcal{P}$  such that  $P(X \neq Y) > 0$ .

It is possible to define an order relation  $\leq$  on  $L^2(\mathcal{P})$ . Classical arguments prove that  $(L^2(\mathcal{P}), c_{2,\mathcal{P}}, \leq)$  is a Banach lattice, (see Appendix A.1 for details). We consider the space of contingent claims  $L^2(\mathcal{P})$  so that under every probability model  $P \in \mathcal{P}$ , we can evaluate the variance of a contingent claim. Properties of random variables are required to be true  $\mathcal{P}$ -quasi surely, i.e.  $P$ -a.s. for every  $P \in \mathcal{P}$ . This indicates that a related stochastic calculus on a probability space is unsuitable.

## 2.2 Scenario-based viable price systems

This subsection is divided into three parts. First, we introduce a new dual space where linear and  $c_{2,\mathcal{P}}$ -continuous functionals are the elements. As discussed in the introduction, we allow sublinear prices as well. This forces us to extend the linear price space, where we discuss two operations on the

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<sup>18</sup>In order to define universal objects, we need the pathwise construction of stochastic integrals, (see Föllmer (1981), Karandikar (1995)).

<sup>19</sup>It is easily verified that  $\mathcal{C}_b(\Omega) \subset \{X \text{ } \mathcal{B}(\Omega)\text{-measurable} : \mathcal{E}^\mathcal{P}(X) < \infty\}$  holds and  $\mathcal{E}^\mathcal{P}(\cdot)$  satisfies the property of a sublinear expectation. For details, see Appendix A.1.1, Peng (2007a) and Appendix B.3.

<sup>20</sup>See Bion-Nadal and Kervarec (2012) for this method.

new price space and take a leaf out of Aliprantis and Tourky (2002).<sup>21</sup> We integrate over the set of priors for the addition operation of functionals.<sup>22</sup> In Proposition 1, we list standard properties of coherent price functionals. The last part in this subsection focuses on the consolidation.

### **Linear and $c_{2,\mathcal{P}}$ -continuous prices systems on $L^2(\mathcal{P})$**

Now, we present the basis for the modified concept of viable price systems. The mutually singular uncertainty generates a different space of contingent claims. This gives us a new topological dual space  $L^2(\mathcal{P})^*$ . The discussion of the dual space is only the first step to get a reasonable notion of viability which accounts for the present type of uncertainty.

We introduce, the topological dual of  $(L^2(\mathcal{P}), c_{2,\mathcal{P}})$ . In Appendix B.1, we give a result, which asserts that the dual space consists of special measures:

$$L^2(\mathcal{P})^* = \left\{ \mu = \int \rho dP : P \in \mathcal{P} \text{ and } \rho \in L^2(P) \right\}.$$

This representation delivers an appropriate form of the dual space. The  $\rho$  in the representation matches with the classical Radon-Nykodym density of the Riesz representation when only one prior  $P$  lies in  $\mathcal{P}$ . The space's description allows for an interpretation of a state price density based on some prior  $P \in \mathcal{P}$ . The stronger capacity norm  $c_{2,\mathcal{P}}(\cdot)$  in comparison to the classical single prior  $L^2(P)$ -norm implies a richer dual space, controlled by the set of priors  $\mathcal{P}$ . Moreover, one element in the dual space chooses implicitly a prior  $P \in \mathcal{P}$  and ignores all other priors. This foreshadows the insufficiency of a linear pricing principle under the present uncertainty model, as indicated in the introduction

### **The price space of sublinear expectations**

In this subsection we introduce a set of sublinear functionals defined on  $L^2(\mathcal{P})$ . The singular prior uncertainty of our model induces the appearance of non-linear price systems.<sup>23</sup> Let  $k(\mathcal{P})$  be the convex closure of  $\mathcal{P}$ . We

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<sup>21</sup>In principle there is a third operation which ignores a subset of priors. This ignorance is in some sense redundant, since we can a priori shrink the set of priors, see Appendix B.1.1 for this operation.

<sup>22</sup>This operation is associated to a weighting of priors.

<sup>23</sup>A subcone of the super order dual is considered in Aliprantis and Tourky (2002). They introduces the mathematical lattice theoretic framework and consider the notion of a semi lattice. In Aliprantis, Florenzano, and Tourky (2005), Aliprantis, Tourky, and Yannelis

refer to this space as the *coherent price space* of  $L^2(\mathcal{P})$  generated by linear  $c_{2,\mathcal{P}}$ -continuous functionals:

$$\mathfrak{P}(\mathcal{P}) = \left\{ \Psi : L^2(\mathcal{P}) \rightarrow \mathbb{R} : \Psi(\cdot) = \sup_{P \in A} E^P[Z_P \cdot] \text{ with } A \subset k(\mathcal{P}), Z_P \in L^2(P) \right\}$$

Elements in  $\mathfrak{P}(\mathcal{P})$  are constructed by a set of  $c_{2,\mathcal{P}}$ -continuous linear functionals  $\{\Pi_P : L^2(\mathcal{P}) \rightarrow \mathbb{R}\}_{P \in \mathcal{P}}$ , which are consolidated by a point-wise maximum. We illustrate this in the following example, for details see Appendix A.1.1.

**Example 2** Let  $\{A_n\}_{n \in \mathbb{N}}$  be a partition of  $\mathcal{P}$ . And let  $\mu_n : \mathcal{M}_1(\Omega) \rightarrow \mathbb{R}$  be a positive measure with support  $A_n$  with  $\mu_n(A_n) = 1$ . The resulting prior  $P_n(\cdot) = \int_{A_n} P(\cdot) d\mu_n$  is given by a weighting operation  $\Gamma_{\mu_n}$ . When we apply  $\Gamma$  to the density we get  $Z_n = \int_{A_n} Z_P d\mu_n$ . Then, these new prior density pairs  $(Z_n, P_n)$  can be consolidated by the supremum operation of the expectations  $E^{P_n}[Z_n \cdot]$ .

A full lattice theoretical discussion of our price space  $\mathfrak{P}(\mathcal{P})$  lies beyond the scope of this paper. The following proposition discusses properties and the extreme case of functionals in the price space  $\mathfrak{P}(\mathcal{P})$ .

**Proposition 1** Every functional in  $\mathfrak{P}(\mathcal{P})$  satisfies the following properties:

1. *Sub-additivity:*  $\Psi(X + Y) \leq \Psi(X) + \Psi(Y)$  for all  $X, Y \in L^2(\mathcal{P})$
2. *Positive homogeneity:*  $\Psi(\lambda X) = \lambda \Psi(X)$  for all  $\lambda \geq 0, X \in L^2(\mathcal{P})$
3. *Monotonicity:* If  $X \geq Y$  then  $\Psi(X) \geq \Psi(Y)$  for all  $X, Y \in L^2(\mathcal{P})$
4. *Constant preserving:*  $\Psi(c) = c$  for all  $c \in \mathbb{R}$
5.  *$c_{2,\mathcal{P}}$ -continuity:* Let  $(X_n)_{n \in \mathbb{N}}$  converge in  $c_{2,\mathcal{P}}$  to some  $X$ , then we have  $\lim_n \Psi(X_n) = \Psi(X)$ .

Moreover, for every  $P \in \mathcal{P}$  and positive measure  $\mu$  with  $\mu(\mathcal{P}) \leq 1$ , we have the following inequalities for every  $X \in L^2(\mathcal{P})$

$$E^P[ZX] \leq \sup_{P' \in \mathcal{P}} E^{P'}[Z_{P'}X] \geq E^{P_\mu}[Z_\mu X], \text{ where } P_\mu(\cdot) = \int_{\mathcal{P}} P(\cdot) d\mu(P).$$

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(2001) general equilibrium models with superlinear price are considered in order to discuss a non-linear theory of value. These cases relate nonlinearity in terms of personalized prices, which may be applied to a differential information economy.



Below, we introduce the consolidation operation  $\Gamma$  for the prior depending price systems.  $\Gamma(\mathcal{P})$  refers to the set of priors in  $\mathcal{P}$  which are relevant. In Example 2, we observe  $\Gamma(\mathcal{P}) = \mathcal{P}$ .

### Marketed spaces and scenario-based price systems

In the spirit of Aliprantis, Florenzano, and Tourky (2005) our commodity-price duality is given by the following pairing  $\langle (L^2(\mathcal{P}), c_{2,\mathcal{P}}), \mathfrak{P}(\mathcal{P}) \rangle$ .

For the single prior framework, viability and the extension of the price system are associated with each other. This structure allows only linear prices and corresponds in our framework to consolidation via the Dirac measure  $\delta_{\{P\}}$  for some  $P \in \mathcal{P}$ . In this case we have  $\Gamma(\mathcal{P}) = \{P\}$ .

We begin by introducing the marketed  $L^2(P)$ -closed subspaces  $M_P \subset L^2(P)$ ,  $P \in \mathcal{P}$ . The underlying idea is that any claim in  $M_P$  can be achieved, whenever  $P \in \mathcal{P}$  is the true probability measure. This input data resembles a partial equilibrium, depending on the prior under consideration.<sup>24</sup> Claims in the marketed space  $M_P$  can be bought and sold, whenever the related prior governs the economy. We illustrate this in the following example.

**Example 3** *Suppose the set of priors is constructed by the procedure in Example 1. Let the marketed space be generated by the quadratic variation of representing an uncertain asset with payoff, at time  $T$ ,  $\langle B \rangle_T$  and a riskless asset 1. We have by construction  $\langle B \rangle_T = \int_0^T \alpha_s ds$   $P^\alpha$ -a.s., the marketed space under  $P^\alpha$  given by*

$$M_{P^\alpha} = \left\{ X \in L^2(P^\alpha) : X = a \cdot \int_0^T \alpha_s ds + b \cdot 1 \text{ } P^\alpha\text{-a.s.}, a, b \in \mathbb{R} \right\}.$$

*But  $\langle B \rangle$  coincides with the  $P$ -quadratic variation under every martingale law  $P \in \mathcal{P}$   $P$ -a.s. Therefore a different  $\hat{\alpha}$  builds a different marketed space  $M_{P^{\hat{\alpha}}}$ . Suppose  $\alpha = \hat{\alpha}$   $P_0$ -a.s. on  $[0, s]$  for some  $s \in ]0, T]$  then we have  $M_{P^\alpha} \cap M_{P^{\hat{\alpha}}}$  consists also of non trivial claims. Note, that  $P^\alpha$  and  $P^{\hat{\alpha}}$  are neither equivalent nor mutually singular.*<sup>25</sup>

<sup>24</sup>One may think that a countable set of scenarios could be sufficient. As we mention in Appendix B 2, the norm can be represented via different countable dense subsets of priors. However, for the marketed space we have a direct prior dependency of all elements in  $\mathcal{P}$ . This implies that different choices of countable and dense scenarios can deliver different price systems, see Definition 1 below.

<sup>25</sup>The event  $\{\omega : \langle B \rangle_r(\omega) = \int_0^r \alpha_t(\omega) dt, r \in [0, s]\}$  has for both priors the same positive mass but the priors restricted to the complement are mutually singular. We refer to Example 3.7 in Epstein and Ji (2012) for a similar example.

We fix linear functionals  $\pi_P$  on  $M_P$ . It is possible that the two components  $\pi_{P_1}, \pi_{P_2} \in \{\pi_P\}_{P \in \mathcal{P}}$  have a common domain, i.e.  $M_{P_1} \cap M_{P_2} \neq \emptyset$ . In this case it is possible to observe different evaluations between different priors, i.e.  $\pi_{P_1}(X) \neq \pi_{P_2}(X)$  with  $X \in M_{P_1} \cap M_{P_2}$ . Moreover, the set  $\{\pi_P\}_{P \in \mathcal{P}}$  of linear scenario-based price functionals inherit the uncertainty of the model. In the single prior setting incompleteness means  $M_P \neq L^2(P)$ . Note that  $\Omega$  is separable by assumption, hence  $L^2(P) = L^2(\Omega, \mathcal{B}(\Omega), P)$  is a separable Hilbert space<sup>26</sup> for each  $P \in \mathcal{P}$  and admits a countable orthonormal basis.  $M_P \otimes M_{P'}$  refers to the linear hull of the involved basis elements in  $M_P$  and  $M_{P'}$ .

**Definition 1** Fix  $L^2(P)$ -closed subspaces  $\{M_P\}_{P \in \mathcal{P}}$  with  $M_P \subset L^2(P)$  and a set  $\{\pi_P\}_{P \in \mathcal{P}}$  of linear scenario-based price functionals  $\pi_P : M_P \rightarrow \mathbb{R}$ . Let the  $\Gamma(\mathcal{P})$ -marketed space be given by  $c_{2,\mathcal{P}}$ -closure of  $\Gamma$ -relevant pasted marketed spaces

$$\mathbb{M}(\Gamma) = \overline{\otimes_{P \in \Gamma(\mathcal{P})} M_P \cap L^2(\mathcal{P})}^{c_{2,\mathcal{P}}}.$$

A price system for  $(\{M_P, \pi_P\}_{P \in \mathcal{P}}, \Gamma)$  is a functional  $\psi : \mathbb{M}(\Gamma) \rightarrow \mathbb{R}$ , where the consolidation operator  $\Gamma$  maps  $\{\pi_P : M_P \rightarrow \mathbb{R}\}_{P \in \mathcal{P}}$  to  $\psi$ .

The  $\Gamma(\mathcal{P})$ -marketed space refers to the space of all *possible* marketed claims in the domain of the consolidation operator  $\Gamma$ , which is a mixture of convex combination and pointwise supremum. For each  $P$ , the related marketed space  $M_P$  consist of contingent claims which can be achieved frictionless, when  $P$  is the true law. We have a set of different price systems  $\{\pi_P : M_P \rightarrow \mathbb{R}\}_{P \in \mathcal{P}}$ . If we want to establish a consolidation of the scenarios in a normative sense we need an additional ingredient in the market, namely  $\Gamma$ . This consolidation determines the set of relevant priors and therefore influences the whole marketed space.

## 2.3 Preferences and the economy

Having discussed the commodity price dual and the role of the consolidation of linear price systems, we introduce agents which are characterized by their

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<sup>26</sup>In terms of Example 2,  $P_0$  is the Wiener measure. In this situation,  $L^2(P_0)$  can be decomposed via the Wiener chaos expansion. The same can be done for the canonical process  $X^\alpha$  related to some  $P^\alpha$ . So we can generate an orthonormal basis for each  $L^2(P^\alpha)$ , with  $\alpha \in \mathcal{D}$ .

preference of trades on  $\mathbb{R} \times L^2(\mathcal{P})$ . There is a single consumption good, a numeraire, which agents will consume at  $t = 0, T$ . Thus, bundles  $(x, X)$  are elements in  $\mathbb{R} \times L^2(\mathcal{P})$ , which are the units at time zero and time  $T$  with uncertain outcome. We call the set of rational preference relations  $\succsim$  on  $\mathbb{R} \times L^2(\mathcal{P})$ ,  $\mathbb{A}(\mathcal{P})$ , which satisfy convexity, strict monotonicity, and  $c_{2,\mathcal{P}}$ -continuity.<sup>27</sup> Let

$$B(x, X, \psi, \mathbb{M}(\Gamma)) = \{(y, Y) \in \mathbb{R} \times \mathbb{M}(\Gamma) : y + \psi(Y) \leq x + \psi(X)\}.$$

denote the *budget set* for a price functional  $\psi : \mathbb{M}(\Gamma) \rightarrow \mathbb{R}$ . We are ready to define the appropriate notion of viability.

**Definition 2** *A price system is scenario based viable if there exists a preference relation  $\succsim \in \mathbb{A}(\mathcal{P})$  and a consumption bundle  $(\hat{x}, \hat{X}) \in \mathbb{R} \times \mathbb{M}(\Gamma)$  with*

$$\begin{aligned} &(\hat{x}, \hat{X}) \in B(0, 0, \psi, \mathbb{M}(\Gamma)), \\ \text{and} \quad &(\hat{x}, \hat{X}) \succsim (x, X), \text{ for all } (x, X) \in B(0, 0, \psi, \mathbb{M}(\Gamma)). \end{aligned}$$

The conditions are necessary and sufficient as a classical model for an economic equilibrium, when we find a preference relation. Now, we relate the viability of  $(\{M_P, \pi_P\}_{P \in \mathcal{P}}, \Gamma)$  with price functionals in  $\mathfrak{P}(\mathcal{P})$  defined on the whole space  $L^2(\mathcal{P})$ . We introduce the notion of strictly positive functionals in  $\mathfrak{P}(\mathcal{P})$ , namely  $\mathfrak{P}(\mathcal{P})_{++}$ . Such a functional  $\Psi : L^2(\mathcal{P}) \rightarrow \mathbb{R}$  is called *strictly positive* if we have  $\Psi(X) > 0$  for every  $X \in L^2(\mathcal{P})_+$  with  $P(X > 0) > 0$  for some  $P \in \mathcal{P}$ .

**Theorem 1** *A price system  $(\{M_P, \pi_P\}_{P \in \mathcal{P}}, \Gamma)$  is scenario based viable if and only if there is an extension of  $\psi : \mathbb{M}(\Gamma) \rightarrow \mathbb{R}$  to all of  $L^2(\mathcal{P})$  in  $\mathfrak{P}(\mathcal{P})_{++}$ .*

This characterization of scenario-based viability takes scenario-based marketed spaces  $\{M_P\}_{P \in \mathcal{P}}$  as given. Moreover, the consolidation operator  $\Gamma$  is a given characteristic of the market. With this in mind one should think that in a general equilibrium system the locally given prices  $\{\pi_P\}_{P \in \mathcal{P}}$  should be part of it. The extension we perceived can be seen as a regulated and coherent price system for every claim in  $L^2(\mathcal{P})$ .

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<sup>27</sup> The class of variational preferences, axiomatized in Maccheroni, Marinacci, and Rustichini (2006), may represent such preferences under mild assumption on the utility index  $u$  and the penalty functional  $c : \mathcal{P} \rightarrow [0, \infty]$ . For instance, the domain of  $c$  must be a subset of  $\mathcal{P}$ .

The proof of the theorem is based on the nonlinear separation of the convex "better off" set, and the budget set. In principle,  $\Gamma$  builds a convex functional such that one of the convex sets lies in the epigraph of  $\Psi$  and the other does not.<sup>28</sup>

In comparison to the single prior case, the degree of incompleteness depends on the prior under consideration.<sup>29</sup> As described in Example 2, this is a natural situation. As such, prior depending prices  $\pi_P$  are also plausible. The expected payoff as a pricing principle depends on the prior under consideration, as well. This concept of scenario-based prices accounts for every  $\Gamma$ -relevant price system simultaneously. We have two operations which constitute the distillation of uncertainty. This consolidation is a characterization of the Walrasian auctioneer, in which case diversification should be encouraged. But this is the sublinearity property.

**Remark 1** *One may ask which  $\Gamma$  is appropriate. Such a question is related to the concept of mechanism design. The market planner can choose a consolidation, which influences the indirect utility of a reported preference relation. However, the full discussion of issues lies beyond the scope of this paper.*

### 3 Security markets and $\mathcal{E}$ -martingales

We extend the primitives with trading dates and trading strategies. We consider a time interval where the market consists of a riskless security and a security with uncertain volatility leading to the set of mutually singular priors. We then introduce a financial market consistent with the volatility, and discuss the modified notion of arbitrage and the equivalent martingale measure. In Section 3.3, Theorem 2 associates scenario-based viability with EsMM sets.

In the last section we consider our model in the  $G$ -framework. Here, the uncertain security process is driven by a  $G$ -Itô process, which shows that the concept of martingale measure sets is not an empty one.

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<sup>28</sup>One may ask if a separation prior by prior is possible as well. This is in general not possible. We illustrate this as follows. A prior depending pricing implies that  $\psi$  must be restricted to  $M_P$ , for each  $P \in \mathcal{P}$ , which we call  $\psi_P$ .  $\pi_P$  as a pricing is not sufficient since,  $M_P \cap M_{\hat{P}} \neq \emptyset$  is possible for some  $P \neq \hat{P}$ , see footnote 22. But an extension of  $\psi_P$  to all of  $L^2(\mathcal{P})$  may now depends on the prior.

<sup>29</sup>This can be seen as an uncertainty in the given partial equilibrium.

**Background: risk neutral asset pricing with one prior**

In order to introduce dynamics and trading dates, we fix a time interval  $[0, T]$  and a filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  on  $(\Omega, \mathcal{F}, P)$ . Fix an  $\mathbb{F}$ -adapted asset price  $\{S_t\}_{t \in [0, T]} = S \in L^2(P \otimes dt)$  and a riskless bond  $S^0 \equiv 1$ . We next review some terminology.

The portfolio process of a strategy  $\eta$  is called  $X(\eta)$ . Simple self-financing strategies are piecewise constant  $\mathbb{F}$ -adapted processes  $\eta$  such that  $dX(\eta) = \eta dS$ , which we call  $\mathcal{A}(P)$ . A  $P$ -arbitrage in  $\mathcal{A}(P)$  is a strategy (with zero initial capital) such that  $X(\eta)_T \geq 0$  and  $P(X(\eta)_T > 0) > 0$ .

A claim is marketed, i.e.  $X \in M$ , if there is a  $\eta \in \mathcal{A}(P)$  such that  $X = \eta_T S_T$ , then we have the (law of one) price  $\pi(X) = \eta_0 S_0$ . An equivalent martingale measure  $Q$  must satisfy that  $S$  is a  $Q$ -martingale and  $Q = \rho P$ , where  $\rho \in L^2(P)$  is a Radon Nykodym-Density with respect to  $P$ . Theorem 2 of Harrison and Kreps (1979) states the following

Under no  $P$ -arbitrage, there is a one to one correspondence between the continuous linear and strictly positive extension of  $\pi : M \rightarrow \mathbb{R}$  to  $L^2(P)$  and the equivalent martingale measure. The relation is given by  $Q(B) = \Pi(1_B)$  and  $\Pi(X) = E^Q[X]$ , where  $B \in \mathcal{F}_T$  and  $X \in L^2(P)$ .

This result can be seen as a preliminary version of the first fundamental theorem of asset pricing.

**3.1 The financial market with uncertain volatility**

We specify the mathematical framework and the modified notions, such as arbitrage. Our probability model is related to the existence of a canonical process with a modified absolutely continuous quadratic variation. We begin by modeling the market and considering the concrete construction for the set of priors. Following, we review the martingale notion for conditional sublinear expectation.

**3.1.1 The dynamics and martingales under sublinear expectation**

The principle idea is to transfer the result of Section 2 into a dynamic setup. The specification in Example 1 of Section 2.1 serves as our uncertainty model. One can directly observe in which sense the quadratic variation creates uncertain volatility from the construction. We introduce the sublinear expectation

$\mathcal{E} : L^2(\mathcal{P}) \rightarrow \mathbb{R}$  given by the supremum of expectations of  $\mathcal{P} = \{P^\alpha : \alpha \in \mathcal{D}\}$ . Moreover, we assume that the set of priors is stable under pasting. For details, we refer to Appendix A.2.1.

As we aim to equip the financial market with a dynamic structure of conditional sub-linear expectation, we introduce the information structure of the financial market as given by an augmented filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ , (see Appendix A.2.1 for details). The setting is based on dynamic sublinear expectation terminology as instantiated by Nutz and Soner (2010).

We give a generalization of Peng's  $G$ -expectation as an example, satisfying the weak compactness of  $\mathcal{P}$  when the sublinear expectation is represented in terms of a supremum of linear expectations. In Section 3.3 and in Appendix B.3, we consider the normal  $G$ -expectation in more details. That said, a possible association of results in Section 2 depends heavily on the weak compactness of the generated set of priors  $\mathcal{P}$ .

**Example 4** *Suppose a trader is confronted with a pool of models describing volatility, as described in Heston (1993). After a statistical analysis of the data two models remain plausible  $P^\alpha$  and  $P^{\hat{\alpha}}$ . Nevertheless, the implications for the trading decision deviate considerably. Even the asset span on its own depends on each scenario, (see Example 3). A mixture of both models does not change this uncertain situation at all. In order to deal with the possibilistic issue let us define the universal extreme cases  $\underline{\sigma}_t = \inf(\alpha_t, \hat{\alpha}_t)$  and  $\bar{\sigma}_t = \sup(\alpha_t, \hat{\alpha}_t)$ . When thinking about reasonable uncertainty management, no scenario should be ignored. The uncertainty model which accounts for all cases between  $\underline{\sigma}$  and  $\bar{\sigma}$  is given by*

$$\mathcal{P} = \{P^\alpha : \alpha_t \in [\underline{\sigma}_t, \bar{\sigma}_t] \text{ for every } t \in [0, T] P_0 \otimes dt \text{ a.e.}\}.$$

*The construction of a sublinear conditional expectation is achieved in Nutz (2010). Here the deterministic bounds of the  $G$ -expectation are replaced by path dependent bounds.*<sup>30</sup>

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<sup>30</sup> This framework is in principle included in Epstein and Ji (2012). In this setting drift and volatility uncertainty are considered simultaneously. Drift uncertainty or  $\kappa$ -ambiguity are well known terms in financial economics. A coherent theory, known as  $g$ -expectation, is available under Brownian information.

The approach is formulated via a correspondence which controls the feasible of Girsanov kernels and the derivative of the quadratic variation at once. For a model concerning drift uncertainty we refer to Section 2.2 of Chen and Epstein (2002).

We introduce an appropriate concept for the dynamics of the continuous time multiple prior uncertainty model. The associated objectives are trading dates, the information structure and the price process (as the carrier of the uncertainty). In order to introduce the price process  $S = \{S_t\}_{t \in [0, T]}$  of an uncertain and long lived security, we must introduce further primitives. Define the time depending set of priors

$$\mathcal{P}(t, P)^o = \{P' \in \mathcal{P} : P = P' \text{ on } \mathcal{F}_t^o\}.$$

This set of priors consists of all extensions  $P : \mathcal{F}_t^o \rightarrow [0, 1]$  from  $\mathcal{F}_t^o$  to  $\mathcal{F} = \mathcal{B}(\Omega)$  in  $\mathcal{P}$ . This is the set of all probability measures in  $\mathcal{P}$  defined on  $\mathcal{F}$  that agree with  $P$  in the events up to time  $t$ . Fix a contingent claim  $X \in L^2(\mathcal{P})$ . In Nutz and Soner (2010), the unique existence of a sublinear expectation  $\{\mathcal{E}(X)_t\}_{t \in [0, T]}$  is proved by the following construction<sup>31</sup>:

$$\mathcal{E}(X)_t^o = \text{ess sup}_{Q' \in \mathcal{P}(t, P)^o} E^{Q'}[X | \mathcal{F}_t] \quad P\text{-a.s.}, \quad \lim_{r \downarrow t} \mathcal{E}(X)_r^o = \mathcal{E}(X)_t.$$

With the sublinearity of the dynamic sublinear conditional expectation we can define a martingale similarly to the single prior setting.<sup>32</sup>

The nonlinearity implies that if a process  $X = \{X_t\}_{t \in [0, T]}$  is a martingale under  $\{\mathcal{E}(\cdot)_t\}_{t \in [0, T]}$  then  $-X$  is not necessarily a martingale. Despite this being the case, we call the process a *symmetric* martingale. In the next subsection we discuss their relationship to asset prices under a modified sublinear expectation.

### 3.1.2 The primitives of the financial market and arbitrage

For the sake of simplicity, we assume that the riskless asset is  $S_t^0 = 1$ , for every  $t \in [0, T]$ , i.e. the interest rate is zero. We call the related abstract financial market  $\mathcal{M}(1, S)$  on the filtered space uncertainty space  $(\Omega, \mathcal{F}, \mathcal{P}; \mathbb{F})$ , whenever the process  $S = \{S_t\}_{t \in [0, T]}$  satisfies

$$S_t \in L^2(\mathcal{P}) \text{ for every } t \in [0, T] \text{ and } \mathbb{F}\text{-adaptedness.}$$

A *simple trading strategy*<sup>33</sup> is a  $\mathbb{F}^o$ -adapted stochastic process  $\{\eta_t\}_{t \in [0, T]}$  in  $L^2(\mathcal{P})$  when there is a finite sequence of dates  $0 < t_0 \leq \dots \leq t_N = T$

<sup>31</sup> Representations of such martingales can be formulated via a 2BSDE. This concept is introduced for example in Soner, Touzi, and Zhang (2011b).

<sup>32</sup>For the multiple prior case with equivalent priors we refer to Riedel (2009).

<sup>33</sup>As mentioned in Harrison and Pliska (1981) simple strategies rule out the introduction of doubling strategies and hence the notion of admissibility.

such that  $\eta = (\eta^{(0)}, \eta^{(1)})$  can be written with  $\eta^i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathcal{P})$  as  $\eta_t = \sum_{i=0}^{N-1} 1_{[t_{i+1}, t_i[}(t) \eta^i$ .

The fraction invested in the riskless asset is denoted by  $\eta_t^{(0)}$ ,  $t \in [0, T]$ . A trading strategy is *self-financing* if  $\eta_{t_{n-1}}^{(0)} S_{t_n}^0 + \eta_{t_{n-1}}^{(1)} S_{t_n} = \eta_{t_n}^{(0)} S_{t_n}^0 + \eta_{t_n}^{(1)} S_{t_n}$  for every  $n = 1, \dots, N$ . The value of the portfolio  $X(\eta)$  takes values in  $L^2(\mathcal{P})$  for every  $t \in [0, T]$ .

The set of simple self-financing trading strategies is denoted by  $\mathcal{A}$ . This financial market  $\mathcal{M}(1, S)$  with trading strategies in  $\mathcal{A}$  is called  $\mathcal{M}(1, S, \mathcal{A})$ . It is well known that a necessary condition for equilibrium is the absence of arbitrage. Therefore, with regard to the equilibrium result of the last section, we introduce arbitrage in the financial market of securities. The modeled uncertainty of the financial market forces us to consider a weaker notion of arbitrage.

Let an event be  $\mathcal{P}$ -quasi surely true if it holds  $P$ -almost surely for each  $P \in \mathcal{P}$ .

**Definition 3** *Let  $\mathcal{R} \subset \mathcal{P}$ . We say there is an  $\mathcal{R}$ -arbitrage opportunity in  $\mathcal{M}(1, S, \mathcal{A})$  if there exist an admissible pair  $\eta \in \mathcal{A}$  such that  $\eta_0 S_0 \leq 0$ ,*

$$\begin{aligned} \eta_T S_T &\geq 0 \quad \mathcal{R} - \text{quasi surely,} \quad \text{and} \\ P(\eta_T S_T > 0) &> 0 \quad \text{for at least one } P \in \mathcal{R}. \end{aligned}$$

The choice of the definition is based on the following observation. This arbitrage strategy is riskless for each  $P \in \mathcal{R}$  and if the prior  $P$  constitutes the market one would gain a profit from with positive probability. With this in mind, our  $\mathcal{P}$ -arbitrage notion can be seen as a weak arbitrage of second order.

We say that a claim  $X^m \in L^2(P)$  is *marketed* in  $\mathcal{M}(1, S, \mathcal{A})$  at time zero under  $P \in \mathcal{P}$  if there is a  $\eta \in \mathcal{A}$  such that  $X^m = \eta_T S_T$   $P$ -almost surely. In this case we say  $\eta$  hedges  $X^m$  and lies in  $M_P$ .  $\eta_0 S_0 = \pi_P(X^m)$  is the price of  $X^m$  in  $\mathcal{M}(1, S, \mathcal{A})$  under  $P \in \mathcal{P}$ .

With Example 3 in mind, fix the marketed spaces  $M_P \subset L^2(P)$ ,  $P \in \mathcal{P}$ . The price of a marketed claim under the prior  $P$  should to be well defined. Let  $\eta, \eta' \in \mathcal{A}(P)$  generating the same claim  $X^m \in M_P$ , i.e.  $\eta_T S_T = \eta'_T S_T$   $P$ -a.s., where  $\mathcal{A}(P)$  refers to self-financing portfolios under  $P$ . We have  $\eta_0 S_0 = \eta'_0 S_0 = \pi_P(X^m)$  under no  $P$ -arbitrage. Note, that this may not be true under no  $\hat{P}$ -arbitrage, with  $P \neq \hat{P} \in \mathcal{P}$ . This is related to the law of one price under a fixed prior. Now, similarly to the single prior case, we define viability



in a financial market. We say that a financial market  $\mathcal{M}(1, S, \mathcal{A})$  is *viable* if it is  $\Gamma(\mathcal{P})$ -arbitrage free and the associated price system  $(\{M_P, \pi_P\}_{P \in \mathcal{P}}, \Gamma)$  is scenario-based viable.

### 3.2 Extensions of price systems and EsMM sets

In Section 2 we introduced the price space of sublinear price functionals generated by a set of linear  $c_{2, \mathcal{P}}$ -continuous functionals. The extension of the price functional is strongly related to the involved linear functionals which constitutes the price systems locally. In this fashion, we introduce a modified notion of fair pricing. In essence, we associate a risk neutral prior to each local and linear extension of a price system. Here, the term local refers to a fixed prior, therefore no uncertainty is present.

In our uncertainty model, the price of a claim equals the (discounted) value under a specific sublinear expectation. Exploration of available information, when multiple priors are present, changes the view of a rational expectation. In economic terms, the notion of symmetric martingales eliminates ambiguity in the valuation. It seems appropriate to introduce a rational pricing principle of sublinear expectations.<sup>34</sup> This motivates the following definition.

**Definition 4** *A set of probability measures  $\mathcal{Q}$  on  $(\Omega, \mathcal{B}(\Omega))$  is called an equivalent symmetric martingale measure set (EsMM set) if the following two conditions hold:*

1. *For every  $Q \in \mathcal{Q}$  there is a  $P \in k(\mathcal{P})$  such that  $P$  and  $Q$  are equivalent to each other, such that  $\frac{dP}{dQ} \in L^2(P)$ .*
2. *The risky asset  $S$  is a symmetric  $\mathcal{E}^{\mathcal{Q}}$ -martingale, where  $\mathcal{E}^{\mathcal{Q}}$  is a sublinear expectation given by the supremum of expectations over  $\mathcal{Q}$ .*

The first condition formulates a direct relation between an elements  $Q$  in the EsMM set  $\mathcal{Q}$  and the primitive priors  $P \in \mathcal{P}$ . The square integrability is a technical condition that guarantees the association to the equilibrium theory of Section 2. The second is the translated martingale condition.<sup>35</sup> The

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<sup>34</sup>The mutually singular priors generate a different view for the pricing of a contingent claim.

<sup>35</sup>It seems possible to proof that if the price process is not a symmetric martingale but a martingale then arbitrage is possible. However, such considerations lies not in scope of this thesis.

rational expectation hypothesis and the idea of a fair gamble should establish maximal neutrality. Under the new sublinear expectation the asset price and hence the portfolio process are symmetric martingales. This implies, as discussed in the introduction, that the value of the claim does not depend on the prior, i.e. the valuation is mean unambiguous.

The case of only one prior is related to the well-known risk-neutral evaluation principle. Here, this principle needs a new requirement due to the more complex uncertainty model. In this sense the symmetry condition is responsible for the *uncertainty neutrality*.

**Remark 2** *Note that in the case of a single prior framework, i.e.  $\mathcal{P} = \{P\}$ , the notion of EsMM sets is reduced to accommodate equivalent martingale measures. In this regard we can think of canonical generalization.*

*On the other hand, classical equivalent martingale measures (EMM) and a linear price theory are still present. Every single valued EsMM set  $\mathcal{Q} = \{Z_P \cdot P\}$  can be seen as an EMM under  $P \in \mathcal{P}$ . Here, the consolidation is given by  $\Gamma = \delta_P$  and we have  $\Gamma(\mathcal{P}) = \{P\}$ .*

The following result motivates the discussion involving maximal risk neutrality and symmetry condition. The one to one mapping of Theorem 2 and hence the choice of the price space are appropriate. In this spirit we show that  $\mathcal{R}$ -arbitrage in  $\mathcal{A}$  with  $\Gamma(\mathcal{P}) = \mathcal{R}$  is inconsistent with an economic equilibrium for agents in  $\mathbb{A}(\mathcal{P})$ . We fix an associated price system using procedure described at the end of Subsection 3.1.

**Theorem 2** *Suppose the financial market model  $\mathcal{M}(1, S, \mathcal{A})$  does not allow any  $\mathcal{P}$ -arbitrage opportunity.*

*Then there is a bijection between EsMM-sets and sublinear  $\Psi : L^2(\mathcal{P}) \rightarrow \mathbb{R}$  in  $\mathfrak{P}(\mathcal{P})$  such that  $\Psi|_{\mathbb{M}(\Gamma)} = \psi$  and  $\Pi_P|_{M_P} = \pi_P$ ,  $P \in \Gamma(\mathcal{P})$ . The relationship is given by*

$$\Psi(X) = \sup_{Q \in \mathcal{R}^*} E^Q[X] = \mathcal{E}^{\mathcal{R}^*}(X),$$

*where  $\mathcal{R} \subset k(\mathcal{P})$  and  $\mathcal{R}^* = \{Z_P \cdot P : P \in \mathcal{R}, Z_P P = \Pi_P\}$  is an EsMM-set.*

Let  $\mathcal{R} \subset \mathcal{P}$  and  $\mathfrak{M}(\mathcal{R})$  be the set of all EsMM-sets  $\mathcal{Q}$  such that the related consolidation  $\Gamma$  satisfies  $\Gamma(\mathcal{P}) = \mathcal{R}$ . Theorem 2 can be seen as the formulation of a one-to-one mapping between

$$\mathfrak{P}(\mathcal{P}) \text{ and } \bigcup_{\mathcal{R} \subset k(\mathcal{P})} \mathfrak{M}(\mathcal{R}).$$

There is a hierarchy of sublinear expectation martingales, related to the chosen consolidation operator  $\Gamma$  and the EsMM-set. We illustrate the relationship between  $\Gamma$  and an EsMM-set in the following example.

**Example 5** *We illustrate the relationship between EsMM-sets and the consolidation operation  $\Gamma$  when a price system is given. For the sake of simplicity, let us assume that  $\{P_1, P_2, P_3, P_4\} = \mathcal{P}$ . Starting with the sublinear price system, we have three price functionals  $\pi_1, \pi_2, \pi_3$  and the consolidation operator  $\Gamma$ . Let us assume that  $\Gamma = (+, \wedge)$  and  $\lambda \in (0, 1)$ . This gives us  $\lambda\pi_1 + (1 - \lambda)\pi_2 = \pi^\lambda$  and  $\Gamma(\pi_1, \pi_2, \pi_3) = \pi^\lambda \wedge \pi_3$ . The resulting EsMM-set is given by  $\mathcal{R}^* = \{Z^\lambda \cdot P^\lambda, Z_3 P_3\} \in \mathfrak{M}(\mathcal{P} \setminus \{P_4\})$ , where  $P^\lambda = \lambda P_1 + (1 - \lambda)P_2$  and  $Z^\lambda = \lambda Z_1 + (1 - \lambda)Z_2$ .*

We close this consideration with some results analogous to those of the single prior setting where we combine Theorem 2 and Theorem 1.

**Corollary 1** *Let  $\mathcal{R} \subset \mathcal{P}$ , such that  $\mathcal{R} = \Gamma(\mathcal{P})$ .*

1. *Scenario-based viability of  $\mathcal{M}(1, S, \mathcal{A})$  is equivalent to the existence of an EsMM-set.*
2. *Market completeness, i.e  $M_P = L^2(P)$  for each  $P \in \mathcal{R}$ , is equivalent to the existence of exactly one EsMM-set in  $\mathfrak{M}(\mathcal{R})$ .*
3. *If  $\mathfrak{M}(\mathcal{R})$  is nonempty, then there exists no  $\mathcal{R}$ -arbitrage.*
4. *If there is a strategy  $\eta \in \mathcal{A}$  with  $\eta_0 S_0, \eta_T S_T \geq 0$   $\mathcal{R}$ -q.s. and  $\mathcal{E}^{\mathcal{R}}(\eta_T S_T) > 0$  then there is an  $\mathcal{R}$ -arbitrage opportunity.*

The result does not depend on the preference of the agent. The expected return under the sublinear expectation  $\mathcal{E}^{\mathcal{Q}}$  equals the riskless asset. Hence, the value of a claim can be considered as the future value in the uncertainty-free world.<sup>36</sup>

### 3.3 A special case: $G$ -expectation

Now, we select a stronger calculus to model the asset prices as a stochastic differential equation driven by a  $G$ -Brownian motion.<sup>37</sup> In this situation the

<sup>36</sup>However, the sublinear expectation depends on  $\Gamma$ .

<sup>37</sup>An illustration of the concept in a discrete time framework is achievable, via an application of the results in Cohen, Ji, and Peng (2011).

volatility of the process concentrates the uncertainty in terms of the derivative of the quadratic variation. The quadratic variation of a  $G$ -Brownian motion creates uncertain volatility.

Again, we review the related result of the single prior framework.

### **Background: Itô processes in the single prior framework**

Now, we specify the asset price in terms of an Itô process

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t, \quad S_0 = 1,$$

driven by a Brownian motion  $B = \{B_t\}_{t \in [0, T]}$  on the given filtered probability space,  $\mu, \sigma$  are processes such that  $S$  is a well defined processes in  $\mathbb{R}_+$ . The filtration is generated by  $B$ . The interest rate is  $r = 0$ . Let  $E^\theta$  be the exponential martingale, given by  $dE_t^\theta = E_t^\theta \theta_t dB_t$ ,  $E_0^\theta = 1$ , with a Novikov consistent kernel  $\theta$  we can apply Girsanov theorem. The following result is from Harrison and Kreps (1979):

The set of equivalent martingale measures is not empty if and only if  $\rho = E_T^\theta \in L^2(P)$ ,  $\theta \in L^2(P \otimes dt)$  and  $S^* = \int \sigma dB$  is a  $P$ -martingale.

$\rho$  can be interpreted as a state price density. The associated market price of risk  $\theta_t = \frac{\mu_t - r}{\sigma_t}$  is the Girsanov or pricing kernel of the state price density.

#### **3.3.1 Security prices as $G$ -Itô processes and sublinear valuation**

Our sublinear expectation is given by the  $G$ -expectation  $E_G : L^2(\mathcal{P}) \rightarrow \mathbb{R}$ .<sup>38</sup> The construction of  $E_G$  on  $L^2(\mathcal{P})$  can be achieved when the sublinear expectation space  $(C([0, T]; \mathbb{R}), \mathcal{C}_b(C([0, T]; \mathbb{R})), E_G)$  as given, (see Appendix B.3 and references therein for more precise treatment).

The Girsanov theorem is precisely what is needed to verify the symmetric  $G$ -martingales property of the price processes  $S$  under some sublinear expectation given by an EsMM-set.

This uncertainty model enables us to apply the necessary stochastic calculus. As such, we model the financial market in the  $G$ -expectation setting, introduced in Peng (2007b) and Peng (2010). Central results, such as a martingale representation, a Girsanov type result, and a well behaved underlying

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<sup>38</sup>It is shown in Theorem 52 of Denis, Hu, and Peng (2011), that this sublinear expectation can be represented by a weakly compact set, when the domain is in  $L_G^1(\Omega)$ .

topology are desired for the foundational grounding of asset pricing.

We select the next rational base, namely an interval  $[\sigma_1, \sigma_2] \subset \mathbb{R}_{++}$ , instead of a constant volatility  $\sigma$ , (see Example 4). The bounds of the interval is a model improving substitution with respect to constant  $\sigma \in \mathbb{R}_{++}$ . We introduce an asset price process driven by a  $G$ -Brownian motion  $\{B_t^G\}_{t \in [0, T]}$ . In Appendix B.3 we present a small primer of the applied results.

The asset price is driven by the following  $G$ -stochastic differential equation

$$dS_t = \mu(t, S_t)d\langle B^G \rangle_t + V(t, S_t)dB_t^G, \quad t \in [0, T], \quad S_0 = 1.$$

Let  $\mu : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $V : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}_+$  be processes such that a unique solution exists.<sup>39</sup> Moreover, let  $V(\cdot, x)$  be a strictly positive process for each  $x \in \mathbb{R}_+$ . The riskless asset has interest rate zero.

The second condition of Definition 4 in Subsection 3.2 highlights how a Girsanov transformation should relate to a symmetric  $G$ -martingale and thus guarantee the non emptiness of the concept. For this purpose we define the related sublinear expectation generated by an EsMM-set,  $\mathcal{Q} = \{ZP : P \in \mathcal{P}\}$  and  $X \in L^2(\mathcal{P})$ :

$$\sup_{Q \in \mathcal{Q}} E^Q[X] = \mathcal{E}^{\mathcal{Q}}(X) = E_G(ZX)$$

Theorem 3 below justifies the choice of this shifted sublinear expectation when the asset price is restrained to a symmetric martingale for an uncertainty-neutral expectation.

Let us consider an exponential martingale  $\mathbf{E}$  under the  $G$ -expectation, with a pricing kernel  $\theta \in M_G^2(0, T)$ , defined in Appendix B.3:

$$d\mathbf{E}_t^\theta = \mathbf{E}_t^\theta \theta(t, S_t)dB_t^G, \quad \mathbf{E}_0^\theta = 1$$

By application of the results in Appendix B.3, we can write  $\mathbf{E}^\theta$  in explicit form

$$\mathbf{E}_t^\theta = \exp \left( -\frac{1}{2} \int_0^t \theta(s, S_s)^2 d\langle B^G \rangle_s - \int_0^t \theta(s, S_s) dB_s^G \right), \quad t \in [0, T].$$

Let the pricing kernel solve  $V(t, S_t)\theta(t, S_t) = \mu(t, S_t)$  for every  $t \in [0, T]$   $\mathcal{P}$ -quasi surely. Before we formulate the last result we define

$$S_t^* = S_0^* + \int_0^t V(s, S_s^*)dB_s^G, \quad t \in [0, T]$$

such that a unique solution on  $(\Omega, \mathcal{H}, E_G)$  exists, see Peng (2010).

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<sup>39</sup>We refer to Chapter 5 in Peng (2010) for existence results of G-SDE's.

**Theorem 3** *The set of  $\mathcal{M}(\mathcal{P})$  of EsMM-sets is not only the empty set if and only if  $S^*$  is an  $E_G$ -martingale and*

$$E_G \left[ \exp \left( \delta \cdot \int_0^T \theta_s^2 d\langle B^G \rangle_s \right) \right] < \infty,$$

for some  $\delta > \frac{1}{2}$ .

With Theorem 2 in mind we can include scenario based viability. Let  $X \in L^2(\mathcal{P})$  be a contingent claim, such that it is priced by  $\mathcal{P}$ -arbitrage with value  $\Psi(X) = E_G(\mathbb{E}_T^\theta X)$ , whenever  $\Gamma$  consists only of a consolidation via the maximum operation.

**Remark 3** *The more precise calculus of the  $G$ -expectation is based on a description of nonlinear partial differential equations. This allows us to create a uniform state price density process in terms of an exponential martingale, based on a  $G$ -martingale representation theorem, (see Appendix B 3).*

*With this in mind, a more elaborated notion of EsMM-sets can be formulated by requiring that the densities  $Z_P$ ,  $P \in \mathcal{P}$  creates a uniform process as a symmetric martingale under sublinear expectation  $\mathcal{P}$ .*

Extensions to continuous trading strategies seem straight forward. Nevertheless, an admissibility condition should be requested, in order to exclude doubling strategies. Considering markets with more than one uncertain security requires a multidimensional Girsanov theorem.<sup>40</sup>

Let us close this subsection with an example on the connection between superreplication of claims and EsMM-sets.

**Example 6** *Under one prior  $P$ , Delbaen (1992) obtained the superreplication price in terms of martingale measures in  $\mathfrak{M}(\{P\})$ :*

$$\begin{aligned} \Lambda(X, P) &= \inf \{ y \geq 0 \mid \exists \theta \in \mathcal{A} : y + \theta_T S_T \geq X \text{ } P - a.s. \} \\ &= \sup_{Q \in \mathfrak{M}(\{P\})} E^Q[X] \end{aligned}$$

*When the uncertainty is given by a set of mutually singular priors, a superreplication price can be derived, see Denis and Martini (2006).*

$$\Lambda(X, \mathcal{P}) = \inf \{ y \geq 0 \mid \exists \theta \in \mathcal{A}(\mathcal{P}) : y + \theta_T S_T \geq X \text{ } \mathcal{P} - q.s. \}$$

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<sup>40</sup>See Osuka (2011).

in terms of an unknown set of martingale laws  $\mathcal{M}$  such that

$$\Lambda(X, \mathcal{P}) = \sup_{Q \in \mathcal{M}} E^Q[X].$$

It turns out that in the  $G$ -framework with simple trading strategies this set is an EsMM-set. When applying our theory to this problem, we get

$$\Lambda(X, \mathcal{P}) = \sup_{P \in \mathcal{P}} \sup_{Q \in \mathfrak{M}(\{P\})} E^Q[X] = E_G[\mathbb{E}_T X],$$

upon applying Theorem 3 and Theorem 3.6 of Vorbrink (2010) and is associated to the maximal EsMM-set in  $\mathfrak{M}(\mathcal{P})$ . However, an easy consequence is that every EsMM-set delivers a price below the superhedging price.

## 4 Discussion and Conclusion

We present a framework and a theory of derivative security pricing where the uncertainty model is given by a set of singular probability measures which incorporate volatility uncertainty. The notion of equivalent martingale measures changes, and the related linear expectation principle becomes a sublinear theory of valuation. The associated arbitrage principle should consider all remaining uncertainty in the consolidation.

The results of this paper may serve as a starting point to obtain a fundamental theorem of asset pricing (FTAP) under mutually singular uncertainty. In Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998), this is achieved for the single prior uncertainty model in great generality. The notion of arbitrage is in principle a separation property of convex sets in a topological space. In this regard, the choice of the underlying topological structure is essential for observing a FTAP. For instance, Levental and Skorohod (1995), establish a FTAP with an *approximate arbitrage* based on a different notion of convergence.

As mentioned in the introduction, Jouini and Kallal (1995) considered security markets with bid-ask spreads and introduced a modified notion of equivalent martingale measures. In this context a FTAP under transaction cost was proved in discrete time by Schachermayer (2004), and in continuous time by Guasoni, Rásonyi, and Schachermayer (2010).

In our setting, two aspects must be kept in mind for deriving a FTAP with mutually singular uncertainty.

Firstly, the spaces of claims and portfolio processes are based on a capacity norm, and thus forces one to argue for the quasi sure analysis, a fact implied in our definition of arbitrage, (see Definition 3). A corresponding notion of free lunch with vanishing *uncertainty* will have to incorporate this more sensitive notion of random variables.

Secondly, the sublinear structure of the price system allows for a nonlinear separation of convex sets. With one prior, the equivalent martingale measure separates achievable claims with arbitrage strategies. In our small meshed structure of random variables this separation is guided by the consolidation operator  $\Gamma$ .

Our preference-free pricing principle gives us a valuation via expected payoffs of different adjusted priors. In comparison to the preference and distribution free results in a perfectly competitive market, see Ross (1976), the implicit assumption is the common knowledge of uncertainty, described by a single probability measure. The uncertainty preface dramatically dictates the consequences for pricing without a utility gradient approach of consumption-based pricing.

The valuation of claims, determined by  $\mathcal{P}$ -arbitrage, contains a new object  $\Gamma$ , which may inspire skepticism. However, note that the consolidation operator  $\Gamma$  should be seen as a tool to regulate financial markets. The valuation of claims in the balance sheet of a bank should depend on  $\Gamma$ . For instance, this may affect fluctuations of opinion in the market as a consequence of uncertainty. In Remark 1 of Section 2 we describe how a good consolidation may be found via consideration of mechanism design. Such considerations may provide a base for the choice of the valuation principle under multiple priors.<sup>41</sup> As a first heuristic, it is possible that utilitarian (convex combination) and Rawlsian (supremum operation) welfare functions may constitute a principle of fair pricing.

Before we close this paper with a discussion on asset pricing under uncertainty and an alternative interpretation of sublinear pricing, we state a technical comment. The suborder dual  $\mathfrak{P}(\mathcal{P})$  of Subsection 2.3 can be elaborated using results on embedding duals in the sub order dual, and could be useful for answer continuity questions about the Riesz-Kantorovich functional.

### **Preferences and Asset pricing**

The uncertainty model in our paper is closely related to Epstein and Wang

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<sup>41</sup>A starting point could be Lopomo, Rigotti, and Shannon (2009).



(1994) and Epstein and Ji (2012) as they consider equilibria with linear prices in their economy. This leads to an indeterminacy in terms of a continuum of linear equilibrium price systems. The relationship between uncertainty and indeterminacy is caused by the constraint to pick one *effective* prior. The Lucas critique<sup>42</sup> applies insofar as it describes the unsuitable usage of a pessimistic investor to fix an effective prior in reduced form. Our approach takes a *preference free* view. We value contingent claims in terms of mean unambiguous asset price processes. In other words, the priors of the uncertainty neutral model yield expectations of the security price that do *not* depend on the chosen "risk neutral" prior. Nevertheless, the idea of a risk neutral valuation principle is not appropriate, as different mutually singular priors delivers different expectations, that cannot be related via a density.

From this point of view, we disarrange the indeterminacy of sublinear prices, and allow for the appearance of a planner to configure the sublinearity. In this sense, the regulator as a policy maker is now able to confront unmeasurable sudden fluctuations in the volatility. A single prior, as a part of the equilibrium output, can create an invisible threat of convention, which may be used to create the illusion of security when faced with an uncertain future. In a model with mutually singular priors, the focus on a single prior creates a hazard. Events with a positive probability under an ignored prior may be a null set under an effective prior in a consumption-based view.

### **Sublinear prices and regulation via consolidation**

In this context, sublinearity is associated with the principle of diversification. In these terms, equilibrium with a sublinear price system covers the concept of Walrasian prices which decentralize with the coincidental awareness of different scenarios. A priori, the instructed Walrasian auctioneer has no knowledge of which prior  $P$  in  $\mathcal{P}$  occurs. The auctioneer assigns to each prior  $P \in \mathcal{P}$  a locally linear price  $\pi_P$ . The degree of discrimination is related to the intensity of nonlinearity. Note that this is a normative category and opens the door to the economic basis of regulation. Each prior is a probabilistic scenario. The auctioneer *consolidates* the price for each possible scenario into one certain valuation. This is also true for an agent in the model, hence the auctioneer should be able to *discriminate under-diversification* in terms of ignorance of priors in this uncertainty model. Further, a von Neumann-Morgenstern utility assumption result in an overconfidence of certainty in

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<sup>42</sup>See Section 3.2 in Epstein and Schneider (2010).

the associated agent.

Since we want to generalize fundamental theorems of asset pricing, we are concerned with the relationship between equivalent martingale measures, viable price systems, and arbitrage. In this setting these concepts must be recast in terms of the multiple prior uncertainty of the model. In contrast, with one prior an equivalent martingale measure is associated with a linear price system. The underlying neoclassical equilibrium concept is a fully positive theory. In the multiple prior setting such a price extension can be regarded as a diversification-neutral valuation principle. Here, diversification refers to a given set of priors  $\mathcal{P}$ . Should the unlucky situation arise that an unconsidered prior governs the market, it is the task of the regulator to robustify these option via an appropriate price system. For instance, uniting two valuations of contingent claims cannot be worse than adding the two uncertain outcomes separately. This is the diversification principle under  $\mathcal{P}$ .

Recalling the quotation of Aliprantis, Tourky, and Yannelis (2001) in the introduction, the degree of sublinearity in our approximation is regulated by the type of consolidation of scenario-dependent linear price systems. These price systems act locally on each scenario  $P \in \mathcal{P}$  in a linear fashion.

## A Appendix: Details and Proofs

### A.1 Section 2

#### A.1.1 Details for Section 2

Let  $L^2(\mathcal{P}) = \mathcal{L}^2(\mathcal{P})/\mathcal{N}$  be the quotient of  $\mathcal{L}^2(\mathcal{P})$  by the  $c_{2,\mathcal{P}}$  null elements. Such null elements are characterized by random variables which are  $\mathcal{P}$ -polar.  $\mathcal{P}$ -polar sets evaluated under every prior are zero or one. But the value may differ between different priors. A property holds *quasi-surely* (q.s.) if it holds outside a polar set. Furthermore, the space  $L^2(\Omega)$  is characterized by

$$L^p(\mathcal{P}) = \{X \in L^0(\Omega) : X \text{ has a q.c. version, } \lim_{n \rightarrow \infty} \mathcal{E}^{\mathcal{P}}(|X|)^2 1_{\{|X| > n\}} = 0\},$$

where  $L^0(\Omega)$  denotes the space of all measurable real-valued functions on  $\Omega$ . A mapping  $X : \Omega \rightarrow \mathbb{R}$  is said to be quasi-continuous if  $\forall \varepsilon > 0$  there exists an open set  $O$  with  $\sup_{P \in \mathcal{P}} P(O) < \varepsilon$  such that  $X|_{O^c}$  is continuous. We say that  $X : \Omega \rightarrow \mathbb{R}$  has a *quasi-continuous version* (q.c.) if there exists a quasi-continuous function  $Y : \Omega \rightarrow \mathbb{R}$  with  $X = Y$  q.s. The mathematical

framework provided enables the analysis of stochastic processes for several mutually singular probability measures simultaneously. All equations are understood in the sense of quasi-sure. This means that a property holds almost-surely for all scenarios  $P \in \mathcal{P}$ .

When recast the order relation taken from Bion-Nadal and Kervarec (2012) we have:  $X \geq 0$  if and only if

$$\begin{aligned} &\text{there is a sequence } \{X_n\}_{n \in \mathbb{N}} \subset \mathcal{C}_b(\Omega), X_n \geq 0 \text{ such that} \\ &\forall Y \in L^2(\mathcal{P}) \text{ of class } X \text{ we have } \lim_{n \rightarrow \infty} c_{2,\mathcal{P}}(Y - X_n) = 0. \end{aligned}$$

Since, for all  $X, Y \in L^2(\mathcal{P})$  with  $|X| \leq |Y|$  imply  $c_{2,\mathcal{P}}(X) \leq c_{2,\mathcal{P}}(Y)$ , we have that  $L^2(\mathcal{P})$  is a Banach lattice.<sup>43</sup>

Following we discuss the different operations for consolidation. Let  $\Pi_P = ZP \in L^2(\mathcal{P})^*$ , with  $P \in \mathcal{P}$ .

Let  $\mu$  be a measure on the Borel measurable space  $(\mathcal{P}, \mathcal{B}(\mathcal{P}))$  with  $\mu(\mathcal{P}) = 1$  and full support on  $\mathcal{P}$ . In this context we can consider the additive case in  $\mathfrak{P}(\mathcal{P})$ , where a new prior is generated:<sup>44</sup>

$$\Gamma_\mu : \prod_{P \in \mathcal{P}} L^2(\mathcal{P})^* \rightarrow \mathfrak{P}(\mathcal{P}), \quad \Gamma_\mu(\{\Pi_P\}_{P \in \mathcal{P}}) = \int_{\mathcal{P}} E^P[Z \cdot] d\mu(P) = E^{P_\mu}[Z \cdot]$$

We can consider the Dirac measure as an example. The related consideration of only one special prior in  $\mathcal{P}$  is in essence the uncertainty model in Harrison and Kreps (1979). The operation in question is given by  $(\Pi_P)_{P \in \mathcal{P}} \mapsto E^P[Z \cdot]$ .<sup>45</sup>

The second operation in  $\mathfrak{P}(\mathcal{P})$  is a point-wise maximum:

$$\Gamma_{\text{sup}} : \prod_{P \in \mathcal{P}} L^2(\mathcal{P})^* \rightarrow \mathfrak{P}(\mathcal{P}), \quad \Gamma_{\text{sup}}(\{\Pi_P\}_{P \in \mathcal{P}}) = \sup_{P \in \mathcal{P}} E^P[Z \cdot] = \mathcal{E}^{\mathcal{P}}(Z \cdot).$$

This is an extreme form of consolidation and can be considered as the highest awareness of all priors. Note that combinations between the maximum and addition operation are possible as indicated in Example 2 and Proposition 1.

<sup>43</sup>This is of interest for existence result of general equilibria.

<sup>44</sup>The related operation of convex functionals would corresponds to the convolution operation. Since we have no assumption on the convexity of  $\mathcal{P}$ , the prior  $P_\mu$  may only lie in the convex hull of  $\mathcal{P}$ .

<sup>45</sup>A different point of view is that this case can be seen as a special case of Riedel (2011).

### A.1.2 Proofs of Section 2

**Proof of Proposition 1** *The 5th claim follows from Theorem 1 in Biagini and Frittelli (2010), whereas the other claims follow directly from the construction of the functionals in  $\mathfrak{P}(\mathcal{P})$ . ■*

For the proof of Theorem 1, we define the shifted preference relationship  $\succsim^0$  such that every feasible net trade is worse off than  $(0, 0) \in B(0, 0, \psi, \mathbb{M}(\Gamma))$ . Obviously, an agent given by  $\succsim^0$  does not trade. Hence, an initial endowment constitutes a no trade equilibrium.

**Proof of Theorem 1** *Let the price system  $(\{M_P, \pi_P\}_{P \in \mathcal{P}}, \Gamma)$  be given and we have a  $\Psi \in \mathfrak{P}(\mathcal{P})$  on  $L^2(\mathcal{P})$  such that  $\Psi|_{\mathbb{M}(\Gamma)} = \psi$ . The relation on  $\mathbb{R} \times L^2(\mathcal{P})$ , given by*

$$(x, X) \succ (x', X') \quad \text{if } x + -\Psi(-X) \geq x' + -\Psi(-X'),$$

*is an element of  $\mathbb{A}(\mathcal{P})$ . This can be justified by the  $c_{2,\mathcal{P}}$ -continuity and concavity of  $-\Psi(-\cdot)$ . The bundle  $(\hat{x}, \hat{X}) = (0, 0)$  satisfies the viability condition of Definition 2, hence  $(\{M_P, \pi_P\}_{P \in \mathcal{P}}, \Gamma)$  is scenario-based viable.*

*In the other direction, let  $\psi : \mathbb{M}(\Gamma) \rightarrow \mathbb{R}$  be a price system,  $\succ \in \mathbb{A}(\mathcal{P})$ , and  $(\hat{x}, \hat{X})$  satisfy the viability condition. We may assume  $(\hat{x}, \hat{X}) = (0, 0)$ , since it is only a geometric deferment. Consider the following sets*

$$\succ^0 = \{(x, X) \in \mathbb{R} \times L^2(\mathcal{P}) : (x, X) \succ (0, 0)\},$$

$$B(0, 0, \psi, \mathbb{M}(\Gamma)) = \{(x, X^m) \in \mathbb{R} \times \mathbb{M}(\Gamma) : x + \psi(X^m) \leq 0\}.$$

*We have that  $B(0, 0, \psi, \mathbb{M}(\Gamma))$  and  $\succ^0$  are convex sets. By the  $c_{2,\mathcal{P}}$ -upper semi continuity of  $\succ$ ,  $\succ^0$  is  $c_{2,\mathcal{P}}$ -open. We apply Lemma 1, stated below in terms of a nonlinear separation theorem. A non zero  $c_{2,\mathcal{P}}$ -continuous sublinear functional on  $\mathbb{R} \times L^2(\mathcal{P})$  with  $\phi(x, X) \geq 0$  for all  $(x, X) \in \succ^0$  and  $\phi(x, X) \leq 0$  for all  $(x, X) \in B(0, 0, \psi, \mathbb{M}(\Gamma))$  are constructed.*

*There is a  $(y, Y)$  with  $\phi(y, Y) < 0$ , since  $\phi$  is non trivial. Strict monotonicity implies  $(1, 0) \succ (0, 0)$ . The continuity  $\succ$  gives us  $(1 + \varepsilon y, \varepsilon Y) \succ (0, 0)$ , for some  $\varepsilon > 0$ , hence*

$$\begin{aligned} -\phi(1 + \varepsilon y, \varepsilon Y) &= -\phi(1, 0) + \varepsilon \phi(y, Y) \leq 0 \\ \text{and } \phi(1, 0) &\geq -\varepsilon \phi(y, Y) > 0 \end{aligned}$$

We have  $\phi(1, 0) > 0$  and after a renormalization let  $\phi(1, 0) = 1$ . Moreover write  $\phi(x, X) = x + \Psi(X) > 0$ , where  $\Psi : L^2(\mathcal{P}) \rightarrow \mathbb{R}$  is a functional in  $\mathfrak{P}(\mathcal{P})$ .

Strict positivity of  $\Psi$  follows from  $(0, x) \succ (0, 0)$ , hence  $(-\varepsilon, x) \succ (0, 0)$ , and therefor  $\Psi(x) - \varepsilon \geq 0$ .

Let  $X^m \in \mathbb{M}(\Gamma)$ , since  $(-\psi(X^m), X^m), (\psi(X^m), -X^m) \in B(0, 0, \psi, \mathbb{M}(\Gamma))$  we have  $0 = \phi(\psi(X^m), X^m) = \psi(X^m) - \Psi(X^m)$  and  $\Psi|_{\mathbb{M}(\Gamma)} = \psi$  follows. ■

The following lemma is applied to the proof of Theorem 1. Let  $\mathfrak{P}|_{\mathbb{M}(\Gamma)}(\mathcal{P})$  be the space of all functionals  $\psi \in \mathfrak{P}(\mathcal{P})$  with domain  $\mathbb{M}(\Gamma)$ .

**Lemma 1** *Let  $\psi \in \mathfrak{P}|_{\mathbb{M}(\Gamma)}(\mathcal{P})$  then there is a  $\Psi \in \mathfrak{P}(\mathcal{P})$  with  $\Psi|_{\mathbb{M}(\Gamma)} = \psi$ .*

Note, that this is a Hahn Banach type result for functionals in  $\mathfrak{P}(\mathcal{P})$ . We illustrate this in the following diagram:

$$\begin{array}{ccc} \{\pi_P : M_P \rightarrow \mathbb{R}\}_{P \in \mathcal{P}} & \xrightarrow{\Gamma} & \psi : \mathbb{M}(\Gamma) \rightarrow \mathbb{R} \\ \text{Hahn Banach} \downarrow & & \downarrow \\ \{\Pi_P : L^2(\mathcal{P}) \rightarrow \mathbb{R}\}_{P \in \mathcal{P}} & \xrightarrow{\Gamma} & \Psi : L^2(\mathcal{P}) \rightarrow \mathbb{R} \end{array}$$

**Proof of Lemma 1** Fix  $\psi : \mathbb{M}(\Gamma) \rightarrow \mathbb{R}$ , given by  $\Gamma(\{\pi_P\}_{P \in \mathcal{P}})$ . We apply Hahn Banach for each  $P \in \mathcal{P}$  with respect to  $\pi_P : M_P \rightarrow \mathbb{R}$ . We have a collection of  $\Pi_P : L^2(\mathcal{P}) \rightarrow \mathbb{R}$  such that  $\Pi|_{M_P} = \pi_P$ . Hence,  $\Psi = \Gamma(\{\Pi_P\}_{P \in \mathcal{P}})$ . By the definition of the price space, we have  $\Psi \in \mathfrak{P}(\mathcal{P})$ . ■

## A.2 Section 3

### A.2.1 Details of Section 3

Next, we discuss the augmentation of our information structure. The unaugmented filtration is given by  $\mathbb{F}^o$ . As mentioned in Subsection 3.1, the set of priors must be stable under pasting, in order to apply the framework of Nutz and Soner (2010). For the sake of completeness we recall this notion.

**Definition 5** *The set of priors is stable under pasting if for every  $P \in \mathcal{P}$ , every  $\mathbb{F}^o$ -stopping time  $\tau$ ,  $B \in \mathcal{F}_\tau^o$  and  $P_1, P_2 \in \mathcal{P}(\mathcal{F}_\tau^o, P)$ , the prior  $P_\tau$  given by*

$$P_\tau = E^P[P_1(A|\mathcal{F}_\tau^o)1_B + P_2(A|\mathcal{F}_\tau^o)1_{B^c}], \quad A \in \mathcal{F}_\tau^o$$

*is a prior in  $\mathcal{P}$ .*

In the multiple prior setting, with a given reference measure this property is equivalent to the well known notion of *time consistency*. However, this is not true if there is no dominant prior. Additionally, the set of priors must be chosen maximally. This is a property which holds for a *fixed* set of random variables. For further consideration, we refer the reader to Section 3 in Nutz and Soner (2010).

The usual condition of a "rich"  $\sigma$ -algebra at time 0 is widely used in mathematical finance. But the economic meaning is questionable. Our uncertainty model of mutually singular priors can be augmented, similarly to the classical case, using the right continuous filtration given by  $\mathbb{F}^+ = \{\mathcal{F}_t^+\}_{t \in [0, T]}$  where

$$\mathcal{F}_t^+ = \bigcap_{s > t} \mathcal{F}_s^o, \text{ for } t \in [0, T[$$

The second step is to augment the minimal right continuous filtration  $\mathbb{F}^+$  by all polar sets of  $(\mathcal{P}, \mathcal{F}_T^o)$ , i.e.  $\mathcal{F}_t = \mathcal{F}_t^+ \vee \mathcal{N}(\mathcal{P}, \mathcal{F}_T^o)$ . This augmentation is strictly smaller than the universal augmentation  $\bigcap_{P \in \mathcal{P}} \overline{\mathbb{F}}^o{}^P$ . This choice is economically reasonable as the initial  $\sigma$ -field contains not all 0-1 limit events. An agent considers this exogenously specified information structure. It describes what information the agent *can* know at each date. This is the analogue to a filtration in the single prior framework satisfying the usual conditions.

According to Appendix B.2, we have a countable set  $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{P}$  such that for every positive random variable  $X$  in  $L^2(\mathcal{P})$  we have

$$\mathcal{E}^{\mathcal{P}}(X) = \sup_{n \in \mathbb{N}} E^{P_n}[X]. \quad (2)$$

We write  $(P_n) \sim \mathcal{P}$  for a set, which allows such a countable reduction.<sup>46</sup> Note that the related Banach spaces are the same, see Bion-Nadal and Kervarec (2012).

The  $\mathcal{P}$ -arbitrage condition can be reformulated with a special prior  $\tilde{P}$  in a simpler form:  $\eta_T S_T \geq 0$   $\tilde{P}$ -a.s and  $\tilde{P}(\eta_T S_T > 0) > 0$ . Without convexity of  $\mathcal{P}$ ,  $\hat{P} \in \mathcal{P}$  is not necessarily true.<sup>47</sup>

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<sup>46</sup>This reduction is heavily related to the weak compactness of  $\mathcal{P}$ , see Bion-Nadal and Kervarec (2012).

<sup>47</sup>Note that the arbitrage definition has only positive random variables under consideration. This allows us to consider an arbitrage controlling prior in the canonical class, see Appendix B.2.

### A.2.2 Proofs of Section 3

For the proof of Theorem 2 we need a result from Appendix B.1. We formulate a generalized Riesz representation Theorem: A linear functional  $\Pi$  on  $L^2(\mathcal{P})$  is  $c_{2,\mathcal{P}}$ -continuous if and only if for every  $X \in L^2(\mathcal{P})$  we have  $\Pi(X) = E^P[Z_P X]$  for some  $P \in \mathcal{P}$  and  $Z_P \in L^2(P)$ .

**Proof of Theorem 2** We fix an EsMM-set  $\mathcal{Q}$ . The related consolidation  $\Gamma$  gives us the set of relevant priors  $\Gamma(\mathcal{P}) \subset \mathcal{P}$ . Let  $Z_P = \frac{dP}{dQ}$ , for each  $Q \in \mathcal{Q}$  and the related  $P \in \mathcal{P}$ . We have  $Z_P \in L^2(P)$ . Let a strictly positive  $\Psi \in \mathfrak{P}(\mathcal{P})_{++}$  be given.

Take a marketed claim  $X^m \in \mathbb{M}(\Gamma)$  and let  $\eta \in \mathcal{A}$  be a self-financing trading strategy that hedges  $X^m$ . This gives us the following equalities, since  $\eta \in \mathcal{A}$ , by the rule for conditional  $\mathcal{E}$ -expectation and since  $S$  is a symmetric  $\mathcal{E}^{\mathcal{Q}}$ -martingale,  $0 \leq t \leq u \leq T$ ,

$$\mathcal{E}_u^{A*}(\eta_t S_t) = \eta_t^+ \mathcal{E}_u^{A*}(S_t) + \eta_t^- \mathcal{E}_u^{A*}(-S_t) = \eta_t^+ S_u - \eta_t^- S_u = \eta_u S_u,$$

where  $\eta = \eta^+ - \eta^-$  with  $\eta^+ - \eta^- \geq 0$   $\mathcal{P}$ -quasi surely. Therefore we achieve

$$\Psi(X^m) = \mathcal{E}_0^{A*}(\eta_T S_T) = \eta_0 S_0 = \psi(X^m).$$

For the other direction let  $\Psi \in \mathfrak{P}(\mathcal{P})_{++}$  with  $\Psi|_{\mathbb{M}(\Gamma)} = \psi$ , related to a set of linear functionals  $\{\pi_P : M_P \rightarrow \mathbb{R}\}_{P \in \mathcal{P}}$  and  $\{\Pi_P : L^2(\mathcal{P}) \rightarrow \mathbb{R}\}_{P \in \mathcal{P}}$ , such that  $\Pi|_{M_P} = \pi_P$ . Now, we define  $\mathcal{Q}$  in terms of  $\Gamma$ .

We discuss the possible cases which can appear. For simplicity we assume  $\mathcal{P} = \{P_1, P_2, P_3\}$ . Let  $P^{k,j} = \frac{1}{2}P^k + \frac{1}{2}P^j$  and  $Z^{k,j} = \frac{1}{2}Z^k + \frac{1}{2}Z^j$ , recall that we can represent each functional  $\Pi_P$  by  $Z_P P$ .

$$1. \frac{1}{2}\Pi_1 + \frac{1}{2}(\Pi_2 \wedge \Pi_3) \text{ becomes } \{Z^{1,2}P^{1,2}, Z_3P_3\} = \mathcal{Q}$$

$$2. (\frac{1}{2}\Pi_1 + \frac{1}{2}\Pi_2) \wedge \Pi_3 \text{ becomes } \{Z_1P_1, Z^{2,3}P^{2,3}\} = \mathcal{Q}$$

Since  $\mathcal{Q} = \{PZ_P : P \in \Gamma(\mathcal{P}), Z_P \in L^2(P)\}$ , the first condition of Definition 4 follows, note that the square integrability of each  $Z_P$  follows from the  $c_{2,\mathcal{P}}$ -continuity of linear functionals which generate  $\Psi$ .

We prove the symmetric martingale property of the asset price process. Let  $B \in \mathcal{F}_t$ ,  $\eta \in \mathcal{A}$  be a self-financing trading strategy and

$$\eta_s = \begin{cases} 1 & s \in [t, u[ \text{ and } \omega \in B \\ 0 & \text{else,} \end{cases} \quad \eta_s^0 = \begin{cases} S_t, & s \in [t, u[ \text{ and } \omega \in B \\ S_u - S_t, & s \in [u, T[ \text{ and } \omega \in B \\ 0 & \text{else.} \end{cases}$$

This strategy yields a portfolio value

$$\eta_T S_T = (S_u - S_t) \cdot 1_B,$$

the claim  $\eta_T S_T$  is marketed at price zero. In terms of the modified sublinear expectation  $\{\mathcal{E}_t^\mathcal{Q}(\cdot)\}_{t \in [0, T]}$ , we have with  $t \leq u$

$$\mathcal{E}_t^\mathcal{Q}((S_t - S_u)1_B) = 0$$

By Theorem 4.7 Xu and Zhang (2010), it follows that  $S_u = \mathcal{E}_t^\mathcal{Q}(S_u)$ .<sup>48</sup> But this means that  $\{S_t\}_{t \in [0, T]}$  is  $\mathcal{E}^\mathcal{Q}$ -martingale. The same argumentation holds for  $-S$ , hence the asset price  $S$  is a symmetric  $\mathcal{E}^\mathcal{Q}$ -martingale. ■

- Proof of Corollary 1**
1. Suppose there is a  $\mathcal{Q} \in \mathcal{M}(\mathcal{P})$  and let  $\eta \in \mathcal{A}$  such that  $\eta_T S_T \geq 0$  and  $P(\eta_T S_T > 0) > 0$  for some  $P \in \mathcal{P}$ . Since for all  $Q \in \mathcal{Q}$  there is a  $P \in k(\mathcal{P})$  such that  $Q \sim P$ , there is a  $Q' \in \mathcal{Q}$  with  $Q'(\eta_T S_T > 0) > 0$ . Hence,  $\mathcal{E}^\mathcal{Q}(\eta_T S_T) > 0$  and by Theorem 2 we observe  $\mathcal{E}^\mathcal{Q}(\eta_T S_T) = \eta_0 S_0$ . This implies that no  $\mathcal{P}$ -arbitrage exists.
  2. In terms of Theorem 1, each  $P \in \mathcal{R}$  admits exactly one extension. With Theorem 2 the result follows.
  3. By Theorem 2 this is equivalent to the non emptiness of  $\mathfrak{P}(\mathcal{P})$ . Fix a  $\Psi \in \mathfrak{P}_{++}(\mathcal{P})$ , with  $\Gamma(\mathcal{P}) = \mathcal{R}$  and a  $\eta \in \mathcal{A}$  such that  $\eta_0 S_0 = 0$  hence  $\Psi(\eta_T S_T) = 0$ . The viability of  $\Psi$  implies  $\eta_T S_T = 0$   $\mathcal{R}$ -q.s. Hence, no  $\mathcal{R}$ -arbitrage exist.
  4. This then follows by the same argument as in Harrison and Pliska (1981) (see the Lemma on p.228). ■

For the proof of Theorem 3, we apply results from stochastic analysis in the  $G$  framework. The results are collected in Appendix B.3.

**Proof of Theorem 3** In accordance to Remark 3, let  $\mathcal{Q}$  be an EsMM-set, given by  $\mathcal{Q} = \{\rho P : P \in \mathcal{P}\}$ , where the density  $\rho$  with  $\rho \in L^2(\mathcal{P})$  and  $E_G[\rho] = -E_G[-\rho]$ . Next define the stochastic process  $(\rho_t)_{t \in [0, T]}$  by  $\rho_t = E_G[\rho | \mathcal{F}_t]$  resulting in a symmetric  $G$ -martingale to which we apply the

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<sup>48</sup>The result is proven for the  $G$ -framework. However the assertion is in our setting true as well, by an application of Theorem 4.10 of Nutz and Soner (2010) instead of Theorem 4.1.42 of Peng (2010).



*martingale representation theorem for  $G$ -expectation, stated in Appendix B.3. Hence, there is a  $\gamma \in M_G^2(0, T)$  such that we can write*

$$\rho_t = 1 + \int_0^t \gamma_s dB_s^G, \quad t \in [0, T], \quad \mathcal{P} - q.s.$$

*By the  $G$ -Itô formula, stated in the Appendix B.3, we have*

$$\ln(\rho_t) = \int_0^t \phi_s dB_s^G + \frac{1}{2} \int_0^t \phi_s^2 d\langle B^G \rangle_s, \quad \mathcal{P} - q.s.$$

*for every  $t \in [0, T]$  in  $L_G^2(\Omega_t)$  and hence*

$$\rho = \mathbb{E}_T^\phi = \exp\left(-\frac{1}{2} \int_0^T \theta_s^2 d\langle B^G \rangle_s - \int_0^T \theta_s dB_s^G\right), \quad \mathcal{P} - q.s.$$

*With this representation of the density process we can apply the Girsanov theorem, stated in Appendix B.3. Set  $\phi_t = \frac{\rho_t}{\gamma_t}$  and consider the process*

$$B_t^\phi = B_t^G - \int_0^t \phi_s ds, \quad t \in [0, T].$$

*By the Girsanov formula for  $G$ -Brownian motion, stated in Appendix B.3, we deduce that  $B^\phi$  is a  $G$ -Brownian motion under the sublinear expectation  $\mathcal{E}^\phi(\cdot) = E_G[\phi \cdot]$  and  $S$  satisfies*

$$S_t = S_0 + \int_0^t V_s dB_s^\phi + \int_0^t (\mu_s + V_s \phi_s) d\langle B^\phi \rangle_s \quad t \in [0, T]$$

*on  $(\Omega, \mathcal{H}, \mathcal{E}^\phi)$ . Since  $V$  is a bounded process, the stochastic integral is a symmetric martingale under  $\mathcal{E}^\phi$ .  $S$  is a symmetric  $\mathcal{E}^\phi$ -martingale if and only if  $\mu_t + V_t \phi_t = 0$ . We have shown that  $\rho$  is simultaneous Radon-Nikodym type density of the EsMM-set  $\mathcal{Q} = \rho\mathcal{P}$ . Hence, the power set of EsMM-sets is not only the empty set since  $\phi = \theta$ . ■*

## B Appendix: Required results

In this Appendix we introduce the mathematical framework more carefully. We also collect all the results applied in Sections 2 and 3.

## B.1 The sub order dual

In this subsection we discuss the mathematical preliminaries for the price space of sublinear functionals for Section 3.

*The topological dual space:*

1. Let  $c_{2,\mathcal{P}}$  be a capacity norm, defined in Section 2.2, on a complete separable metric space  $\Omega$ . Every continuous linear form  $l$  on  $L^2(\mathcal{P})$  admits a representation:

$$l(X) = \int X d\mu \quad \forall X \in L^2(\mathcal{P}),$$

where  $\mu$  is a bounded signed measure defined on a  $\sigma$ -algebra containing the Borel  $\sigma$ -algebra of  $\Omega$ . If  $l$  is a non-negative linear form, the measure  $\mu$  is non-negative finite.

2. We have  $L^2(\mathcal{P})^* = \{\mu = ZP : P \in \mathcal{P} \text{ and } Z \in L^2(\mathcal{P})_+\}$ .

Note that the capacity norm defined in (1) is a Prohorov capacity. We apply Proposition 3 from Bion-Nadal and Kervarec (2012). The second assertion can be proven via a modification of Theorem I.30 in Kervarec (2008), where the case of  $L^1(\mathcal{P})$  is treated.

### B.1.1 Semi lattices and their intrinsic structure

We begin with the most simple operation of consolidation, ignoring a subset of priors and giving a weight to the others.

**Integration:**

Let  $\mu \in \mathcal{M}_{\leq 1}(\mathcal{P})$  be the positive measure  $\mu$  such that  $\mu(\mathcal{P}) \leq 1$ . In our case the underlying space is  $\Pi_{P \in \mathcal{P}} L^2(\mathcal{P})^*$  such that the density component is invariant, when considering the representation  $l(X) = \int ZX dP$ . So let  $N \subset \mathcal{B}(\mathcal{P})$  be a Borel measurable set, a by a measure  $\mu \in \mathcal{M}_{\leq 1}(\mathcal{P})$  is given by

$$\Gamma(\mu, N) : \times_{P \in \mathcal{P}} L^2(\mathcal{P})^* \rightarrow L^2(\mathcal{P})^*, \{\pi_P\}_{P \in \mathcal{P}} \mapsto \int_N 1 d\mu(P) \cdot Z.$$

The size of  $N$  determines the degree of ignorance, related to the exclusion of the prior in the countable reduction. A measure with mass less than implies an ignorance The Dirac measure is a projection to one certain probability

model.

Next, we consider the supremum operation of functionals. Note that this gives us the connection to sublinear expectations.

**The point-wise supremum:**

The operation of point-wise maximum preserves the convexity. We review a result which gives an iterated application of the Hahn-Banach Theorem.

*Representation of sublinear functionals* Frittelli (2000): Let  $\psi$  be a sublinear functional on a topological vector space  $V$ , then

$$\psi(X) = \max_{x^* \in P_\psi} x^*(X),$$

where  $P_\psi = \{x^* \in X^* : x^*(X) \leq \psi(X) \text{ for all } X \in V\} \neq \emptyset$

The maximum operation can also be associated to a lattice structure. In economic terms this is related to a normative choice of the super hedging intensity. The diversification valuation operator consolidation is set to *one nonlinear* valuation functional. Note that the operation preserves monotonicity.

## B.2 The set of probability models

The model of multiple priors motivates the introduction of the following mapping

$$c : \mathcal{B}(\Omega) \rightarrow [0, 1], \quad c(A) = \sup_{P \in \mathcal{P}} P(A).$$

It is easy to prove that  $c(\cdot)$  is a Choquet capacity.<sup>49</sup> The capacity notion may be used for an alternative formulation of Theorem 2.

Fix  $(\Omega, \mathcal{B}(\Omega)) = (C_0([0, T], \mathcal{B}(C_0([0, T])))$ . We refer to Bion-Nadal and Ker-varec (2012) where the state space consists of a cadlag path. We give a criterion for the weak compactness of  $\mathcal{P}$ . Let  $\sigma^1, \sigma^2 : [0, T] \rightarrow \mathbb{R}$  be two measures with a Holder continuous distribution function  $t \mapsto \sigma^i([0, t]) = \sigma^i(t)$ .

A probability measure  $P$  on  $\Omega$  is a martingale probability measure if the coordinate process is a martingale with regard to the canonical (raw) filtration. Let  $\sigma^1, \sigma^2$  be two measures with a Holder continuous distribution function

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<sup>49</sup>For a general treatment, see again Denis, Hu, and Peng (2011) and the references therein.

$$t \mapsto \sigma^i([0, t]) = \sigma(t).$$

*Criterion for weak compactness of priors*, Denis, Kervarec, et al. (2007): Let  $\mathcal{P}(\sigma^1, \sigma^2)$  be the set of martingale probability measures with

$$d\sigma^1(t) \leq d\langle B \rangle_t^P \leq d\sigma^2(t),$$

where  $\langle B \rangle^P$  is the quadratic variation of  $B$  under  $P$ . Then the set  $\mathcal{P}(\sigma^1, \sigma^2)$  is weakly compact.

Now, we discuss the concept of countable reduction. We apply the following result in Section 2.

*Countable reduction*, Bion-Nadal and Kervarec (2010): Let  $c_{2,\mathcal{P}}$  be given by a weakly compact set of probability measures  $\mathcal{P}$ . Then there is a countable set  $(P_n)_{n \in \mathbb{N}} \subset \mathcal{P}$  such that for all  $X \in L^2(\mathcal{P})$

$$c_{2,\mathcal{P}}(X) = \sup_{n \in \mathbb{N}} E^{P_n}[|X|^2]^{\frac{1}{2}}.$$

The associated Banach spaces are the same. This assertion holds, since the closure of  $\mathcal{P}$  has a countable dense subset (for the weak\*-topology or in probabilistic terms the vague topology).

Following, we introduce an equivalence class associated with the  $c_{2,\mathcal{P}}$ -norm on  $\mathcal{P}$ . We start with some single prior considerations, taken from Bion-Nadal and Kervarec (2010). Note that  $L^2(\{P\}) = L^2(P)$ , let  $Q \in L^1(P)^*$  and remember

$$Q \sim P \text{ if and only if } \left( \forall X \in L^1(P)_+, X = 0 \text{ in } L^1(P) \Leftrightarrow \int X dQ = 0 \right).$$

Whenever  $\mathcal{P}$  is weakly relatively compact, we can associate a probability measure  $P$  to  $L(\mathcal{P})$ , characterizing the (quasi sure) null elements in the positive cone  $L^2(\mathcal{P})_+$ . Let  $\mathcal{M}^+(c_{2,\mathcal{P}})$  be the set of non-negative finite measures on  $(\Omega, \mathcal{B}(\Omega))$  defining an element of  $L^2(\mathcal{P})^*$ . Define on  $\mathcal{M}^+(c_{2,\mathcal{P}})$  the relation  $R_{c_{2,\mathcal{P}}}$  by:

$$\mu R_{c_{2,\mathcal{P}}} \nu \text{ if and only if } \left( \forall X \in L^2(\mathcal{P})_+, \int X d\mu = 0 \Leftrightarrow \int X d\nu = 0 \right)$$

It follows that  $R_{c_{2,\mathcal{P}}}$  is an equivalence relation on  $\mathcal{P}$ . We are able to say more about the dual space of  $L^2(\mathcal{P})$ .

*Reference measure for the positive cone*, Bion-Nadal and Kervarec (2010): There is a unique  $R_{c_2, \mathcal{P}}$  equivalence class in  $\mathcal{M}^+(c_2, \mathcal{P})$  such that  $\mu \in \mathcal{M}^+(c_2, \mathcal{P})$  belongs to this class if and only if

$$\forall X \in L^2(\mathcal{P})_+, \{\mu(X) = 0\} \text{ if and only if } \{X = 0 \text{ in } L^2(\mathcal{P})\}.$$

This class is referred as the canonical  $c_2, \mathcal{P}$ -class. For every countable weakly relatively compact set  $(P^n)_{n \in \mathbb{N}}$  such that (1) holds, for  $\alpha_n > 0$ , for each  $n \in \mathbb{N}$ , such that  $\sum_{n \in \mathbb{N}} \alpha_n = 1$  the probability measure  $\sum_{n \in \mathbb{N}} \alpha_n P^n$  belongs to the canonical  $c_2, \mathcal{P}$ -class.

This gives us an easy definition of  $\mathcal{P}$ -arbitrage, as mentioned in Section 3.2.

### B.3 Stochastic analysis with $G$ -Brownian motion

We introduce the notion of sublinear expectation for the  $G$ -Brownian motion. This includes the concept of  $G$ -expectation, the Itô calculus with  $G$ -Brownian motion and related results concerning the representation of  $G$ -expectation and (symmetric)  $G$ -martingales. For a more precise detour we refer to the Appendix of Vorbrink (2010) and to references therein.

At the end of this section we present a Girsanov theorem for  $G$ -Brownian motion, which we apply in Theorem 3 of Subsection 3.4.

Let  $\Omega \neq \emptyset$  be a given set. Let  $\mathcal{H}$  be a linear space of real valued functions defined on  $\Omega$  with  $c \in \mathcal{H}$  for all constants  $c$  and  $|X| \in \mathcal{H}$  if  $X \in \mathcal{H}$ . Note that in our model we choose  $\mathcal{C}_b(\Omega) = \mathcal{H}$  and  $\Omega = \Omega_T = C_0([0, T])$ .

A sublinear expectation  $\hat{E}$  on  $\mathcal{H}$  is a functional  $\hat{E} : \mathcal{H} \rightarrow \mathbb{R}$  satisfying monotonicity, constant preserving, sub-additivity and positive homogeneity. The triple  $(\Omega, \mathcal{H}, \hat{E})$  is called a *sublinear expectation space*. For the construction of the  $G$ -expectation, the notion of independence and  $G$ -normal distributions we refer to Peng (2010).

A process  $(B_t)_{t \geq 0}$  on a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{E})$  is called a  *$G$ -Brownian motion* if the following properties are satisfied:

- (i)  $B_0 = 0$ .
- (ii) For each  $t, s \geq 0$ :  $B_{t+s} - B_t \sim B_t$  and  $\hat{E}[|B_t|^3] \rightarrow 0$  as  $t \rightarrow 0$ .
- (iii) The increment  $B_{t+s} - B_t$  is independent from  $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$  for each  $n \in \mathbb{N}$  and  $0 \leq t_1 \leq \dots \leq t_n \leq t$ .
- (iv)  $\hat{E}[B_t] = -\hat{E}[-B_t] = 0 \quad \forall t \geq 0$ .

The following observation is important for the characterization of  $G$ -martingales. The space  $C_{l,Lip}(\mathbb{R}^n)$ , where  $n \geq 1$  is the space of all real-valued continuous functions  $\varphi$  defined on  $\mathbb{R}^n$  such that  $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \forall x, y \in \mathbb{R}^n$ . We define

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) | n \in \mathbb{N}, t_1, \dots, t_n \in [0, T], \varphi \in C_{l,Lip}(\mathbb{R}^n)\}.$$

The Itô integral can also be defined for the following processes: Let  $H_G^0(0, T)$  be the collection of processes  $\eta$  having the following form: For a partition  $\{t_0, t_1, \dots, t_N\}$  of  $[0, T]$ ,  $N \in \mathbb{N}$ , and  $\xi_i \in L_{ip}(\Omega_{t_i}) \quad \forall i = 0, 1, \dots, N-1$ , let  $\eta$  ( see Song (2011)) be given by

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t) \quad \forall t \leq T.$$

For  $\eta \in H_G^0(0, T)$  let  $\|\eta\|_{M_G^2} := \left( E_G \left[ \int_0^T |\eta_s|^2 ds \right] \right)^{\frac{1}{2}}$  and denote by  $M_G^2(0, T)$  the completion of  $H_G^0(0, T)$  under this norm.

As before we can construct Itô's integral  $I$  on  $H_G^0(0, T)$  and extend it to  $M_G^2(0, T)$  continuously, hence  $I : M_G^2(0, T) \rightarrow L^2(\mathcal{P})$ .

The next result is an Itô formula. The presentation of basic notions on stochastic calculus with respect to  $G$ -Brownian motion lies beyond the scope of this appendix.

*Itô-formula*, Li and Peng (2011): Let  $\Phi \in C^2(\mathbb{R})$  and  $dX_t = \mu_t d\langle B^G \rangle_t + V_t dB_t^G$ ,  $t \in [0, T]$ ,  $\mu, V \in M_G^2(0, T)$  are bounded processes. Then we have for every  $t \geq 0$ :

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial \Phi(X_u) V_u dB_u^G + \frac{1}{2} \int_s^t \partial^2 \Phi(X_u) \mu_u + \partial^2 \Phi(X_u) V_u^2 d\langle B^G \rangle_u.$$

Next, we introduce a in  $G$ -framework of martingales. A process  $M = \{M_t\}_{t \in [0, T]}$  with values in  $L^2(\mathcal{P})$  is called  $G$ -martingale if  $E_G(M_t | \mathcal{F}_s) = M_s$  for all  $s, t$  with  $s \leq t \leq T$ . If  $M$  and  $-M$  are both  $G$ -martingales  $M$  is called a symmetric  $G$ -martingale. This terminology also applies to general sublinear expectations as those in Section 3.2.

By means of the characterization of the conditional  $G$ -expectation we have that  $M$  is a  $G$ -martingale if and only if for all  $0 \leq s \leq t \leq T, P \in \mathcal{P}$ ,

$$M_s = \text{ess sup}_{Q' \in \mathcal{P}(s, P)} E^{Q'}[M_t | \mathcal{F}_s] \quad P - a.s.$$

In Song (2011), this identity declares that a  $G$ -martingale  $M$  can be seen as a multiple prior martingale which is a supermartingale for any  $P \in \mathcal{P}$  and a martingale for an optimal measure.

*Characterization for  $G$ -martingales*, Soner, Touzi, and Zhang (2011a): Let  $x \in \mathbb{R}, z \in M_G^2(0, T)$  and  $\eta \in M_G^1(0, T)$ . Then the process

$$M_t := x + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds, \quad t \leq T,$$

is a  $G$ -martingale.

In particular, the nonsymmetric part  $-K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds$ ,  $t \in [0, T]$ , is a  $G$ -martingale which is different compared to classical probability theory since  $\{-K_t\}_{t \in [0, T]}$  is continuous, and non-increasing with a quadratic variation equal to zero.  $M$  is a symmetric  $G$ -martingale if and only if  $K \equiv 0$ .

*Martingale representation*, Song (2011): Let  $\xi \in L_G^2(\Omega_T)$ . Then the  $G$ -martingale  $X$  with  $X_t := E_G[\xi | \mathcal{F}_t]$ ,  $t \in [0, T]$ , has the following unique representation

$$X_t = X_0 + \int_0^t z_s dB_s - K_t$$

where  $K$  is a continuous, increasing process with  $K_0 = 0$ ,  $K_T \in L_G^\alpha(\Omega_T)$ ,  $z \in H_G^\alpha(0, T)$ ,  $\forall \alpha \in [1, 2)$ , and  $-K$  a  $G$ -martingale. Here,  $H_G^\alpha(0, T)$  is the completion of  $H_G^0(0, T)$  under the norm  $\|\eta\|_{H_G^\alpha} := \left( E_G \left[ \int_0^T |\eta_s|^2 ds \right]^{\frac{\alpha}{2}} \right)^{\frac{1}{\alpha}}$ .

If  $\xi$  is bounded from above we get that  $z \in M_G^2(0, T)$  and  $K_T \in L_G^2(\Omega_T)$ , see Song (2011).

Finally we establish a Girsanov type theorem with  $G$ -Brownian motion. Consequently we establish the result, and we discuss some heuristics in terms of a  $G$ -Doleans Dade exponential. Define the density process by  $\mathbf{E}^\theta$  as the unique solution of  $d\mathbf{E}_t^\theta = \mathbf{E}_t^\theta \theta_t dB_t^G$ ,  $\mathbf{E}_0^\theta = 1$ . The proof of the Girsanov theorem is based on a Levy martingale characterization theorem for  $G$ -Brownian motion.

*Girsanov for  $G$ -expectation*, Xu, Shang, and Zhang (2011): Assume the following Novikov type condition: There is an  $\varepsilon > 0$  such that

$$E_G \left[ \exp \left( \left( \frac{1}{2} + \varepsilon \right) \int_0^T \theta_s^2 d\langle B^G \rangle_s \right) \right] < \infty$$

Then  $B_t^\theta = B_t^G - \int_0^t \theta_s \langle B^G \rangle_s$  is a  $G$ -Brownian motion under the sublinear expectation  $\mathcal{E}^\theta(\cdot)$  given by,  $\mathcal{E}^\theta(X) = E_G[\mathbb{E}_T^\theta \cdot X]$ ,  $\mathcal{P}^\theta = \mathbb{E}_T^\theta \mathcal{P}$  with  $X \in L^2(\mathcal{P}^\theta)$ .

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