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The evolution of R&D networks

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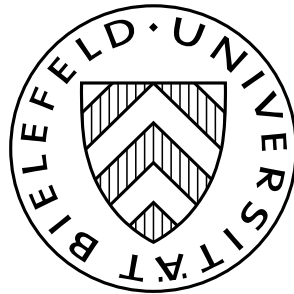
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467

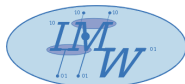
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The Evolution of R&D Networks

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Abstract

In this paper, we study a standard Cournot model where firms are able to form bilateral collaboration agreements which lower marginal cost. While a static analysis of such a model can be found in Goyal and Joshi [5], we introduce an evolutionary model. Stable networks (in the static sense) exhibit the dominant group architecture and can be characterized with respect to the size of the group. However, in contrast to Goyal and Joshi [5], we find that the group size of connected firms in stochastically stable networks is generically unique and monotonically decreasing in cost of link formation. Further, there exists a lower bound on the group size of connected firms such that a non-empty network can be stochastically stable.

JEL Classifications: C72, C73, L13, O30

Keywords: R&D Networks, Oligopoly, Stochastic Stability

1 Introduction

The formation of R&D networks, where firms cooperate with respect to their innovative activities, is an important feature of many industries (see e.g. Hagedoorn [7], Powell et al. [13], Roijakkers and Hagedoorn [14]). In many cases the firms cooperating on the R&D level are competitors in the market, which gives rise to intricate strategic considerations when selecting R&D cooperation partners. Given the empirical evidence of R&D collaborations it is important to gain a sound understanding of the factors determining the structure of R&D networks. From a theoretical perspective Goyal and Joshi [5] have studied the structure of pairwise Nash stable (PNS) R&D

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networks in a seminal contribution. They consider a setting where links reduce marginal production costs of firms, which compete a la Cournot. They show that the pairwise Nash stable (PNS) networks exhibit the dominant group architecture (with one completely connected group and all other firms isolated). However, a wide range of these types of networks (with respect to the size of the dominant group) may be PNS. And although, the sizes of the dominant group are sensitive to the cost of link formation, there is no unique prediction with respect to the networks which will be observed. Moreover surprisingly, the minimal size of the component in a non-empty network is increasing in the cost of link formation for a certain cost range. The analysis in Goyal and Joshi [5] is static and although dynamic models of R&D network formation have recently been provided in different economic frameworks (see e.g. Baum et al. [1], König et al. [11, 12]), a dynamic analysis of the standard Cournot setting considered in Goyal and Joshi [5] is so far missing.

In this paper, we fill this gap and focus on the dynamics of R&D networks in a Cournot oligopoly. We assume in this two stage game, where decisions about links are made in the first stage and quantities are chosen in the second, that interaction on the collaboration network is faster than the dynamics of the networks. This implies in our model that the unique Nash equilibrium of the second stage, i.e. the equilibrium choice of quantities, is immediately established.¹ For the evolution of collaboration links, we employ the dynamic model of network formation by Jackson and Watts [9]. In this framework, each link is considered one by one and the decision makers play a myopic best-reply to the current state with high probability and make mistakes with low probability. The resulting stochastically stable networks select among the pairwise stable networks, and are those which are observed most of the time in this dynamic model.

In the main result of our paper, we characterize the set of stochastically stable networks. Trivially they also exhibit the dominant group architecture. The size of the dominant group is monotonically decreasing in the cost of link formation, solving the puzzle of non-monotonicity in Goyal and Joshi [5]. Moreover, we find a generically unique prediction with respect to the group size of the stochastically stable networks. Further, we show that there exists a threshold of the dominant group size, below which only the empty network can be stochastically stable. This result has interesting connections to analytical findings on efficient networks; e.g. in a similar two stage game, Westbrook [16] studies the efficient networks and also concludes that either the empty network is efficient or there exists a lower threshold on the size of the dominant group for efficient networks that are pairwise Nash stable. Finally, it is worth noting that stochastically stable networks may lie outside

¹In a model with multiple second stage equilibria a slow-fast dynamic needs to be modeled explicitly as e.g. in Dawid and MacLeod [4].

the set of PNS networks which are considered in Goyal and Joshi [5].

2 The Model

A set of $N = \{1, \dots, n\}$ ex ante identical firms participates in a two stage game. To exclude uninteresting cases we assume $n \geq 3$. Firms first form bilateral agreements of collaboration. We denote by $g^n := \{\{i, j\} | i, j \in N, i \neq j\}$ the set of all possible collaboration agreements, which we call the complete network. The set of all undirected networks is given by $G = \{g : g \subseteq g^n\}$. For notational convenience we denote by $ij = ji := \{i, j\} \in g$ a collaboration link between firm i and firm j in network g . Given a network $g \in G$, the neighbors of player i are represented by the set $N_i(g) := \{j \in N \mid ij \in g\}$. We denote by $\eta_i(g) := |N_i(g)|$ the degree of firm i and by $\eta_{-i} := \sum_{j \neq i} \eta_j$ the sum of all other firms' degree. For a network $g \in G$ and a set of links $l \subseteq g^n \setminus g$ (which is also a network) let $g + l := g \cup l$ be the network obtained by adding the links l to network g . Similarly, let $g - l := g \setminus l$ denote the network obtained by deleting the set of links $l \subseteq g$ from network $g \in G$. Collaboration links can be interpreted as R&D agreements lowering marginal costs of producing the homogeneous good. However, maintenance of links is costly, with constant cost f per formed link.

In the second stage, firms compete in the market by choosing quantities.² We assume that marginal cost of producing the homogeneous good is constant for each firm and for $i \in N$ given by $c_i(g) = \gamma_0 - \gamma\eta_i(g)$ with $\gamma < \frac{\gamma_0}{n-1}$. Let $q_i \in \mathbb{R}_+$ be the quantity chosen by firm i and let $q = (q_1, \dots, q_n) \in \mathbb{R}_+^n$ be the profile of quantities chosen. We assume that market demand is linear and given by $P(q) = \max[0, \alpha - \sum_{j \in N} q_j]$. Assuming positive prices, the profit of firm $i \in N$ in the second stage can be derived to be, $\tilde{\pi}_i(q, g) = (\alpha - \sum_{j \in N} q_j)q_i - q_i c_i(g)$. Taking the network g as given, firms try to maximize profits. The interior Cournot equilibrium can be calculated to be,

$$q_i(g)^* := \frac{(\alpha - \gamma_0) + n\gamma\eta_i(g) - \gamma \sum_{j \neq i} \eta_j(g)}{n + 1},$$

which is strictly positive assuming $\alpha - \gamma_0 - \gamma(n-1)(n-2) > 0$. Thus, in equilibrium of the second stage, profits are $\tilde{\pi}_i(g) = (q_i(g)^*)^2$. Adjusting for the cost of link formation and noting that payoff only depends on the degree distribution, we denote, abusing notation:

$$\pi_i(\eta_i, \eta_{-i}) := \pi_i(g) := \frac{((\alpha - \gamma_0) + n\gamma\eta_i(g) - \gamma\eta_{-i}(g))^2}{(n + 1)^2} - \eta_i(g)f. \quad (1)$$

²A more detailed derivation of the second stage equilibria can be found in Goyal and Joshi [5].

So far the static model is in line with Goyal and Joshi [5]. We now present a dynamic model of network formation. We assume that adjustment in the quantity choice stage is fast compared to the rate by which changes in the network occur. Thus, we consider a dynamic model of network formation such that the (unique) equilibrium in the second stage is immediately adapted for each change in the network.³ To model the network dynamics we employ the stochastic process introduced by Jackson and Watts [9]: time is discrete $t = 0, 1, \dots$ and at $t = 0$ an arbitrary network is given. We denote the network at time $t \in \mathbb{N}$ by g_t . At each point in time t , one link is selected by a probability distribution which is identical and independent over time with full support, i.e. $p(ij) > 0$ for all $ij \in g^n$. If the selected link is already contained in g_t then both firms decide to keep or delete the link and if not both firms decide whether to add or not to add the link. These decisions are myopic and based on marginal payoffs from the given link, $\Delta_i^+(\eta_i, \eta_{-i}) := \pi_i(\eta_i + 1, \eta_{-i} + 1) - \pi_i(\eta_i, \eta_{-i})$ and $\Delta_i^-(\eta_i, \eta_{-i}) := \pi_i(\eta_i, \eta_{-i}) - \pi_i(\eta_i - 1, \eta_{-i} - 1)$ which can be calculated to be:⁴

$$\begin{aligned}\Delta_i^+(\eta_i, \eta_{-i}) &= \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) + \gamma(n-1) + 2\gamma n \eta_i - 2\gamma \eta_{-i} \right] - f \\ \Delta_i^-(\eta_i, \eta_{-i}) &= \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) - \gamma(n-1) + 2\gamma n \eta_i - 2\gamma \eta_{-i} \right] - f\end{aligned}$$

The link $ij \notin g_t$ is then added if $\Delta_i^+(\eta_i, \eta_{-i}) > 0$ and $\Delta_j^+(\eta_j, \eta_{-j}) \geq 0$, while it is not added else. Similarly a link $ij \in g_t$ is kept if $\Delta_k^-(\eta_k, \eta_{-k}) \geq 0$ for both $k \in \{i, j\}$, while it is deleted else. With high probability $1 - \epsilon$ the decision of the players is implemented while with low probability ϵ the decision is reversed, which can be interpreted as firms making a mistake or a mutation. The such defined stochastic process is an ergodic Markov process on the state space of G with unique limit distribution μ^ϵ depending on the probability of mistakes. The networks $g \in G$ such that $\lim_{\epsilon \rightarrow 0} \mu^\epsilon(g) > 0$ are called *stochastically stable* (see e.g. Young [17]). By construction the absorbing states of the unperturbed process (for $\epsilon = 0$) are the pairwise stable networks (PS), i.e. the networks $g \in G$ such that for all $i \in N : \Delta_i^-(\eta_i(g), \eta_{-i}(g)) \geq 0$ and for all pairs $i, j \in N$ with $ij \notin g$: $\Delta_i^+(\eta_i(g), \eta_{-i}(g)) > 0 \Rightarrow \Delta_j^+(\eta_j(g), \eta_{-j}(g)) < 0$.⁵ The condition for pairwise stability is weaker than that for pairwise Nash stability, used in Goyal

³Although it is well known that the equilibrium in multi-firm Cournot oligopolies is unstable under a standard best response dynamics due to overshooting (see Theocharis [15]), assuming a certain degree of inertia in the dynamics makes the equilibrium stable (see Dawid [3]) and in our analysis it is implicitly assumed that the inertia in quantity adjustment is sufficiently large such that the unique Cournot equilibrium is reached for any given R&D network.

⁴For notational convenience we will drop the dependence of $\eta_i(g_t)$ on g_t whenever the reference is clear.

⁵The definition of Jackson and Wolinsky [10] is adapted here to our framework.

and Joshi [5], since PNS requires that in addition to the PS conditions that $\pi_i(\eta_i, \eta_{-i}) - \pi_i(0, \eta_{-i} - \eta_i) \geq \eta_i f$ holds for all $i \in N$.

3 The Evolution of Collaboration Networks

In order to study the set of stochastically stable networks, we first characterize the set of pairwise stable networks. Note that

$$\Delta_i^+(\eta_i + k, \eta_{-i} + k) - \Delta_i^+(\eta_i, \eta_{-i}) = \frac{2k\gamma^2(n-1)^2}{(n+1)^2} > 0 \quad (2)$$

which implies that $\pi_i(g)$ is convex in own links and

$$\Delta_i^+(\eta_i, \eta_{-i} + 2k) - \Delta_i^+(\eta_i, \eta_{-i}) = -\frac{4k\gamma^2(n-1)}{(n+1)^2} < 0 \quad (3)$$

which implies that $\pi_i(g)$ satisfies the strategic substitutes property.⁶ From the convexity property (and ex ante identical firms), it follows directly that only networks with dominant group architecture such that there exists one completely connected group of firms of size k and all other firms isolated, denoted by g^k , can be pairwise stable.⁷ In the following the pairwise stable networks g^k are characterized in terms of the size of the dominant group.

Proposition 1. *There exist numbers $(0 <) F_0 < F_1 < F_2$ with the following properties:*

1. for $f < F_0$ the complete network g^n is the unique PS network,
2. for $F_0 \leq f < F_1$, there exists $\underline{k}(f) \in \mathbb{N}$, $1 < \underline{k}(f) < n$ such that $PS = \{g^{\underline{k}(f)}, \dots, g^n\}$,
3. for $f = F_1$ we have $PS = \{g^1, \dots, g^n\}$,
4. for $F_1 < f \leq F_2$, there exists $\underline{k}(f), \bar{k}(f) \in \mathbb{N} : 1 < \underline{k}(f) \leq \frac{n+2}{2} \leq \bar{k}(f) < n$ and $\underline{k}(f) + \bar{k}(f) = (n+2)$ such that $PS = \{g^1, g^{\underline{k}(f)}, \dots, g^{\bar{k}(f)}\}$,
5. for $f > F_2$ the empty network g^1 is the unique PS network.

The pattern of pairwise stable networks exhibits similar structure as the pattern of pairwise Nash stable networks in Goyal and Joshi [5]. In fact the

⁶Convexity captures a positive externality of own links and strategic substitutes captures a negative externality of other firms' links on marginal utility of a given link. For formal definitions of these properties see, among others, Goyal and Joshi [6] and Hellmann [8].

⁷See Goyal and Joshi [6], Lemma 4.1 for an analogous statement for pairwise Nash stable (PNS) networks. The proof trivially also holds for pairwise stable networks.

set of PS networks contains the set of PNS networks.⁸ Two properties are notable when comparing the two sets. First, the cost threshold such that the complete network stops being PS coincides with the threshold such that the empty network starts becoming PS. Second, for the non-monotonicity part of $k(f)$, i.e. $F_1 < f < F_2$, the minimal k and the maximal k such that g^k , $k \neq 1$ is PS are symmetric around $\frac{n+2}{2}$. These two observations do not hold for the PNS networks in Goyal and Joshi [6]. For an illustration of the PS and PNS networks, see also Figure 1.

Proposition 1 completely characterizes the set of pairwise stable networks. The only other possible recurrent classes of the unperturbed process (s.t. $\epsilon = 0$) are closed cycles.⁹ The following Lemma shows, that there do not exist closed cycles in our model.

Lemma 1. *In the model of collaboration networks where payoff satisfies (1), there does not exist a closed cycle.*

Thus the only recurrent classes are the singleton states of pairwise stable networks. We now employ the techniques by Jackson and Watts [9] to find the stochastically stable networks. Since the set of stochastically stable networks is the set of networks with minimal stochastic potential, this requires the computation of the stochastic potential of a network. The stochastic potential is defined as the sum of all transition costs of the minimal cost (directed) tree connecting all networks, where the transition cost between two networks is given by the minimal number of mutations to move from one network to another. Since all other states are transient, we may restrict the construction of the minimal cost tree to the set of pairwise stable networks, i.e. we construct minimal resistance trees for each g^k . To denote the transition costs for $k \geq 2$, let $c^+(k)$ denote the minimal number of mutations necessary to move from g^k to g^{k+1} and let $c^-(k)$ denote the minimal number of mutations necessary to move from g^k to g^{k-1} . Moreover, denoting

$$\kappa(k) := \arg \min_{\tilde{k} \in \{0 \dots k\}} \left(\Delta_i^+(\tilde{k}, k(k-1) + \tilde{k}) \geq 0 \right), \quad (4)$$

we get $c^+(k) = \kappa(k)$ and $c^-(k+1) = k - \kappa(k)$, which is proved in Lemma 2.

Lemma 2. *Let $k \geq 2$ and let g^k and g^{k+1} be pairwise stable. Then the minimal number of mistakes to move from g^k to g^{k+1} is given by $c^+(k) = \kappa(k)$ and minimal number of mistakes to move from g^{k+1} to g^k is given by $c^-(k+1) = k - \kappa(k)$.*

Lemma 2 shows that the number of mistakes necessary to move between two dominant group networks, g^k , g^{k+1} is determined by $\kappa(k)$. For a PS network g^k , the number $\kappa(k)$ is the minimal number of links an isolated firm

⁸This holds trivially due to the definition of PS and PNS, see Bloch and Jackson [2].

⁹For a definition of improving paths and closed cycles, see Jackson and Watts [9].

needs to be given in order to have an incentive to form a link. Note that a firm in the dominant group always has an incentive to form a link.

From Proposition 1 we have for $f < F_1$ that if g^k and $g^{\tilde{k}}$ are PS for $k < \tilde{k}$ then also $g^{k'}$ is PS for all $k' \in \mathbb{N}$ such that $k < k' < \tilde{k}$. The only case of there being a gap (in terms of the size k) between two pairwise stable networks g^k is for g^1 and $g^{\underline{k}(f)}$ if $F_1 < f < F_2$. Thus we get that the stochastic potential of a network g^k with $k \geq 2$ is given by

$$r(g^k) = c(g^1, g^{\underline{k}(f)}) + \sum_{l=\underline{k}(f)}^{k-1} c^+(l) + \sum_{k+1}^{\bar{k}(f)} c^-(l),$$

where $\underline{k}(f)$ and $\bar{k}(f)$ is the minimal respectively maximal number $k \in \{2, \dots, n\}$ such that g^k is pairwise stable and $c(g^1, g^{\underline{k}(f)})$ is the minimal number of mistakes to move from the empty network to $g^{\underline{k}(f)}$, which is set to 0 if the empty network is not pairwise stable. Denoting by $\Delta^r(k)$ the difference in stochastic potentials between two networks, g^k and g^{k+1} , $k \geq 2$, we get:

$$\Delta^r(k) := r(g^{k+1}) - r(g^k) = 2\kappa(k) - k.$$

To characterize stochastically stable networks in Proposition 2, we show first that $\Delta^r(k)$ is weakly decreasing in $k \in \mathbb{N}$ up to $k = \frac{n-1}{4}$ and then weakly increasing. Thus, the network(s) g^k which satisfy the necessary condition, $\Delta^r(k) \geq 0$ and $\Delta^r(k-1) \leq 0$,¹⁰ are the only candidates for stochastic stability besides the empty and complete network. In the following we characterize the stochastically stable networks.

Proposition 2. *There exist numbers $F_0^*, F_1^* \in \mathbb{R}$ such that $F_0 < F_0^* < F_1 < F_1^* < F_2^* < F_2$ such that:*

1. *for $f < F_0^*$ the complete network g^n is uniquely stochastically stable.*
2. *for $F_0^* < f < F_1^*$ there exists a function $k^*(f) : [F_0^*, F_1^*] \mapsto \{\frac{n-1}{4}, n-1\}$ such that either the network g^{k^*} is uniquely stochastically stable or g^{k^*} and g^{k^*+1} are the only stochastically stable networks. Moreover, $k^*(f)$ is weakly decreasing in f .*
3. *for $F_1^* \leq f \leq F_2^*$ the empty network and the network g^{k^*} (respectively the networks g^{k^*} and g^{k^*+1}) are stochastically stable.*
4. *For $f > F_2^*$ the empty network g^1 is uniquely stochastically stable.*

The proof is presented in the appendix. It may be helpful to illustrate the result of Proposition 2 by Figure 1.

¹⁰Since $\Delta^r(k) \geq 0$ implies $r(g^{k+1}) \geq r(g^k)$ and $\Delta^r(k-1) \leq 0$ implies $r(g^{k-1}) \geq r(g^k)$, and the stochastically stable networks are those which minimize stochastic potential.

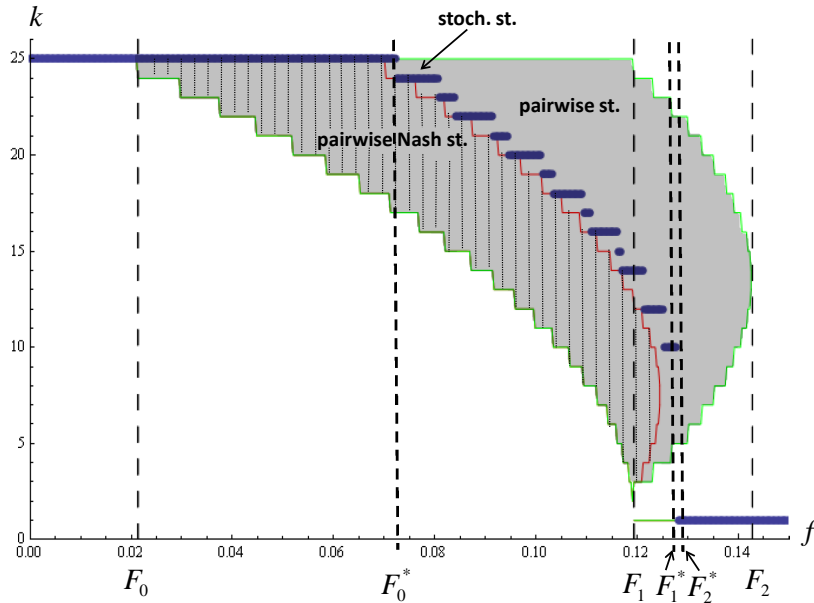


Figure 1: The set of pairwise stable (gray area), pairwise Nash stable (ruled area) and stochastically stable networks (blue).

The stochastically stable networks follow a clear pattern. First, the size of the connected component in stochastically stable networks is (weakly) decreasing with cost of link formation, although the sizes of PS and PNS networks exhibit a non-monotonicity property for a certain cost range. Second, there exists a lower bound of the component size of stochastically stable networks. Third, as Figure 1 indicates, the stochastically stable networks may lie outside the set of PNS networks characterized by Goyal and Joshi [5].¹¹ Thus, the dynamics of network formation introduced by Jackson and Watts [9] may converge to networks which are not pairwise Nash stable, i.e. where firms would be better off deleting all their links. The reason for this is that multiple link decisions are not considered in the dynamic model by Jackson and Watts [9]. A motivation for such a dynamics may be in our context that link revision opportunities only arrive at certain times due to long lasting contracts (for existing links) or occasionally meetings between firms (to create new links). Therefore a model where each link is considered one by one and firms behave myopically is reasonable. From a more general perspective, the observation that stochastically stable networks might not be pairwise Nash stable, shows that this concept can in general not be supported by a dynamic foundation, which has the usual properties of evolutionary dynamics, that changes in the state from one period to the next

¹¹The parameter constellation underlying this figure is $n = 25, \gamma_0 = 2, \gamma = 0.05, \alpha = 35$.

are local.

Comparing the stochastically stable networks characterized here with the welfare maximizing networks (see Westbrock [16]) shows that no general statements can be made whether the stochastically stable networks are more or less dense than the efficient ones. Numerical calculations not presented here show that for different values of link formation costs both cases can occur.

4 Conclusion

Considering a stochastic evolutionary process of network formation for collaboration networks between firms who later compete in a Cournot oligopoly, we find that the long–run equilibria, i.e. the stochastically stable networks, exhibit interesting properties. First, we get a generically unique selection of the pairwise stable networks. Second, the size of the dominant group is monotonically decreasing in the cost of link formation. For a certain cost range, static stability notions, like pairwise stable and pairwise Nash stable networks, do not exhibit such a monotonicity property. Third, there exists a lower threshold on the size of the dominant group such that below that threshold only the empty network is pairwise stable. This may be interpreted in a way such that there needs to be a number of firms to join a certain project in order for the project to succeed in the long–run. Interestingly, our fourth observation of the long–run equilibria is that they usually are not contained in the set of pairwise Nash stable networks. Thus, even though firms may be better off leaving the dominant group, in the long–run the large networks survive.

Acknowledgments

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References

- [1] Baum, J., Cowan, R., and Jonard, N. (2010). Network-independent partner selection and the evolution of innovation networks. *Management Science*, 56:2094–2110.
- [2] Bloch, F. and Jackson, M. (2006). Definitions of equilibrium in network formation games. *International Journal of Game Theory*, 34(3):305–318.
- [3] Dawid, H. (1999). On the stability of monotone discrete selectiondynamics with inertia. *Mathematical Social Sciences*, 37:265–280.
- [4] Dawid, H. and MacLeod, W. B. (2008). Hold-up and the evolution of investment and bargaining norms. *Games and Economic Behavior*, 62(1):26–52.
- [5] Goyal, S. and Joshi, S. (2003). Networks of collaboration in oligopoly. *Games and Economic Behavior*, 43(1):57–85.
- [6] Goyal, S. and Joshi, S. (2006). Unequal connections. *International Journal of Game Theory*, 34(3):319–349.
- [7] Hagedoorn, J. (2002). Inter-firm r&d partnerships: an overview of major trends and patterns since1960. *Research Policy*, 31:477–492.
- [8] Hellmann, T. (2012). On the existence and uniqueness of pairwise stable networks. *International Journal of Game Theory*, forthcoming.
- [9] Jackson, M. O. and Watts, A. (2002). The evolution of social and economic networks. *Journal of Economic Theory*, 106(2):265–295.
- [10] Jackson, M. O. and Wolinsky, A. (1996). A Strategic Model of Social and Economic Networks. *Journal of Economic Theory*, 71(1):44–74.
- [11] König, M. D., Battiston, S., Napoletano, M., and Schweitzer, F. (2011). Recombinant knowledge and the evolution of innovation networks. *Journal of Economic Behavior and Organization*, 79:145–164.
- [12] König, M. D., Battiston, S., Napoletano, M., and Schweitzer, F. (2012). The efficiency and stability of r&d networks. *forthcoming in Games and Economic Behavior*.
- [13] Powell, W. W., White, D. R., Koput, K. W., and Owen-Smith, J. (2005). Network dynamics and field evolution: The growth of interorganizational collaboration in the life sciences. *American Journal of Sociology*, 110:1132–1205.

- [14] Roijackers, N. and Hagedoorn, J. (2006). Inter-firm r&d partnering in pharmaceutical biotechnology since 1975: Trends, patterns, and networks. *Research Policy*, 35:431–446.
- [15] Theocharis, R. (1960). On the stability of the cournot solution on the oligopoly problem. *Review of Economic Studies*, 27:133–134.
- [16] Westbrook, B. (2010). Natural concentration in industrial research collaboration. *RAND Journal of Economics*, 41(2):351–371.
- [17] Young, H. P. (1998). *Individual Strategy and Social Structure*. Princeton University Press, Princeton.

APPENDIX

Proof of Proposition 1

Since only the networks of type g^k (with one completely connected component of size k and all other firms isolated) can be pairwise stable,¹² we only have to consider incentives to add a link for an isolated player or incentives to delete a link for a connected player. In a network g^k , an isolated firm has no incentive to add a link if $\Delta_i^+(0, k(k-1)) < 0$ and a connected firm has no incentive to delete a link if $\Delta_i^-(k-1, (k-1)^2) \geq 0$. Note that we have

$$\begin{aligned} & \Delta_i^+(0, k(k-1)) < 0 \\ \Leftrightarrow & \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) + \gamma(n-1) - 2\gamma k(k-1) \right] < f \end{aligned} \quad (5)$$

and

$$\begin{aligned} & \Delta_i^-(k-1, (k-1)^2) > 0 \\ \Leftrightarrow & \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) - \gamma(n-1) + 2\gamma(k-1)(n+1-k) \right] > f \end{aligned} \quad (6)$$

Thus, the complete network is stable as long as $\Delta_i^-(n-1, (n-1)^2) \geq 0$ which implies that $f \leq F_1 := \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) + \gamma(n-1) \right]$ and the empty network is stable as long as $\Delta_i^+(0, 0) \leq 0$ which implies that $f \geq \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) + \gamma(n-1) \right] = F_1$. Moreover, note that $\Delta_i^+(0, k(k-1))$ is strictly decreasing for $k \geq 1$ and $\Delta_i^-(k-1, (k-1)^2)$ is strictly increasing for $k < \frac{n+2}{2}$ and strictly decreasing for $k > \frac{n+2}{2}$. In particular we then get that for $f = F_1$ all networks g^k are pairwise stable. Since $\Delta_i^+(0, k(k-1))$ is strictly decreasing for $k \geq 1$, the complete network is uniquely pairwise stable if $\Delta_i^+(0, (n-1)(n-2)) > 0$ which implies that $f < F_0 := \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) + \gamma(n-1)(5-2n) \right]$. Moreover, since $\Delta_i^-(k-1, (k-1)^2)$ attains its maximum at and is symmetric around $k = \frac{n+2}{2}$ and since k can only adopt natural numbers between 1 and $n-1$, we get that the empty network is uniquely pairwise stable if $\Delta_i^-\left(\lceil \frac{n+2}{2} \rceil - 1, (\lceil \frac{n+2}{2} \rceil - 1)^2\right) < 0$ which implies that $f > F_2 := \frac{\gamma(n-1)}{(n+1)^2} \left[2(\alpha - \gamma_0) - \gamma(n-1) + 2\gamma(\lceil \frac{n}{2} \rceil)(n - \lceil \frac{n+2}{2} \rceil) \right]$. The remainder of the statement follows straightforwardly from the slope of $\Delta_i^+(0, k(k-1))$ and $\Delta_i^-(k-1, (k-1)^2)$.¹³ \square

Proof of Lemma 1 We show that from any network $g \in G$ there exists an improving path to a pairwise stable network.¹⁴ Without loss of gener-

¹²Goyal and Joshi [5], Lemma 4.1 trivially also holds for pairwise stability.

¹³See also Figure 1 for an illustration. The proof here is rather kept concise. Goyal and Joshi [5] provide a more elaborate proof for the result on PNS networks, see Goyal and Joshi [5], Proposition 4.1.

¹⁴For the definitions of improving paths and cycles, see Jackson and Watts [9]

ality the players are ordered to size of η_i , such that $\eta_1 \geq \eta_2 \geq \dots \geq \eta_n$ (otherwise reorder according to a permutation). By convexity and strategic substitutes we have that also $\Delta_i^+(\eta_i(g), \eta_{-i}(g)) \geq \Delta_j^+(\eta_j(g), \eta_{-j}(g))$ and $\Delta_i^-(\eta_i(g), \eta_{-i}(g)) \geq \Delta_j^-(\eta_j(g), \eta_{-j}(g))$ for all $i > j$. We now employ the following algorithm which preserves the order: If there exists players $j \in N$ who want to delete a link, i.e. such that $\Delta_j^-(\eta_j(g), \eta_{-j}(g)) < 0$, then start with the player with the largest number k for whom $\eta_k(g) > 0$, i.e. start with the player $k \in N$ with smallest positive η_k which implies $\Delta_k^-(\eta_k(g), \eta_{-k}(g)) < 0$ since Δ_k^- is lowest among all players with positive $\eta_k(g)$. By convexity, k has an incentive to then delete all of his links. Continue with players deleting all of their links according to this order (starting from the last player such that $\Delta_k^-(\eta_k(g), \eta_{-k}(g)) < 0$ and $\eta_k(g) > 0$) until this is no longer possible. Then this network is pairwise stable or there exists players who want to add links. Due the predefined order and the fact that $\Delta_i^+(\eta_i, \eta_{-i})$ and $\Delta_i^-(\eta_i, \eta_{-i})$ are increasing in η_i and decreasing in η_{-i} this can only be the players with connections. Now start with the players first in order and add links for any player until one recipient declines a connection. Either that network is pairwise stable or we apply the procedure again by deleting links from the last player in order such that $\eta_k > 0$. Since the number of players is finite, the algorithm finally terminates at a pairwise stable network g^k . \square

Proof of Lemma 2 Note that in any pairwise stable network g^k , $k \neq n - 1$ any connected player wants to form a link with an isolated player $i \in N$ by Proposition 1. However, since g^k is assumed to be pairwise stable, we have that $\Delta_i^+(0, k(k - 1)) < 0$, i.e. the isolated players decline the connection. Note that there are two ways to increase player i 's incentive to form a link, by deleting links between two connected players because of strategic substitutes or build links between i and connected players. However, the effect of the former is dominated by the latter since (2) and (3) holds. Thus, player i wants to form links by himself as soon as she has formed $\arg \min_{\tilde{k} \in \{0 \dots k\}} \left(\Delta_i^+(\tilde{k}, k(k - 1) + \tilde{k}) \geq 0 \right)$ links. Note that because of convexity in own links and strategic substitutes the connected players still want a link with player i , implying that there exists a zero resistance path to the network g^{k+1} . The other direction is analogous. \square

Proof of Proposition 2 If there exists a unique pairwise stable network, then it follows directly that it is stochastically stable. Hence, Proposition 1 directly implies that the fully connected network is stochastically stable for $f < F_0$ and the empty network for $f > F_2$. Hence we restrict attention to $F_0 \leq f \leq F_2$. In this range there are several PS networks and the subset of the PS networks with minimal stochastic potential gives the set of stochastically stable networks (see e.g. Young [17]).

Let us first compare the stochastic potential of networks g^k with $k \geq \underline{k}$. The stochastic potential of a PS network g^k , $k \in [\underline{k}(f), \bar{k}(f)]$ is given by $r(g^k) = c(g^1, g^k) + \sum_{l=\underline{k}}^{k-1} c^+(l) + \sum_{l=k+1}^{\bar{k}} c^-(l)$, where $c(g^1, g^{\underline{k}(f)})$ is the minimal number of mistakes necessary to move from the empty network g^1 to $g^{\underline{k}(f)}$, which is zero if g^1 is not pairwise stable and where $\bar{k}(f) = n$, if the complete network is pairwise stable. By Lemma 2 we have for the difference in stochastic potential between two adjacent PS networks $\Delta^r(k) = r(g^{k+1}) - r(g^k) = 2\kappa(k) - k$, where $\kappa(k)$ is given by (4). In order to characterize the discrete $\kappa(k)$ let $h(k)$ be the implicit function $h(k) := \{h \in \mathbb{R} : \Delta_i^+(h, k(k-1) + h) = 0\}$. Since $\Delta_i^+(h, k(k-1) + h)$ is strictly increasing in h the solution is unique for every $k \in \{1, \dots, n-1\}$. Solving for h we get

$$h(k) = \frac{k^2 - k}{n-1} + \frac{f - 2(\alpha - \gamma_0) - \gamma(n-1)}{2\gamma(n-1)}, \quad (7)$$

and hence $\kappa(k) = \lceil h(k) \rceil$, if $0 \leq h(k) \leq k$. Otherwise if $h(k) < 0$ then $\kappa(k) = 0$ and if $h(k) > k$ then $\kappa(k) = k$. Taking the continuous approximation of Δ^r we get $\tilde{\Delta}^r(k) = 2h(k) - k$ which yields $\frac{\partial \tilde{\Delta}^r(k)}{\partial k} = 2h'(k) - 1 = \frac{(4k-2)-n-1}{n-1}$. Thus, $\tilde{\Delta}^r(k)$ is strictly decreasing/increasing for $k < / > (n+1)/4$, has a global minimum at $k = (n+1)/4$ and $\tilde{\Delta}^r(k) < 0$ for $k = (n+1)/4$. Considering the continuous approximation $\tilde{\Delta}^r(k)$ the main intuition of the proof can be seen straightforwardly. For small enough f , we have $\tilde{\Delta}^r(k) < 0$ for all k implying that the complete network is stochastically stable. Otherwise there exists a unique $k^*(f)$ with $\tilde{\Delta}^r(k^*(f)) = 0$ and $\tilde{\Delta}^r(k) > 0$ for all $k > k^*(f)$. This means that $k^*(f)$ is a local minimizer of the stochastic potential $r(g^k)$. Moreover, $k^*(f) \geq \frac{n+1}{4}$. The only other candidate for a global minimizer is $k = 1$, i.e. the empty network. Since k is the size of the dominant group, k can only be an integer. Moreover, the number of mistakes $\kappa(k)$ can only take on integer values. In the following we therefore prove the statement by considering $\Delta^r(k) = 2\kappa(k) - k = 2\lceil h(k) \rceil - k$. Note that a necessary condition for a minimizer $k^*(f)$ of $r(g^k)$ is that $\Delta^r(k^*(f)) \geq 0$ (since this implies that $r(g^{k^*(f)+1}) \geq r(g^{k^*(f)})$) and $\Delta^r(k^*(f) - 1) \leq 0$ (since this implies that $r(g^{k^*(f)}) \leq r(g^{k^*(f)-1})$).

We first show the following auxiliary Lemmas which are helpful in restricting the set of possible minimizers of the stochastic potential. We then show the statement.

Lemma 3. *There does not exist a stochastically stable network g^k such that $2 \leq k \leq \frac{n-1}{4}$.*

Proof. Note that $h(k+1) - h(k) = \frac{2k}{n-1} \leq \frac{1}{2}$ if and only if $k \leq \frac{n-1}{4}$ which yields $\lceil h(k+1) \rceil - \lceil h(k) \rceil \leq 1$ for all $k \leq \frac{n-1}{4}$ and from $\lceil h(k) \rceil - \lceil h(k-1) \rceil = 1$ it follows that $\lceil h(k+1) \rceil - \lceil h(k) \rceil = 0$ for all $k \leq \frac{n-1}{4}$ and from $\lceil h(k+1) \rceil - \lceil h(k) \rceil = 1$ it follows that $\lceil h(k) \rceil - \lceil h(k-1) \rceil = 0$ for all $k \leq \frac{n-1}{4}$.

Suppose now that there exists a $2 \leq k \leq \frac{n-1}{4}$ such that g^k is stochastically stable. Necessary for stochastic stability of g^k is that $\Delta^r(k) \geq 0$ and $\Delta^r(k-1) \leq 0$.

First, let k be odd. Then $\Delta^r(k) \geq 0 \Leftrightarrow \lceil h(k) \rceil \geq \frac{k}{2}$ implies that both inequalities must be strict, since $\frac{k}{2} \notin \mathbb{Z}$. Then because of $k \leq \frac{n-1}{4}$ either $\lceil h(k-1) \rceil = \lceil h(k) \rceil$ which trivially implies that $\lceil h(k-1) \rceil > \frac{k-1}{2}$ or $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ which implies that $\lceil h(k-1) \rceil > \frac{k}{2} - 1$, and, hence $\lceil h(k-1) \rceil \geq \frac{k}{2} - \frac{1}{2}$ since $\lceil \frac{k}{2} - 1 \rceil = \frac{k}{2} - \frac{1}{2}$. Thus, $\Delta^r(k) \geq 0$ for k odd implies $\Delta^r(k-1) \geq 0$. Moreover if $\Delta^r(k-1) = 0$ then $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ and thus we have $\lceil h(k-1) \rceil = \lceil h(k-2) \rceil$ implying that $\Delta^r(k-2) > 0$, and hence, $r(g^{k-2}) < r(g^k)$, contradicting stochastic stability of g^k .

Now let k be even and suppose $\Delta^r(k) \geq 0$. First, consider $\Delta^r(k) > 0 \Leftrightarrow \lceil h(k) \rceil > \frac{k}{2}$. Thus, $\lceil h(k) \rceil \geq \frac{k}{2} + 1$ since $\frac{k}{2} \in \mathbb{Z}$. As above let $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ (the other case $\lceil h(k-1) \rceil = \lceil h(k) \rceil$ trivially implies $\Delta^r(k-1) > 0$.) Then $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1 \geq \frac{k}{2} > \frac{k-1}{2}$, implying $\Delta^r(k-1) > 0$. Finally suppose that $\Delta^r(k) = 0$. First, if $\lceil h(k-1) \rceil = \lceil h(k) \rceil$ then $\Delta^r(k-1) > 0$ and we are in the case above, where k is odd. Second if $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ then we must have $\lceil h(k+1) \rceil = \lceil h(k) \rceil$, implying that $\Delta(g^{k+1}) < 0$ which implies that $r(g^{k+2}) < r(g^{k+1}) = r(g^k)$, contradicting stochastic stability of g^k . \square

Lemma 4. *Assume that $\min_{k \in \{1, \dots, n-1\}} \Delta^r(k) < 0$ and $\Delta^r(n-1) \geq 0$. Then, there either exists a unique $k^*(f) \in \{\frac{n-1}{4}, \dots, n-1\}$ with $\Delta^r(k^* - 1) < 0$, $\Delta^r(k^*) > 0$, $\Delta^r(k) \leq 0$, $\forall \frac{n-1}{4} \leq k < k^* - 1$ and $\Delta^r(k) \geq 0$, $\forall k^* < k \leq n-1$ or a unique $k^*(f) \in \{\frac{n-1}{4}, \dots, n-1\}$ with $\Delta^r(k^*) = 0$, $\Delta^r(k^* - 1) < 0$, $\Delta^r(k^* + 1) > 0$, $\Delta^r(k) \leq 0$, $\forall \frac{n-1}{4} \leq k < k^* - 1$ and $\Delta^r(k) \geq 0$, $\forall k^* + 1 < k \leq n-1$. Furthermore, $k^*(f)$ is weakly decreasing with respect to f .*

Proof. We show first that $\Delta^r(k) > 0$ implies $\Delta^r(l) \geq 0$ for all $l > k$. Suppose that there is a $k > \frac{n-1}{4}$ such that $\Delta^r(k) > 0 \Leftrightarrow \lceil h(k) \rceil > \frac{k}{2}$. If k is even then $\frac{k}{2} \in \mathbb{Z}$ and hence $\lceil h(k) \rceil > \frac{k+1}{2}$ implying $\lceil h(k+1) \rceil \geq \lceil h(k) \rceil > \frac{k+1}{2}$, and thus $\Delta^r(k+1) > 0$. If k is odd then $\frac{k}{2} \notin \mathbb{Z}$ and hence $\lceil h(k) \rceil \geq \frac{k+1}{2}$ implying $\lceil h(k+1) \rceil \geq \lceil h(k) \rceil \geq \frac{k+1}{2}$, and thus $\Delta^r(k+1) \geq 0$. Note however that if $\Delta^r(k+1) = 0$ then it must be that $\lceil h(k+1) \rceil = \lceil h(k) \rceil$ implying that $\lceil h(k+2) \rceil \geq \lceil h(k+1) \rceil + 1$,¹⁵ and, hence, $\Delta^r(k+2) > 0$. Thus if $\Delta^r(k) > 0$ then $\Delta^r(l) \geq 0$ for all $l > k$.

Second, we note that $\min_{k \in \{\frac{n-1}{4}, \dots, n-1\}} \Delta^r(k) < 0$. Assume to the contrary that $\Delta^r(k) \geq 0$ for all $k \in \{\frac{n-1}{4}, \dots, n-1\}$. From Lemma 3 it then follows that $\Delta^r(k) \geq 0$ for all $k \in \{1, \dots, \frac{n-1}{4}\}$, since $\Delta^r(k) \geq 0$ for $k = \frac{n-1}{4}$ contradicting the the assumption $\min_{k \in \{1, \dots, n-1\}} \Delta^r(k) < 0$.

¹⁵Since for $k \geq \frac{n-1}{4}$, $h(k+1) - h(k) = \frac{2k}{n-1} \geq \frac{1}{2}$ and thus $\lceil h(k+1) \rceil - \lceil h(k) \rceil = 0$ implies $\lceil h(k+2) \rceil - \lceil h(k+1) \rceil = 1$.

Given that $\min_{k \in \{\frac{n-1}{4}, \dots, n-1\}} \Delta^r(k) < 0$ define $k^*(f)$ by $k^*(f) := 1 + \max [k \in \{\frac{n-1}{4}, \dots, n-1\} | \Delta^r(k) < 0]$.¹⁶ If $\Delta^r(k^*) > 0$ the statements of the first of the two cases given in the text of the Lemma follow directly from our arguments above. If $\Delta^r(k^*) = 0$, it follows, due to the definition of $\Delta^r(k)$, from $\Delta^r(k^*) = 0$ that $\Delta^r(k^* + 1) \neq 0$, and due to the definition of k^* we must have $\Delta^r(k^* + 1) > 0$. Similarly, we must have $\Delta^r(k^* - 1) < 0$. Hence, we obtain the statements concerning the second case given in the Lemma. Finally, the claim that k^* is weakly decreasing with respect to f follows from the observation that $h(k)$ is increasing in f , which implies that $\Delta^r(k)$ is weakly increasing in f . Accordingly, k^* decreases (weakly) as f is increased. \square

In order to prove the claims of the Proposition, we first observe that for sufficiently small values of f , where $\Delta^r(n-1) < 0$, we have that $\Delta^r(k) < 0$ for all $k \in \{1, \dots, n-1\}$ implying that the stochastic potential is minimized for the complete network and the complete network is the unique stochastically stable network.

In what follows we therefore focus on values of f where $\Delta^r(n-1) \geq 0$. We consider first the case $F_0 \leq f < F_1$. As shown in Proposition 1 the set of candidates for stochastically stable networks is given by $\{g^{\underline{k}(f)}, \dots, g^n\}$, where $\underline{k}(f)$ is the smallest k such that $\Delta_i^+(0, k(k-1)) < 0$. This property implies that for $k^*(f) < \underline{k}(f)$ we must have $c^+(k^*(f)) = 0$. Given that we have $c^-(k) > 0$ for all $k \in \{2, \dots, n\}$ and $f < F_1$, this implies $\Delta^r(k^*(f)) = c^+(k^*(f)) - c^-(k^*(f) + 1) < 0$, which contradicts Lemma 4. Hence, we must have $k^*(f) \geq \underline{k}(f)$ and therefore $\min_{\{k \in \{1, \dots, n-1\}\}} \Delta^r(k) < 0$. Direct application of Lemma 4 now establishes that among the PS networks the minimal stochastic potential is attained for g^{k^*} , if $\Delta^r(k^*) > 0$, or for each of the networks g^{k^*} and g^{k^*+1} , if $\Delta^r(k^*) = 0$.

Considering $F_1 \leq f < F_2$ we observe first that $r(g^1) - r(g^{\underline{k}(f)}) = c(g^{\underline{k}(f)}, g^1) - c(g^1, g^{\underline{k}(f)})$ is (weakly) decreasing in f and negative for sufficiently large f . On the one hand, we have that $c(g^1, g^{\underline{k}(f)})$ is (weakly) increasing in f , which follows because if $\Delta_i^+(\eta_i, \eta_{-i}) < 0$ for some f , then $\Delta_i^+(\eta_i, \eta_{-i}) < 0$ for all $f' > f$. Moreover, $\underline{k}(f)$ is increasing in f . The same argument implies that $c(g^{\underline{k}(f)}, g^1) = c^-(\underline{k}(f))$ is (weakly) decreasing in f . Obviously, we have $r(g^1) = 0$ for sufficiently large f , which implies that $r(g^1) - r(g^{\underline{k}(f)}) < 0$ for sufficiently large f . From the arguments above it follows that there exists an interval $[\tilde{f}_l, \tilde{f}_h]$ such that

$$r(g^1) - r(g^{\underline{k}(f)}) \begin{cases} > 0 & f < \tilde{f}_l \\ = 0 & f \in [\tilde{f}_l, \tilde{f}_h] \\ < 0 & f > \tilde{f}_h. \end{cases}$$

¹⁶For convenience we will drop the dependence on f .

As the next step of the proof we establish that $g^{\underline{k}(f)}$ is never stochastically stable. To this end, we show that $k^*(f) \geq \underline{k}(f)$ for all $f \leq \tilde{f}_h$. Given the (weak) monotonicity of $k^*(f)$ and $\underline{k}(f)$ it suffices to show this claim for $f = \tilde{f}_h$. Assume that $k^*(\tilde{f}_h) < \underline{k}(\tilde{f}_h)$. Then, $k^*(f)$ is not pairwise stable and thus we have $c(g^{k^*}, g^1) = 0$. Furthermore, due to the definition $k^*(\tilde{f}_h)$ it follows from Lemma 4 that $r(g^{\underline{k}(\tilde{f}_h)}) > r(g^{k^*(\tilde{f}_h)})$. This implies

$$r(g^1) \leq r(g^{k^*(\tilde{f}_h)}) + c(g^{k^*}, g^1) = r(g^{k^*(\tilde{f}_h)}) < r(g^{\underline{k}(\tilde{f}_h)})$$

and we obtain a contradiction to $r(g^1) = r(g^{\underline{k}(\tilde{f}_h)})$. Hence $k^*(f) \geq \underline{k}(f)$ for all $f \leq \tilde{f}_h$. Since, by definition g^{k^*} always has a lower stochastic potential than $g^{\underline{k}}$, this shows that the only candidates for stochastically stable networks are g^1 and g^{k^*} (sometimes together with g^{k^*+1}). Considering the difference in stochastic potential between these two networks we have

$$r(g^1) - r(g^{k^*(f)}) = c(g^{\underline{k}(f)}, g^1) - c(g^1, g^{\underline{k}(f)}) + \sum_{k=\underline{k}}^{k^*-1} (-\Delta^r(k))$$

We know already that the first term is (weakly) decreasing in f . For the sum, we know that for each k the term $(-\Delta^r(k))$ is decreasing in f . Furthermore, the number of summands (weakly) decreases for increasing f and each summand is non-negative, because of $k < k^*$. Altogether, we obtain that $r(g^1) - r(g^{k^*(f)})$ is weakly decreasing with respect to f . Arguments analogous to above establish that the difference is negative for sufficiently large f . The claims of the Proposition follow now directly by setting $F_0^* = \min[f | k^*(f) = n - 1]$, $F_1^* = \min[f | r(g^1) - r(g^{k^*(f)}) = 0]$, $F_2^* = \max[f | r(g^1) - r(g^{k^*(f)}) = 0]$. \square