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Continuous-time public good contribution under uncertainty

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Continuous-Time Public Good Contribution under Uncertainty*

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Abstract. We study a continuous-time problem of optimal public good contribution under uncertainty for an economy with a finite number of agents. Each agent can allocate his wealth between private consumption and repeated but irreversible contributions to increase the stock of some public good. We study the corresponding social planner problem and the case of strategic interaction between the agents and we characterize the optimal investment policies by a set of necessary and sufficient stochastic Kuhn-Tucker conditions. Suitably combining arguments from Duality Theory and the General Theory of Stochastic Processes, we prove an abstract existence result for a Nash equilibrium of our public good contribution game. Also, we show that our model exhibits a dynamic *free rider* effect. We explicitly evaluate it in a symmetric Black-Scholes setting with Cobb-Douglas utilities and we show that uncertainty and irreversibility of public good provisions do not affect free-riding.

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JEL subject classification: C02, C61, C62, C73.
Keywords: irreversible investment, singular stochastic control, first order conditions for optimality, stochastic games, Nash equilibrium, free-riding

1 Introduction

We study a very general stochastic continuous-time problem of optimal public good contribution under portfolio constraints for an economy with a fixed number of agents. Each agent chooses how to allocate his wealth between private consumption and repeated but irreversible contributions to increase the stock of some public good. In order to determine the (unique) efficient allocation we first consider the corresponding social planner problem. As in other settings, we will later see that its solution cannot be obtained by strategic interaction between the agents because of a classical *free rider effect*: agents enjoy the contributions of others but do not take into account other's benefits when making their own contributions (see, e.g., Cornes and Sandler

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[20] or Laffont [35]). Our analysis is supported by establishing a set of necessary and sufficient stochastic Kuhn-Tucker conditions for optimal investment policies, which in turn lead to the identification of a universal signal process that triggers the optimal public good contributions. We provide an explicit solution for a symmetric Black-Scholes-type setting with Cobb-Douglas utilities, which enables a detailed evaluation of the free-rider effect. In this case the level of uncertainty and irreversibility of public good provisions surprisingly do not influence free-riding. Finally we combine arguments from Duality Theory (cf., e.g., Rockafellar [46] for an overview) and the General Theory of Stochastic Processes (cf. Dellacherie and Meyer [21]) to prove an abstract existence result for a Nash equilibrium of our dynamic public good contribution game.

In the economic literature there is a long tradition of research on public good contribution and free rider problems started by the static analyses of Olson [42] and Samuelson [47], and further developed by Bergstrom, Blume and Varian [13], Groves and Ledyard [28] (in the context of a general equilibrium model), Palfrey and Rosenthal [43], [44], among others. Later on, free riding has received increasing attention and the first papers on public good contribution were generalized in many directions. We recall some of them. The dynamic free rider effect is formalized by Varian [50] in a complete information game of voluntary contribution to a public good. It is shown in Varian [50] that if agents contribute sequentially (that is, each agent contributes after observing the contributions made by the earlier agents), then the total contribution generated never exceeds the total contribution made by agents playing simultaneously, i.e. without observing the contribution made by any other player; Markov equilibria in dynamic free rider problems are tackled for instance in the early papers Fershtman and Nitzan [26] and Levhari and Mirman [36] in which the authors study equilibria in linear strategies for a differential game with quadratic costs and show the implications of the free rider effect for the steady state; a dynamic private provision of a discrete public good with imperfect information about individual actions is considered in Marx and Matthews [40]; a direct extension to a Bayesian setting of the model in Varian [50] is addressed in Bag and Roy [4]; irreversibility constraints on the public good contribution are introduced in the literature on 'monotone games' by assuming that players' individual actions can only increase over time. We refer to Lockwood and Thomas [37], Matthews [41] and, more recently, to Battaglini, Nunnari and Palfrey [12], among others. Several papers also considered public good provision problems under uncertainty. In Austen-Smith [2], for example, the authors argue that if uncertainty is modeled as a risk (additive uncertainty), then risk-aversion (concavity of utility) increases contributions to the public good. In Gradstein, Nitzan and Slutsky [27] and Sandler, Sterbenz and Posnett [48] it is instead shown that in a general equilibrium setting risk-aversion is not sufficient to guarantee that contributions to a public good increase. The provision would only increase if the marginal utility was concave as well. On the other hand, Eichberger and Kelsey [23] conclude that *free riders do not like uncertainty* when the latter is modeled as Knightian uncertainty rather than risk. Recently, the originally deterministic setting of Fershtman and Nitzan [26] has been extended by Wang [51]. A diffusion term is included in the controlled dynamics of project value and it is shown that the free rider effect is emphasized by uncertainty. Subgame consistent cooperative solutions for public good provisions by asymmetric agents with transferable payoffs in a stochastic differential game framework are finally considered in Yeung and Petrosyan [52].

Here we study a general stochastic, continuous-time public good contribution problem for an economy with a finite number of agents and with an irreversibility constraint on the public good contribution. In a symmetric Black-Scholes setting with Cobb-Douglas utilities we are able to

explicitly evaluate the free rider effect and thus to study the role played by the irreversibility of the public good contributions and the uncertainty in the model. From the mathematical point of view our problem falls into the class of continuous-time, optimal stochastic control problems with both monotone and absolutely continuous control processes. We analyze it by a first order condition approach that may be thought of as a stochastic, infinite-dimensional generalization of the classical Kuhn-Tucker conditions of real analysis. Our method does not require any Markovian or diffusive hypothesis, and in this sense it represents a substitute in non-Markovian frameworks for the Hamilton-Jacobi-Bellman equation. In the latest years several papers tackled singular stochastic control problems by means of such an approach. We refer to Bank and Riedel [10], [11] for an intertemporal utility maximization problem with Hindy, Huang and Kreps preferences; to Bank [6], Chiarolla and Ferrari [16], Ferrari [25] and Riedel and Su [45] for the irreversible investment problem of a monopolistic firm with both limited and unlimited resources; to Chiarolla, Ferrari and Riedel [17] for the social planner problem in a market with N firms and limited resources; to Steg [49] for a general capital accumulation game with open loop strategies.

We start analyzing the public good contribution problem by taking the point of view of a fictitious social planner who aims to maximize the expected total welfare of the economy. Assuming prices of the public good and the private consumption are given by discounted exponential martingales, we prove existence and uniqueness of the social planner's optimal policy. The optimal investment strategy is completely characterized in terms of necessary and sufficient stochastic Kuhn-Tucker conditions and it is given in terms of the unique solution of a backward stochastic equation in the spirit of Bank and El Karoui [7]. We then consider strategic interaction between the agents in our economy and we show that any Nash equilibrium is again the solution of a set of first order conditions for optimality. Suitably combining Duality Theory and the General Theory of Stochastic Processes, we are able to apply the Kakutani-Fan-Glicksberg Theorem to prove an abstract existence result for a Nash equilibrium (see our Definition 4.1 and Theorem 2.6 below).

From the economic point of view, it is worth to note that our model exhibits a dynamic free rider effect. We study it in detail comparing the explicit forms of the social planner's optimal policy and the Nash equilibrium in a symmetric Black-Scholes framework with Cobb-Douglas utilities. As a new interesting result, we show that irreversibility of the public good contributions and the level of uncertainty in the model do not influence free-riding.

The paper is organized as follows. In Section 2 we set up the model and we state our main findings. In Section 3 we consider the social planner problem, proving existence and uniqueness of its solution and introducing the stochastic Kuhn-Tucker conditions for optimality. The public good contribution game is addressed in Section 4 where we prove an abstract existence result for Nash equilibrium and discuss the dynamic free rider effect. Finally we refer to Appendix A for some technical proofs.

2 Model and Main Results

We consider a continuous-time stochastic economy with a finite number of agents over a fixed time horizon $T < +\infty$. Each agent, indexed by $i = 1, \ldots, n$, chooses how to allocate his wealth w^i between private consumption x^i and arbitrary but nondecreasing cumulative contributions C^i

to increase the stock of some public good. One may think that the agents are financed entirely by their labour or by holding a portfolio of financial instruments. Hence they are part of a more complex financial market that, however, we do not model explicitly. We assume a continuous revelation of information about an exogenous source of uncertainty and we allow the agents to condition their decisions on the accumulated information. Formally, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness. For the moment we do not make any Markovian assumption.

Let $\psi_x(t)$ and $\psi_c(t)$ denote the price of the private good and the price of the contribution to the public good at time t, respectively. Each agent can make his own investment choice by picking (x^i, C^i), $i = 1, \dots, n$, in the nonempty, convex set

$$\begin{aligned}\mathcal{B}_{w^i} := \Big\{ &(x^i, C^i) : \Omega \times [0,T] \mapsto \mathbb{R}^2_+ \text{ adapted, s.t. } C^i \text{ is right-continuous, nondecreasing,} \\ &C^i(0-) = 0 \ \mathbf{P}\text{-a.s., and } \mathbf{E}\Big[\int_0^T \psi_x(t)x^i(t)dt + \int_0^T \psi_c(t)dC^i(t)\Big] \le w^i \Big\}.\end{aligned} \tag{2.1}$$

Here $\mathbf{E}[\int_0^T \psi_x(t)x^i(t)dt + \int_0^T \psi_c(t)dC^i(t)] \le w^i$ defines the budget constraint of agent i, and prices are actually understood as state-price deflators. The agents are assumed to derive some expected, time-separable utility from the private good and the aggregate public good process $C := \sum_{i\in\{1,\dots,n\}} C^i$. Given a combination of strategies from $\prod_{i=1}^n \mathcal{B}_{w^i}$, agent i's utility is

$$U^i(x^i, C^i; C^{-i}) := \mathbf{E}\left[\int_0^T e^{-\int_0^t r(s)\,ds} u^i(x^i(t), C(t))\,dt\right], \tag{2.2}$$

where $C^{-i} := \sum_{j\in\{1,\dots,n\}\setminus i} C^j$, r is an exogenous discount factor and the random fields $u^i : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ describe instantaneous utilities. In the economic literature on public good contribution it is customary to assume quasilinear utilities (see, e.g., [50]). Here, instead, we work with general concave utilities as specified in the following

Assumption 1.

i. The optional process $r := \{r(t), t \in [0,T]\}$ is such that $0 < \kappa_r \le r(\omega, t) \le k_r$, $\mathbf{P}$-a.s., $t \in [0,T]$, for some constants κ_r and k_r.

ii. The optional processes $\psi_c := \{\psi_c(t), t \in [0,T]\}$ and $\psi_x := \{\psi_x(t), t \in [0,T]\}$ are such that $\psi_c(t) := e^{\alpha_c(t)}\mathcal{E}_c(t)$ and $\psi_x(t) := e^{\alpha_x(t)}\mathcal{E}_x(t)$, for some continuous and uniformly bounded processes α_c and α_x, and for some exponential martingales $\mathcal{E}_c$ and $\mathcal{E}_x$.

iii. For any $\omega \in \Omega$, the mapping $(x, c) \mapsto u^i(\omega, x, c)$ is increasing and strictly concave on $\mathbb{R}^2_+$, as well as twice continuously differentiable on the open cone $\mathbb{R}^2_{++}$. Moreover, it satisfies the Inada conditions

$$u^i_x(\omega, 0+, c) = +\infty \quad \text{and} \quad u^i_x(\omega, +\infty, c) = 0$$

for any $c > 0$.

iv. For $(x, c) \in \mathbb{R}^2_+$ fixed, $\omega \mapsto u^i(\omega, x, c)$ is progressively measurable.

v. *The family* $\left(e^{-\int_0^t r(\omega,s)\,ds}u^i(\omega,x(\omega,t),C(\omega,t)),\ \ (x,C)\in\mathcal{B}_w\right)$ *is* $\mathbf{P}\otimes dt$*-uniformly integrable for any* $w\in\mathbb{R}_+$.

It is easy to see that Assumption 1.ii. is satisfied, for example, by the classical benchmark case of a geometric Brownian motion. The Inada conditions guarantee that there will be an interior solution for optimal private consumption. Note that since u^i is concave in c, $e^{-\int_0^t r(\omega,s)\,ds}u^i_c(\omega,x(\omega,t),C(\omega,t))$ is $\mathbf{P}\otimes dt$-integrable for any $(x,C)\in\mathcal{B}_{w^i}$. Finally, under Assumption 1 the payoff in (2.2) is well defined and finite for any $i=1,\dots,n$.

We now review our main findings.

Our first main result characterizes the efficient allocation in terms of an index process l^* that is independent of agents and serves as a signal process for the aggregate level of the public good (see Section 3). Such an index process has appeared in other contexts as well, as for example for Hindy–Huang–Kreps preferences (see Bank and Riedel [10]) and irreversible investment (cf. Riedel and Su [45]). It can be characerized by a backward equation.

The level $l^*(t)$ can be viewed as the level the society would like to have if it started from scratch at time t. As investment into the public good is irreversible, the actual stock of public good is the running maximum of the index process. With the help of the first-order conditions, we can also express the private demand as a function of its price and the index process.

Theorem 2.1. *Let Assumption 1 hold and define for every* $i=1,\dots,n$ $g^i(\cdot,c)$ *as the inverse of* $u^i_x(\cdot,c)$*, as well as* $h^i(\psi,c):=u^i_c(g(\psi,c),c)$ *for any* $\psi,c>0$*. Suppose each* h^i *satisfies the Inada conditions*

$$h^i(\psi,0+)=+\infty \quad \text{and} \quad h^i(\psi,+\infty)=0.$$

Then the unique solution of the social planner's problem (3.1) is

$$\begin{cases} C_*(t)=\sum_{i=1}^n C^i_*(t)=(\sup_{0\le u\le t} l^*(u))\vee 0 \\ x^i_*(t)=g^i\Big(\frac{\lambda}{\gamma^i}\psi_x(t),C_*(t)\Big),\quad i=1,\dots,n \end{cases} \tag{2.3}$$

for a suitable Lagrange multiplier $\lambda>0$ *and where the optional, upper right-continuous process* $l^*(t)$ *uniquely solves*

$$\mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t),\ \sup_{\tau\le u\le t} l^*(u)\Big)\,dt\Big|\mathcal{F}_\tau\Big]=\lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}} \tag{2.4}$$

for any stopping time $\tau\in[0,T]$*,* $\mathbf{P}$*-a.s.*

By the flexibility of private consumption, x^i_* equates price and marginal utility at the optimum. The Inada condition on h^i, which is a strictly decreasing function by the strict concavity of u^i, is used to guarantee that the backward equation (2.4) for l^* involving marginal utility with respect to C_* has a solution.

For economic applications, the following homogeneous model allowing an explicit solution is quite important. Utilities are of the Cobb–Douglas type, and prices are exponential Lévy processes, including the important special case of geometric Brownian motion. Our model has utilities $u^i(x,c)=\frac{x^\alpha c^\beta}{\alpha+\beta}$, $i=1,\dots,n$, for some $\alpha,\beta\in(0,1)$ such that $\alpha+\beta<1$, and prices

$\psi_x(t) = e^{-rt}\mathcal{E}_x(t)$, $\psi_c(t) = e^{-rt}\mathcal{E}_c(t)$, for some exponential Lévy processes[1] $\mathcal{E}_c$ and $\mathcal{E}_x$ and an interest rate $r > 0$.

Proposition 2.2. *Define the processes*

$$\gamma(t) := \frac{1}{A}\Big[\left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t)\inf_{0\le s\le t}\left(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s)\mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s)\right)\Big]^{-\frac{1}{1-\alpha}}, \tag{2.5}$$

$$\theta(t) := \sup_{0\le s\le t}\left(\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)\right), \tag{2.6}$$

and the constants

$$l_0 := \frac{nw}{\mathbb{E}\Big[\int_0^\infty \psi_x(t)\gamma(t)dt + \int_0^\infty \psi_c(t)d\theta(t)\Big]} \tag{2.7}$$

and

$$A := \mathbf{E}\Big[\int_0^\infty \delta e^{-ru}\inf_{0\le s\le u}\left(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\right)du\Big] \tag{2.8}$$

with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$.

Then, if l_0 and A are finite, the social planner's optimal solution is such that

$$C_*(t) = l_0\theta(t) \tag{2.9}$$

and

$$x_*^i(t) = \frac{1}{n}l_0\gamma(t), \quad i = 1,\dots,n, \tag{2.10}$$

with

$$\lambda = \frac{1}{n^\alpha}A^{1-\alpha}l_0^{\alpha+\beta-1}.$$

We then move on to study the public good contribution game (cf. Section 4 below). We are able to characterize the unique equilibrium with the help of suitable first order conditions. In the specific Cobb–Douglas case with Lévy stochastics, we solve explicitly the symmetric variant of the game.

Proposition 2.3. *Take $u^i(x,c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$, $i = 1,\dots,n$, for some $\alpha,\beta \in (0,1)$ such that $\alpha+\beta < 1$, and $\psi_x(t) = e^{-rt}\mathcal{E}_x(t)$, $\psi_c(t) = e^{-rt}\mathcal{E}_c(t)$, for some exponential Lévy processes $\mathcal{E}_c$ and $\mathcal{E}_x$. Define the processes*

$$\gamma(t) := \frac{1}{A}\Big[\left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t)\inf_{0\le s\le t}\left(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s)\mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s)\right)\Big]^{-\frac{1}{1-\alpha}}, \tag{2.11}$$

$$\theta(t) := \sup_{0\le s\le t}\left(\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)\right), \tag{2.12}$$

[1]The martingale property of Assumption 1 is without loss of generality in this case, one just has to correct r by the Lévy exponents of $\mathcal{E}_x$ and $\mathcal{E}_c$, respectively. In any case, the martingale property of $\mathcal{E}_x$ and $\mathcal{E}_c$ is not needed in the proof of the following results.

and the constants

$$\kappa := \frac{w}{\mathbb{E}\Big[\int_0^\infty \psi_x(t)\gamma(t)dt + \frac{1}{n}\int_0^\infty \psi_c(t)d\theta(t)\Big]} \tag{2.13}$$

and

$$A := \mathbf{E}\Big[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\Big(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\Big)\,du\Big] \tag{2.14}$$

with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$.

Then, if κ *and* A *are finite, the symmetric Nash equilibrium of game (4.1) is given by*

$$\hat{C}^i(t) = \frac{\kappa}{n}\theta(t), \quad i = 1,\dots,n, \tag{2.15}$$

$$\hat{x}^i(t) = \kappa\gamma(t), \quad i = 1,\dots,n, \tag{2.16}$$

with

$$\lambda^i = A^{1-\alpha}\kappa^{\alpha+\beta-1}, \quad i = 1,\dots,n.$$

The results of Propositions 2.2 and 2.3 above allows us to explicitly evaluate the free-rider effect (see Section 4.1 below).

Proposition 2.4. *Let* C_* *be the optimal aggregated public good contribution for the social planner problem (cf. (2.9)) and let* $\hat{C}$ *denote its Nash-equilibrium value (cf. (2.15)). Assume* $\psi_c(t) = e^{-rt}$ *and* $\psi_x(t) = e^{-rt}\mathcal{E}_x(t) \equiv e^{-rt+\sigma W(t)}$, $\sigma > 0$, *for a one-dimensional Brownian motion* W *and for some* r *such that* $\sqrt{2r} > \frac{\sigma\alpha}{1-\alpha-\beta}$. *Then, for any* $n \ge 1$ *one has*

$$\frac{C_*(t)}{\hat{C}(t)} = \frac{\kappa}{l_0} = \frac{\alpha+\beta}{n\alpha+\beta} \le 1, \tag{2.17}$$

where κ *and* l_0 *are as in (2.13) and (2.7), respectively.*

We observe that the ratio $C_*/\hat{C}$, the underprovision of the public good due to free-riding, does not depend on σ, the volatility of the Brownian motion W. Thus, in our model

Corollary 2.5. *The degree of free-riding does not depend on the level of uncertainty.*

This seems to be in contrast to the idea that uncertainty might have some effect on the free rider effect (cf. Austen-Smith [2], Eichberger and Kelsey [23] and Wang [51], among others). Moreover, we show that also irreversibility of public good provisions do not have any effect on free-riding. These two results represent the main economically interesting conclusions of our paper.

Finally, under the reasonable and common assumption that the private good is square-integrable, we also provide an abstract existence result for a Nash equilibrium (cf. Section 4.2).

Theorem 2.6. *Under Assumptions 1 and 2 (see Section 4.2 below), there exists a Nash equilibrium* $(\hat{x}^i, \hat{C}^i)_{i\in\{1,\dots,n\}} \in \prod_{i=1}^n \mathcal{B}_{w^i}$ *for the game:*

$$U^i(\hat{x}^i, \hat{C}^i; \hat{C}^{-i}) \ge U^i(x^i, C^i; \hat{C}^{-i}) \quad \textit{for all} \;\; (x^i, C^i) \in \mathcal{B}_{w^i}, x^i \in \mathbb{L}^2(d\mu_x),$$

$i = 1,\dots,n$.

3 The Social Planner Problem

We start our analysis by studying a social planner problem for the economy described in Section 2. Throughout this section, denote by $(\underline{x}, \underline{C})$ a vector with components $(x^1, \dots, x^n, C^1, \dots, C^n)$ and introduce the nonempty, convex set

$$\mathcal{B}_w := \Bigg\{ (\underline{x}, \underline{C}) : \Omega \times [0,T] \mapsto \mathbb{R}^{2n}_+ \text{ adapted s.t. } C^i \text{ is right-continuous, nondecreasing,}$$
$$C^i(0-) = 0,\ i = 1, \dots, n,\ \mathbf{P}\text{-a.s. and } \sum_{i=1}^n \mathbf{E}\Big[\int_0^T \psi_x(t) x^i(t) dt + \int_0^T \psi_c(t) dC^i(t) \Big] \le w \Bigg\}$$

with $w := \sum_{i=1}^n w^i$. We say that $(\underline{x}, \underline{C})$ is admissible if $(\underline{x}, \underline{C}) \in \mathcal{B}_w$. Suppose that there exists a fictitious social planner aiming to maximize the aggregate expected utility by allocating efficiently the available wealth. In mathematical terms, this amounts to solving the optimization problem with value function

$$V_{SP} := \sup_{(\underline{x}, \underline{C}) \in \mathcal{B}_w} U_{SP}(\underline{x}, \underline{C}) = \sup_{(\underline{x}, \underline{C}) \in \mathcal{B}_w} \sum_{i=1}^n \gamma^i U^i(x^i, C^i; C^{-i}) \tag{3.1}$$

with $U^i(x^i, C^i; C^{-i})$ as in (2.2) and for positive weights γ^i, $i = 1, \dots, n$, such that $\sum_{i=1}^n \gamma^i = 1$.

Theorem 3.1. *Under Assumption 1 there exists a unique $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$ that solves the social planner's problem (3.1).*

Proof. Recall that $\psi_x(t) = e^{\alpha_x(t)} \mathcal{E}_x(t)$ and $\psi_c(t) = e^{\alpha_c(t)} \mathcal{E}_c(t)$, for some continuous and bounded processes α_x and α_c, and for some exponential martingales $\mathcal{E}_x$ and $\mathcal{E}_c$ (cf. Assumption 1.ii.). Let $\tilde{\mathbf{E}}_c[\cdot]$ and $\tilde{\mathbf{E}}_x[\cdot]$ be the expectations under the measures $\tilde{\mathbf{P}}_c$ and $\tilde{\mathbf{P}}_x$ with Radon-Nikodym derivative $\mathcal{E}_c(T)$ and $\mathcal{E}_x(T)$, respectively, with respect to $\mathbf{P}$. Since $\mathcal{E}_x(T) > 0$ and $\mathcal{E}_c(T) > 0$ a.s., the measure $\mathbf{P}$ is equivalent to $\tilde{\mathbf{P}}_c$ and $\tilde{\mathbf{P}}_x$. Denote by $\mathcal{V}$ the space of all optional random measures on $[0,T]$ endowed with the weak-topology in the probabilistic sense, by $\mathbb{L}^1(d\mu_x)$ the space of all functions integrable with respect to the measure $d\mu_x := d\tilde{\mathbf{P}}_x \otimes dt$ and set $x(t) := \sum_{i=1}^n x^i(t)$. Then $\mathcal{B}_w \subset \mathbb{L}^1(d\mu_x)^n \otimes \mathcal{V}^n$. Indeed, for any $i = 1, \dots, n$, and for some constant $K_1 > 0$

$$\begin{aligned} w &\ge \mathbf{E}\Big[\int_0^T \psi_x(t) x^i(t) dt \Big] = \mathbf{E}\Big[\int_0^T e^{\alpha_x(t)} \mathbf{E}[\mathcal{E}_x(T) | \mathcal{F}_t]\, x^i(t) dt \Big] \\ &= \mathbf{E}\Big[\mathcal{E}_x(T) \int_0^T e^{\alpha_x(t)} x^i(t) dt \Big] \ge K_1 \tilde{\mathbf{E}}_x \Big[\int_0^T x^i(t) dt \Big], \end{aligned} \tag{3.2}$$

where Girsanov's Theorem implies the last step. Also, each component of $\underline{C}$ is the cumulative distribution of an optional random measure; i.e., it is an adapted, nondecreasing process with right-continuous paths.

Let now $\{(\underline{x}_m, \underline{C}_m)\}_{m \in \mathbb{N}} \subset \mathcal{B}_w$ be a maximizing sequence; that is, a sequence such that

$$\lim_{m \to \infty} \sum_{i=1}^n \gamma^i U^i(x^i_m, C^i_m; C^{-i}_m) = V_{SP}.$$

The two sequences $\{\tilde{\mathbf{E}}_x[\int_0^T x_m^i(t)dt]\}_{m\in\mathbb{N}}$ and $\{\tilde{\mathbf{E}}_c[C_m^i(T)]\}_{m\in\mathbb{N}}$ are uniformly bounded in m for every $i = 1, \ldots, n$, because of (3.2) and since, analogously,

$$\begin{aligned} w &\geq \mathbf{E}\Big[\int_0^T \psi_c(t)dC^i(t)\Big] = \mathbf{E}\Big[\int_0^T e^{\alpha_c(t)}\mathbf{E}[\mathcal{E}_c(T)|\mathcal{F}_t]\, dC^i(t)\Big] = \mathbf{E}\Big[\mathcal{E}_c(T)\int_0^T e^{\alpha_c(t)}dC^i(t)\Big] \\ &= \tilde{\mathbf{E}}_c\Big[\int_0^T e^{\alpha_c(t)}dC^i(t)\Big] \geq K_2\tilde{\mathbf{E}}_c[C^i(T)], \end{aligned}$$

where the second equality follows from [30], Theorem 1.33, and with $K_2 > 0$ a suitable constant. Hence by Komlòs' theorem (see Komlòs [34] and Kabanov [32], Lemma 3.5, for a version of Komlòs' theorem for optional random measures), for every $i = 1, \ldots, n$ there exist two subsequences $\{\tilde{x}_m^i\}_{m\in\mathbb{N}} \subset \{x_m^i\}_{m\in\mathbb{N}}$ and $\{\tilde{C}_m^i\}_{m\in\mathbb{N}} \subset \{C_m^i\}_{m\in\mathbb{N}}$ such that

$$X_k^i(t) := \frac{1}{k+1}\sum_{m=0}^{k} \tilde{x}_m^i \to x_*^i(t), \;\; d\mu_x\text{-a.e.} \tag{3.3}$$

and

$$I_k^i(t) := \frac{1}{k+1}\sum_{m=0}^{k} \tilde{C}_m^i \to C_*^i(t), \;\; \tilde{\mathbf{P}}_c\text{-a.s., for every point of continuity of } C_*^i(\cdot) \text{ and } t = T \tag{3.4}$$

as $k \to \infty$ for some $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable process x_*^i and for some $C_*^i \in \mathcal{V}$, $i = 1, \ldots, n$. From now on we will denote by C_*^i the right-continuous modification of C_*^i. Notice that having $\lim_{k\to\infty} I_k^i(t) = C_*^i(t)$ $\tilde{\mathbf{P}}_c$-a.s. for every point of continuity of $C_*^i(\cdot)$ and for $t = T$ means that the sequence of optional random measures on $[0,T]$ $dI_k^i(\cdot)$ converges weakly a.s. to $dC_*^i(\cdot)$; that is,

$$\lim_{k\to\infty}\int_0^T f(t)dI_k^i(t) = \int_0^T f(t)dC_*^i(t), \;\; \tilde{\mathbf{P}}_c - a.s., \tag{3.5}$$

for every continuous and bounded function $f(\cdot)$ (see, e.g., Billingsley [15]). We now claim that the Komlòs' limit $(\underline{x}_*, \underline{C}_*) := (x_*^1, \ldots, x_*^n, C_*^1, \ldots, C_*^n)$ belongs to $\mathcal{B}_w$ and that it is optimal for the social planner's problem (3.1). Indeed, $(\underline{X}_k, \underline{I}_k) := (X_k^1, \ldots, X_k^n, I_k^1, \ldots, I_k^n) \in \mathcal{B}_w$ by convexity of $\mathcal{B}_w$, and (3.3), (3.5) and Fatou's Lemma imply

$$\begin{aligned} w &\geq \liminf_{k\to\infty}\sum_{i=1}^n \mathbf{E}\Big[\int_0^T \psi_x(t)X_k^i(t)dt + \int_0^T \psi_c(t)dI_k^i(t)\Big] \\ &= \liminf_{k\to\infty}\sum_{i=1}^n \Big(\tilde{\mathbf{E}}_x\Big[\int_0^T e^{\alpha_x(t)}X_k^i(t)dt\Big] + \tilde{\mathbf{E}}_c\Big[\int_0^T e^{\alpha_c(t)}dI_k^i(t)\Big]\Big) \\ &= \sum_{i=1}^n \tilde{\mathbf{E}}_x\Big[\int_0^T e^{\alpha_x(t)}x_*^i(t)dt\Big] + \tilde{\mathbf{E}}_c\Big[\int_0^T e^{\alpha_c(t)}dC_*^i(t)\Big] \\ &= \sum_{i=1}^n \mathbf{E}\Big[\int_0^T \psi_x(t)x_*^i(t)dt + \int_0^T \psi_c(t)dC_*^i(t)\Big]; \end{aligned}$$

that is, $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$. Recall now that $\mathbf{P}_x \sim \mathbf{P}$ and $\mathbf{P}_c \sim \mathbf{P}$. Then (3.4) and (3.3) also hold $\mathbf{P}$-a.s. and $d\mathbf{P} \otimes dt$-a.e., respectively, and therefore we may write

$$\sum_{i=1}^n \gamma^i U^i(x^i_*, C^i_*; C^{-i}_*) = \lim_{k\to\infty} \sum_{i=1}^n \gamma^i U^i(X^i_k, I^i_k; I^{-i}_k) = V_{SP}$$

by the uniform integrability assumed in Assumption 1.v. and because $(\underline{X}_k, \underline{I}_k)$ is a maximizing sequence by concavity of each U^i. Hence $(\underline{x}_*, \underline{C}_*)$ is optimal.

Finally, uniqueness of $(\underline{x}_*, \underline{C}_*)$ follows as usual from strict concavity of the utility functions u^i, $i = 1, \dots, n$ and from convexity of $\mathcal{B}_w$. □

Remark 3.2. *Notice that all the arguments employed in the proof of Theorem 3.1 still work in the infinite-horizon case $T = +\infty$, under the further assumption that $\mathcal{E}_x$ and $\mathcal{E}_c$ are $\mathbf{P}$-uniformly integrable martingales.*

We now aim to characterize the social planner's optimal policy by means of a set of first order conditions for optimality. This approach has been used in various instances to solve singular stochastic control problems of the monotone follower type (see Bank [6], Bank and Riedel [10], Chiarolla, Ferrari and Riedel [17], Riedel and Su [45] and Steg [49], among others), and it may be thought of as a stochastic, infinite dimensional generalization of the classical Kuhn-Tucker method. In the previous papers the optimal policy is constructed as the running supremum of a desirable value. Such level of satisfaction is the optional solution of a stochastic backward equation in the spirit of Bank-El Karoui (cf. Bank and El Karoui [7], Theorem 3) and it may be represented in terms of the value functions of a family of standard optimal stopping problems.

For any $(\underline{x}, \underline{C}) \in \mathcal{B}_w$ define the *Lagrangian functional* of problem (3.1) as

$$\begin{aligned}\mathcal{L}^w(\underline{x}, \underline{C}; \lambda) :=& \sum_{i=1}^n \gamma^i U^i(x^i, C^i; C^{-i}) + \lambda\Big\{ w - \sum_{i=1}^n \mathbf{E}\Big[\int_0^T \psi_x(t) x^i(t) dt + \int_0^T \psi_c(t) dC^i(t) \Big]\Big\} \\ =& \sum_{i=1}^n \gamma^i \mathbf{E}\left[\int_0^T e^{-\int_0^t r(s)\,ds} u^i(x^i(t), C(t))\, dt \right] \\ &+ \lambda\Big\{ w - \mathbf{E}\Big[\int_0^T \psi_x(t) x(t) dt + \int_0^T \psi_c(t) dC(t) \Big]\Big\}\end{aligned}$$

for some Lagrange multiplier $\lambda > 0$, and where again $x(t) := \sum_{i=1}^n x^i(t)$ and $C(t) := \sum_{i=1}^n C^i(t)$. Moreover, let $\mathcal{T}$ be the set of all $\mathcal{F}_t$-stopping times with values in $[0, T]$ a.s. and denote by $\nabla_c \mathcal{L}^w$ the Lagrangian functional's supergradient with respect the aggregated public good; that is, the unique optional process given by

$$\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)(\tau) := \mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i\, u^i_c(x^i(t), C(t))\, dt \Big| \mathcal{F}_\tau \Big] - \lambda \psi_c(\tau) \mathbb{1}_{\{\tau < T\}}$$

for any $\tau \in \mathcal{T}$.

On the other hand, an additional consumption of the private good x^i affects marginal utility only at those times at which consumption actually occurs. It means that

$$\nabla_x \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)(\tau) := \gamma^i e^{-\int_0^\tau r(s) ds} u^i_x(x^i(\tau), C(\tau)) - \lambda \psi_x(\tau), \quad \tau \in \mathcal{T}.$$

Remark 3.3. *Following Bank and Riedel [10], the quantity $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)(t)$ may be interpreted as the marginal expected profit resulting from an additional infinitesimal investment at time t when the investment plan is $(\underline{x}, \underline{C})$ and the Lagrange multiplier is λ. Mathematically, $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)$ is the Riesz representation of the Lagrangian gradient at C. More precisely, for any arbitrary but fixed $\lambda > 0$, define $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)$ as the optional projection of the product-measurable process*

$$\Phi(\omega, t) := \int_t^T e^{-\int_0^s r(u)\,du} \sum_{i=1}^n \gamma^i\, u_c^i(x^i(s), C(s))\, ds - \lambda \psi_c(t) \mathbb{1}_{\{t<T\}}$$

for $\omega \in \Omega$ and $t \in [0, T]$. Hence $\nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda)$ is uniquely determined up to $\mathbb{P}$-indistinguishability and it holds

$$\mathbb{E}\Big\{ \int_{[0,T)} \nabla_c \mathcal{L}^w(\underline{x}, \underline{C}; \lambda) dC(t) \Big\} = \mathbb{E}\Big\{ \int_{[0,T)} \Phi(t) dC(t) \Big\}$$

for all admissible C (cf. Jacod [30], Theorem 1.33).

Proposition 3.4. *Let Assumption 1 hold. An admissible policy $(\underline{x}_*, \underline{C}_*)$ is optimal for the social planner's problem (3.1) if and only if there exists a Lagrange multiplier $\lambda > 0$ such that the following first order conditions hold true for any stopping time $\in \mathcal{T}$*

$$\begin{cases} \mathbf{E}\Big[\int_0^T \psi_x(t) x_*(t) dt + \int_0^T \psi_c(t) dC_*(t) \Big] = w, \\ \mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_c^i(x_*^i(t), C_*(t))\, dt \Big| \mathcal{F}_\tau \Big] \le \lambda \psi_c(\tau) \mathbb{1}_{\{\tau<T\}}, \ \ \mathbf{P}-a.s., \\ \mathbf{E}\Big[\int_0^T \Big(\mathbf{E}\Big[\int_t^T e^{-\int_0^s r(u)\,du} \sum_{i=1}^n \gamma^i u_c^i(x_*^i(s), C_*(s))\, ds \Big| \mathcal{F}_t \Big] - \lambda \psi_c(t) \Big) dC_*(t) \Big] = 0, \\ \gamma^i e^{-\int_0^\tau r(s) ds} u_x^i(x_*^i(\tau), C_*(\tau)) \le \lambda \psi_x(\tau), \ \ \mathbf{P}-a.s. \text{ with equality whenever } x_*^i(\tau) > 0. \end{cases} \tag{3.6}$$

The proof of Proposition 3.4 is given in Appendix A, Section A.1. It generalizes that of Bank and Riedel [10], Theorem 3.2, to the present setting of a multidimensional optimal consumption problem (with both classical absolutely continuous and monotone controls) and it resembles the arguments employed to prove optimality of the classical Kuhn-Tucker conditions of real analysis. Indeed, concavity of the utility functions u^i, $i = 1, \ldots, n$ yields sufficiency, whereas the proof of the necessity part is a bit more delicate. One has indeed to linearize the original problem (3.1) around its optimal solution $(\underline{x}_*, \underline{C}_*)$ and then to show that $(\underline{x}_*, \underline{C}_*)$ solves the linearized problem as well. Finally, one must prove that any solution to the linearized problem (and therefore $(\underline{x}_*, \underline{C}_*)$ as well) satisfies some flat-off conditions similar to the third and the fourth ones of (3.6).

Notice that because of the Inada conditions (cf. Assumption 1.iii.) the fourth one of (3.6) is binding at any $\tau \in \mathcal{T}$, i.e.

$$\gamma^i e^{-\int_0^\tau r(s) ds} u_x^i(x_*^i(\tau), C_*(\tau)) = \lambda \psi_x(\tau).$$

Recalling that $(x,c)\mapsto u^i(x,c)$ is strictly concave, and denoting by $g^i(\cdot,c)$ the inverse of $u^i_x(\cdot,c)$, we may write

$$x^i_*(\tau)=g^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^\tau r(s)ds}\psi_x(\tau),C_*(\tau)\Big),\quad \tau\in\mathcal{T}. \tag{3.7}$$

Then, by plugging (3.7) into (3.6) we obtain the equivalent formulation

$$\begin{cases}\mathbf{E}\Big[\int_0^T\psi_x(t)x_*(t)dt+\int_0^T\psi_c(t)dC_*(t)\Big]=w,\\ \mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^ih^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t),C_*(t)\Big)\,dt\Big|\mathcal{F}_\tau\Big]\le\lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}},\ \ \mathbf{P}-a.s.,\\ \mathbf{E}\Big[\int_0^T\Big(\mathbf{E}\Big[\int_t^T e^{-\int_0^s r(u)\,du}\sum_{i=1}^n\gamma^ih^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^s r(u)du}\psi_x(s),C_*(s)\Big)\,ds\Big|\mathcal{F}_t\Big]-\lambda\psi_c(t)\Big)dC_*(t)\Big]=0,\\ e^{-\int_0^\tau r(s)ds}\gamma^iu^i_x(x^i_*(\tau),C_*(\tau))=\lambda\psi_x(\tau),\ \ \mathbf{P}-a.s.,\end{cases} \tag{3.8}$$

for any $\tau\in\mathcal{T}$ and with $h^i(\psi,c):=u^i_c(g^i(\psi,c),c)$.

Although the first order conditions of Proposition 3.4 (or those in (3.8)) completely characterize the optimal policy, they are not binding at all times and so they cannot be directly used to determine C_* and consequently $\underline{x}_*$ by (3.7). As usual in the literature on monotone follower problems (see, e.g., Chiarolla and Ferrari [16], Chiarolla and Haussmann [18] and [19], El Karoui and Karatzas [24], Karatzas and Shreve [33] or Riedel and Su [45]), the optimal policy consists of keeping the controlled process close to some barrier (which is the free boundary of the associated optimal stopping problem in a Markovian setting, cf. Ferrari [25]) in a 'minimal way'. Here we derive the social planner's optimal investment into the public good C_* in terms of the running supremum of the unique optional solution l^* to the backward stochastic equation[2] presented in our Theorem 2.1 in Section 2, which naturally arises from the first order conditions (3.8).

We can now provide the proof.

Proof of Theorem 2.1. Existence of a unique optional, upper right-continuous solution l^* to (2.4) is shown in Appendix A, Proposition A.2. To show optimality of $(\underline{x}_*(t),C_*(t))$ as in (2.3) it suffices to verify that it is admissible and it satisfies the sufficient and necessary first order conditions (3.8). C_* as in (2.3) is adapted with right-continuous sample paths, since l^* is optional and upper right-continuous, and x^i_*, $i=1,2,\dots,n$, is adapted and positive, since g^i is continuous and positive. Moreover, for any $\tau\in\mathcal{T}$ we have

$$\begin{aligned}&\mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^ih^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t),(\sup_{0\le u\le t}l^*(u))\vee 0\Big)\,dt\Big|\mathcal{F}_\tau\Big]\\ &\le\mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n\gamma^ih^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t),\sup_{\tau\le u\le t}l^*(u)\Big)\,dt\Big|\mathcal{F}_\tau\Big]=\lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}},\end{aligned} \tag{3.9}$$

[2]Cf. Bank and El Karoui [7], Theorem 1 and Theorem 3.

where the first inequality follows from the fact that $c \mapsto h^i(\psi, c)$ is strictly decreasing, whereas (2.4) implies the last equality. On the other hand, if $\tau \in \mathcal{T}$ is a time of investment, i.e. such that $dC_*(\tau) > 0$, we have $(\sup_{0\le u\le t} l^*(u)) \vee 0 = \sup_{\tau\le u\le t} l^*(u)$ and equality holds in (3.9). Therefore the second line of (3.8) is satisfied as well. The optimal private good consumption x^i_* of (2.3) is then determined by means of (3.7). □

The process l^* may be found numerically by backward induction on a discretized version of problem (2.4) (see Bank and Föllmer [8], Section 4). In some cases, when $T = +\infty$, (2.4) has a closed form solution as in the case of a Cobb-Douglas utility function (see Section 3.1 below).

3.1 Explicit Results for a Symmetric Economy with Cobb-Douglas Utility

In this section we aim to explicitly solve the social planner's problem in the symmetric case, that is when all the agents have the same utility function. To do so, also suppose that $T = +\infty$, $r(t) = r$ a.s. for all $t \ge 0$ and $w^i = w$ for all $i = 1, \dots, n$. Moreover, we may assume that the social planner does not prefer any agent more than any other; that is, $\gamma^i = \frac{1}{n}$ for every $i = 1, \dots, n$.

The explicit solution has been presented in Proposition 2.2 in Section 2.

Proof of Proposition 2.2. Recall that $h^i(\psi, c) = u^i_c(g^i(\psi, c), c)$, where $g^i(\cdot, c)$ is the inverse of $u^i_x(\cdot, c)$. For any $\lambda > 0$, simple algebra leads to $h^i(\frac{\lambda}{\gamma^i} e^{rt}\psi_x(t), C(t)) = \delta(n\lambda\mathcal{E}_x(t))^{\frac{\alpha}{\alpha-1}} C^{\frac{\alpha+\beta-1}{1-\alpha}}(t)$ with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$. Set $C_*(t) = \sup_{0\le s\le t} l^*(s) \vee 0$ for some progressively measurable process $l^*(t)$ to be found and then (2.4) becomes

$$\mathbf{E}\left[\int_\tau^\infty \delta e^{-rs}(n\lambda\mathcal{E}_x(s))^{\frac{\alpha}{\alpha-1}}\Big(\sup_{\tau\le u\le s} l^*(u)\Big)^{\frac{\alpha+\beta-1}{1-\alpha}} ds \,\Big|\, \mathcal{F}_\tau\right] = \lambda e^{-r\tau}\mathcal{E}_c(\tau),$$

i.e.,

$$\mathbf{E}\left[\int_0^\infty \delta e^{-ru}(n\lambda)^{\frac{\alpha}{\alpha-1}} \frac{\mathcal{E}_x{}^{\frac{\alpha}{\alpha-1}}(u+\tau)}{\mathcal{E}_c(\tau)} \inf_{0\le s\le u}\Big(l^{*\,\frac{\alpha+\beta-1}{1-\alpha}}(s+\tau)\Big) du \,\Big|\, \mathcal{F}_\tau\right] = \lambda. \tag{3.10}$$

Make now the *ansatz* $l^*(t) := l_0\mathcal{E}_c{}^{\frac{1-\alpha}{\alpha+\beta-1}}(t)\mathcal{E}_x{}^{\frac{\alpha}{\alpha+\beta-1}}(t)$ for some constant l_0, and use independence and stationarity of Lévy increments to rewrite (3.10) as

$$\frac{1}{n^{\frac{\alpha}{1-\alpha}}} l_0^{\frac{\alpha+\beta-1}{1-\alpha}} \mathbf{E}\left[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\Big(\mathcal{E}_c(s)\mathcal{E}_x^{\frac{\alpha}{\alpha-1}}(u-s)\Big) du\right] = \lambda^{\frac{1}{1-\alpha}}.$$

By setting $A := \mathbf{E}[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\Big(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\Big) du]$ (cf. (2.8)) and by solving the previous equation for λ one easily obtains

$$\lambda := \frac{1}{n^\alpha} A^{1-\alpha} l_0^{\alpha+\beta-1}.$$

On the other hand, $x^i_*(t) = [n\lambda\left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t)C_*^{-\beta}(t)]^{\frac{1}{\alpha-1}}$ by (3.7) and therefore

$$x^i_*(t) = \frac{1}{A}(n\lambda)^{-\frac{1}{1-\alpha}}\left[\left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t) l_0^{-\beta} \inf_{0\le s\le t}\left(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s)\mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s)\right)\right]^{-\frac{1}{1-\alpha}};$$

that is,

$$x_*^i(t) = \frac{1}{n} l_0 \gamma(t) \tag{3.11}$$

with $\gamma(t)$ as in (2.5).

To determine l_0 we make use of the budget constraint $\mathbb{E}[\int_0^\infty \psi_x(t)x_*(t)dt + \int_0^\infty \psi_c(t)dC_*(t)] = nw$. In fact, recalling that $x_*(t) := \sum_{i=1}^n x_*^i(t)$, from (3.11) we find

$$l_0 \mathbf{E}\left[\int_0^\infty \psi_x(t)\gamma(t)dt + \int_0^\infty \psi_c(t)d\theta(t)\right] = nw, \tag{3.12}$$

since $C_*(t) = \sup_{0\le s\le t} l^*(s) = l_0 \sup_{0\le s\le t}(\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)) = l_0\theta(t)$ with $\theta(t)$ as in (2.6). By solving (3.12) for l_0 (2.7) follows. Notice that l_0 of (2.7) and A of (2.8) are finite under some further specifications of the model as it is shown in the proof of Proposition 2.4 in Section 4.1 below. □

Remark 3.5. *As usual in monotone follower problems, the optimal aggregated public good level $C_*(t)$ (cf. (2.5) and (2.9)) is a singular process since it increases only on a set of zero Lebesgue measure. Moreover, the ratio $\frac{x_*^i(t)}{w}$ is independent of n, since $\frac{x_*^i(t)}{w} = \gamma(t)(\mathbf{E}[\int_0^\infty \psi_x(t)\gamma(t)dt + \int_0^\infty \psi_c(t)d\theta(t)])^{-1}$, whereas $C_*(t) \sim n$ and $\lambda \sim n^{-(1-\beta)}$. That is a typical behaviour for a Cobb-Douglas utility function.*

Corollary 3.6. *Assume $\mathcal{E}_x(t) = \mathcal{E}_c(t) = e^{\sigma W(t)} =: \mathcal{E}(t)$, $\sigma > 0$, for a one-dimensional Brownian motion W and take $\sqrt{2r} > \frac{\sigma(\alpha+\beta)}{1-\alpha-\beta}$. Then one has*

$$V_{SP} = \Xi \left[\frac{1}{\sqrt{2r} + \frac{\sigma\alpha}{1-\alpha}}\right]\left[\frac{1}{\sqrt{2r} - \frac{\sigma(\alpha+\beta)}{1-\alpha-\beta}}\right],$$

where

$$\Xi := \frac{2}{A^\alpha}\frac{l_0^{\alpha+\beta}}{(\alpha+\beta)n^\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}. \tag{3.13}$$

Proof. If $\mathcal{E}_x(t) = \mathcal{E}_c(t) \equiv \mathcal{E}(t) := e^{\sigma W(t)}$, by Proposition (2.2) (cf. (2.9) and (2.10)) straightforward calculations lead to

$$(x_*^i)^\alpha(t) = \frac{1}{A^\alpha}\frac{l_0^\alpha}{n^\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} \mathcal{E}^{-\frac{\alpha}{1-\alpha}}(t)\left(\sup_{0\le s\le t}\mathcal{E}^{-\frac{\beta}{1-\alpha-\beta}}(s)\right)^{\frac{\alpha}{1-\alpha}}$$

and

$$C_*^\beta(t) = l_0^\beta \sup_{0\le s\le t}\mathcal{E}^{-\frac{\beta}{1-\alpha-\beta}}(s)$$

with l_0 and A as in (2.7) and (2.8), respectively. Notice that A and l_0 are both finite since $A \le \delta\mathbf{E}[\int_0^\infty e^{-ru-\frac{\sigma\alpha}{1-\alpha}W(u)}du] = \int_0^\infty e^{-[r-\frac{1}{2}(\frac{\sigma\alpha}{1-\alpha})^2]u}du < \infty$ (for $\sqrt{2r} > \frac{\sigma(\alpha+\beta)}{1-\alpha-\beta}$), and $l_0 \le nw(\mathbf{E}[\int_0^\infty \psi_x(t)\gamma(t)dt])^{-1} \le nw\beta\alpha^{-1} < \infty$.

Recall (2.2) and (3.1). Then, with Ξ as in (3.13), it follows that

$$
\begin{aligned}
V_{SP} :=& \sum_{i=1}^{n} \gamma_i \frac{1}{\alpha+\beta} \mathbf{E}\left[\int_0^\infty e^{-rt}(x_*^i)^\alpha(t) C_*^\beta(t) dt\right] \\
=& \frac{\Xi}{r} \mathbf{E}\left[\int_0^\infty r e^{-rt} \mathcal{E}^{-\frac{\alpha}{1-\alpha}}(t) \sup_{0\le s\le t}\left(\mathcal{E}^{-\frac{\beta}{(1-\alpha)(1-\alpha-\beta)}}(s)\right) dt\right] \\
=& \frac{\Xi}{r} \mathbf{E}\left[\int_0^\infty r e^{-rt} e^{-\frac{\sigma\alpha}{1-\alpha}W(t)}\, e^{-\frac{\sigma\beta}{(1-\alpha)(1-\alpha-\beta)}\inf_{0\le u\le t} W(u)} dt\right] \\
=& \frac{\Xi}{r} \mathbf{E}\left[e^{-\frac{\sigma\alpha}{1-\alpha}W(\tau_r)}\, e^{-\frac{\sigma\beta}{(1-\alpha)(1-\alpha-\beta)}\inf_{0\le u\le \tau_r} W(u)}\right], \qquad (3.14)
\end{aligned}
$$

where τ_r is an independent exponentially distributed random time with parameter r. Now, by using the Excursion Theory for Lévy processes (cf. Bertoin [14]), $W(t) - \sup_{0\le u\le t} W(u)$ is independent of $\sup_{0\le u\le t} W(u)$, and by the Duality Theorem, $W(t) - \sup_{0\le u\le t} W(u)$ has the same distribution as $\inf_{0\le u\le t} W(u)$. Hence, we may write from (3.14)

$$
\begin{aligned}
V_{SP} :=& \frac{\Xi}{r} \mathbf{E}\left[e^{-\frac{\sigma\alpha}{1-\alpha}W(\tau_r)}\, e^{-\frac{\sigma\beta}{(1-\alpha)(1-\alpha-\beta)}\inf_{0\le u\le \tau_r} W(u)}\right] \\
=& \frac{\Xi}{r} \mathbf{E}\left[e^{\frac{\sigma\alpha}{1-\alpha}[-W(\tau_r)-\sup_{0\le u\le \tau_r}(-W(u))]}\, e^{[\frac{\sigma\alpha}{1-\alpha}+\frac{\sigma\beta}{(1-\alpha)(1-\alpha-\beta)}]\sup_{0\le u\le \tau_r}(-W(u))}\right] \\
=& \frac{\Xi}{r} \mathbf{E}\left[e^{\frac{\sigma\alpha}{1-\alpha}\inf_{0\le u\le \tau_r}(-W(u))}\right] \mathbf{E}\left[e^{\frac{\sigma(\alpha+\beta)}{1-\alpha-\beta}\sup_{0\le u\le \tau_r}(-W(u))}\right], \\
=& \frac{\Xi}{r} \mathbf{E}\left[e^{-\frac{\sigma\alpha}{1-\alpha}\sup_{0\le u\le \tau_r} W(u)}\right] \mathbf{E}\left[e^{\frac{\sigma(\alpha+\beta)}{1-\alpha-\beta}\sup_{0\le u\le \tau_r}(-W(u))}\right], \\
=& \frac{\Xi}{r}\left[\frac{\sqrt{2r}}{\sqrt{2r}+\frac{\sigma\alpha}{1-\alpha}}\right]\left[\frac{\sqrt{2r}}{\sqrt{2r}-\frac{\sigma(\alpha+\beta)}{1-\alpha-\beta}}\right],
\end{aligned}
$$

where the last equality follows from the fact that $\sup_{0\le u\le \tau_r} W(u) \sim \sup_{0\le u\le \tau_r}(-W(u)) \sim Exp(\sqrt{2r})$ (see, e.g., Bertoin [14], Chapter VII). □

4 The Public Good Contribution Game

In Section 3 we have taken the point of view of a fictitious social planner aiming to efficiently maximize the social welfare. Here we aim to study strategic interaction between the agents of our economy. We shall see that our model exhibits a dynamic free rider effect (cf. Varian

[50], among others) that we will analyze in detail in Section 4.1 below. Determining agent i's optimal choice of a strategy against a given process C^{-i} specifying aggregate contributions by the opponents amounts to solving the stochastic control problem with value function

$$V^i(C^{-i}) := \sup_{(x^i,C^i)\in\mathcal{B}_{w^i}} U^i(x^i, C^i; C^{-i}), \qquad i = 1, 2, \ldots, n, \tag{4.1}$$

where $\mathcal{B}_{w^i}$ and U^i are as in (2.1) and (2.2), respectively. The description of the game is completed by the introduction of a standard Nash equilibrium concept.

Definition 4.1. $(\hat{x}^1, \ldots, \hat{x}^n, \hat{C}^1, \ldots, \hat{C}^n)$ *is a* Nash equilibrium *if for all* $i \in \{1, \ldots, n\}$, $(\hat{x}^i, \hat{C}^i) \in \mathcal{B}_{w^i}$ *and* $U^i(\hat{x}^i, \hat{C}^i, \hat{C}^{-i}) = V^i(\hat{C}^{-i})$.

While this equilibrium notion does not limit the ability of any agent to optimize against given strategies of the others, it does limit the extent of dynamic interaction that can take place. Although agents do react to the evolving exogenous uncertainty, they take the contribution processes of others as given and do not react to deviations from announced (equilibrium) play. Therefore, one might term such an equilibrium as one in *precommitment strategies*. Unfortunately there are serious conceptual difficulties in defining a related game with more explicit feedback strategies as argued by Back and Paulsen [3], which is why we consider simple Nash equilibria here.

As in the social planner's case we shall first characterize solutions of the best reply problems (4.1) by means of a stochastic Kuhn-Tucker approach. The next Proposition accomplishes this. Its proof may be obtained by adopting arguments similar to those employed to prove Proposition 3.4.

Proposition 4.2. *Let* $\hat{C}^{-i}$ *be given and Assumption 1 hold. Then* $(\hat{x}^i, \hat{C}^i) \in \mathcal{B}_{w^i}$ *attains* $V^i(\hat{C}^{-i})$ *(cf. (4.1)) if and only if there exists a Lagrange multiplier* $\lambda^i > 0$ *such that for any stopping time* $\tau \in \mathcal{T}$ *the following first order conditions hold true*

$$\begin{cases} \mathbf{E}\Big[\int_0^T \psi_x(t)\hat{x}^i(t)dt + \int_0^T \psi_c(t)d\hat{C}^i(t)\Big] = w^i, \\ \mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds} u_c^i(\hat{x}^i(t), \hat{C}(t))\,dt \Big| \mathcal{F}_\tau\Big] \le \lambda^i \psi_c(\tau), \ \ \mathbf{P}-a.s., \\ \mathbf{E}\Big[\int_0^T \Big(\mathbf{E}\Big[\int_t^T e^{-\int_0^s r(u)\,du} u_c^i(\hat{x}^i(s), \hat{C}(s))\,ds \Big| \mathcal{F}_t\Big] - \lambda^i\psi_c(t)\Big) d\hat{C}^i(t)\Big] = 0, \\ e^{-\int_0^\tau r(u)\,du} u_x^i(\hat{x}^i(\tau), \hat{C}(\tau)) \le \lambda^i \psi_x(\tau), \ \ \mathbf{P}-a.s. \text{ with equality whenever } \hat{x}^i(\tau) > 0. \end{cases} \tag{4.2}$$

The Inada conditions (cf. Assumption 1.iii.) imply that the fourth one of (4.2) is always

binding. Hence, we may equivalently rewrite (4.2) as

$$\begin{cases} \mathbf{E}\Big[\int_0^T \psi_x(t)\hat{x}^i(t)dt + \int_0^T \psi_c(t)d\hat{C}^i(t)\Big] = w^i, \\ \mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds} h^i(\lambda^i e^{\int_0^t r(s)\,ds}\psi_x(t), \hat{C}(t))\,dt \Big| \mathcal{F}_\tau\Big] \leq \lambda^i \psi_c(\tau), \ \ \mathbf{P}-a.s., \\ \mathbf{E}\Big[\int_0^T \Big(\mathbf{E}\Big[\int_t^T e^{-\int_0^s r(u)\,du} h^i(\lambda^i e^{\int_0^s r(u)\,du}\psi_x(s), \hat{C}(s))\,ds \Big| \mathcal{F}_t\Big] - \lambda^i\psi_c(t)\Big) d\hat{C}^i(t)\Big] = 0, \\ e^{-\int_0^\tau r(u)\,du} u_x^i(\hat{x}^i(\tau), \hat{C}(\tau)) = \lambda^i \psi_x(\tau), \ \ \mathbf{P}-a.s., \end{cases}$$

where again $h^i(\psi, c) := u_c^i(g^i(\psi, c), c)$ with $g^i(\cdot, c)$ the inverse of $u_x^i(\cdot, c)$.

As in Section 3.1, we can use the first order conditions to explicitly solve the best reply problems (4.1) in the symmetric case, i.e. in the case that all the agents have the same utility function. To do so we also need to assume that $T = +\infty$, $r(t) = r$ a.s. for all $t \geq 0$ and $w^i = w$ for all $i = 1, \ldots, n$. Then we obtain the explicit Nash equilibrium presented in Proposition 2.3 in Section 2. The proof, employing arguments similar to those used for the proof of Proposition (2.2), is given in Appendix A, Section A.3, for the sake of completeness.

4.1 The Free Rider Effect

In this section we shall assume the same symmetric setting of Section 3.1 to study the so called *free-rider effect*.

Let x_*^i be the optimal private consumption in the social planner's problem (cf. (2.10)), and let $\hat{x}^i$ denote the Nash equilibrium private consumption (cf. (2.16)). Then one has

$$\begin{aligned} x_*^i(t) &= \frac{w\gamma(t)}{\mathbf{E}\Big[\int_0^\infty \psi_x(t)\gamma(t)dt + \int_0^\infty \psi_c(t)d\theta(t)\Big]} \\ &\leq \frac{w\gamma(t)}{\mathbf{E}\Big[\int_0^\infty \psi_x(t)\gamma(t)dt + \frac{1}{n}\int_{[0,\infty)} \psi_c(t)d\theta(t)\Big]} = \hat{x}^i(t), \end{aligned}$$

with equality for $n = 1$. It follows that in a strategic context each agent spends more for the private consumption than what would be suggested by the social planner. On the other hand, we have $\kappa \leq l_0$ (with κ as in (2.13), l_0 as in (2.7) and equality if $n = 1$) which implies that the social planner's optimal cumulative contribution into the public good (2.9) is bigger than the corresponding Nash equilibrium counterpart (2.15). That is, our model shows a dynamic free rider effect.

The evaluation of the free rider effect can be made even more explicit in a Black-Scholes setting and with the public good taken as a numéraire. The result is stated in Proposition 2.4 of Section 2 and we can now present the proof.

Proof of Proposition 2.4. From (2.9) and (2.15) it easily follows that

$$\frac{C_*(t)}{\hat{C}(t)} = \frac{\kappa}{l_0} = \frac{\mathbf{E}\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + \int_0^\infty e^{-rt}d\theta(t)\Big]}{\mathbf{E}\Big[n\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + \int_0^\infty e^{-rt}d\theta(t)\Big]}$$

$$= \frac{\mathbf{E}\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + r\int_0^\infty e^{-rt}\theta(t)dt\Big]}{\mathbf{E}\Big[n\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt + r\int_0^\infty e^{-rt}\theta(t)dt\Big]}, \tag{4.3}$$

with $\gamma(t)$ and $\theta(t)$ as in (2.11) and (2.12), respectively. Then, in order to obtain (2.17), we need to evaluate

$$\mathbf{E}\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt\Big] \qquad \text{and} \qquad \mathbf{E}\Big[\int_0^\infty re^{-rt}\theta(t)dt\Big].$$

We have

$$\mathbf{E}\Big[\int_0^\infty re^{-rt}\theta(t)dt\Big] = \mathbf{E}\Big[\int_0^\infty re^{-rt}\sup_{0\le s\le t}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)\,dt\Big]$$

$$\begin{aligned}
&= \mathbf{E}\Big[e^{-\frac{\sigma\alpha}{1-\alpha-\beta}\inf_{0\le s\le\tau_r}W(s)}\Big] = \mathbf{E}\Big[e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\le s\le\tau_r}(-W(s))}\Big] \\
&= \frac{\sqrt{2r}}{\sqrt{2r}-\frac{\sigma\alpha}{1-\alpha-\beta}},
\end{aligned} \tag{4.4}$$

where τ_r is an independent exponentially distributed random time and where the last equality follows from $\sup_{0\le s\le\tau_r}(-W(s)) \sim Exp(\sqrt{2r})$ (cf., e.g., Bertoin [14], Chapter VII). On the other hand, recall γ as in (2.11) and exploit the fact from Excursion Theory for Lévy processes that $W(t) - \sup_{0\le u\le t}W(u)$ is independent of $\sup_{0\le u\le t}W(u)$ and the Duality Theorem saying that $W(t) - \sup_{0\le u\le t}W(u)$ has the same distribution as $\inf_{0\le u\le t}W(u)$ to find

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt\Big] \\
&= \frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}}\mathbf{E}\Big[\int_0^\infty re^{-rt}e^{-\frac{\sigma\alpha}{1-\alpha}W(t)}e^{-\frac{\sigma\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}\inf_{0\le u\le t}W(u)}dt\Big] \\
&= \frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}}\mathbf{E}\Big[e^{\frac{\sigma\alpha}{1-\alpha}(-W(\tau_r))}e^{\frac{\sigma\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}\sup_{0\le u\le\tau_r}(-W(u))}\Big] \\
&= \frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}}\mathbf{E}\Big[e^{\frac{\sigma\alpha}{1-\alpha}[\tilde{W}(\tau_r))-\sup_{0\le u\le\tau_r}\tilde{W}(u)]}e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\le u\le\tau_r}\tilde{W}(u)}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} \mathbf{E}\left[e^{\frac{\sigma\alpha}{1-\alpha}\inf_{0\le u\le\tau_r}\tilde{W}(u)}\right]\mathbf{E}\left[e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\le u\le\tau_r}\tilde{W}(u)}\right]\\
&= \frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} \mathbf{E}\left[e^{-\frac{\sigma\alpha}{1-\alpha}\sup_{0\le u\le\tau_r}W(u)}\right]\mathbf{E}\left[e^{\frac{\sigma\alpha}{1-\alpha-\beta}\sup_{0\le u\le\tau_r}\tilde{W}(u)}\right]\\
&= \frac{1}{rA}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}} \left[\frac{\sqrt{2r}}{\sqrt{2r}+\frac{\sigma\alpha}{1-\alpha}}\right]\left[\frac{\sqrt{2r}}{\sqrt{2r}-\frac{\sigma\alpha}{1-\alpha-\beta}}\right],
\end{aligned}
$$

where we have defined the Brownian motion $\tilde{W} := -W$ and where we have used once more $\sup_{0\le s\le\tau_r} W(s) \sim \sup_{0\le s\le\tau_r}\tilde{W}(s) \sim Exp(\sqrt{2r})$. Again, if τ_r is an independent exponentially distributed random time one has

$$
\begin{aligned}
A &= \mathbf{E}\left[\int_0^\infty \delta\, e^{-rt} \inf_{0\le s\le t} \mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(t-s)\,dt\right] = \frac{\delta}{r}\mathbf{E}\left[e^{-\frac{\sigma\alpha}{1-\alpha}\sup_{0\le s\le\tau_r}W(\tau_r-s)}\right]\\
&= \frac{\delta}{r}\mathbf{E}\left[e^{-\frac{\sigma\alpha}{1-\alpha}\sup_{0\le s'\le\tau_r}W(s')}\right] = \frac{\delta}{r}\left[\frac{\sqrt{2r}}{\sqrt{2r}+\frac{\sigma\alpha}{1-\alpha}}\right]
\end{aligned}
$$

with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha}}$ as $\sup_{0\le s\le\tau_r}(-W(s)) \sim Exp(\sqrt{2r})$. Therefore

$$
\mathbf{E}\left[\int_0^\infty e^{-rt}\mathcal{E}_x(t)\gamma(t)dt\right] = \frac{\alpha}{\beta}\left[\frac{\sqrt{2r}}{\sqrt{2r}-\frac{\sigma\alpha}{1-\alpha-\beta}}\right]. \tag{4.5}
$$

Finally, by plugging (4.4) and (4.5) into (4.3), some simple algebra leads to (2.17). □

It follows from (2.17) that free-riding is not influenced by the uncertain status of the economy in the sense that the ratio $C_*/\hat{C}$ does not depend on σ, the volatility of the Brownian motion W. In contrast to the idea that uncertainty might have some effect on the free rider effect (cf. Austen-Smith [2], Eichberger and Kelsey [23] and Wang [51], among others) in our model we can conlude that

Corollary 4.3. *The degree of free-riding does not depend on the level of uncertainty.*

We now evaluate the role that irreversibility of the public good contribution has in the free rider effect. To do so we compare the ratio (2.17) with the analogous one we shall obtain by assuming instead perfect reversibility of C; i.e., by assuming that each agent can adjust contribution in the public good freely at every point of time.

Proposition 4.4. *Assume perfect reversibility of the public good contribution. Denote by $C_\star$ the optimal aggregated public good contribution made by the social planner and by $\widetilde{C}$ its Nash equilibrium value. Then, under the same hypotheses of Proposition (2.4), one has*

$$
\frac{C_\star(t)}{\widetilde{C}(t)} = \frac{\alpha+\beta}{n\alpha+\beta} \tag{4.6}
$$

for any $n \ge 1$.

Proof. We only sketch the proof. Under perfect reversibility of the public good contribution, the optimal investment criterion is to equate the marginal operating profit with the user cost of capital (see, e.g., Jorgensen [31]). Hence the first-order conditions for optimality in the social planner's problem read

$$\begin{cases} \dfrac{\alpha}{\alpha+\beta}(x_\star^i)^{\alpha-1}(t)C_\star^\beta(t) = \lambda_\star n\mathcal{E}_x(t), \\[2ex] \dfrac{\beta}{\alpha+\beta}(x_\star^i)^{\alpha}(t)C_\star^{\beta-1}(t) = \lambda_\star r, \\[2ex] \mathbf{E}\Big[\displaystyle\int_0^\infty \psi_x(t)\sum_{i=1}^n x_\star^i(t)\,dt + r\int_0^\infty e^{-rt}C_\star(t)\,dt\Big] = nw, \end{cases} \tag{4.7}$$

whereas for the Nash equilibrium they are

$$\begin{cases} \dfrac{\alpha}{\alpha+\beta}(\widetilde{x}^i)^{\alpha-1}(t)\widetilde{C}^\beta(t) = \widetilde{\lambda}\mathcal{E}_x(t), \\[2ex] \dfrac{\beta}{\alpha+\beta}(\widetilde{x}^i)^{\alpha}(t)\widetilde{C}^{\beta-1}(t) = \widetilde{\lambda}r, \\[2ex] \mathbf{E}\Big[\displaystyle\int_0^\infty \psi_x(t)\widetilde{x}^i(t)\,dt + r\int_0^\infty e^{-rt}\frac{1}{n}\widetilde{C}(t)\,dt\Big] = w. \end{cases} \tag{4.8}$$

By solving systems (4.7) and (4.8) one easily obtains

$$\begin{cases} x_\star^i(t) = \left(\dfrac{r\alpha}{\beta}\right)\left[\lambda_\star\left(\dfrac{\alpha+\beta}{\alpha}\right)\left(\dfrac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}}(n\mathcal{E}_x(t))^{-\frac{(1-\beta)}{1-\alpha-\beta}}, \\[2ex] C_\star(t) = \left[\lambda_\star\left(\dfrac{\alpha+\beta}{\alpha}\right)\left(\dfrac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}}(n\mathcal{E}_x(t))^{-\frac{\alpha}{1-\alpha-\beta}}, \\[2ex] \lambda_\star^{-\frac{1}{1-\alpha-\beta}} = \dfrac{nw}{rn^{-\frac{\alpha}{1-\alpha-\beta}}\left(\frac{\alpha+\beta}{\beta}\right)\left[\left(\frac{r\alpha}{\beta}\right)^{1-\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)\right]^{-\frac{1}{1-\alpha-\beta}}\mathbf{E}\Big[\displaystyle\int_0^\infty e^{-rt}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t)\,dt\Big]}, \end{cases}$$

and

$$\begin{cases} \widetilde{x}^i(t) = \left(\dfrac{r\alpha}{\beta}\right)\left[\lambda_\star\left(\dfrac{\alpha+\beta}{\alpha}\right)\left(\dfrac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}}\mathcal{E}_x^{-\frac{(1-\beta)}{1-\alpha-\beta}}(t), \\[2ex] \widetilde{C}(t) = \left[\lambda_\star\left(\dfrac{\alpha+\beta}{\alpha}\right)\left(\dfrac{r\alpha}{\beta}\right)^{1-\alpha}\right]^{-\frac{1}{1-\alpha-\beta}}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t), \\[2ex] \widetilde{\lambda}^{-\frac{1}{1-\alpha-\beta}} = \dfrac{nw}{rn^{-\frac{\alpha}{1-\alpha-\beta}}\left(\frac{n\alpha+\beta}{n\beta}\right)\left[\left(\frac{r\alpha}{\beta}\right)^{1-\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{-\frac{1}{1-\alpha-\beta}}\right]^{-\frac{1}{1-\alpha-\beta}}\mathbf{E}\Big[\displaystyle\int_0^\infty e^{-rt}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t)\,dt\Big]}, \end{cases}$$

with $\mathbf{E}\left[\int_0^\infty e^{-rt}\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(t)\,dt\right] < \infty$ since $\sqrt{2r} > \frac{\alpha}{1-\alpha-\beta}$. Then (4.6) follows. □

Corollary 4.5. *For any $n \geq 1$ one has*

$$\frac{C_*(t)}{\hat{C}(t)} = \frac{C_\star(t)}{\widetilde{C}(t)} \leq 1.$$

That is, irreversibility of the public good contributions does not influence the degree of free-riding.

In conclusion, we have shown that in our model, for a symmetric economy with Cobb-Douglas utilities, the degree of underprovision of the public good due to free-riding does not depend on irreversibility of the public good contributions or the level of uncertainty, when the latter is given by an exogenous one-dimensional Brownian motion.

4.2 An Existence Result for Nash Equilibria

We conclude the analysis of the game by proving an abstract existence result for a Nash equilibrium (cf. Definition 4.1). To do so we will combine arguments from Conjugate Duality Theory (cf. Rockafellar [46]) and from the General Theory of Stochastic Processes (cf. Dellacherie and Meyer [21]) to apply the Kakutani-Fan-Glicksberg Theorem.

Recall that $\psi_x(t) = e^{\alpha_x(t)}\mathcal{E}_x(t)$ and $\psi_c(t) = e^{\alpha_c(t)}\mathcal{E}_c(t)$ for some continuous and bounded processes α_x and α_c, and for some exponential martingales $\mathcal{E}_x$ and $\mathcal{E}_c$ (cf. Assumption 1.ii.). Moreover, as in the proof of Theorem 3.1, let $\tilde{\mathbf{E}}_c[\cdot]$ and $\tilde{\mathbf{E}}_x[\cdot]$ denote the expectations under the measures $\tilde{\mathbf{P}}_c$ and $\tilde{\mathbf{P}}_x$ with Radon-Nikodym derivative $\mathcal{E}_c(T)$ and $\mathcal{E}_x(T)$, respectively, with respect to $\mathbf{P}$. Since $\mathcal{E}_x(T) > 0$ and $\mathcal{E}_c(T) > 0$ a.s., the measure $\mathbf{P}$ is equivalent to $\tilde{\mathbf{P}}_c$ and $\tilde{\mathbf{P}}_x$. Finally, define $\mathbb{L}^2(d\mu_x)$ as the space of all functions square-integrable with respect to the product measure $d\mu_x := d\tilde{\mathbf{P}}_x \otimes dt$. From now on we make the following

Assumption 2. $x^i \in \mathbb{L}^2(d\mu_x)$, *for any* $i = 1, \ldots, n$.

Since ψ_x is a discounted martingale, an application of Girsanov's Theorem shows that Assumption 2 means that $\mathbf{E}[\int_0^T \psi_x(t)|x^i(t)|^2 dt] < \infty$ for any $i = 1, \ldots, n$; that is, $(x^i)^2$ has a finite price. Such a condition is quite common in financial models where one usually takes the commodity space to be $\mathbb{L}^2$ (on some measure space) and the consumption set to be the positive cone $\mathbb{L}^2_+$ (see Duffie and Zame [22] or Hildenbrand and Sonnenschein [29], Chapter 34, among others).

Taking account of the additional constraint, we define the new set of admissible strategies for agent $i = 1, \ldots, n$ and arbitrary but fixed $w^i \in \mathbb{R}_+$ by

$$\begin{aligned}\mathcal{A}_{w^i} := \Big\{&(x^i, C^i) : \Omega \times [0,T] \mapsto \mathbb{R}^2_+ \text{ adapted s.t. } C \text{ is right-continuous, nondecreasing,}\\ &C^i(0-) = 0 \ \mathbf{P}\text{-a.s.}, \ \mathbf{E}\left[\int_0^T \psi_x(t)x^i(t)dt\right] + \mathbf{E}\left[\int_0^T \psi_c(t)dC^i(t)\right] \leq w^i \text{ and}\\ &\tilde{\mathbf{E}}_x\left[\int_0^T |x^i(t)|^2 dt\right] < \infty\Big\}.\end{aligned}$$

Notice that $\mathcal{A}_{w^i} \subseteq \mathcal{S}_{w^i} \otimes \mathcal{K}_{w^i}$, with

$$\mathcal{S}_{w^i} := \Big\{C^i : (0, C^i) \in \mathcal{A}_{w^i}\Big\}, \qquad \mathcal{K}_{w^i} := \Big\{x^i : (x^i, 0) \in \mathcal{A}_{w^i}\Big\}.$$

It is convenient to rewrite the maximal sets of contribution to the public good or of private consumption, respectively, under the measures $\tilde{\mathbf{P}}_c$ and $\tilde{\mathbf{P}}_x$; that is

$$\begin{aligned}\mathcal{S}_{w^i} := \Big\{ & C^i : \Omega \times [0,T] \mapsto \mathbb{R}_+ \text{ adapted s.t. } C \text{ is right-continuous, nondecreasing,} \\ & C^i(0-) = 0 \ \tilde{\mathbf{P}}_c\text{-a.s. and } \tilde{\mathbf{E}}_c\Big[\int_0^T e^{\alpha_c(t)} dC^i(t)\Big] \le w^i \Big\},\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\mathcal{K}_{w^i} := \Big\{ & x^i : \Omega \times [0,T] \mapsto \mathbb{R}_+ \text{ adapted s.t. } \tilde{\mathbf{E}}_x\Big[\int_0^T e^{\alpha_x(t)} x^i(t) dt\Big] \le w^i \text{ and} \\ & \tilde{\mathbf{E}}_x\Big[\int_0^T |x^i(t)|^2 dt\Big] < \infty \Big\}.\end{aligned} \tag{4.10}$$

Proving existence of a Nash equilibrium with payoffs

$$V^i(C^{-i}) := \sup_{(x^i, C^i) \in \mathcal{A}_{w^i}} U^i(x^i, C^i; C^{-i}), \quad i = 1, \dots, n,$$

and U^i as in (2.2) usually amounts to applying some fixed point argument, and hence to proving compactness of the set of admissible strategies and continuity (or even upper semicontinuity) of the payoff functionals. By employing a duality approach, we start proving compactness of $\mathcal{K}_{w^i}$ and $\mathcal{S}_{w^i}$ (cf. (4.9) and (4.10)) in some suitable topologies and closedness of $\mathcal{A}_{w^i}$ with respect to the associated product topology. From now on let $i = 1, \dots, n$ be arbitrary but fixed.

Proposition 4.6. *The set $\mathcal{K}_{w^i}$ is a compact subset of $\mathbb{L}^2(d\mu_x)$ with respect to the weak-topology $\sigma(\mathbb{L}^2(d\mu_x), \mathbb{L}^2(d\mu_x))$.*

Proof. The set $\mathcal{K}_{w^i}$ is the polar of the singleton $\left\{\frac{e^{\alpha_x}}{w^i}\right\} \subset \mathbb{L}^2(d\mu_x)$. Hence, the Banach-Alaoglu Theorem (see, e.g., Aliprantis and Border [1]) implies the thesis. □

To take care of $\mathcal{S}_{w^i}$ we may adopt arguments by Martins-da-Rocha and Riedel [38] and [39]. Let $\mathcal{M}_+$ be the set of all positive, nondecreasing and càdlàg functions from $[0,T]$ into $\mathbb{R}_+$. The set of ($\tilde{\mathbf{P}}_c$-equivalent classes of) mappings $C : \Omega \mapsto \mathcal{M}_+$ such that $C(t)$ is $\mathcal{F}_t$-adapted for any $t \in [0,T]$ and $C(T) \in \mathbb{L}^1(\tilde{\mathbf{P}}_c)$ is denoted by E_+. The linear span of E_+ is E. It follows that any $C \in E_+$ may be identified with an optional random measure on $[0,T]$ that we denote by dC. Moreover, if $Z \in E$, then there exist $C_1, C_2 \in E_+$ such that $Z = C_1 - C_2$. We endow E with the total variation norm

$$||Z||_E := \tilde{\mathbf{E}}_c\Big[\int_0^T d|Z|(t)\Big], \qquad Z \in E. \tag{4.11}$$

Let $B(T)$ denote the space of bounded functions on $[0,T]$ and $\mathbb{L}^\infty(\tilde{\mathbf{P}}_c, B(T))$ the space (up to indistinguishability) of all progressively measurable processes $\psi : \Omega \times [0,T] \mapsto \mathbb{R}$ such that $\omega \mapsto \sup_{t \in [0,T]} \psi(\omega, t) \in \mathbb{L}^\infty(\tilde{\mathbf{P}}_c)$.

A natural duality $\langle \cdot, \cdot \rangle$ on $\mathbb{L}^\infty(\tilde{\mathbf{P}}_c, B(T)) \times E$ is

$$\langle \psi, Z \rangle := \tilde{\mathbf{E}}_c \Big[\int_0^T \psi(t) dZ(t) \Big]$$

and we denote by F the space of bounded processes $\psi \in \mathbb{L}^\infty(\tilde{\mathbf{P}}_c, B(T))$ that are optional, i.e. measurable with respect to the optional sigma-field $\mathcal{O}$ generated, for example, by all the càdlàg processes (see Dellacherie and Meyer [21], among others). On the other hand, F_+ is the order dual cone

$$F_+ := \{\psi \in F : \langle \psi, C \rangle \geq 0, \ \ \forall C \in E_+\}.$$

Notice that, $\psi \in F_+$ if and only if $\psi(t) \geq 0$ for every $t \in [0, T]$ (cf. Martins-da-Rocha and Riedel [38], Proposition 1). Moreover, if $\psi \in F_+$, then the duality product $\langle \psi, C \rangle$, $C \in E_+$, is the value of the cumulative contribution C under the price ψ. According to Martins-da-Rocha and Riedel [38], Proposition 1, the pair $\langle F, E \rangle$ is a Riesz dual pair and thus the following result holds.

Proposition 4.7. *The set $\mathcal{S}_{w^i}$ is a compact subset of E with respect to the weak*-topology $\sigma(E, F)$.*

Proof. Notice that $\mathcal{S}_{w^i} \subset E_+ \subset E$ and that it is the polar of the singleton $\left\{ \dfrac{e^{\alpha_c}}{w^i} \right\} \subset F_+ \subset F$. The Banach-Alaoglu Theorem (see, e.g., Aliprantis and Border [1]) gives the sought result. □

Since $\mathcal{A}_{w^i}$ is closed with respect to the associated product topology on $\mathcal{S}_{w^i} \otimes \mathcal{K}_{w^i}$, we finally obtain

Proposition 4.8. *The set $\mathcal{A}_{w^i}$ is a compact subset of $\mathbb{L}^2(d\mu_x) \otimes E_+$ for the weak-topology $\sigma(\mathbb{L}^2(d\mu_x), \mathbb{L}^2(d\mu_x))$ and the weak*-topology $\sigma(E, F)$.*

We now deal with the upper semicontinuity of the utility functional U^i, $i = 1, \ldots, n$, of (2.2) under the product of the weak-topology $\sigma(\mathbb{L}^2(d\mu_x), \mathbb{L}^2(d\mu_x))$ and the weak*-topology $\sigma(E, F)$. The uniform boundedness principle for weak and weak* topologies (cf., e.g., Aliprantis and Border [1]) will be a key tool for the proof.

Proposition 4.9. *For any C^{-i} fixed, $i = 1, \ldots, n$, the mapping $(x^i, C^i) \mapsto U^i(x^i, C^i; C^{-i})$ with*

$$U^i(x^i, C^i; C^{-i}) = \mathbf{E} \Big[\int_0^T e^{-rs} u^i(x^i(s), C^i(s) + C^{-i}(s))\, ds \Big]$$

is upper semicontinuous on $\mathbb{L}^2(d\mu_x) \otimes E_+$ for the weak-topology $\sigma(\mathbb{L}^2(d\mu_x), \mathbb{L}^2(d\mu_x))$ and the weak-topology $\sigma(E, F)$.*

Proof. To prove upper semicontinuity of $U^i(\cdot, \cdot; C^{-i})$ for any given C^{-i}, it suffices to show that the set

$$\mathcal{G}^a_{w^i} := \{(x^i, C^i) \in \mathbb{L}^2(d\mu_x) \otimes E_+ : \ U^i(x^i, C^i; C^{-i}) \geq a\}$$

is closed for any $a \in \mathbb{R}$. Therefore, fix $a \in \mathbb{R}$ and take $\{(x^i_k, C^i_k)\}_{k \in \mathbb{N}} \subset \mathcal{G}^a_{w^i}$ such that $x^i_k \rightharpoonup x^i$ in the weak-topology $\sigma(\mathbb{L}^2(d\mu_x), \mathbb{L}^2(d\mu_x))$ and $C^i_k \rightharpoonup^* C^i$ in the weak*-topology $\sigma(E, F)$, for some $(x^i, C^i) \in \mathbb{L}^2(d\mu_x) \otimes E_+$. We want to show that $(x^i, C^i) \in \mathcal{G}^a_{w^i}$. First of all, since $x^i_k \rightharpoonup x^i$ in the weak-topology $\sigma(\mathbb{L}^2(d\mu_x), \mathbb{L}^2(d\mu_x))$, x^i_k is strongly bounded in $\mathbb{L}^2(d\mu_x)$ and hence, by the

Banach-Saks Theorem [5], it admits a subsequence converging strongly in $\mathbb{L}^2(d\mu_x)$ to x^i in the Cesàro sense. Without loss of generality we may also assume that (up to a further subsequence still denoted by x^i_k) the convergence holds $d\mu_x$-a.e. in the Cesàro sense and hence $d\mathbf{P} \otimes dt$-a.e. as $\tilde{\mathbf{P}}_x \sim \mathbf{P}$.

On the other hand, $C^i_k \rightharpoonup^* C^i$ in the weak*-topology $\sigma(E, F)$, and therefore $\{C^i_k\}_{k\in\mathbb{N}}$ is strongly bounded in E; that is, $\sup_{k\geq 1} ||C^i_k||_E = \sup_{k\geq 1} \tilde{\mathbf{E}}_c[C^i_k(T)] < \infty$ (cf. (4.11)). By a version of Komlòs' theorem for optional random measures (cf. Kabanov [32], Lemma 3.5), it follows that that there exists a subsequence of $\{C^i_k\}_{k\in\mathbb{N}}$ (still denoted by $\{C^i_k\}_{k\in\mathbb{N}}$) that converges weakly $\tilde{\mathbf{P}}_c$-a.s. in the Cesàro sense to an optional random measure dC^i; i.e.,

$$I^i_j(t) := \frac{1}{j}\sum_{\ell=1}^{j} C^i_l(t) \to C^i(t), \quad \tilde{\mathbf{P}}_c - \text{a.s., for any } t \text{ of continuity of } C^i(\cdot) \text{ and for } t = T \quad (4.12)$$

as $j \to \infty$. Still denoting by C^i the right-continuous version of C^i, then (4.12) also holds $d\tilde{\mathbf{P}}_c \otimes dt$-a.e., as C^i is nondecreasing and right-continuous. Hence, it holds $d\mathbf{P} \otimes dt$-a.e. being $\tilde{\mathbf{P}}_c \sim \mathbf{P}$.

Then, joint continuity and concavity of u^i imply

$$\begin{aligned}
U^i(x^i, C^i; C^{-i}) &= \mathbf{E}\Big[\int_0^T e^{-rs} u^i\Big(\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} x^i_j(s), \lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} C^i_j(s)\Big)\, ds\Big] \\
&= \mathbf{E}\Big[\int_0^T e^{-rs} \limsup_{k\to\infty} u^i\Big(\frac{1}{k}\sum_{j=1}^{k} x^i_j(s), \frac{1}{k}\sum_{j=1}^{k} C^i_j(s)\Big)\, ds\Big] \\
&\geq \mathbf{E}\Big[\int_0^T e^{-rs} \limsup_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} u^i(x^i_j(s), C^i_j(s))\, ds\Big] \\
&\geq \limsup_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \mathbf{E}\Big[\int_0^T e^{-rs} u^i(x^i_j(s), C^i_j(s))\, ds\Big] \geq a,
\end{aligned}$$

where the second inequality follows by Fatou's Lemma thanks to Assumption 1.v., whereas the last step is due to the fact that $(x^i_k, C^i_k) \in \mathcal{G}^a_{w^i}$. Therefore $\mathcal{G}^a_{w^i}$ is closed and this concludes the proof. □

We may now prove the main result of this section, which is Theorem 2.6 (cf. Section 2).

Proof of Theorem 2.6. $\mathcal{A}_{w^i}$ is compact by Proposition 4.8. Moreover, the best reply correspondence $r(C)$, with $r_i(C^{-i}) := \arg\max_{(x^i,C^i)\in\mathcal{A}_{w^i}} U^i(x^i, C^i; C^{-i})$, $i = 1, \ldots, n$, maps $\prod_{i=1}^n \mathcal{B}_{w^i}$ into itself and it is non-empty, convex, and upper hemicontinuous thanks to Proposition 4.9. The Kakutani-Fan-Glicksberg Theorem (see, e.g., Aliprantis and Border [1], Corollary 17.55) finally implies the thesis. □

A Proofs and Technical Results

A.1 Proof of Proposition 3.4

In this section we prove Proposition 3.4. The proof is a generalization of that in Bank and Riedel [10], Theorem 3.2, to the case of a multivariate optimal consumption problem. Sufficiency easily follows from concavity of the utility functions u^i, $i = 1, \dots, n$. On the other hand, the next Lemma accomplishes the proof of the necessity part. Necessity is proved by linearizing the original problem (3.1) around its optimal solution $(\underline{x}_*, \underline{C}_*)$, by showing that $(\underline{x}_*, \underline{C}_*)$ solves the linearized problem as well and that it satisfies some flat-off conditions similar to those of (3.6).

Recall the notation $x(t) := \sum_{i=1}^n x^i(t)$ and $C(t) := \sum_{i=1}^n C^i(t)$.

Lemma A.1. *Let $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$ be optimal for problem (3.1) and set*

$$\Psi_*(t) := \mathbf{E}\Big[\int_t^T e^{-\int_0^s r(u)\,du} \sum_{i=1}^n \gamma^i\, u_c^i(x_*^i(s), C_*(s))\, ds \Big| \mathcal{F}_t\Big]. \tag{A-1}$$

Then $(\underline{x}_*, \underline{C}_*)$

i. solves the linear optimization problem

$$\sup_{(\underline{x},\underline{C})\in\mathcal{B}_w} \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)ds} \sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t)) x^i(t) dt + \int_0^T \Psi_*(t) dC(t)\Big]; \tag{A-2}$$

ii. satisfies

$$\begin{cases} \Big(e^{-\int_0^t r(s)ds}\gamma^i u_x^i(x_*^i(t), C_*(t)) - M\psi_x(t)\Big)\hat{x}^i(t) = 0, \quad i = 1, \dots, n, \\ \mathbf{E}\Big[\int_0^T \Big(\Psi_*(t) - M\psi_c(t)\Big) dC_*(t)\Big] = 0, \end{cases} \tag{A-3}$$

with

$$M := \operatorname*{ess\,sup}_{(\omega,t)} \Bigg[\max\Bigg\{\frac{e^{-\int_0^t r(s)ds}\gamma^1 u_x^1(x_*^1(t), C_*(t))}{\psi_x(t)}, \dots \\ \dots, \frac{e^{-\int_0^t r(s)ds}\gamma^n u_x^n(x_*^n(t), C_*(t))}{\psi_x(t)}, \frac{\Psi_*(t)}{\psi_c(t)}\Bigg\}\Bigg]. \tag{A-4}$$

Proof. The proof splits into two steps.

Step 1. Let $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$ be optimal for problem (3.1). For $(\underline{x}, \underline{C}) \in \mathcal{B}_w$ and $\epsilon \in [0, 1]$, define the admissible strategy $(\underline{x}_\epsilon, \underline{C}_\epsilon)$ with $\underline{x}_\epsilon(t) := \epsilon\underline{x}(t) + (1-\epsilon)\underline{x}_*(t)$ and such that $C_\epsilon(t) = \epsilon C(t) + (1-\epsilon)C_*(t)$. Notice that $\underline{x}_\epsilon(t)$ and $C_\epsilon(t)$ converge to $\underline{x}_*(t)$ and $C_*(t)$, respectively, a.s. for $t \in [0, T]$ when $\epsilon \downarrow 0$. Now, optimality of $(\underline{x}_*, \underline{C}_*)$, concavity of u^i and an application of

Fubini's Theorem allow us to write

$$\begin{aligned}
0 &\geq \frac{1}{\epsilon}[U_{SP}(\underline{x}_\epsilon, \underline{C}_\epsilon) - U_{SP}(\underline{x}_*, \underline{C}_*)] \\
&\geq \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t))(x^i(t) - x_*^i(t))\,dt\Big] \\
&\quad + \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_c^i(x_\epsilon^i(t), C_\epsilon(t))(C(t) - C_*(t))\,dt\Big] \\
&= \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t))(x^i(t) - x_*^i(t))\,dt\Big] \\
&\quad + \mathbf{E}\Big[\int_0^T \Phi_\epsilon(t)(dC(t) - dC_*(t))\Big],
\end{aligned}$$

where $\Phi_\epsilon(t) := \int_t^T e^{-\int_0^s r(u)\,du} \sum_{i=1}^n \gamma^i\, u_c^i(x_\epsilon^i(s), C_\epsilon(s))\,ds$. One has

$$\begin{aligned}
&\liminf_{\epsilon\downarrow 0} \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t)) x^i(t) dt\Big] \\
&\geq \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t)) x^i(t) dt\Big],
\end{aligned}$$

and

$$\liminf_{\epsilon\downarrow 0} \mathbf{E}\Big[\int_0^T \Phi_\epsilon(t) dC(t)\Big] \geq \mathbf{E}\Big[\int_0^T \Phi_*(t) dC(t)\Big],$$

with $\Phi_* := \Phi_0$, by Fatou's Lemma. If now

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t)) x_*^i(t) dt\Big] \\
&= \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t)) x_*^i(t) dt\Big],
\end{aligned} \tag{A-5}$$

and

$$\lim_{\epsilon\downarrow 0} \mathbf{E}\Big[\int_0^T \Phi_\epsilon(t) dC_*(t)\Big] = \mathbf{E}\Big[\int_0^T \Phi_*(t) dC_*(t)\Big], \tag{A-6}$$

then

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t)) x^i(t)\,dt\Big] + \mathbf{E}\Big[\int_0^T \Phi_*(t) dC(t)\Big] \\
&\leq \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)\,ds} \sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t)) x_*^i(t)\,dt\Big] + \mathbf{E}\Big[\int_0^T \Phi_*(t) dC_*(t)\Big].
\end{aligned}$$

By replacing Φ_* with its optional projection Ψ_* as in (A-1) (cf. Jacod [30], Theorem 1.33) it follows that $(\underline{x}_*, \underline{C}_*)$ is optimal for problem (A-2) as well.

To conclude the proof we must prove (A-5) and (A-6). First of all, notice that $\int_0^T \Phi_\epsilon(t)dC_*(t) = \int_0^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_c^i(x_\epsilon^i(t), C_\epsilon(t))C_*(t)dt$ by Fubini's Theorem. Hence, to have (A-5) and (A-6) it suffices to show that the families $(\Gamma_\epsilon^1)_{\epsilon\in[0,\frac{1}{2}]}$ and $(\Gamma_\epsilon^2)_{\epsilon\in[0,\frac{1}{2}]}$ given by

$$\Gamma_\epsilon^1(t) := e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t))x_*^i(t) \text{ and } \Gamma_\epsilon^2(t) := e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_c^i(x_\epsilon^i(t), C_\epsilon(t))C_*(t)$$

are $\mathbf{P}\otimes dt$-uniformly integrable. Concavity of u^i and the fact that $x_\epsilon^i(t) \geq \frac{1}{2}x_*^i(t)$ a.s. for $\epsilon \in [0, \frac{1}{2}]$ and every $t \in [0, T]$ lead to

$$\Gamma_\epsilon^1(t) \leq 2e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u_x^i(x_\epsilon^i(t), C_\epsilon(t))x_\epsilon^i(t) \leq 2e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i u^i(x_\epsilon^i(t), C_\epsilon(t)). \tag{A-7}$$

The last term in the right-hand side of (A-7) is $\mathbf{P}\otimes dt$-uniformly integrable by Assumption 1.v. Then (A-5) holds by Vitali's Convergence Theorem. Similar arguments show that $(\Gamma_\epsilon^2)_{\epsilon\in[0,\frac{1}{2}]}$ is $\mathbf{P}\otimes dt$-uniformly integrable as well.

Step 2. We now show that the flat-off conditions (A-3) hold for any solution $(\underline{\hat{x}}, \underline{\hat{C}})$ of the linear problem (A-2). Then, by Step 1, they also hold for $(\underline{x}_*, \underline{C}_*)$.

Notice that for every $(\underline{x}, \underline{C}) \in \mathcal{B}_w$ one has

$$\begin{aligned}&\mathbf{E}\Big[\int_0^T \sum_{i=1}^n e^{-\int_0^t r(s)ds}\gamma^i u_x^i(x_*^i(t), C_*(t))x^i(t)dt + \int_0^T \Psi_*(t)dC(t)\Big]\\ &\leq M\mathbf{E}\Big[\int_0^T \sum_{i=1}^n \psi_x(t)x^i(t)dt + \int_0^T \psi_c(t)dC(t)\Big] = Mw\end{aligned} \tag{A-8}$$

by definition of M (cf. (A-4)). Obviously, if $(\underline{x}, \underline{C})$ satisfies (A-3) we then have equality in (A-8). On the other hand, if

$$\sup_{(\underline{x},\underline{C})\in\mathcal{B}_w} \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))x^i(t)dt + \int_0^T \Psi_*(t)dC(t)\Big] = Mw, \tag{A-9}$$

then equality holds through (A-8) and we obtain (A-3).

It therefore remains to prove (A-9). To this end take $K < M$ and define the stopping times

$$\begin{cases} \tau_K^i := \inf\{t \in [0,T) : e^{-\int_0^t r(s)ds}\gamma^i u_x^i(x_*^i(t), C_*(t)) \leq K\psi_x(t)\} \wedge T, \quad i = 1, \ldots, n,\\ \sigma_K := \inf\{t \in [0,T) : \Psi_*(t) > K\psi_c(t)\} \wedge T,\end{cases}$$

together with the investment strategies

$$x_K^i(t) := \alpha \mathbb{1}_{[0,\tau_K^i]}(t), \qquad C_K(t) := \alpha \mathbb{1}_{[\sigma_K,T]}(t),$$

for some α such that $\mathbf{E}[\int_0^T \sum_{i=1}^n \psi_x(t)x_K^i(t)dt + \int_0^T \psi_c(t)dC_K(t)] = w$. We then have

$$\begin{aligned}
Mw &\geq \sup_{(\underline{x},\underline{C})\in\mathcal{B}_w} \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))x^i(t)dt + \int_0^T \Psi_*(t)dC(t)\Big] \\
&\geq \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))x_K^i(t)dt + \int_0^T \Psi_*(t)dC_K(t)\Big] \\
&\geq K\,\mathbf{E}\Big[\int_0^T \sum_{i=1}^n \psi_x(t)x_K^i(t)dt + \alpha\psi_c(\sigma_K)\mathbb{1}_{\{\sigma_K<T\}}\Big] \\
&\geq K\,\mathbf{E}\Big[\int_0^T \sum_{i=1}^n \psi_x(t)x_K^i(t)dt + \int_0^T \psi_c(t)dC_K(t)\Big] = Kw,
\end{aligned}$$

which yields (A-9) by letting $K \uparrow M$. □

We are now able to prove Proposition 3.4.

Proof of Proposition 3.4. Sufficiency follows from concavity of utility function u^i, $i = 1, \ldots, n$, (cf. Assumption 1). Indeed, for $(\underline{x}_*, \underline{C}_*) \in \mathcal{B}_w$ satisfying (3.6) and for $(\underline{x}, \underline{C})$ any other admissible policy we may write

$$\begin{aligned}
U_{SP}(\underline{x}_*, \underline{C}_*) - U_{SP}(\underline{x}, \underline{C}) \geq& \mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))(x_*^i(t) - x^i(t))dt\Big] \\
&+\mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u_c^i(x_*^i(t), C_*(t))(C_*(t) - C(t))dt\Big] \\
=&\mathbf{E}\Big[\int_0^T e^{-\int_0^t r(s)ds}\sum_{i=1}^n \gamma^i u_x^i(x_*^i(t), C_*(t))(x_*^i(t) - x^i(t))dt\Big] \\
&+\mathbf{E}\Big[\int_0^T \Big(\int_t^T e^{-\int_0^s r(u)\,du}\sum_{i=1}^n \gamma^i u_c^i(x_*^i(s), C_*(s))\,ds\Big)(dC_*(t) - dC(t))\Big] \\
\geq& \lambda(w - w) = 0,
\end{aligned}$$

where (3.6) and Fubini's Theorem lead to the second inequality, whereas the last one is implied by the first and the fourth of (3.6) and by the budget constraint. Finally, Lemma A.1 yields the proof of the necessary part. □

A.2 Proposition A.2

Proposition A.2. *Under Assumption 1 and with h^i as in the hypothesis of Theorem 2.1 there exists an optional process l^* which solves the backward stochastic equation*

$$\mathbf{E}\Big[\int_\tau^T e^{-\int_0^t r(s)\,ds}\sum_{i=1}^n \gamma^i h^i\Big(\frac{\lambda}{\gamma^i}e^{\int_0^t r(u)du}\psi_x(t), \sup_{\tau\leq u\leq t} l^*(u)\Big)\,dt\Big|\mathcal{F}_\tau\Big] = \lambda\psi_c(\tau)\mathbb{1}_{\{\tau<T\}} \qquad \text{(A-10)}$$

for any $\tau \in \mathcal{T}$. Moreover, l^* has upper right-continuous sample paths and then it is unique up to indistinguishability.

Proof. Recall that the mapping $c \mapsto h^i(\psi, c)$ is the composition of $c \mapsto g^i(\psi, c)$ and $c \mapsto u_c^i(x, c)$, and hence it is continuous, strictly decreasing and it satisfies the Inada conditions hypothesized in Theorem 2.1. These properties are inherited by the function $\sum_{i=1}^n \gamma^i h^i(\psi, \cdot)$, being $\gamma^i > 0$, $i = 1, \ldots, n$. Moreover, for any given $\lambda > 0$ the process $\lambda\psi_c(t)\mathbb{1}_{\{t<T\}}$ is of class (D), lower semicontinuous in expectation and it vanishes at T, since $\mathcal{E}_c$ is a uniformly integrable martingale and α_c is continuous and bounded. Suitably applying Bank and El Karoui [7], Theorem 3 (see also the example in Bank and El Karoui [7], Section 3.1), we have existence of an optional signal process l^* solving (A-10). Then, easily adopting arguments similar to those in Bank and Küchler [9], proof of Theorem 1, one can show that such l^* is upper right-continuous and therefore it is unique up to indistinguishability by Bank and El Karoui [7], Theorem 1, and Meyer's optional section theorem (see, e.g., Dellacherie and Meyer [21], Theorem IV.86). □

A.3 Proof of Proposition 2.3

Recall that $h^i(\psi, c) := u_c^i(g^i(\psi, c), c)$ with $g^i(\cdot, c)$ the inverse of $u_x^i(\cdot, c)$. For any $\lambda^i > 0$, straightforward computations lead to $h^i(\lambda^i e^{rt}\psi_x(t), C(t)) = \delta(\lambda^i \mathcal{E}_x(t))^{\frac{\alpha}{\alpha-1}} C^{\frac{\alpha+\beta-1}{1-\alpha}}(t)$, with $\delta := \frac{\beta}{\alpha}\left(\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\alpha-1}}$. Set $C_*^i(t) = \sup_{0\le s\le t} l^*(s) \vee 0$ for some progressively measurable process $l^*(t)$ solving

$$\mathbf{E}\left[\int_\tau^\infty \delta e^{-rs}(\lambda^i \mathcal{E}_x(s))^{\frac{\alpha}{\alpha-1}}\left(n \sup_{\tau\le u\le s} l^*(u)\right)^{\frac{\alpha+\beta-1}{1-\alpha}} ds \middle| \mathcal{F}_\tau\right] = \lambda^i e^{-r\tau}\mathcal{E}_c(\tau),$$

i.e.,

$$\mathbf{E}\left[\int_0^\infty \delta e^{-ru}(\lambda^i)^{\frac{\alpha}{\alpha-1}} \frac{\mathcal{E}_x^{\frac{\alpha}{\alpha-1}}(u+\tau)}{\mathcal{E}_c(\tau)} \inf_{0\le s\le u}\left(n l^{*\frac{\alpha+\beta-1}{1-\alpha}}(s+\tau)\right) du \middle| \mathcal{F}_\tau\right] = \lambda^i. \tag{A-11}$$

Now define $l^*(t) := \frac{\kappa}{n}\mathcal{E}_c^{\frac{1-\alpha}{\alpha+\beta-1}}(t)\mathcal{E}_x^{\frac{\alpha}{\alpha+\beta-1}}(t)$ for some constant κ and use independence and stationarity of Lévy increments to rewrite (A-11) as

$$\kappa^{\frac{\alpha+\beta-1}{1-\alpha}}\mathbf{E}\left[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\left(\mathcal{E}_c(s)\mathcal{E}_x^{\frac{\alpha}{\alpha-1}}(u-s)\right) du\right] = (\lambda^i)^{\frac{1}{1-\alpha}}. \tag{A-12}$$

By defining $A := \mathbf{E}[\int_0^\infty \delta e^{-ru} \inf_{0\le s\le u}\left(\mathcal{E}_c(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha}}(u-s)\right) du]$ (cf. (2.14)), and by solving (A-12) for λ^i one obtains

$$\lambda^i := A^{1-\alpha}\kappa^{\alpha+\beta-1}.$$

But now $x_*^i(t) = [\lambda^i\left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t)C_*^{-\beta}(t)]^{\frac{1}{\alpha-1}}$, and therefore

$$x_*^i(t) = \frac{1}{A}(\lambda^i)^{-\frac{1}{1-\alpha}}\left[\left(\frac{\alpha+\beta}{\alpha}\right)\mathcal{E}_x(t) l_0^{-\beta} \inf_{0\le s\le t}\left(\mathcal{E}_c^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}}(s)\mathcal{E}_x^{\frac{\alpha\beta}{1-\alpha-\beta}}(s)\right)\right]^{-\frac{1}{1-\alpha}};$$

that is,

$$x_*^i(t) = \kappa\gamma(t) \tag{A-13}$$

with $\gamma(t)$ as in (2.11).

To determine κ we use the budget constraint $\mathbb{E}[\int_0^\infty \psi_x(t)x_*^i(t)dt + \int_0^\infty \psi_c(t)dC_*^i(t)] = w$. Indeed, by (A-13) we have

$$\kappa \mathbf{E}\left[\int_0^\infty \psi_x(t)\gamma(t)dt + \frac{1}{n}\int_0^\infty \psi_c(t)d\theta(t)\right] = w, \tag{A-14}$$

since $C_*^i(t) = \sup_{0\le s\le t} l^*(s) = \frac{\kappa}{n}\sup_{0\le s\le t}(\mathcal{E}_c^{-\frac{1-\alpha}{1-\alpha-\beta}}(s)\mathcal{E}_x^{-\frac{\alpha}{1-\alpha-\beta}}(s)) = \kappa\theta(t)$ with $\theta(t)$ as in (2.12). Now the result follows by solving (A-14) for κ.

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