

Trockel, Walter

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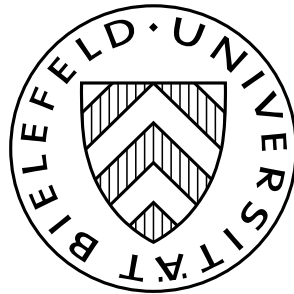
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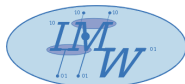
# Robustness of Intermediate Agreements for the Discrete Raiffa Solution

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Walter Trockel



IMW · Bielefeld University  
Postfach 100131  
33501 Bielefeld · Germany



email: [imw@wiwi.uni-bielefeld.de](mailto:imw@wiwi.uni-bielefeld.de)  
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# Robustness of Intermediate Agreements for the Discrete Raiffa Solution\*

Walter Trockel<sup>a,b</sup>

a Murat Sertel Institute for Advanced Economic Studies, Istanbul  
Bilgi University, Turkey

b IMW, Bielefeld University, Postbox 100131, 33501 Bielefeld, Germany

<http://www.imw.uni-bielefeld.de/members/wtrockel.php>

## Abstract

These notes consist of two parts. In the first one I present a counter example to Proposition 3 of Anbarci & Sun (2013). In the second part I propose a correction based on an axiom similar to one used by Salonen (1988) in the first axiomatization of the Discrete Raiffa solution.

\*Counterexample and correction for Proposition 3 in N. Anbarci and C.-j. Sun (2013):  
Robustness of Intermediate Agreements and Bargaining Solutions  
Games and Economic Behavior, 77, 367-376

# 1 Basic Definitions and Axioms

This section is mainly an extract of relevant parts of the respective section in Anbarci and Sun (2013) supplemented by some remarks and an axiom from Salonen (1988).

## 1.1 Basic Definitions

An  $n$ -person (bargaining) problem is a pair  $(S, d)$ , where  $S \subset \mathbb{R}^n$  is the set of utility possibilities that the players can achieve through cooperation and  $d \in S$  is the disagreement point, which is the utility allocation that results if no agreement is reached. For all  $S$ , let

$$IR(S, d) := \{x \in S \mid x \geq d\}$$

be the set of individually rational utility allocations.

Let  $\Sigma$  be the class of all  $n$ -person problems satisfying the following:

- (1) The set  $S$  is compact, convex and comprehensive.
- (2)  $x > d$  for some  $x \in S$

It will be convenient to consider also  $\tilde{\Sigma}$ , the set of all bargaining problems satisfying just(1).

Denote the ideal point of  $(S, d) \in \tilde{\Sigma}$  as  $\tilde{b}(S, d) = (\tilde{b}_i(S, d))_{i=1, \dots, n}$  where  $\tilde{b}_i(S, d) := \max\{x_i \in \mathbb{R} \mid x \in IR(S, d)\}$ ; the midpoint of  $(S, d) \in \tilde{\Sigma}$  is  $\tilde{m}(S, d) := 1/n \tilde{b}(S, d) + (1 - 1/n)d$ . Denote by  $b$  and  $m$  the restrictions of  $\tilde{b}$  and  $\tilde{m}$  to  $\Sigma \subset \tilde{\Sigma}$ , respectively.

A solution on  $\tilde{\Sigma}$  is a function  $\tilde{f} : \tilde{\Sigma} \rightarrow \mathbb{R}^n$  such that for all  $(S, d) \in \tilde{\Sigma}$  we have  $\tilde{f}(S, d) \in S$ . The restriction of  $\tilde{f}$  to  $\Sigma \subset \tilde{\Sigma}$  is denoted  $f$  and is called a solution on  $\Sigma$ .

Consider any bargaining problem  $(S, d) \in \tilde{\Sigma}$ . The game  $(H^S, d) \in \tilde{\Sigma}$  defined by  $H^S := \text{co}\{d, \tilde{b}_1(S, d)e_1, \dots, \tilde{b}_n(S, d)e_n\}$ , with  $e_i$ ,  $i = 1, \dots, n$  the canonical unit vectors of  $\mathbb{R}^n$ , is the “largest hyperplane game contained” in  $(S, d)$ . The game  $(H^S, \tilde{m}(S, d))$  is an element of  $\tilde{\Sigma} \setminus \Sigma$ .

Given any bargaining problem  $(S, d) \in \Sigma$  and a solution  $f : \Sigma \rightarrow \mathbb{R}^n$   $D(S, d, f) := \{d' \in IR(S, d) \mid f(S, d') = f(S, d)\}$ .

Notice, that this definition employed by Anbarci and Sun (2013) makes use of the assumption that  $(S, d') \in \Sigma$ . Therefore for the game  $H^S(S, d) \in \Sigma$  the set  $D(H^S, d, m)$  does not contain  $d' := m(H^S, d) = m(S, d)$  as an element. Therefore, the solution  $m : \Sigma \rightarrow \mathbb{R}^n$  on  $\Sigma$  has the property that  $m(S, d) \notin D(S, d, m)$ .

## 1.2 Axioms

First I introduce the three axioms of Anbarci and Sun (2013) relevant for my analysis. Then I formulate for the present context and in the present terminology of Anbarci and Sun an axiom due to Salonen (1988) that is crucial for the announced correction in the second part of this note. Let  $f : \Sigma \rightarrow \mathbb{R}^n$  be a solution on  $\Sigma$ .

### Midpoint Domination (MD)

For any  $(S, d) \in \Sigma : f(S, d) \geq m(S, d)$

### Independence of Non-Midpoint-Dominating Alternatives (INMD):

For all  $(S, d), (T, d) \in \Sigma$  if  $IR(S, m(S, d)) = IR(T, m(T, d))$  then  $f(S, d) = f(T, d)$ .

As Anbarci and Sun (2013) stress the hypothesis of INMD implies:

$b(S, d) = b(T, d)$  and  $m(S, d) = m(T, d)$ . Therefore the solution  $m : \Sigma \rightarrow \mathbb{R}^n$  on  $\Sigma$  satisfies INMD.

### Robustness of Intermediate Agreements in the $(d, b)$ -Box (RIA-Box):

For all  $(S, d), (T, d) \in \Sigma$  such that  $S \subset T$  and  $b(S, d) = b(T, d)$ :

$$(D(S, d, f) \cup \{f(S, d)\}) \cap (D(T, d, f) \cup \{f(T, d)\}) \setminus \{d\} \neq \emptyset.$$

According to Proposition 3 of Anbarci and Sun (2013) these three axioms (MD, INMD, RIA-Box) determine uniquely the Discrete Raiffa Solution  $DR$  on  $\Sigma$ , which they define as follows:

For any  $(S, d) \in \Sigma$  consider the non-decreasing sequence  $(m_t)_{t \in \mathbb{N}_0}$

with  $m_t \in S$  for all  $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $m_0 := m(S, d)$  and  $m_t := m(S, m_{t-1})$  for all  $t \in \mathbb{N}$ . Then  $DR(S, m) := \lim_{t \rightarrow \infty} m_t$ .

Salonen (1988) was the first article to my best knowledge that provided in his Theorem 2 an axiomatization of the Discrete Raiffa solution on the set  $\tilde{\Sigma}$  of bargaining problems. The three axioms he is using are anonymity, covariance under affine transformations and an axiom, that he called Second Decomposability axiom.

In the context of  $\Sigma$  rather than  $\tilde{\Sigma}$  and the terminology of Anbarci and Sun this axiom can be restated as:

*SD* : For all  $(S, d), (T, d) \in \Sigma$  such that  $S \subset T$  and  $b(S, d) = b(T, d)$  there exists a bargaining problem  $(A, d) \in \Sigma$  such that:  
 $b(A, d) = b(S, d) (= b(T, d))$  and  $f(A, d) \in D(S, d, f) \cap D(T, d, f)$

In fact,  $(H^S, d) = (H^T, d)$  can serve as such an  $(A, d) \in \Sigma$ .

Based on this insight an axiomatization of the Raiffa solution via repeated application of the three “standard axioms” Pareto optimality, covariance and symmetry has been derived in Trockel (2009).

## 2 A Counterexample to Proposition 3

Let  $n := 2$ . The mapping  $m : \Sigma \rightarrow \mathbb{R}^2 : (S, d) \mapsto m(S, d) \in S$  is a solution on  $\Sigma$ . It provides the counterexample.

I have to verify that  $m$  satisfies each of the three axioms of Proposition 3 of Anbarci and Sun (2013), namely *MD*, *INMD* and *RIA-Box*.

### 2.1 MD

$m$  satisfies the required weak inequality in *MD* as equality.

### 2.2 INMD

It has already been remarked before and in fact, by Anbarci and Sun, that  $m$  satisfies *INMD*.

## 2.3 RIA - Box

In order to establish *RIA*-Box for  $m$  we need to verify that:

$$(D(S, d, m) \cup \{m(S, d)\}) \cap (D(T, d, m) \cup \{m(T, d)\}) \setminus \{d\} \neq \emptyset$$

for all  $(S, d), (T, d) \in \Sigma$  with  $S \subset T$  and  $b(S, d) = b(T, d)$ .

First, by  $b(S, d) = b(T, d)$  we get  $m(S, d) = m(T, d) > d$ .

Therefore we are done, if  $D(S, d, m)$  and  $D(T, d, m)$  are both well-defined (possibly empty) sets.

Consider first  $D(T, d, m) := \{d' \in IR(T, d) \mid m(T, d') = m(T, d)\}$   $IR(T, d)$  is well defined as  $(T, d) \in \Sigma \subset \tilde{\Sigma}$  and  $d \in IR(T, d)$  thus  $D(T, d, m) \neq \emptyset$ . Notice, that  $m(T, d) \notin D(T, d, m)$  because by  $H^T \equiv H^S \subseteq S \subset T$  we have  $m(T, m(T, d)) \neq m(T, d)$ .

Next I consider  $D(S, d, m) = \{d' \in IR(S, d) \mid m(S, d') = m(S, d)\}$ .

Again  $IR(S, d)$  is well defined as  $(S, d) \in \Sigma \subset \tilde{\Sigma}$  and  $d \in IR(S, d)$  thus  $D(S, d, m) \neq \emptyset$ . In the special case that  $S = H^S$  we have  $(S, m(S, d)) \in \tilde{\Sigma} \setminus \Sigma$ . Then  $\tilde{m}(S, m(S, d)) = m(S, d) \notin D(S, d, m)$ , as  $m(S, m(S, d))$  is not defined.

Summarizing we get, that  $D(T, d, m)$  and  $D(S, d, m)$  are non empty sets and that for  $\bar{m} := m(T, d) = m(S, d)$ :

$$\bar{m} \in (D(T, d, m) \cup \{\bar{m}\}) \cap ((D(S, d, m) \cup \{\bar{m}\}) \setminus \{d\}).$$

This proves the correctness of the counterexample.

## 3 Correction

Next I will discuss how the axiom *RIA*-Box can be modified in order to exclude the solution  $m$  but not *DR*.

First I reformulate *RIA*-Box as follows:

Consider an arbitrary pair  $(S, d), (T, d) \in \Sigma$  such that

$$S \subset T, b(S, d) = b(T, d).$$

$$\mathbf{A} := (D(S, d, f) \cap (D(T, d, f)) \setminus \{d\}$$

$$\mathbf{B} := (D(S, d, f) \cap \{f(T, d)\}) \setminus \{d\}$$

$$\mathbf{C} := (\{f(S, d)\} \cap D(T, d, f)) \setminus \{d\}$$

$$\mathbf{D} := (\{f(S, d)\} \cap \{f(T, d)\}) \setminus \{d\}$$

Clearly,  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{D}$  may vary with the considered solution  $f$ !

*RIA*-Box is equivalent to:  $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D} \neq \emptyset$ .

Next I want to see, first for  $m$ , then for *DR* which of the sets  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  is always, never or sometimes (depending on the choice of  $(S, d), (T, d)$ ) non-empty.

### 3.1 $f := m$

A: It suffices for me to verify that  $\mathbf{A}$  may be empty.

Take  $T := [0, 1]^2, S := H^T, d := (0, 0)$ . It is easy to see that  $D(S, d, m) = co \{d, m(S, d)\} \setminus \{m(S, d)\}$  while  $D(T, d, m) = \{d\}$ . Therefore  $\mathbf{A} = \emptyset$ .

B:  $m(T, d) = m(S, d) \notin D(S, d, m)$  has been established already in the previous section. So  $\mathbf{B}$  is always empty.

C: Suppose  $\mathbf{C} \neq \emptyset$ . Then  $m(S, d) \in \{d' \in IR(T, d) \mid m(T, d') = m(T, d)\}$ . This is equivalent to  $m(T, m(S, d)) = m(T, d)$ . As  $S \subset T$  we have in fact  $m(T, m(S, d)) > m(T, d) > d$ . Therefore  $\mathbf{C} = \emptyset$ .

D:  $\mathbf{D}$  is always non-empty as  $m(T, d) = m(S, d) > d$

The solution  $m : \Sigma \rightarrow \mathbb{R}^2$  satisfies *RIA*-Box if and only if  $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D}$  is always non-empty. As  $\mathbf{B} \cup \mathbf{C}$  is always empty for  $m$  *RIA*-Box holds for  $m$  if and only if  $\mathbf{A} \cup \mathbf{D}$  is always non-empty. But this is the case as  $D$  is always non-empty for  $m$ . It may happen that  $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} = \emptyset$  for  $m$ . Therefore requiring  $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} = \emptyset$  for all  $(S, d), (T, d)$  with  $S \subset T, b(S, d) = b(T, d)$  would eliminate  $m$  as a solution. Clearly, then already  $\mathbf{A} \cup \mathbf{C} \neq \emptyset$  would do so.



### 3.2 $f = DR$

$\mathbf{A}, \mathbf{C}, \mathbf{D}$  may be empty or non-empty for suitable choices of  $(S, d), (T, d) \in \Sigma, S \subset T$ , with  $b(S, d) = b(T, d)$ . Suppose  $\mathbf{B} \neq \emptyset$ . This is only possible if  $f(T, d) = f(S, d) \in D(S, d, f)$ . But  $f(S, d) \in D(S, d, f)$  requires that  $f(S, f(S, d)) = f(S, d)$ . However, as (by known Pareto optimality of  $DR$ )  $(S, f(S, d)) \in \tilde{\Sigma} \setminus \Sigma$  the term  $f(S, f(S, d))$  is not defined for  $f := DR : \Sigma \rightarrow \mathbb{R}^2$ . So we get  $\mathbf{B}$  is always empty.

So we may replace  $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D} \neq \emptyset$  which is equivalent to  $RIA$ -Box, without any effect on  $DR$  by  $\mathbf{A} \cup \mathbf{C} \cup \mathbf{D} \neq \emptyset$ .

I have to make sure now that skipping the  $\mathbf{D}$  in this assumption, which makes the assumption formally stronger, does not exclude  $DR$ . Consider  $f := DR$ .

If  $\mathbf{D} = \emptyset$  then  $\mathbf{A} \cup \mathbf{C} \cup \mathbf{D} \neq \emptyset$  is equivalent to  $\mathbf{A} \cup \mathbf{C} \neq \emptyset$ .

If  $\mathbf{D} \neq \emptyset$  for some  $(S, d), (T, d)$  with  $S \subset T, b(S, d) = b(T, d)$  then  $f(T, d) = f(S, d) \neq d$  and  $S \neq H^S$  because  $f(H^S, d) = m(H^S, d) = m(S, d)$  is not Pareto optimal in  $T$ , while  $f(T, d) = f(S, d)$  is. But by the definition of  $DR$  the point  $f(H^S, d) = m(H^S, d)$  is the first term  $m_0$  in the sequence of points  $m_t \in D(S, d, f)$  converging to  $f(S, d)$  and the first term in the sequence of points  $m'_t \in D(T, d, f)$  converging to  $f(T, d)$ . Hence,  $m_0 = m(S, d) \in A$ . So  $D \neq \emptyset$  implies  $A \neq \emptyset$ . Hence it can never happen that  $\mathbf{A} \cup \mathbf{C} \cup \mathbf{D} \neq \emptyset$  for some  $(S, d), (T, d)$  while  $\mathbf{A} \cup \mathbf{C} = \emptyset$  for  $f = DR$ .

So  $DR$  satisfies: “ $\mathbf{A} \cup \mathbf{C} \cup \mathbf{D}$  always non empty” if and only if it satisfies: “ $\mathbf{A} \cup \mathbf{C}$  always non empty”.

So the strengthening of  $RIA$ -Box has no impact on  $DR$  and clearly it does not allow other solutions to enter that are already excluded by  $RIA$ -Box.

So let me rewrite the sloppy assumption: “ $\mathbf{A} \cup \mathbf{C}$  always non empty” in a formal way more similar to  $RIA$ -Box.

“ $\mathbf{A} \cup \mathbf{C}$  always non empty” means:

**$RIA^*$ -Box:**

For all  $(S, d), (T, d) \in \Sigma$  with  $S \subset T$  and  $b(S, d) = b(T, d)$ :

$$(D(S, d, f) \cup \{f(S, d)\}) \cap D(T, d, f) \setminus \{d\} \neq \emptyset$$

So this axiom requires that there is an intermediate agreement point  $p$  for the larger bargaining problem  $(T, d)$  that is either the solution of the smaller problem  $(S, d)$  or also an intermediate agreement point for  $(S, d)$ .

This axiom is somewhat weaker than Salonen's axiom  $SD$  that requires in addition that  $p = f(A, d)$  for some bargaining problem  $(A, d) \in \Sigma$  with  $b(A, d) = b(T, d) = b(S, d)$  and  $A \subseteq S$ .

Going through the analysis we have proved (for  $n = 2$ ):

**Proposition:**

$DR$  is the unique solution on  $\Sigma$  that satisfies  $MD$ ,  $INMD$  and  $RIA^*$ -Box.

Remark:

We may replace in this proposition  $RIA^*$ -Box by Salonen's axiom  $SD$ , which therefore turns out not only in combination with anonymity and covariance, but also in combination with  $MD$  and  $INMD$  to axiomatize the Discrete Raiffa solution.

## References

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