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# Pareto improving price regulation when the asset market is incomplete <sup>123</sup>

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## **Abstract**

When the asset market is incomplete, competitive equilibria are constrained suboptimal, and there is scope for Pareto improving interventions. Price regulation, which operates anonymously, on market variables, can be such a Pareto improving policy, even when the welfare effects of rationing are taken into account.

Key words: incomplete asset market, fix-price equilibria, Pareto improvement.

JEL classification numbers: D45, D52, D60.

# 1 Introduction

When the asset market is incomplete, competitive equilibrium allocations generically fail to satisfy the criterion of constrained Pareto optimality which recognizes the incompleteness of the asset market. Geanakoplos and Polemarchakis (1986) showed that, generically, there exist reallocations of portfolios that yield Pareto improvements in welfare after prices in spot commodity markets adjust to attain equilibrium; Citanna, Kajii and Villanacci (1998) developed the general method for the determination of improving interventions.

Expansions of the asset market do not necessarily lead to Pareto improvements: Hart (1975) gave an example of financial innovation that leads to a Pareto deterioration; Cass and Citanna (1998), Elul (1995) and Hara (1997) identified conditions for Pareto improving financial innovation.

The failure of constrained optimality casts doubt on non-intervention with competitive markets.

Nevertheless, apart from informational requirements, Kajii (1994) argued that the heterogeneity of individuals and the requirement of anonymity may interfere with improving interventions.

Citanna, Polemarchakis and Tirelli (2001) showed that the taxation of trades in assets can generically implement a Pareto improvement; but it requires that the number of individuals not exceed the number of traded assets.

An alternative to the reallocation of asset portfolios or the taxation of trades in assets is the direct regulation of prices in spot commodity markets. An intervention in spot market prices is not an intervention in individual choice variables but in market variables. As such, it satisfies the requirement of anonymity.

To address the issue of Pareto improving price regulation, one needs a notion of equilibrium that allows for trading at non-competitive prices. An extension of the fix-price equilibrium of Drèze (1975) serves this purpose; alternative specifications, in Barro and Grossman (1971), Bénassy (1975) or Younès (1975) should not affect the argument.

Laroque (1978, 1981) showed that the behavior of fix-price equilibria in the neighborhood of competitive equilibria is particularly complicated. There are robust examples for which, at regulated prices close to competitive prices, there are no fix-price equilibria close to competitive equilibria. Here, one

considers the class of economies, evidently restrictive, that satisfy conditions sufficient for the local uniqueness of fix-price equilibria; a robust example, which satisfies the stronger conditions in Schulz (1983), yields globally unique fix-price equilibria.

Pareto improving price regulation is generically possible; the deviation of prices from their competitive equilibrium values can be chosen independently of the state of the world <sup>1</sup> as long as the number of commodities exceeds the number of individuals.

The conditions under which the result holds, that the number of instruments (contingent commodities) exceed the number of objectives (individuals), imply that the result complements the one of Geanakoplos and Polemarchakis (1986) and Citanna, Polemarchakis and Tirelli (2001).

Antecedents of this result are the argument in Polemarchakis (1979), where fixed wages that need not match shocks in productivity may yield higher expected utility in spite of the loss of output in an economy of overlapping generations; and the argument in Drèze and Gollier (1993), which employs the capital asset pricing model to determine optimal schedules of wages that differ from the marginal productivity of labor. Kalmus (1997) gave an example of Pareto improving price regulation.

Of serious concern are the informational requirements needed to determine, even compute, improving interventions. In the case of price regulation they involve knowledge of marginal utilities of income and excess demands for commodities across states. Geanakoplos and Polemarchakis (1990) and Kübler, Chiappori, Ekeland and Polemarchakis (2001) are only first steps towards an analysis of the informational requirements of active policy.

## 2 The economy

The economy is that of the standard two-period general equilibrium model with numéraire assets and incomplete asset markets. Assets exchange before and commodities after the state of the world realizes.

States of the world are  $s \in \mathcal{S} = \{1, \dots, S\}$  and commodities are  $l \in \mathcal{L} = \{1, \dots, L + 1\}$ . At state  $s$ , commodity  $(L + 1, s)$  is numéraire. Assets are  $a \in \mathcal{A} = \{1, \dots, A + 1\}$ . Asset  $A + 1$  is numéraire. The payoffs of assets are

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<sup>1</sup>John Geanakoplos insisted on this point.

denominated in the numéraire commodity,  $(L + 1, s)$ , at every state of the world.

A bundle of commodities at a state of the world is  $x_s = (\dots, x_{l,s}, \dots, x_{(L+1),s})$ ; across states of the world,  $x = (\dots, x_s, \dots)$ . A portfolio of assets is  $y = (\dots, y_a, \dots, y_{A+1})$ .

The payoffs of an asset across states of the world are  $R_{a\cdot} = (\dots, R_{a,s}, \dots)'$ ; at a state of the world, payoffs of assets are  $R_{\cdot s} = (\dots, R_{a,s}, \dots)$ ; across states of the world, the asset payoff matrix is  $R = (\dots, R_{a\cdot}, \dots)$ .

The asset payoff matrix has full column rank, and the numéraire asset has positive payoffs:  $R_{A+1\cdot} > 0$ .

Individuals are  $i \in \mathcal{I} = \{1, \dots, I\}$ . A utility function,  $u^i$ , that satisfies standard conditions of continuity, monotonicity, quasi-concavity and, when required, smoothness and boundary behavior on the domain of non-negative bundles of commodities and the endowment,  $e^i$ , a strictly positive bundle, describe an individual.

The utility functions and consumption sets of individuals as well as the matrix of asset payoffs do not vary. The allocation of endowments,  $\omega = (\dots, e^i, \dots)$ , identifies an economy, and the set of economies coincides with the strictly positive orthant of the commodity space; a property holds generically if it holds for an open set of economies of full Lebesgue measure.

Prices of commodities at a state of the world are  $(\dots, p_{l,s}, \dots, 1)$ ; across states of the world,  $p = (\dots, p_s, \dots) \gg 0$ ; the price of the numéraire commodity at a state of the world is  $p_{L+1,s} = 1$ ; the domain of prices of commodities is  $\mathcal{P}$ . Prices of assets are  $q = (\dots, q_a, \dots, 1)$ ; The price of the numéraire asset is  $q_{A+1} = 1$ . The domain of prices of assets is  $\mathcal{Q}$ .

It is often convenient to truncate prices of commodities and prices of assets by deleting the prices of the numéraires. Commodities or assets other than the numéraire are  $\check{\mathcal{L}}$  or  $\check{\mathcal{A}}$ ; the domain of prices of commodities or assets other than the numéraires is  $\check{\mathcal{P}}$  or  $\check{\mathcal{Q}}$ .

At arbitrary terms of trade, a competitive equilibrium, typically, does not exist. In commodities and assets other than the numéraire, rationing on net trades, uniform across individuals, serves to attain market clearing. Rationing in the supply (demand) of commodities other than the numéraire is  $\underline{z} \leq 0$  ( $\bar{z} \geq 0$ ). Rationing in the supply (demand) of assets other than the numéraire is  $\underline{y} \leq 0$  ( $\bar{y} \geq 0$ ).

At prices and rationing  $(p, q, \underline{z}, \bar{z}, \underline{y}, \bar{y})$ , the budget set of an individual is

$$\beta^i(p, q, \underline{z}, \bar{z}, \underline{y}, \bar{y}) = \left\{ (x, y) : \begin{array}{l} qy \leq 0, \\ p_s(x_s - e_s^i) \leq R_s \cdot y, \quad s \in \mathcal{S}, \\ \underline{z}_{l,s} \leq x_{l,s} - e_{l,s}^i \leq \bar{z}_{l,s}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \\ \underline{y}_a \leq y \leq \bar{y}_a, \quad a \in \check{\mathcal{A}} \end{array} \right\};$$

his optimization problem is to choose a utility maximizing consumption bundle and asset portfolio,  $d^i(p, q, \underline{z}, \bar{z}, \underline{y}, \bar{y})$ , in his budget set.

An individual is effectively rationed in his supply (demand) for a commodity or an asset if he could increase his utility when the rationing constraint in the supply (demand) of that commodity or asset is removed. There is effective supply (demand) rationing in the market for a commodity or an asset if at least one individual is effectively rationed in his supply (demand) for this commodity or asset. At a competitive equilibrium, there is neither effective supply rationing nor effective demand rationing in the market for any commodity or asset. In this sense, a competitive equilibrium is a special case of a fix-price equilibrium.

**Definition 1 (Fix-price equilibrium)** *A fix-price equilibrium at prices  $(p, q)$  is a pair  $((x^*, y^*), (\underline{z}^*, \bar{z}^*, \underline{y}^*, \bar{y}^*))$ , such that*

1. *for every individual,  $(x^{i*}, y^{i*}) \in d^i(p, q, \underline{z}^*, \bar{z}^*, \underline{y}^*, \bar{y}^*)$ ,*
2.  *$\sum_{i=1}^I x^{i*} = \sum_{i=1}^I e^i$  and  $\sum_{i=1}^I y^{i*} = 0$ ,*
3. *for every  $l \in \check{\mathcal{L}}$ , if for some  $i'$   $x_{l,s}^{i'*} - e_{l,s}^{i'} = \underline{z}_{l,s}^*$ , then for all  $i \in \mathcal{I}$   $x_{l,s}^{i*} - e_{l,s}^i < \bar{z}_{l,s}^*$ , while if for some  $i'$   $x_{l,s}^{i'*} - e_{l,s}^{i'} = \bar{z}_{l,s}^*$ , then for all  $i \in \mathcal{I}$   $x_{l,s}^{i*} - e_{l,s}^i > \underline{z}_{l,s}^*$ ,*
4. *for every  $a \in \check{\mathcal{A}}$ , if for some  $i'$   $y_a^{i'*} = \underline{y}_a^*$ , then for all  $i \in \mathcal{I}$   $y_a^{i*} < \bar{y}_a^*$ , while if for some  $i'$   $y_a^{i'*} = \bar{y}_a^*$ , then for all  $i \in \mathcal{I}$   $y_a^{i*} > \underline{y}_a^*$ .*

At prices  $(p, q)$ , fix-price equilibria exist — Herings and Polemarchakis (1998) gives a proof.

A sign vector,

$$r = (r_{1,1}, \dots, r_{L,S}, r_1, \dots, r_A),$$

describes the state of markets at a fix-price equilibrium. If there is effective supply rationing in the market for a commodity or an asset, the associated component of the sign vector is -1, if there is effective demand rationing it is +1, and if there is no effective rationing it is 0.

For a sign vector  $r$ , the set  $\mathcal{PQ}(r)$  is the set of prices  $(p, q) \in \mathcal{P} \times \mathcal{Q}$ , for which there exists a fix-price equilibrium at prices  $(p, q)$  with state of the markets  $r$ . For prices  $(p, q) \in \mathcal{P} \times \mathcal{Q}$ , the set of fix-price equilibrium allocations is  $\mathcal{D}(p, q)$ , and, for a sign vector  $r$ , the set of fix-price equilibrium allocations with state of the markets  $r$  is  $\mathcal{D}(p, q, r)$ .

A neighborhood of  $\alpha$  is  $\mathcal{N}_\alpha$ . If  $((p^*, q^*), (x^*, y^*))$  is a competitive equilibrium, the allocation  $(x^*, y^*)$  is locally unique as a fix-price equilibrium allocation if there exists a neighborhood  $\bar{\mathcal{N}}_{x^*, y^*}$  such that for every  $\mathcal{N}_{x^*, y^*} \subset \bar{\mathcal{N}}_{x^*, y^*}$  there exists a neighborhood  $\mathcal{N}_{p^*, q^*}$  with  $\mathcal{D}(p, q) \cap \mathcal{N}_{x^*, y^*}$  a singleton for every  $(p, q) \in \mathcal{N}_{p^*, q^*}$ .

If a competitive equilibrium allocation is locally unique as a fix-price equilibrium allocation, then, for prices close to competitive equilibrium prices, there is exactly one fix-price equilibrium allocation close to the competitive allocation.

For a locally unique competitive equilibrium allocation, for each sign vector  $r$ , the function  $(\hat{x}^r, \hat{y}^r) : \bar{\mathcal{N}}_{p^*, q^*} \cap \mathcal{PQ}(r) \rightarrow \mathbb{R}^{I(L+1)S+I(A+1)}$  associates the unique fix-price equilibrium allocation in  $\bar{\mathcal{N}}_{x^*, y^*} \cap \mathcal{D}(p, q, r)$  to  $(p, q)$ .

Comparative statics require a differentiable form of local uniqueness.

**Definition 2 (Differentiable local uniqueness)** *If  $((p^*, q^*), (x^*, y^*))$  is a competitive equilibrium, the allocation  $(x^*, y^*)$  is differentiably locally unique as a fix-price equilibrium allocation if it is locally unique and there is a neighborhood  $\mathcal{N}_{p^*, q^*}$  such that, for every sign vector  $r$ , the function  $(\hat{x}^r, \hat{y}^r)|_{\mathcal{N}_{p^*, q^*} \cap \mathcal{PQ}(r)}$  is differentiable.*

Laroque and Polemarchakis (1978) proved, for a complete asset market, that, generically, the set of fix-price equilibrium allocations can be represented by a finite number of continuously differentiable functions of prices. Nevertheless, the results in Laroque (1978) and the examples in Madden (1982) show that competitive equilibria need not be locally unique as fix-price equilibria. Even though fix-price equilibrium allocations exist for all prices, there may be robust local non-existence, and therefore local non-uniqueness as a fix-price equilibrium, at competitive prices.



**Assumption 1** *For endowments in  $\Omega^*$ , an open set of full Lebesgue measure, if  $((p^*, q^*), (x^*, y^*))$  is a competitive equilibrium, then the competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation.*

By an argument similar to the one in the proof of Theorem 1, Laroque (1981), Herings and Polemarchakis (1998) characterizes economies where competitive equilibrium allocation are differentiably locally unique as a fix-price equilibrium allocation.

Local uniqueness of fix-price equilibrium allocations at competitive equilibria is not too strong a requirement; it is less demanding than the requirement of uniqueness of fix-price equilibrium allocations at prices in a neighborhood of competitive prices, for which a sufficient condition is given in Schulz (1983). This guarantees a certain degree of generality of the results.

The function  $(\hat{x}, \hat{y})$  associates the unique fix-price equilibrium allocation in  $\mathcal{N}_{x^*, y^*}$  to  $(p, q) \in \mathcal{N}_{p^*, q^*}$ . At a locally unique fix-price equilibrium,

$$v^i(p, q) = u^i(\hat{x}^i(p, q)), \quad (p, q) \in \mathcal{N}_{p^*, q^*}$$

defines the indirect utility function of an individual; it is differentiable when the fix-price equilibrium is differentiably locally unique

$$\partial_{p_{l,s}} v^i(p^*, q^*) = -\partial_{x_{L+1,s}} u^i(x^{i*})(x_{l,s}^{i*} - e_{l,s}^i), \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

The effect of a change in the spot market price of commodity  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$  is equal to minus the marginal utility of the numéraire commodity in state  $s$  multiplied by the excess demand of commodity  $(l, s)$  at the competitive equilibrium. Proposition 4.8 implies that the indirect welfare effects of a change in prices, generated by the induced change in the rationing constraints and individuals' choices, equal zero.

## Pareto improving price regulation

If the asset market is incomplete,  $A + 1 < S$ , generically, competitive equilibrium allocations are not Pareto optimal.

Price regulation can Pareto improve on a competitive equilibrium  $((p^*, q^*), (x^*, y^*))$  if there exist prices of commodities  $p$  such that a fix-price equilibrium of commodities at prices of commodities and assets  $(p, q^*)$  Pareto

dominates the allocation  $x^*$ . The ambiguity introduced by the possibility of multiple fix-price equilibrium allocations at prices  $(p, q^*)$  is circumvented by considering local variations at competitive equilibrium allocations that are differentially locally unique as fix-price equilibria.

**Definition 3 (Pareto improving price regulation)** *Price regulation can Pareto improve upon a competitive equilibrium,  $((p^*, q^*), (x^*, y^*))$ , that is locally unique as a fix-price equilibrium if there exists an infinitesimal variation in commodity prices,  $d\check{p}$ , such that  $\sum_{(l,s) \in \tilde{\mathcal{L}} \times \mathcal{S}} \partial_{p_{l,s}} v^i(p^*, q^*) dp_{l,s} > 0$ ,  $i \in \mathcal{I}$ .*

*Uniform price regulation can improve upon a competitive equilibrium if  $d\check{p}_s = d\check{p}_{s'}$ ,  $s, s' \in \mathcal{S}$ .*

Generically, it is possible to make every individual better off by choosing appropriate price regulations on the spot markets when asset markets are incomplete. One needs at least as many instruments,  $LS$ , as individuals,  $I$ .

Uniform price regulation is effective when  $L \geq I$ , which reflects again that the number of instruments has to exceed the number of objectives. This complements the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when  $2L \leq I \leq L(S - 1) + 1$ .

**Proposition 1** *If the asset market is incomplete, and if  $LS \geq I > 1$ , then, generically, price regulation can improve upon any competitive equilibrium. If  $L \geq I > 1$ , then, generically, uniform price regulation can improve upon any competitive equilibrium.*

The proof of proposition 1 follows the approach developed in Geanakoplos and Polemarchakis (1986) and Citanna, Kajii and Villanacci (1998); Herings and Polemarchakis (1998) spells out the argument.

An example illustrates.

### 3 An example

There are two individuals, three states of the world, two commodities, and two assets.

The utility function of individual,  $i$ , has an additively separable representation,  $u^i = \sum_{s \in \mathcal{S}} \pi_s u_s^i$ , with state dependent cardinal utility

$$u_s^i(x_s) = \alpha_s^i \ln x_{1,s} + \beta_s^i x_{2,s}, \quad \alpha_s^i > 0, \beta_s^i > 0,$$

and a strictly positive probability measure  $(\pi_1, \pi_2, \pi_3)$  over the states of the world; his endowment,  $e^i = (e_1^i, e_2^i, e_3^i)$ , is strictly positive.

The payoffs of assets are  $R_1 = (0, 1, 1)'$ , and  $R_2 = (1, 0, 0)'$ ; which allows for the following interpretation: consumption at state of the world 1 is concurrent with the trade in assets, while the only asset available, traded against consumption, is an indexed bond with state-independent payoffs.

The parameters in the utility functions of individuals and their endowments are such that

$$\pi = \frac{\pi_1 \beta_1^1}{\pi_2 \beta_2^1 + \pi_3 \beta_3^1} = \frac{\pi_1 \beta_1^2}{\pi_2 \beta_2^2 + \pi_3 \beta_3^2},$$

and, for  $\gamma_s^i = \alpha_s^i / \beta_s^i$ ,

$$\begin{aligned} & \max \left\{ -e_{2,s}^1 + \frac{\gamma_s^1 e_{1,s}^2 - \gamma_s^2 e_{1,s}^1}{e_{1,s}^1 + e_{1,s}^2} : s = 2, 3, -\pi e_{2,1}^2 + \pi \frac{\gamma_1^2 e_{1,1}^1 - \gamma_1^1 e_{1,1}^2}{e_{1,1}^1 + e_{1,1}^2} \right\} \\ & \leq \min \left\{ \pi e_{2,1}^1 + \pi \frac{\gamma_1^2 e_{1,1}^1 - \gamma_1^1 e_{1,1}^2}{e_{1,1}^1 + e_{1,1}^2}, e_{2,s}^2 + \frac{\gamma_s^1 e_{1,s}^2 - \gamma_s^2 e_{1,s}^1}{e_{1,s}^1 + e_{1,s}^2} : s = 2, 3 \right\}, \end{aligned}$$

which eliminates equilibria at the boundaries of their consumption sets <sup>2</sup>.

Fix-price equilibrium exists for all prices of commodities,  $p$ , and price of the indexed bond  $q = 1/\pi$ . For individual  $i$ ,  $\gamma_s^i / e_{1,s}^i \leq \gamma_s^{i'} / e_{1,s}^{i'}$ ; there are four

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<sup>2</sup>A possible choice of parameters is, for instance,

$$\begin{aligned} \pi_1 &= 1, \pi_2 = \pi_3 = \frac{1}{2}, \\ \alpha_1^1 &= \beta_1^1 = 1, \alpha_2^1 = \beta_2^1 = \frac{4}{3}, \alpha_3^1 = \beta_3^1 = \frac{2}{3}, \\ \alpha_1^2 &= \beta_1^2 = 1, \alpha_2^2 = \beta_2^2 = \frac{2}{3}, \alpha_3^2 = \beta_3^2 = \frac{4}{3}, \\ e_1^1 &= (1, 1)', e_2^1 = (1, 1)', e_3^1 = (2, 1)', \\ e_1^2 &= (1, 1)', e_2^2 = (2, 1)', e_3^2 = (1, 1)'. \end{aligned}$$

different cases: (i)  $0 < p_s \leq \gamma_s^i/e_{1,s}^i$ , (ii)  $\gamma_s^i/e_{1,s}^i \leq p_s \leq (\gamma_s^i + \gamma_s^{i'})/(e_{1,s}^i + e_{1,s}^{i'})$ , (iii)  $(\gamma_s^i + \gamma_s^{i'})/(e_{1,s}^i + e_{1,s}^{i'}) \leq p_s \leq \gamma_s^{i'}/e_{1,s}^{i'}$ , and (iv)  $\gamma_s^{i'}/e_{1,s}^{i'} \leq p_s$ .

Competitive equilibrium prices are given by  $p_s^* = \frac{\gamma_s^1 + \gamma_s^2}{e_{1,s}^1 + e_{1,s}^2}$ ,  $s = 1, 2, 3$ , and  $q^* = \frac{1}{\pi}$ . Those prices belong to the intersection of cases (ii) and (iii).

If  $\gamma_s^i/e_{1,s}^i \leq p_s \leq (\gamma_s^i + \gamma_s^{i'})/(e_{1,s}^i + e_{1,s}^{i'})$ , case (ii), there is aggregate excess demand for commodity 1, but individual  $i$  supplies the commodity, and trade takes place, with individual  $i'$  rationed on his demand of the commodity. Equilibria obtain for  $\bar{z}_{1,s}^* = e_{1,s}^i - \gamma_s^i/p_s$ ,  $x_{1,s}^{i*} = \gamma_s^i/p_s$ ,  $x_{1,s}^{i'*} = e_{1,s}^{i'} + e_{1,s}^i - \gamma_s^i/p_s$ , and  $y^{i'*} = -y^{i*}$ . At  $s = 1$ ,  $x_{2,1}^{i*} = p_1 e_{1,1}^i + e_{2,1}^i - \gamma_1^i - (1/\pi)y^{i*}$ ,  $x_{2,1}^{i'*} = e_{2,1}^{i'} - p_1 e_{1,1}^i + \gamma_1^i + (1/\pi)y^{i*}$ ,  $y^{i*} \leq \pi(p_1 e_{1,1}^i + e_{2,1}^i - \gamma_1^i)$ , and  $y^{i*} \geq -\pi(e_{2,1}^{i'} - p_1 e_{1,1}^i + \gamma_1^i)$ . At  $s = 2$  or  $s = 3$ ,  $x_{2,s}^{i*} = p_s e_{1,s}^i + e_{2,s}^i - \gamma_s^i + y^{i*}$ ,  $x_{2,s}^{i'*} = e_{2,s}^{i'} - p_s e_{1,s}^i + \gamma_s^i - y^{i*}$ ,  $y^{i*} \geq -p_s e_{1,s}^i - e_{2,s}^i + \gamma_s^i$ , and  $y^{i*} \leq e_{2,s}^{i'} - p_s e_{1,s}^i + \gamma_s^i$ . The remaining parameters of the rationing scheme are set so as not to be binding.

If  $(\gamma_s^i + \gamma_s^{i'})/(e_{1,s}^i + e_{1,s}^{i'}) \leq p_s \leq \gamma_s^{i'}/e_{1,s}^{i'}$ , case (iii) there is aggregate excess supply of commodity 1, and individual  $i$  supplies the commodity, rationed by the demand of individual  $i'$ . Equilibria obtain for  $\bar{z}_{1,s}^* = e_{1,s}^{i'} - \gamma_s^{i'}/p_s$ ,  $x_{1,s}^{i*} = e_{1,s}^i + e_{1,s}^{i'} - \gamma_s^{i'}/p_s$ ,  $x_{1,s}^{i'*} = \gamma_s^{i'}/p_s$ , and  $y^{i'*} = -y^{i*}$ . At  $s = 1$ ,  $x_{2,1}^{i*} = e_{2,1}^i - p_1 e_{1,1}^{i'} + \gamma_1^{i'} - (1/\pi)y^{i*}$ ,  $x_{2,1}^{i'*} = p_1 e_{1,1}^{i'} + e_{2,1}^i - \gamma_1^{i'} + (1/\pi)y^{i*}$ ,  $y^{i*} \leq \pi(e_{2,1}^i - p_1 e_{1,1}^{i'} + \gamma_1^{i'})$ , and  $y^{i*} \geq -\pi(p_1 e_{1,1}^{i'} + e_{2,1}^i - \gamma_1^{i'})$ . At  $s = 2$  or  $s = 3$ ,  $x_{2,s}^{i*} = e_{2,s}^i - p_s e_{1,s}^{i'} + \gamma_s^{i'} + y^{i*}$ ,  $x_{2,s}^{i'*} = p_s e_{1,s}^{i'} + e_{2,s}^i - \gamma_s^{i'} - y^{i*}$ ,  $y^{i*} \geq -e_{2,s}^i + p_s e_{1,s}^{i'} - \gamma_s^{i'}$ , and  $y^{i*} \leq p_s e_{1,s}^{i'} + e_{2,s}^i - \gamma_s^{i'}$ . The remaining parameters of the rationing scheme are set so as not to be binding.

The utility attained by each individual at a fix-price equilibrium is unambiguously determined by the prices of commodities. At prices  $p$ , the utility of individual  $i$  at the fix-price equilibrium is  $v^i(p) = \sum_{s \in \mathcal{S}} \pi_s v_s^i(p_s)$ , where,

if  $\gamma_s^i/e_{1,s}^i \leq p_s \leq (\gamma_s^i + \gamma_s^{i'})/(e_{1,s}^i + e_{1,s}^{i'})$ , case (ii), then

$$v_s^i(p_s) = \alpha_s^i \ln\left(\frac{\gamma_s^i}{p_s}\right) + \beta_s^i(p_s e_{1,s}^i + e_{2,s}^i - \gamma_s^i),$$

$$v_s^{i'}(p_s) = \alpha_s^{i'} \ln\left(e_{1,s}^{i'} + e_{1,s}^i - \frac{\gamma_s^i}{p_s}\right) + \beta_s^{i'}(e_{2,s}^{i'} - p_s e_{1,s}^i + \gamma_s^i),$$

while, if  $(\gamma_s^i + \gamma_s^{i'})/(e_{1,s}^i + e_{1,s}^{i'}) \leq p_s \leq \gamma_s^{i'}/e_{1,s}^{i'}$ , case (iii), then

$$v_s^i(p_s) = \alpha_s^i \ln(e_{1,s}^i + e_{1,s}^{i'} - \frac{\gamma_s^{i'}}{p_s}) + \beta_s^i (e_{2,s}^i - p_s e_{1,s}^{i'} + \gamma_s^{i'}),$$

$$v_s^{i'}(p_s) = \alpha_s^{i'} \ln(\frac{\gamma_s^{i'}}{p_s}) + \beta_s^{i'} (p_s e_{1,s}^{i'} + e_{2,s}^{i'} - \gamma_s^{i'}).$$

Substitution of the competitive equilibrium prices in either case (ii) or case (iii) gives the utility levels at the competitive equilibrium. The indirect utility function is differentiable at competitive prices, and the derivative is given by

$$\partial_{p_s} v^i(p^*) = \pi_s \beta_s^i \frac{\gamma_s^{i'} e_{1,s}^i - \gamma_s^i e_{1,s}^{i'}}{\gamma_s^i + \gamma_s^{i'}} = -\pi_s \beta_s^i (x_s^{i*} - e_s^i),$$

$$\partial_{p_s} v^{i'}(p^*) = \pi_s \beta_s^{i'} \frac{\gamma_s^i e_{1,s}^{i'} - \gamma_s^{i'} e_{1,s}^i}{\gamma_s^{i'} + \gamma_s^i} = -\pi_s \beta_s^{i'} (x_s^{i'*} - e_s^{i'}).$$

For  $v_s = \pi_s(\gamma_s^2 e_{1,s}^1 - \gamma_s^1 e_{1,s}^2)/(\gamma_s^1 + \gamma_s^2)$ , it holds that

$$V = \begin{pmatrix} \partial v^1(p^*) \\ \partial v^2(p^*) \end{pmatrix} = \begin{pmatrix} \beta_1^1 v_1 & \beta_2^1 v_2 & \beta_3^1 v_3 \\ -\beta_1^2 v_1 & -\beta_2^2 v_2 & -\beta_3^2 v_3 \end{pmatrix}.$$

If the matrix  $V$  has full row rank, then price regulation can Pareto improve the competitive equilibrium allocation. If the ratios of the marginal utilities of income of the individuals are not the same across all states of the world,  $\beta_1^1/\beta_1^2 \neq \beta_2^1/\beta_2^2$  or  $\beta_3^1/\beta_3^2 \neq \beta_2^1/\beta_2^2$ , for the matrix  $V$  to have full row rank it is sufficient that  $v_s \neq 0$ , for every state of the world. Since  $v_s = 0$  if and only if  $e_{1,s}^1/e_{1,s}^2 = \gamma_s^1/\gamma_s^2$ , generically in the endowments of individuals it is possible to Pareto improve on the competitive allocation <sup>3</sup>.

Because of the constant marginal utility of the numéraire commodity, here, an ad hoc argument shows that variations in endowments of individuals suffice to render Pareto improving price regulation generic.

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<sup>3</sup>For the specification of parameters given in footnote 12,

$$V = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}.$$

Both individuals benefit if the price of commodity 1 in states 2 and 3 is fixed below its competitive equilibrium value.

With the number of non-numéraire commodities 1, less than the number of individuals, 2, it is not always possible to Pareto improve on the competitive equilibrium by state - independent price regulation.

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## Appendix 1: existence of fix-price equilibria

A compact, convex subset of the consumption set that contains the aggregate initial endowment in its interior is  $\widetilde{\mathcal{X}}^i$ . The assumptions on utility functions and on the asset return matrix imply that all  $S + 1$  budget inequalities in the definition of the budget set hold with equality at the optimal choice of an individual. The rationing inequalities do not necessarily hold with equality. The budget set related to  $\widetilde{\mathcal{X}}^i$  with all budget inequalities required to hold with equality is  $\widetilde{\beta}^i$  and the corresponding demand function  $\widetilde{d}^i$ . Since prices are fixed at  $(p, q)$ , they are omitted in the notation.

The demand functions  $\widetilde{d}^i$ ,  $i \in \mathcal{I}$ , are continuous.

If  $(\underline{z}_n, \underline{z}_n, \underline{y}_n, \underline{y}_n)$  is a sequence that converges to  $(\underline{z}, \underline{z}, \underline{y}, \underline{y})$ , then the sequence  $(\widetilde{d}^i(\underline{z}_n, \underline{z}_n, \underline{y}_n, \underline{y}_n))$  has a convergent subsequence, with limit  $(\widehat{x}, \widehat{y}) \in \widetilde{\beta}^i(\underline{z}, \underline{z}, \underline{y}, \underline{y})$ .

If there exists  $(\widetilde{x}, \widetilde{y}) \in \widetilde{\beta}^i(\underline{z}, \underline{z}, \underline{y}, \underline{y})$ , such that  $u^i(\widetilde{x}) > u^i(\widehat{x})$ , and  $\widetilde{\mathcal{L}}_-$ ,  $\widetilde{\mathcal{L}}_+$ ,  $\widetilde{\mathcal{A}}_-$ , and  $\widetilde{\mathcal{A}}_+$ , is the sets of non-numeraire commodities and non-numeraire assets for which  $\widetilde{x}_{l,s} - e_{l,s}^i$  is negative, positive,  $\widetilde{y}_a$  is negative, and positive, respectively, then, for

$$\lambda^n =$$

$$\min \left\{ 1, \frac{\underline{z}_{l,s}^n}{\underline{x}_{l,s}^n - e_{l,s}^i}, (l, s) \in \widetilde{\mathcal{L}}_-, \frac{\underline{z}_{l,s}^n}{\underline{x}_{l,s}^n - e_{l,s}^i}, (l, s) \in \widetilde{\mathcal{L}}_+, \frac{\underline{y}_a^n}{\underline{y}_a^n}, a \in \widetilde{\mathcal{A}}_-, \frac{\underline{y}_a^n}{\underline{y}_a^n}, a \in \widetilde{\mathcal{A}}_+ \right\},$$

$\widetilde{x}^n = e^i + \lambda^n(\widetilde{x} - e^i)$ , and  $\widetilde{y}^n = \lambda^n \widetilde{y}$ ,  $(\widetilde{x}^n, \widetilde{y}^n) \in \widetilde{\beta}^i(\underline{z}^n, \underline{z}^n, \underline{y}^n, \underline{y}^n)$ . Evidently,  $\lim_{n \rightarrow \infty} \lambda^n = 1$ , and  $\lim_{n \rightarrow \infty} (\widetilde{x}^n, \widetilde{y}^n) = (\widetilde{x}, \widetilde{y})$ . By the continuity of  $u^i$ ,  $\widetilde{x}^n$  is strictly preferred to the consumption bundle in  $\widetilde{d}^i(\underline{z}_n, \underline{z}_n, \underline{y}_n, \underline{y}_n)$ , a contradiction.

Since there is no rationing in the market of the numéraire asset nor in the market of the numéraire commodities, the argument for equilibrium existence is not trivial.

If  $((x^*, y^*), (\underline{z}^*, \underline{z}^*, \underline{y}^*, \underline{y}^*))$  is a fix-price equilibrium at prices  $(p, q)$ , then  $\widetilde{x}_{l,s}^{*i'} < \sum_{i=1}^I e_{l,s}^i + \varepsilon$ , with  $\varepsilon$  some fixed positive number. Since  $R$  has full column rank, this implies that there is  $\alpha > 0$  such that  $\|y^{*i}\|_\infty < \alpha$  for any  $y^{*i}$  consistent with a fix-price equilibrium at prices  $(p, q)$ .

The functions  $(\underline{z}, \underline{z}) : \mathcal{C}^{LS} \rightarrow -\mathbb{R}_+^{LS} \times \mathbb{R}_+^{LS}$  and  $(\underline{y}, \underline{y}) : \mathcal{C}^A \rightarrow -\mathbb{R}_+^A \times \mathbb{R}_+^A$ , where  $\mathcal{C}^K = \{r \in \mathbb{R}^K : 0 \leq r_k \leq 1\}$  is the unit cube of dimension  $K$ , are

defined by

$$\underline{z}_{l,s}(r) = -\min\{2r_{l,s}(\sum_{i=1}^I e_{l,s}^i + \varepsilon), \sum_{i=1}^I e_{l,s}^i + \varepsilon\}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\bar{z}_{l,s}(r) = \min\{(2 - 2r_{l,s})(\sum_{i=1}^I e_{l,s}^i + \varepsilon), \sum_{i=1}^I e_{l,s}^i + \varepsilon\}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\underline{y}_a(\rho) = -\min\{2\rho_a\alpha, \alpha\}, \quad a \in \check{\mathcal{A}},$$

$$\bar{y}_a(\rho) = \min\{(2 - 2\rho_a)\alpha, \alpha\}, \quad a \in \check{\mathcal{A}}.$$

The excess demand function  $\tilde{z} : \mathcal{C}^{LS} \times \mathcal{C}^A \rightarrow \mathbb{R}^{LS} \times \mathbb{R}^A$  is

$$\tilde{z}_{l,s}(r, \rho) = \sum_{i=1}^I \tilde{d}_{l,s}^i(\underline{z}(r), \bar{z}(r), \underline{y}(\rho), \bar{y}(\rho)) - \sum_{i=1}^I e_{l,s}^i, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}$$

$$\tilde{z}_a(r, \rho) = \sum_{i=1}^I \tilde{d}_a^i(\underline{z}(r), \bar{z}(r), \underline{y}(\rho), \bar{y}(\rho)), \quad a \in \check{\mathcal{A}}.$$

If  $(r^*, \rho^*) \in \mathcal{C}^{LS} \times \mathcal{C}^A$  is such that  $\tilde{z}(r^*, \rho^*) = 0$ , then  $((x^*, y^*), (\underline{z}^*, \bar{z}^*, \underline{y}^*, \bar{y}^*))$ , where  $(x^{*i}, y^{*i}) = \tilde{d}^i(\underline{z}^*, \bar{z}^*, \underline{y}^*, \bar{y}^*)$ ,  $i \in \mathcal{I}$ ,  $(\underline{z}^*, \bar{z}^*) = (\underline{z}(r^*), \bar{z}(r^*))$ ,  $(\underline{y}^*, \bar{y}^*) = (\underline{y}(r^*), \bar{y}(r^*))$ , is a fix-price equilibrium. Conditions 1 and 2 of Definition 1 are satisfied for non-numeraire commodities and assets. The construction of the functions  $(\underline{z}, \bar{z})$  and  $(\underline{y}, \bar{y})$  takes care of Conditions 3 and 4.

The set  $\tilde{z}(\mathcal{C}^{LS} \times \mathcal{C}^A)$  is compact. Let the set  $\mathcal{ZY}$  be a compact, convex set that contains  $\tilde{z}(\mathcal{C}^{LS} \times \mathcal{C}^A)$ . The correspondence  $\mu : \mathcal{ZY} \rightarrow \mathcal{C}^{LS} \times \mathcal{C}^A$  is defined by

$$\mu(z, y) = \arg \max \left\{ \sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} r_{l,s} z_{l,s} + \sum_{a \in \check{\mathcal{A}}} \rho_a y_a : r \in \mathcal{C}^{LS}, \rho \in \mathcal{C}^A \right\}.$$

The correspondence  $\varphi : \mathcal{ZY} \times \mathcal{C}^{LS} \times \mathcal{C}^A \rightarrow \mathcal{ZY} \times \mathcal{C}^{LS} \times \mathcal{C}^A$  is defined by  $\varphi(z, y, r, \rho) = \{\tilde{z}(r, \rho)\} \times \mu(z, y)$ . It is a non-empty, compact, convex valued, upper hemi-continuous correspondence defined on a non-empty, compact, convex set. By Kakutani's fixed point theorem,  $\varphi$  has a fixed point,  $(z^*, y^*, r^*, \rho^*)$ .

If, for some  $a \in \check{\mathcal{A}}$ ,  $y_a^* < 0$ , then, by the definition of  $\mu$ ,  $\rho_a^* = 0$ , so  $y_a^* \geq 0$ , a contradiction. If, for some  $a \in \check{\mathcal{A}}$ ,  $y_a^* > 0$ , then, by the definition of  $\mu$ ,  $\rho_a^* = 1$ , so  $y_a^* \leq 0$ , a contradiction. Consequently,  $y_a^* = 0$ , for all  $a \in \check{\mathcal{A}}$ . Moreover,  $y_{A+1}^* = -\sum_{a \in \check{\mathcal{A}}} q_a y_a^* = 0$ .

If, for some  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ ,  $z_{l,s}^* < 0$ , then, by the definition of  $\mu$ ,  $r_{l,s}^* = 0$ , so  $z_{l,s}^* \geq 0$ , a contradiction. If, for some  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ ,  $z_{l,s}^* > 0$ , then, by the definition of  $\mu$ ,  $r_{l,s}^* = 1$ , so  $z_{l,s}^* \leq 0$ , a contradiction. Consequently,  $z_{l,s}^* = 0$ , for all  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ . Moreover, for every  $s \in \mathcal{S}$ ,  $z_{L+1,s}^* = -\sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} p_{l,s} z_{l,s}^* + R_s y^* = 0$ .

It follows that  $0 \in \tilde{z}(r^*, \rho^*)$ , so a fix-price equilibrium at prices  $(p, q)$  exists.

## Appendix 2: local comparative statics

In the optimization problem an individual faces when determining his demand, the Lagrange multipliers corresponding to the rationing constraints in the markets for commodities (assets) are  $\pi$  ( $\rho$ ). The individual optimization problem leads to the study of a modified demand function,  $\hat{d}^i$ . At prices and Lagrange multipliers  $(p, q, \pi, \rho)$ ,  $\hat{d}^i$  is defined by the solution to the optimization problem

$$\begin{aligned} \max \quad & u^i(x) - \sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} \pi_{l,s} x_{l,s} - \sum_{a \in \mathcal{A}} \rho_a y_a, \\ \text{s.t.} \quad & qy \leq 0, \\ & p_s(x_s - e_s^i) \leq R_s y, \quad s \in \mathcal{S}. \end{aligned}$$

The set of  $(p, q, \pi, \rho)$  on which each individual optimization problem has a solution is  $\mathcal{N}$ . Whenever  $(p^*, q^*)$  are competitive equilibrium prices,  $\mathcal{N}$  is a neighborhood of  $(p^*, q^*, 0, 0)$ .

The modified demand function,  $\hat{d}^i$ ,  $i \in \mathcal{I}$ , is continuously differentiable on  $\mathcal{N}$ .

At a competitive equilibrium,  $((p^*, q^*), (x^*, y^*))$ ,  $z_{l,s}^-$ ,  $z_{l,s}^+$ ,  $y_a^-$  and  $y_a^+$ , defined by

$$\begin{aligned} z_{l,s}^- &= \min_{i \in \mathcal{I}} x_{l,s}^{i*} - e_{l,s}^i, & z_{l,s}^+ &= \max_{i \in \mathcal{I}} x_{l,s}^{i*} - e_{l,s}^i, & (l, s) &\in \check{\mathcal{L}} \times \mathcal{S}, \\ y_a^- &= \min_{i \in \mathcal{I}} y_a^{i*}, & y_a^+ &= \max_{i \in \mathcal{I}} y_a^{i*}, & a &\in \mathcal{A}, \end{aligned}$$

determine the minimal and the maximal excess demands on both the spot

and the asset markets. If

$$\underline{\mathcal{I}}_{l,s} = \{i \in \mathcal{I} : x_{l,s}^{i*} - e_{l,s}^i = z_{l,s}^-\}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\overline{\mathcal{I}}_{l,s} = \{i \in \mathcal{I} : x_{l,s}^{i*} - e_{l,s}^i = z_{l,s}^+\}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\underline{\mathcal{I}}_a = \{i \in \mathcal{I} : y_a^{i*} = y_a^-\}, \quad a \in \check{\mathcal{A}},$$

$$\overline{\mathcal{I}}_a = \{i \in \mathcal{I} : y_a^{i*} = y_a^+\}, \quad a \in \check{\mathcal{A}},$$

then, in a neighborhood of the competitive equilibrium, only individuals in  $\underline{\mathcal{I}}_{l,s}$  ( $\overline{\mathcal{I}}_{l,s}$ ) may be rationed on supply (demand) in the spot market  $(l, s)$ , and only individuals in  $\underline{\mathcal{I}}_a$  ( $\overline{\mathcal{I}}_a$ ) on supply (demand) in the asset market  $a$ .

For an open set of endowments with full Lebesgue measure  $\Omega \subset \mathbb{R}_{++}^{I(L+1)S}$ , for any competitive equilibrium  $((p^*, q^*), (x^*, y^*))$  of  $\mathcal{E}$ ,  $|\underline{\mathcal{I}}_{l,s}| = |\overline{\mathcal{I}}_{l,s}| = 1$ ,  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ , and  $|\underline{\mathcal{I}}_a| = |\overline{\mathcal{I}}_a| = 1$ ,  $a \in \check{\mathcal{A}}$ .

For a generic set of economies, there is exactly one individual in each market with the minimal excess demand and exactly one individual with the maximal excess demand. For the remainder, the allocation of endowments in the economy  $\mathcal{E}$  belong to the set  $\Omega$ , which permits the study of the local structure of the set fix-price equilibria in the neighborhood of a competitive equilibrium  $((p^*, q^*), (x^*, y^*))$  of  $\mathcal{E}$ .

For every individual, the function  $c^i : \mathbb{R}^{LS} \times \mathbb{R}^A \rightarrow \mathbb{R}^{LS} \times \mathbb{R}^A$  is defined by

$$c_{l,s}^i(\pi, \rho) = \begin{cases} \pi_{l,s}, & \text{if } \pi_{l,s} \leq 0 \text{ and } \{i\} = \underline{\mathcal{I}}_{l,s} \text{ or } \pi_{l,s} \geq 0 \text{ and } \{i\} = \overline{\mathcal{I}}_{l,s}, \\ 0, & \text{otherwise,} \end{cases}$$

$$c_a^i(\pi, \rho) = \begin{cases} \rho_a, & \text{if } \rho_a \leq 0 \text{ and } \{i\} = \underline{\mathcal{I}}_a \text{ or } \rho_a \geq 0 \text{ and } \{i\} = \overline{\mathcal{I}}_a, \\ 0, & \text{otherwise.} \end{cases}$$

The function  $c$  relates the Lagrange multipliers  $(\pi, \rho)$  to the fix-price equilibria in the neighborhood of the competitive equilibrium. The aggregate modified excess demand function for commodities and assets other than the

numéraire is  $\hat{z} : \mathcal{N} \rightarrow \mathbb{R}^{L+S+A}$  defined by

$$\hat{z}_{l,s}(p, q, \pi, \rho) = \sum_{i \in \mathcal{I}} \hat{d}_{l,s}^i(p, q, c^i(\pi, \rho)) - \sum_{i \in \mathcal{I}} e^i, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\hat{z}_a(p, q, \pi, \rho) = \sum_{i \in \mathcal{I}} \hat{d}_a^i(p, q, c^i(\pi, \rho)), \quad a \in \check{\mathcal{A}}.$$

For the study of fix-price equilibria in the neighborhood of the competitive equilibrium, it is sufficient to restrict attention to the zero points of  $\hat{z}$ . Neighborhoods  $\mathcal{N}_{x^{i*}, y^{i*}}^i$  are such that, for every  $(x, y) \in \mathcal{N}_{x^*, y^*} = \times_{i \in \mathcal{I}} \mathcal{N}_{x^{i*}, y^{i*}}^i$ , for all  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ , for all  $a \in \check{\mathcal{A}}$ ,

$$x_{l,s}^{i'} - e_{l,s}^{i'} < 0 \text{ and } x_{l,s}^{i'} - e_{l,s}^{i'} < x_{l,s}^i - e_{l,s}^i, \quad i \neq i', i' \in \underline{\mathcal{I}}_{l,s}$$

$$x_{l,s}^{i'} - e_{l,s}^{i'} > 0 \text{ and } x_{l,s}^{i'} - e_{l,s}^{i'} > x_{l,s}^i - e_{l,s}^i, \quad i \neq i', i' \in \overline{\mathcal{I}}_{l,s}$$

$$y_a^{i'} < 0 \text{ and } y_a^{i'} < y_a^i, \quad i \neq i', i' \in \underline{\mathcal{I}}_a$$

$$y_a^{i'} > 0 \text{ and } y_a^{i'} > y_a^i, \quad i \neq i', i' \in \overline{\mathcal{I}}_a.$$

If  $((p^*, q^*), (x^*, y^*))$  is a competitive equilibrium, and  $(x, y) \in \mathcal{N}_{x^*, y^*}$ , then  $(x, y) \in \mathcal{D}(p, q)$  if and only if there is  $(p, q, \pi, \rho) \in \mathcal{N}$  such that  $\hat{d}^i(p, q, c^i(\pi, \rho)) = (x^i, y^i)$ ,  $i \in \mathcal{I}$ , and  $\hat{z}(p, q, \pi, \rho) = (0, 0)$ .

The function  $\hat{z}$  is Lipschitz continuous because of the differentiability of the functions  $\hat{d}^i$  and the Lipschitz continuity of the functions  $c^i$ . It is differentiable at each  $(p, q, \pi, \rho) \in \mathcal{N}$  where all components of  $\pi$  and  $\rho$  are non-zero. For each sign vector  $r$  without zero components,

$$\mathcal{N}^r = \{(p, q, \pi, \rho) \in \mathcal{N} : \pi_{l,s} r_{l,s} > 0, (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \rho_a r_a > 0, a \in \check{\mathcal{A}}\}.$$

The function  $\hat{z}$  is differentiable on  $\mathcal{N}^r$ . The limit of its Jacobian,  $\lim_{n \rightarrow \infty} \partial \hat{z}(p^n, q^n, \pi^n, \rho^n) = \partial \hat{z}^r(p^*, q^*, 0, 0)$ , along a sequence  $((p^n, q^n, \pi^n, \rho^n) \in \mathcal{N}^r)$  that converges to  $(p^*, q^*, 0, 0)$ , exists;

$$\partial_{\check{p}, \check{q}} \hat{z}_{l,s}^r(p^*, q^*, 0, 0) = \sum_{i \in \mathcal{I}} \partial_{\check{p}, \check{q}} \hat{d}_{l,s}^i(p^*, q^*, 0, 0) = \partial_{\check{p}, \check{q}} z_{l,s}(p^*, q^*),$$

$$\partial_{\check{p}, \check{q}} \hat{z}_a^r(p^*, q^*, 0, 0) = \sum_{i \in \mathcal{I}} \partial_{\check{p}, \check{q}} \hat{d}_a^i(p^*, q^*, 0, 0) = \partial_{\check{p}, \check{q}} z_a(p^*, q^*),$$

where  $z(p, q)$  denotes the unconstrained total excess demand function for commodities and assets other than the numeraire at prices  $(p, q)$ . It follows that the Jacobian with respect to  $(\check{p}, \check{q})$  is independent of  $r$  at a competitive equilibrium.

**Lemma 1** *If  $((p^*, q^*), (x^*, y^*))$  is a competitive equilibrium such that  $\partial z(p^*, q^*)$  is of full rank, then, for each sign vector  $r$  without zero components, the tangent cone at  $(p^*, q^*)$  to the set of price systems having a local fix-price equilibrium with state of the markets  $r$  is*

$$\{(p, q) \in \mathcal{P} \times \mathcal{Q} : (\check{p}, \check{q}) = (\partial z(p^*, q^*))^{-1} \partial_{\pi, \rho} \hat{z}^r(p^*, q^*, 0, 0)(\pi, \rho),$$

$$\pi_{l,s} r_{l,s} > 0, (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \rho_a r_a > 0, a \in \check{\mathcal{A}}\}.$$

**Proof** The restriction of  $\hat{z}$  to  $\mathcal{N}^r$  extends to a differentiable function  $\tilde{z} : \mathcal{N} \rightarrow \mathbb{R}^{LS+A}$  as follows. For  $i \in \mathcal{I}$ , the function  $\tilde{c}^i$  is defined by  $\tilde{c}_{l,s}^i(\pi, \rho) = \pi_{l,s}$  if  $i \in \underline{\mathcal{I}}_{l,s}$ ,  $r_{l,s} = -1$ , or  $i \in \bar{\mathcal{I}}_{l,s}$ ,  $r_{l,s} = +1$ ,  $\tilde{c}_{l,s}^i(\pi, \rho) = 0$  otherwise, and  $\tilde{c}_a^i(\pi, \rho) = \rho_a$  if  $i \in \underline{\mathcal{I}}_a$ ,  $r_a = -1$ , or  $i \in \bar{\mathcal{I}}_a$ ,  $r_a = +1$ , and  $\tilde{c}_a^i(\pi, \rho) = 0$  otherwise. The function  $\tilde{z}$  is defined as  $\hat{z}$  with  $c$  replaced by  $\tilde{c}$ . Since  $\partial z(p^*, q^*)$  is of full rank, it follows by the implicit function theorem that the solution to  $\tilde{z}(p, q, \pi, \rho) = (0, 0)$  determines  $p$  and  $q$  as a function of  $\pi$  and  $\rho$  in a neighborhood of  $(0, 0)$ . The derivative of this function at  $(0, 0)$  with respect to  $\pi$  and  $\rho$  is given by  $(\partial z(p^*, q^*))^{-1} \partial_{\pi, \rho} \tilde{z}(p^*, q^*, 0, 0)$ . The expression in the proposition follows immediately if one takes into account that only  $\pi$ 's and  $\rho$ 's satisfying  $\pi_{l,s} r_{l,s} > 0$ ,  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ , and  $\rho_a r_a > 0$ ,  $a \in \check{\mathcal{A}}$ , should be considered.  $\square$

Proposition 2 in Geanakoplos and Polemarchakis (1986) shows that the assumption that  $\partial z(p^*, q^*)$  has full rank at every competitive equilibrium holds generically in endowments. Lemma 3.2 characterizes the tangent cones to the regions in the price space having a fix-price equilibrium with state of the markets  $r$  in the neighborhood of a competitive equilibrium. It guarantees neither that the closures of these tangent cones cover the price space nor that the tangent cones are full-dimensional nor that the tangent cones do not intersect. If this were the case, local uniqueness would result.

In general, an increase in a price causes a different individual to be rationed as a decrease in a price. Since  $\partial_{\pi, \rho} \hat{z}^r$ , and therefore the tangent cone, depend on  $\partial_{\pi, \rho} \hat{d}^i$  for the individual  $i$  that is rationed, the fact that the tangent cones need not fit nicely together does not come as a surprise. In abstract terms, the fact that different individuals get rationed at different prices in the neighborhood of a competitive equilibrium, creates non-differentiabilities in the function  $\hat{z}$  at competitive prices. At a point of non-differentiability, the implicit function theorem need not apply, and local uniqueness may fail.

The generalized Jacobian of a Lipschitz continuous function  $f$  at a point  $x$  is the convex hull of all matrices that are the limits of the sequence  $(\partial f(x^n))$ , where  $(x^n)$  is a convergent sequence with  $\lim_{n \rightarrow \infty} x^n = x$  and  $f$  is differentiable at  $x^n$ .

A restriction of the fundamentals of the economy, the utility functions of individuals and the matrix of asset payoffs is required to guarantee that, generically, competitive equilibrium allocations are differentially locally unique as fix-price equilibrium allocations

If a function  $f$  is Lipschitz continuous,  $f(\hat{x}, \hat{y}) = 0$ , and every matrix  $M$  in  $\partial_x f(\hat{x}, \hat{y})$  has full rank, then there exist a neighborhood  $\mathcal{N}_{\hat{x}, \hat{y}}$ , a neighborhood  $\mathcal{N}_{\hat{y}}$ , and a Lipschitz continuous function  $g$  on  $\mathcal{N}_{\hat{y}}$  such that  $(x, y) \in \mathcal{N}_{\hat{x}, \hat{y}}$  and  $f(x, y) = 0$  if and only if  $y \in \mathcal{N}_{\hat{y}}$  and  $x = g(y)$ .

**Assumption 2** *For endowments in  $\Omega^*$ , an open set of full Lebesgue measure, if  $((p^*, q^*), (x^*, y^*))$  is a competitive equilibrium, then the determinants of the matrices  $\partial_{\pi, \rho} \hat{z}^r(p^*, q^*, 0, 0)$ , with  $r$  sign vectors without zero components, are either all equal to  $-1$  or all equal to  $+1$ .*

By an argument similar to the one in the proof of Theorem 1, Laroque (1981), the competitive equilibrium allocation is differentially locally unique as a fix-price equilibrium allocation.

**Remark** An example of an economy that satisfies differentiable local uniqueness for all endowments and, à fortiori, satisfies assumption 3.

There are three states of the world, two commodities and two assets. The utility functions have an additively separable representation  $u^i = \sum_{s \in \mathcal{S}} \pi_s u_s^i$  with

$$u_s^i(x_s) = \alpha^i \ln x_{1,s} + (1 - \alpha^i) x_{2,s}, \quad 0 < \alpha^i < 1,$$

and a uniform probability measure  $\pi$  over the states of the world. The payoffs of the assets are  $R_1 = (1, 0, 0)'$ , and  $R_2 = (0, 1, 0)'$ . Endowments are chosen such that  $|\mathcal{I}_{l,s}| = |\bar{\mathcal{I}}_{l,s}| = 1$ ,  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ , and  $|\mathcal{I}_a| = |\bar{\mathcal{I}}_a| = 1$ ,  $a \in \check{\mathcal{A}}$ , so they belong to a set of full Lebesgue measure by Lemma 4.4.

Competitive equilibrium prices are  $(p^*, q^*)$ . All partial derivatives are evaluated at  $(p^*, q^*, 0, 0)$ . It holds that  $\partial_{\pi_{1,s}} \hat{z}^r = \partial_{\pi_{1,s}} \hat{d}^{i(1,s)}$ , where  $\{i(1, s)\} = \mathcal{I}_{1,s}$  if  $r_{1,s} = -1$ , and  $\{i(1, s)\} = \bar{\mathcal{I}}_{1,s}$  if  $r_{1,s} = +1$ . An increase in  $\pi_{1,s}$  corresponds to the introduction of demand rationing or the disappearance of supply rationing on commodity  $(1, s)$ , which decreases the demand for commodity

$(1, s)$ , so  $\partial_{\pi_{1,s}} \hat{z}_{1,s}^r$  is negative. The change in income spent on commodity  $(1, s)$  equals  $p_{1,s}^* \partial_{\pi_{1,s}} \hat{z}_{1,s}^r$ . The individual  $i(1)$  is the one affected by rationing in the asset market, so  $\{i(1)\} = \underline{1}$  if  $r_1 = -1$ , and  $\{i(1)\} = \bar{1}$  if  $r_1 = +1$ . From the properties of the Cobb-Douglas utility function, it follows that

$$\begin{aligned}\partial_{\pi_{1,1}} \hat{d}_{1,2}^{i(1,1)} &= \frac{-\alpha_1^{i(1,1)} p_{1,1}^* q_1^* \partial_{\pi_{1,1}} \hat{z}_{1,1}^r}{p_{1,2}^* q_2^* (2 - \alpha_1^{i(1,1)})}, \partial_{\pi_{1,1}} \hat{d}_{1,3}^{i(1,1)} = 0, \partial_{\pi_{1,1}} \hat{d}_1^{i(1,1)} = \frac{p_{1,1}^* \partial_{\pi_{1,1}} \hat{z}_{1,1}^r}{(2 - \alpha_1^{i(1,1)})}, \\ \partial_{\pi_{1,2}} \hat{d}_{1,1}^{i(1,2)} &= \frac{-\alpha_1^{i(1,2)} p_{1,2}^* q_1^* \partial_{\pi_{1,2}} \hat{z}_{1,2}^r}{p_{1,1}^* q_2^* (2 - \alpha_1^{i(1,2)})}, \partial_{\pi_{1,2}} \hat{d}_{1,3}^{i(1,2)} = 0, \partial_{\pi_{1,2}} \hat{d}_1^{i(1,2)} = \frac{-p_{1,2}^* q_2^* \partial_{\pi_{1,2}} \hat{z}_{1,2}^r}{q_1^* (2 - \alpha_1^{i(1,2)})}, \\ \partial_{\pi_{1,3}} \hat{d}_{1,1}^{i(1,3)} &= 0, \partial_{\pi_{1,3}} \hat{d}_{1,2}^{i(1,3)} = 0, \partial_{\pi_{1,3}} \hat{d}_1^{i(1,3)} = 0, \\ \partial_{\rho_1} \hat{d}_{1,1}^{i(1)} &= \frac{\alpha_1^{i(1)} \partial_{\rho_1} \hat{z}_1^r}{p_{1,1}^*}, \partial_{\rho_1} \hat{d}_{1,2}^{i(1)} = \frac{-\alpha_1^{i(1)} q_1^* \partial_{\rho_1} \hat{z}_1^r}{p_{1,2}^* q_2^*}, \partial_{\rho_1} \hat{d}_{1,3}^{i(1)} = 0.\end{aligned}$$

The sign of the determinant of  $\partial_{\pi,\rho} \hat{z}^r$  does not change by premultiplying it by the strictly positive row vector  $(p_{1,1}^* q_1^*, p_{1,2}^* q_2^*, 1, q_1^*)$  and postmultiplying it by the strictly positive column vector  $((2 - \alpha_1^{i(1,1)}) / -p_{1,1}^* q_1^* \partial_{\pi_{1,1}} \hat{z}_{1,1}^r, (2 - \alpha_1^{i(1,2)}) / -p_{1,2}^* q_2^* \partial_{\pi_{1,2}} \hat{z}_{1,2}^r, 1 / -\hat{z}_{1,3}^r, 1 / -q_1^* \partial_{\rho_1} \hat{z}_1^r)'$ . The resulting matrix is given by

$$\begin{bmatrix} \alpha_1^{i(1,1)} - 2 & \alpha_1^{i(1,2)} & 0 & -\alpha_1^{i(1)} \\ \alpha_1^{i(1,1)} & \alpha_2^{i(1,2)} - 2 & 0 & \alpha_1^{i(1)} \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{bmatrix}$$

and its determinant equals

$$(4 - 2\alpha_1^{i(1,1)} - 2\alpha_1^{i(1,2)})(1 - \alpha_1^{i(1)}) > 0.$$

The determinant of  $\partial_{\pi,\rho} \hat{z}^r$  is positive, irrespective of the sign vector  $r$ . It follows that the competitive equilibrium allocation is differentially locally unique as a fix-price equilibrium allocation.  $\square$

As in Laroque (1981), whenever there are two sign vectors without zero components  $r^1$  and  $r^2$  such that the determinants of  $\partial_{\pi,\rho} \hat{z}^{r^1}(p^*, q^*, 0, 0)$  and



$\partial_{\pi, \rho} \hat{z}^{r^2}(p^*, q^*, 0, 0)$  have opposite signs and  $\partial z(p^*, q^*)$  has full rank, then for every neighborhood  $\mathcal{N}_{x^*, y^*}$  there exists for every neighborhood  $\mathcal{N}_{p^*, q^*}$  a price system  $(p, q) \in \mathcal{N}_{p^*, q^*}$  with at least two fix-price equilibrium allocations in  $\mathcal{N}_{x^*, y^*}$ . Assumption 3 is “almost necessary” for the differentiable local uniqueness of competitive equilibrium allocations.

Local uniqueness of fix-price equilibrium allocations at competitive equilibria is not too strong a requirement. It is less demanding than the requirement of uniqueness of fix-price equilibrium allocations at prices in a neighborhood of competitive prices. The latter requirement has a very clear interpretation and guarantees a certain degree of generality of our results.

The function  $(\hat{x}, \hat{y}) : \mathcal{N}_{p^*, q^*} \rightarrow \mathbb{R}^{I(L+1)S+I(A+1)}$  associates the unique fix-price equilibrium allocation in  $\mathcal{N}_{x^*, y^*}$  to  $(p, q) \in \mathcal{N}_{p^*, q^*}$ . The indirect utility function of an individual at a locally unique fix-price equilibrium is defined by

$$v^i(p, q) = u^i(\hat{x}^i(p, q)), \quad (p, q) \in \mathcal{N}_{p^*, q^*}.$$

**Lemma 2** *If  $((p^*, q^*), (x^*, y^*))$  is a competitive equilibrium, then the indirect utility function  $v^i : \mathcal{N}_{p^*, q^*} \rightarrow \mathbb{R}$  is differentiable and*

$$\partial_{p_{l,s}} v^i(p^*, q^*) = -\partial_{x_{L+1,s}} u^i(x^{i*})(x_{l,s}^{i*} - e_{l,s}^i), \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

**Proof** For every sign vector  $r$ , the restriction of  $v^i$  to  $\mathcal{N}_{p^*, q^*} \cap \mathcal{PQ}(r)$ , denoted  $v^{i^r}$ , is differentiable. From the differentiation of the budget constraints

$$q\hat{y}^{i^r}(p, q) = 0 \text{ and } p_s(\hat{x}_s^{i^r}(p, q) - e_s^i) = R_s \cdot \hat{y}^{i^r}(p, q), \quad s \in \mathcal{S},$$

with respect to  $p_{l,\bar{s}}$ , and the first order conditions for individual optimization at a competitive equilibrium,

$$\partial_{x_{l,s}^i} u^i(x^{i*}) = \partial_{x_{L+1,s}^i} u^i(x^{i*}) p_{l,s}^*, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

and

$$\sum_{s \in \mathcal{S}} \partial_{x_{L+1,s}^i} u^i(x^{i*}) R_s = \mu^i q^*, \quad \text{for some } \mu^i > 0,$$

it follows that

$$\partial_{p_{l,\bar{s}}} v^{i^r}(p^*, q^*) = -\partial_{x_{L+1,\bar{s}}^i} u^i(x^{i*})(x_{l,\bar{s}}^{i*} - e_{l,\bar{s}}^i).$$

Since the derivative is independent of the sign vector  $r$ , the result follows.  $\square$

### Appendix 3: Pareto improving price regulation

Price regulation can Pareto improve on a competitive equilibrium  $((p^*, q^*), (x^*, y^*))$  if there exist prices of commodities  $p$  such that a fix-price equilibrium of commodities at prices of commodities and assets  $(p, q^*)$  Pareto dominates the allocation  $x^*$ . The ambiguity introduced by the possibility of multiple fix-price equilibrium allocations at prices  $(p, q^*)$  is circumvented by considering local variations at competitive equilibrium allocations that are differentially locally unique as fix-price equilibria.

**Definition 4 (Pareto improving price regulation)** *A competitive equilibrium  $((p^*, q^*), (x^*, y^*))$  can be Pareto improved by price regulation if it is differentially locally unique as fix-price equilibrium and there exists an infinitesimal variation in commodity prices  $d\check{p}$  such that  $\sum_{(l,s) \in \mathcal{L} \times \mathcal{S}} \partial_{p_{l,s}} v^i(p^*, q^*) dp_{l,s} > 0$ ,  $i \in \mathcal{I}$ .*

*The competitive equilibrium can be Pareto improved by uniform price regulation if it can be Pareto improved by a price regulation with  $d\check{p}_s = d\check{p}_{s'}$ ,  $s, s' \in \mathcal{S}$ .*

Pareto improvement by price regulation is possible only if the asset market is incomplete. Another necessary requirement is that the economy allows for heterogeneous individuals.

**Assumption 3**  $A + 1 < S$  and  $I > 1$ .

The function  $\varphi$  is defined by

$$\varphi(x, \tilde{\lambda}, \tilde{p}, e) = \begin{pmatrix} \partial u^i(x^i) - \tilde{\lambda}^i \tilde{p}, & i \in \mathcal{I} \\ \sum_{s \in \mathcal{S}} \tilde{p}_s (x_s^i - e_s^i), & i \in \mathcal{I} \\ \sum_{i \in \mathcal{I}} (x_{l,s}^i - e_{l,s}^i), & (l, s) \in \mathcal{L} \times \mathcal{S} \setminus \{(L+1, S)\} \\ \sum_{s \in \mathcal{S}} n_s \tilde{p}_s (x_s^i - e_s^i), & i \in \mathcal{I} \setminus \{1\} \end{pmatrix},$$

where the Lagrangian multiplier  $\tilde{\lambda}^i \in \mathbb{R}$  does not vary with the state of the world, the prices of commodities  $\tilde{p} \in \mathbb{R}_{++}^{(L+1)S-1} \times \{1\}$  are discounted prices, with only the price of commodity  $(L+1, S)$  normalized to 1, and

$n \neq 0$  is a fixed vector such that  $nR = 0$ . In the standard reformulation of the incomplete markets model in discounted prices, it is assumed that one individual is unconstrained, so that his marginal utility at an optimal choice is proportional to the price system. Pareto optimality implies that the marginal utility vectors of all agents should be proportional to the price system. The function  $\varphi$  is completed by specifying budget constraints and market clearing conditions, and one condition for every individual but one that recognizes the incompleteness of markets:  $\sum_{s \in \mathcal{S}} n_s \tilde{p}_s (x_s^i - e_s^i) = 0$ . The existence of  $n \neq 0$  such that  $nR = 0$  follows from market incompleteness. It follows that the function  $\varphi$  vanishes at a Pareto optimal competitive equilibrium.

For a function  $f$  that depends on a vector of variables  $\alpha$  and on endowments  $e$ ,  $f_e(\alpha)$  is the function that results from fixing  $e$ ; for instance,  $\varphi_e(x, \tilde{\lambda}, \tilde{p}) = \varphi(x, \tilde{\lambda}, \tilde{p}, e)$ .

**Lemma 3** *Generically, competitive equilibrium allocations are not Pareto optimal.*

**Proof** A necessary condition for  $x$  to be a Pareto optimal competitive equilibrium allocation for an economy  $e$  is that  $\varphi_e(x, \tilde{\lambda}, \tilde{p}) = 0$ . Since the dimension of the domain of  $\varphi_e$  is lower than the dimension of the range, whenever  $\varphi_e$  is transverse to 0, a solution to  $\varphi_e(x, \tilde{\lambda}, \tilde{p}) = 0$  does not exist. By a standard argument,  $\varphi$  is transverse to 0. By the transversal density theorem, the set of economies for which  $\varphi_e$  is transverse to 0 has full Lebesgue measure. By a standard argument, this set can be chosen to be open.  $\square$

The function  $\psi : \Xi \times \Omega^* \rightarrow \mathbb{R}^N$  is defined by

$$\psi(\xi, e) = \begin{pmatrix} \partial_{x_s^i} u^i(x^i) - \lambda_s^i p_s, & i \in \mathcal{I}, s \in \mathcal{S} \\ p_s(x_s^i - e_s^i) - R_s y^i, & i \in \mathcal{I}, s \in \mathcal{S} \\ \lambda^i R - \mu^i q, & i \in \mathcal{I} \\ \sum_{i \in \mathcal{I}} (x_{l,s}^i - e_{l,s}^i), & (l, s) \in \check{\mathcal{L}} \times \mathcal{S} \\ \sum_{i \in \mathcal{I}} y_a^i, & a \in \check{\mathcal{A}} \\ qy^i, & i \in \mathcal{I} \end{pmatrix},$$

$\xi = (x, \lambda, y, \mu, \check{p}, \check{q})$  and  $\Xi = \mathbb{R}_{++}^{I(L+1)S} \times \mathbb{R}_{++}^{IS} \times \mathbb{R}^{I(A+1)} \times \mathbb{R}^I \times \check{P} \times \check{Q}$ . The dimension of  $\Xi$  is  $N$ . When  $\xi^*$  is consistent with a competitive equilibrium, it is necessarily the case that  $\psi_e(\xi^*) = 0$ .

The function  $h : \Xi \times \mathbb{R}^I \times \Omega^* \rightarrow \mathbb{R}^{LS+1}$  is defined by

$$h(\xi, \alpha, e) = \begin{pmatrix} \sum_{i \in \mathcal{I}} \alpha^i \lambda_s^i (x_{l,s}^i - e_{l,s}^i), & (l, s) \in \check{\mathcal{L}} \times \mathcal{S} \\ \sum_{i \in \mathcal{I}} (\alpha^i)^2 - 1 \end{pmatrix}.$$

A competitive equilibrium can be Pareto improved by price regulation if the matrix of partial derivatives of the indirect utility functions with respect to prices has full rank<sup>4</sup>. By Proposition 4.8, this matrix is guaranteed to have full rank if there is no solution to  $\psi_e(\xi) = 0$  in combination with  $h_e(\xi, \alpha) = 0$ .

The function  $\tilde{\psi} : \Xi \times \mathbb{R}^I \times \Omega^* \rightarrow \mathbb{R}^{N+LS+1}$  is defined by

$$\tilde{\psi}(\xi, \alpha, e) = \begin{pmatrix} \psi(\xi, e) \\ h(\xi, \alpha, e) \end{pmatrix}.$$

If  $\tilde{\psi}$  is transverse to 0, then it follows from the transversal density theorem that for a subset of endowments of full Lebesgue measure,  $\tilde{\psi}_e$  is transverse to 0. If  $LS \geq I$ , then the dimension of the range of  $\tilde{\psi}_e$  exceeds that of the domain. Transversality of  $\tilde{\psi}_e$  implies that there are no solutions to the associated system of equations. It is possible to Pareto improve all competitive equilibria by price regulation.

**Proposition 2** *If  $LS \geq I$ , then, generically, all competitive equilibria of  $\mathcal{E}$  can be Pareto improved by price regulation.*

**Proof** One fixes  $(\bar{l}, \bar{s}) \in \check{\mathcal{L}} \times \mathcal{S}$  and  $\Omega^{**}$ , an open subset of endowments in  $\Omega^*$  of full Lebesgue measure, such that no competitive equilibrium of the associated economy  $\mathcal{E}$  is Pareto optimal. The function  $\hat{\psi} : \Xi \times \Omega^{**} \rightarrow \mathbb{R}^{N+1}$  is defined by

$$\hat{\psi}(\xi, e) = \begin{pmatrix} \psi(\xi, e) \\ \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} (x_{l,s}^i - e_{l,s}^i) \end{pmatrix}.$$

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<sup>4</sup>If the matrix of partial derivatives has full rank, it is possible to generate any desired marginal change in utilities by means of price regulation.

If  $\widehat{\psi}(\xi, e) = 0$ , then the matrix  $\widehat{M}$  of partial derivatives of  $\widehat{\psi}$  evaluated at  $(\xi, e)$  has full row rank: if  $v'\widehat{M} = 0$ , then  $v = 0$ . The components of  $v$  are  $v_{1,i,l,s}$ ,  $i \in \mathcal{I}$ ,  $(l, s) \in \mathcal{L} \times \mathcal{S}$ ,  $v_{2,i,s}$ ,  $i \in \mathcal{I}$ ,  $s \in \mathcal{S}$ ,  $v_{3,i,a}$ ,  $i \in \mathcal{I}$ ,  $a \in \mathcal{A}$ ,  $v_{4,l,s}$ ,  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ ,  $v_{5,a}$ ,  $a \in \check{\mathcal{A}}$ ,  $v_{6,i}$ ,  $i \in \mathcal{I}$ , and  $v_9$ , according to the labelling of the equations defining  $\widehat{\psi}$ .

If  $v$  is such that  $v'\widehat{M} = 0$ , then  $0 = v'\partial_{e_{L+1,s}^i} \widehat{\psi}(\xi, e) = -v_{2,i,s}$ ,  $i \in \mathcal{I}$ ,  $s \in \mathcal{S}$ . It follows that, for  $i \in \mathcal{I}$ ,

$$0 = v'\partial_{e_{l,s}^i} \widehat{\psi}(\xi, e) = -v_{4,l,s}, \quad (l, s) \in (\check{\mathcal{L}} \setminus \{\bar{l}\}) \times \mathcal{S},$$

$$0 = v'\partial_{e_{l,s}^i} \widehat{\psi}(\xi, e) = -v_{4,\bar{l},s} - v_9 \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} = 0, \quad s \in \mathcal{S} \setminus \{\bar{s}\},$$

$$0 = v'\partial_{e_{l,\bar{s}}^i} \widehat{\psi}(\xi, e) = -v_{4,\bar{l},\bar{s}}.$$

Consequently, if  $v_{4,\bar{l},\widehat{s}} = 0$  for some  $\widehat{s} \in \mathcal{S} \setminus \{\bar{s}\}$ , then  $v_9 = 0$  and  $v_{4,\bar{l},s} = 0$ , for all  $s \in \mathcal{S} \setminus \{\bar{s}\}$ . If, on the contrary,  $v_{4,\bar{l},s} \neq 0$ , for all  $s \in \mathcal{S} \setminus \{\bar{s}\}$ , then

$$\frac{\lambda_s^i}{\lambda_{\bar{s}}^i} = -\frac{v_{4,\bar{l},s}}{v_9} = \frac{\lambda_s^{i'}}{\lambda_{\bar{s}}^{i'}}, \quad i, i' \in \mathcal{I}, \quad s \in \mathcal{S} \setminus \{\bar{s}\}.$$

Hence, for  $i, i' \in \mathcal{I}$ , for  $s^1, s^2 \in \mathcal{S}$ ,  $\lambda_{s^1}^i / \lambda_{s^2}^i = (\lambda_{s^1}^i / \lambda_{\bar{s}}^i)(\lambda_{\bar{s}}^i / \lambda_{s^2}^i) = (\lambda_{s^1}^{i'} / \lambda_{\bar{s}}^{i'})(\lambda_{\bar{s}}^{i'} / \lambda_{s^2}^{i'}) = \lambda_{s^1}^{i'} / \lambda_{s^2}^{i'}$ . The economy  $e$  has then a Pareto optimal competitive equilibrium induced by  $\xi$ , contradicting  $e \in \Omega^{**}$ . Consequently,  $v_{4,\bar{l},s} = 0$ ,  $s \in \mathcal{S} \setminus \{\bar{s}\}$ , and  $v_9 = 0$ .

For  $i \in \mathcal{I}$ , and  $(l, s) \in \mathcal{L} \times \mathcal{S}$ ,

$$0 = v'\partial_{x_{l,s}^i} \widehat{\psi}(\xi, e) = v'_{1,i,\cdot} \partial_{x_{l,s}^i} \partial u^i(x^i).$$

It is possible to represent a utility function satisfying A2 by one with  $\partial^2 u^i(x^i)$  negative definite on a bounded subset of the consumption set. Then it follows that  $v_{1,i,\cdot} = 0$ . For  $i \in \mathcal{I}$ ,  $0 = v'\partial_{y_{A+1}^i} \widehat{\psi}(\xi, e) = v_{8,i}$ . Also, for  $a \in \check{\mathcal{A}}$ ,  $0 = v'\partial_{y_a^i} \widehat{\psi}(\xi, e) = v_{5,a}$ . Finally,  $0 = v'\partial_{\lambda_s^i} \widehat{\psi}(\xi, e) = v'_{3,i,\cdot} R'_{s,\cdot}$ ,  $i \in \mathcal{I}$ ,  $s \in \mathcal{S}$ . Since  $R$  has full column rank it follows that  $v_{3,i,a} = 0$ ,  $i \in \mathcal{I}$ ,  $a \in \mathcal{A}$ .

Therefore,  $v = 0$ ,  $\widehat{M}$  has full row rank  $N + 1$ , and  $\widehat{\psi}$  is transverse to 0. The set of endowments such that  $\widehat{\psi}_e$  is transverse to zero is denoted  $\widehat{\Omega}_{\bar{l},\bar{s}}$ . By the transversal density proposition,  $\Omega^{**} \setminus \widehat{\Omega}_{\bar{l},\bar{s}}$  has Lebesgue measure zero.

For  $e \in \widehat{\Omega}_{l,\bar{s}}$ , the dimension of the range of  $\widehat{\psi}_e$  exceeds that of the domain, so  $(\widehat{\psi}_e)^{-1}(\{0\}) = \emptyset$ .

The set  $\widehat{\Omega} = \cap_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} \widehat{\Omega}_{l,s}$  is of full Lebesgue measure and, by a standard argument, open. Redefine the function  $\widetilde{\psi}$  such that endowments belong to  $\Omega^* \cap \widehat{\Omega}$ . For  $(\xi, \alpha, e)$  such that  $\widetilde{\psi}(\xi, \alpha, e) = 0$ ,  $\widetilde{M}$  is the matrix of partial derivatives of  $\psi$  evaluated at  $(\xi, \alpha, e)$ .

If  $v$  is such that  $v' \widetilde{M} = 0$ , and the components of  $v$  are denoted by  $v_{1,i,l,s}$ ,  $v_{2,i,s}$ ,  $v_{3,i,a}$ ,  $v_{4,l,s}$ ,  $v_{5,a}$ ,  $v_{6,i}$ ,  $v_{7,l,s}$ , and  $v_8$ , then,  $0 = v' \partial_{e_{L+1,s}^i} \widetilde{\psi}(\xi, \alpha, e) = -v_{2,i,s}$ ,  $i \in \mathcal{I}$ ,  $s \in \mathcal{S}$ . Hence,

$$0 = v' \partial_{e_{l,s}^i} \widetilde{\psi}(\xi, \alpha, e) = -v_{4,l,s} - \alpha^i \lambda_s^i v_{7,l,s}, \quad i \in \mathcal{I}, (l, s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

Since  $\sum_{i \in \mathcal{I}} (\alpha^i)^2 = 1$ , there is  $i'$  such that  $\alpha^{i'} \neq 0$ . If there is  $\bar{s} \in \mathcal{S}$  such that, for  $i \in \mathcal{I} \setminus \{i'\}$ ,  $\alpha^{i'} \lambda_{\bar{s}}^{i'} - \alpha^i \lambda_{\bar{s}}^i = 0$ , then, for any  $l \in \check{\mathcal{L}}$ ,

$$\begin{aligned} 0 &= \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \alpha^i \lambda_s^i (x_{l,s}^i - e_{l,s}^i) = \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\alpha^{i'} \lambda_s^{i'}}{\lambda_s^{i'}} \lambda_s^i (x_{l,s}^i - e_{l,s}^i) \\ &= \alpha^{i'} \lambda_{\bar{s}}^{i'} \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} (x_{l,s}^i - e_{l,s}^i). \end{aligned}$$

Since  $\alpha^{i'} \neq 0$ ,  $\sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} (\lambda_s^i / \lambda_{\bar{s}}^i) (x_{l,s}^i - e_{l,s}^i) = 0$ , a contradiction since  $e \in \widehat{\Omega}$ . Consequently, for every  $s \in \mathcal{S}$ , there is  $i \in \mathcal{I} \setminus \{i'\}$  such that  $\alpha^{i'} \lambda_s^{i'} - \alpha^i \lambda_s^i \neq 0$ . For  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ ,  $(\alpha^{i'} \lambda_s^{i'} - \alpha^i \lambda_s^i) v_{7,l,s} = 0$ , so  $v_{7,l,s} = 0$ , and, thus  $v_{4,l,s} = 0$ . Also,  $0 = v' \partial_{\alpha^{i'}} \widetilde{\psi}(\xi, \alpha, e) = 2\alpha^{i'} v_8$ , so, since  $\alpha^{i'} \neq 0$ ,  $v_8 = 0$ . It follows as in the first part of the proof that  $v_{1,i,l,s} = 0$ ,  $i \in \mathcal{I}$ ,  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ , that  $v_{6,i} = 0$ ,  $i \in \mathcal{I}$ , that  $v_{5,a} = 0$ ,  $a \in \check{\mathcal{A}}$ , and that  $v_{3,i,a} = 0$ ,  $i \in \mathcal{I}$ ,  $a \in \mathcal{A}$ .

Therefore,  $\widetilde{M}$  has rank  $N + LS + 1$  and  $\widetilde{\psi}$  intersects 0 transversally. If  $\widetilde{\Omega}$  is the set of economies such that  $\widetilde{\psi}_e$  is transverse to 0, then  $\Omega^* \setminus \widetilde{\Omega}$  has Lebesgue measure zero by the transversal density theorem. Openness follows by a standard argument.  $\square$

Generically, it is possible to make every individual better off by choosing appropriate price regulations on the spot markets when asset markets are incomplete. One needs at least as many instruments,  $LS$ , as individuals,  $I$ . Proposition 1 makes clear that this is all one needs. This is not the case in the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when  $2L \leq I \leq L(S - 1) + 1$ .

A competitive equilibrium can be Pareto improved by uniform price regulation if the matrix of partial derivatives of the indirect utility functions with respect to uniform price regulation has full rank.

The function  $k : \Xi \times \mathbb{R}^I \times \Omega^* \rightarrow \mathbb{R}^{L+1}$  is defined by

$$k(\xi, \alpha, e) = \begin{pmatrix} \sum_{s \in \mathcal{S}} h_{l,s}(x, \lambda, \alpha, e), & l \in \check{\mathcal{L}} \\ \sum_{i \in \mathcal{I}} (\alpha^i)^2 - 1 \end{pmatrix}.$$

The matrix of partial derivatives of the indirect utility functions with respect to uniform price regulation is guaranteed to have full rank if there is no solution to  $\psi_e(\xi) = 0$  in combination with  $k_e(\xi, \alpha) = 0$ .

**Corollary 1** *If  $L \geq I$ , then, generically, all competitive equilibria of  $\mathcal{E}$  can be Pareto improved by uniform price regulation.*

**Proof** The argument follows that in the proof of Proposition 5.3. The equations related to  $h$  that characterize Pareto improving price regulation are replaced by the equations related to  $k$  that characterize Pareto improvements by uniform price regulation. This defines a function  $\bar{\psi}$ . The matrix  $\bar{M}$  gives the partial derivatives of  $\bar{\psi}$  evaluated at some  $(\xi, \alpha, e)$  with  $\bar{\psi}(\xi, \alpha, e) = 0$ . If  $v' \bar{M} = 0$ , by considering the partial derivatives with respect to  $e_{l,s}^i$ , it follows that  $v_{2,i,s} = 0$ ,  $i \in \mathcal{I}$ ,  $s \in \mathcal{S}$ , and  $v_{4,l,s} + \alpha^i \lambda_s^i v_{7,l} = 0$ ,  $i \in \mathcal{I}$ ,  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ . If  $i'$  is such that  $\alpha^{i'} \neq 0$ , and if  $\bar{s} \in \mathcal{S}$  such that, for  $i \in \mathcal{I} \setminus \{i'\}$ ,  $\alpha^{i'} \lambda_{\bar{s}}^{i'} - \alpha^i \lambda_{\bar{s}}^i = 0$ , then

$$\begin{aligned} 0 &= \sum_{i \in \mathcal{I}} \alpha^i \sum_{s \in \mathcal{S}} \lambda_s^i (x_{l,s}^i - e_{l,s}^i) = \alpha^{i'} \lambda_{\bar{s}}^{i'} \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} (x_{l,s}^i - e_{l,s}^i) \\ &= \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} (x_{l,s}^i - e_{l,s}^i) = \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} (x_{l,s}^i - e_{l,s}^i), \quad l \in \check{\mathcal{L}}, \end{aligned}$$

which contradicts  $e \in \hat{\Omega}$ . It follows that  $v_{4,l,s} = 0$ ,  $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ , and  $v_{7,l} = 0$ ,  $l \in \check{\mathcal{L}}$ . The remainder of the proof follows the argument in the proof of Proposition 6.  $\square$

Uniform price regulation is effective when  $L \geq I$ , which reflects again that the number of instruments has to exceed the number of objectives. It complements the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when  $2L \leq I \leq L(S-1) + 1$ .