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Ellsberg Games

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Ellsberg Games^{*}

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We propose a framework for normal form games where players can use Knightian uncertainty strategically. In such *Ellsberg games*, ambiguity-averse players may render their actions objectively ambiguous by using devices such as Ellsberg urns, in addition to the standard mixed strategies. While Nash equilibria remain equilibria in the extended game, there arise new *Ellsberg equilibria* with distinct outcomes, as we illustrate by negotiation games with

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three players. We characterize Ellsberg equilibria in two-person conflict and coordination games. These equilibria turn out to be consistent with experimental deviations from Nash equilibrium play.

Key words and phrases: Knightian Uncertainty in Games, Strategic Ambiguity, Ellsberg Games

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1 Introduction

Game theory is the science of strategic interaction between rational players. It rests on the foundations that John von Neumann and Oskar Morgenstern laid in Berlin and later in Princeton¹ almost a century ago. John von Neumann was the first to recognize that a meaningful theory required the introduction of randomizing devices: in games of conflict, like poker or other “Gesellschaftsspiele” (parlor games) that von Neumann studied in his seminal paper (?), some sort of device is needed as a formal counterpart of the incentive to conceal one’s own behavior²; a predictable player is easily exploited in strategic conflicts.

In the real world, there are not only probabilistically precise random devices like a fair coin or a die³ that von Neumann introduced; beside these tools for which the laws of probability apply, there are other *probabilistically imprecise* devices on which a player can base his action.

A famous case in point are the Ellsberg experiments where one ball is drawn from an urn whose composition is unknown to the agent. Taking up the common example, we might draw a ball from an urn with 100 red and blue balls, and we know that at least 30 of them are red, at least 20 are blue. This gives us *objective*, but *imprecise* information about the chances to draw a blue ball. Indeed, they are between twenty and seventy percent.

We ask here what happens to the basic results of game theory if we allow players to use such objective, but probabilistically imprecise devices to conceal their own behavior in strategic conflicts. More specifically, we allow players to choose a combination of *sets of probabilities* and an act.

When one extends the basic static game by using such new forms of “mixed” strategies, one has to decide how players evaluate the payoffs. As the reader knows, von Neumann adopted simply the expected value in order to apply the laws of probability; later, he and Oskar Morgenstern justified this assumption by developing the axiomatic foundation for expected utility. In a similar spirit, we adopt a pessimistic expected approach as it has been axiomatized in a recent paper by Gajdos, Hayashi, Tallon, and Vergnaud (2008): players evaluate a profile of strategies encoded by imprecise probabilistic devices by

¹where parts of this text were written, too.

²or, as von Neumann puts it: “Die Verschiedenheit der zwei Größen Max Min und Min Max bedeutet eben, daß von den zwei Spielern S_1 und S_2 nicht jeder gleichzeitig der klügere sein kann.” (p.304, op.cit.).

³that exist only in approximation anyway: like a perfect circle, there is also no perfectly fair coin in reality. It is a useful mathematical idealization, of course.

computing the minimal expected utility derived from all possible priors and acts.

This approach is parsimonious and restrictive at the same time: parsimonious, as it coincides with expected utility as long as all players choose to use a classic strategy with precise probabilistic information, restrictive, as it assumes uncertainty-averse players who do not love to use such devices with imprecise probabilistic information.

Our game model is, indeed, an extension of the classic game, in contrast to some belief-based extension that we discuss in more detail below. Classic mixed strategies are evaluated as before, as we said; players being ambiguity-averse, it is easy to show that standard Nash equilibria remain equilibria in our game. *There is no incentive to introduce imprecise probabilistic devices unilaterally.* This would not be the case if we allowed for ambiguity-loving, or optimistic, players. We thereby avoid the existence problem that plagues other formulations of games under Knightian uncertainty, see Ritzberger usw.

Our equilibrium concept that we christen Ellsberg equilibrium in honor of the experiments mentioned above follows the usual idea of Nash: in equilibrium, no player has an incentive to deviate unilaterally by using another imprecise probabilistic device.

As for Nash equilibrium, we have a certain indifference principle: if players really use a set of priors, not a singleton in equilibrium, then they are indifferent between all potential probability distributions that may govern the act's outcome. This is due to the fact that the player himself does not know the outcome of the imprecise probabilistic device he has chosen. As in von Neumann's original motivation for a mixed strategy, it is important *not to know* the outcome as this prevents the possibility to be outguessed by the other players. The indifference principle facilitates finding Ellsberg equilibria in concrete examples as we amply demonstrate later on.

In a peace negotiation game taken from Greenberg (2000), we illustrate the new equilibrium concept. In that game, two small countries negotiate over war and peace with a third player, the superpower, that can punish either country once war breaks out. As the superpower has no possibility to guess who is responsible for war if one country randomizes and the other goes to war, it punishes both with equal probability in Nash equilibrium, which in turn makes it optimal *ex ante* for the small countries to go to war. Greenberg argues that this outcome is not plausible in his game as the superpower should "remain silent". Indeed, such an informal description of being imprecise can be found in many political and diplomatic texts. "Many different strategies are used to orient toward conflicting interactional goals; some examples include avoiding interaction altogether, remaining silent, or changing the topic." says Eric Eisenberg, e.g., in an

article entitled “Ambiguity as strategy in organizational communication” (Eisenberg (1984)), and he points out that applying one’s resources of ambiguity is key in successful communication when conflicts of interest are present. We show that there exists indeed Ellsberg equilibria in which the superpower uses imprecise probabilistic devices and peace results in all scenarios⁴.

Our approach to games has its most natural and fruitful applications to conflicts where players are at least to some degree in opposition to each other. We consequently perform a detailed study of two-person 2×2 games with conflicting interests, as Matching Pennies, or similar competitive situations. We discuss first two new phenomena, *immunization against ambiguity* (or *hedging* as it is called in decision theory) and *nonlinearity of payoffs* that arise in Ellsberg games.

With imprecise probabilistic devices, the opponents create (Knightian) uncertainty for a player; safety thus becomes an issue, and, as one might anticipate, the minimax strategy starts playing a role⁵ Indeed, the minimax strategy is frequently the unique best reply to imprecise probabilistic devices; this is important as it destroys many potential equilibria that might seem intuitive at first sight, and thus sharpens the equilibrium predictions.

We derive all Ellsberg equilibria of 2×2 conflict games. In zero-sum games, Ellsberg equilibria are value-preserving. Indeed, in zero-sum games, no new equilibria emerge. But if we move away, we obtain a richer class of equilibria in which players use sets of probabilities. The boundaries of these sets are determined by the Nash equilibrium, the minimax equilibrium, and the boundaries of the simplex. As in zero-sum games, the Nash and minimax equilibrium coincide, this explains also why no new equilibria emerge in such games.

For completeness, we also show that imprecise probabilistic devices can appear in coordination games.

We are also able to solve for nontrivial Ellsberg equilibria in the more complex case of 3×3 games with a competitive structure (such as Rock Scissors Paper and its modifications).

While our predictions are broader than the classical unique Nash equilibrium, they remain restrictive, and, at least in principle, testable. Our results do allow to explain

⁴The game has been discussed from various points of view in the literature. In particular, Mukerji and Tallon (2004) describe what we understand as an equilibrium in ambiguous beliefs that would also support the peace outcome. The point of the example is not meant to be new, but to illustrate our equilibrium concept, which is new.

⁵The rationale for using minimax rather than Nash equilibrium strategies in some two player games has been noted by Aumann and Maschler (1972); it comes up in a more complex setting here again.

the experimental findings of Goeree and Holt (2001) who show that humans tend to deviate from Matching Pennies in asymmetric modified Matching Pennies games, but tend to play Nash equilibrium in symmetric Matching Pennies. This corresponds and is consistent with our Ellsberg equilibria.

Relation to the Literature Several authors introduce Knightian uncertainty into complete-information normal form games. Most of the literature has focused on the extension of the beliefs interpretation of mixed strategies as it is expressed in Aumann and Brandenburger (1995), w.g. We discuss their concepts and compare them to our approach.

Dow and Werlang (1994), Lo (1996), Marinacci (2000), Eichberger and Kelsey (2000) and Eichberger, Kelsey, and Schipper (2009) all extend the interpretation of Nash equilibrium as an equilibrium in beliefs. For example, Dow and Werlang (1994) interpret their non-additive (Choquet) probabilities as uncertain beliefs about the other player's action. A pair (P_1, P_2) of non-additive probabilities is then a Nash equilibrium under Knightian uncertainty if each action in a support of player 1's belief P_1 is optimal given that he uses P_2 to evaluate his expected payoff, and similarly for player 2. We thus have here a first version of an equilibrium in beliefs. This approach is refined by Marinacci (2000) and extended to n -person games by Eichberger and Kelsey (2000).

Lo (1996) introduces the concept of equilibrium in beliefs under uncertainty where the beliefs are represented by multiple priors over other players' mixed strategies. Each player i has a set of beliefs B_i over what the other players do, so over ΔS_{-i} . The profile (B_i) then forms a beliefs equilibrium if player j puts positive weight only on strategies of player i that maximize i 's minimal expected payoff given the belief set B_i . This concept allows for disagreement of players' beliefs, and for correlation. Lo therefore introduces the refinement of a beliefs equilibrium with agreement in which player j and k agree about player i 's actions and the beliefs of i over j and k are independent. Lo proves the nice result that every beliefs equilibrium contains a Bayesian beliefs equilibrium (where the belief sets are singletons). As a corollary, he obtains a precursor of Bade (2011)'s main theorem (which we discuss in a later paragraph): in two player games, every beliefs equilibrium contains a Nash equilibrium.

Note that all the equilibrium concepts discussed above do not specify which action will actually be played in equilibrium. In Lo (1996) players can play any pure or mixed strategy that is a best response to their belief set, in the other equilibrium notions mentioned, players only have access to pure strategies in the support of the capacities.

This stands in contrast to Ellsberg equilibrium, where the equilibrium strategy is fixed by the Ellsberg urn chosen. The strategy is a best response to the belief, and the belief coincides with the strategy played.

Klibanoff (1996), Lehrer (2008) and Lo (2009) propose an approach similar to beliefs equilibrium. Uncertainty is present in players' beliefs that are represented by sets of distributions. Equilibrium is defined as a profile of beliefs and an objectively mixed (or pure) strategy for each player, which is the strategy that he plays in equilibrium. These strategies need to be contained in the belief sets. Accordingly, players have to anticipate their opponents' strategy correctly in the sense that the truth is part of their belief. This consistency requirement is weaker than in Nash equilibrium (and weaker than in Ellsberg equilibrium!) and typically the strategies in equilibrium are not best responses to the actual strategies played. Klibanoff (1996) proposes a refinement where only correlated rationalizable beliefs are allowed.⁶ Lehrer (2008) develops a model of decision making under uncertainty with partially-specified probabilities, these are used to represent the players' uncertain beliefs about their opponents. Lo (2009) establishes formal epistemic foundations for an equilibrium concept with ambiguity-averse preferences. He finds that epistemically stochastic independence is not necessary for a generalized Nash equilibrium concept. A correlated Nash equilibrium is a pair $\langle \sigma, \Phi \rangle$ consisting of a profile of beliefs Φ_i and a profile of mixed strategies σ_i where, for consistency, each strategy a_i in the support of σ_i is a best response to the belief Φ_i .

Bade (2011) goes a first step in another direction, away from the beliefs interpretation of Nash equilibrium. She allows players to use acts in the sense of Anscombe-Aumann and players are uncertainty-averse over such acts. In an ambiguous act equilibrium, players play best responses as in Nash equilibrium, but under the generalized framework. A large class of ambiguity-averse preferences are covered. The possible priors for an ambiguous act are part of the players' preferences in her setup. Bade then adds some appropriate consistency properties (agreement on null events) to exclude unreasonably divergent beliefs, and she imposes the rather strong assumption that preferences are *strictly* monotone, following Klibanoff (1996) here. This excludes beliefs on the boundary of strategy sets; such degenerated beliefs are sometimes important, though. For example, it excludes Ellsberg urns with full ambiguity where it is only known that the probability for a red ball is between 0 and 1. Bade's main theorem establishes that under her assumptions, in two-person games the support of ambiguous act equilibria and the

⁶Lo (1996) requires every probability distribution in the belief sets to be a best response, therefore every beliefs equilibrium with agreement is a refinement of equilibrium with uncertainty aversion and rationalizable beliefs (this is shown in Lo (1996), Proposition 9).

support of Nash equilibria coincide.

Note that Aumann (1974), Epstein (1997) and Azrieli and Teper (2011) (amongst others) have also defined games that have Anscombe-Aumann acts as strategies, but to different ends. Aumann (1974) defines such a general game, then imposes Savage expected utility and analyses properties of correlated and subjective equilibrium. Epstein (1997) analyzes games very similar to Bade's, but is mainly interested in rationalizability and iterated deletion of strictly dominated strategies in the generalized framework. Azrieli and Teper (2011) define an extension of an incomplete-information game.

In difference to Bade (2011)'s setup we let ambiguity be an objective instrument that is not derived from subjective preferences. Players can credibly commit to play an Ellsberg urn with a given and known degree of ambiguity. In Ellsberg games players use devices that create ambiguity, thus we extend the objective random devices interpretation of Nash equilibrium. The articles cited above impose non-expected utility representations derived from subjective preferences, like maxmin expected utility by Gilboa and Schmeidler (1989), Choquet expected utility by Schmeidler (1989), or they fix only certain axioms to allow for a large class of ambiguity-averse preferences. To model the preferences in Ellsberg games we use the representation results by Gajdos, Hayashi, Tallon, and Vergnaud (2008) on attitude towards imprecise information which capture the objective ambiguity we have in mind.

The following papers consider strategic or objective ambiguity from another perspective. Stauber (2011) considers incomplete information games with ambiguous beliefs and analyzes the robustness of equilibria of these games. Finally, Perchet (2012) works in the context of repeated games with the possibility of monitoring. The information about the opponents' play is "ambiguously" disturbed and hence the notion of Nash equilibrium generalized. He also generalizes the Lemke-Howson algorithm to compute equilibria of two-player games. In an unpublished working paper, ? develop the concept of 'objectively ambiguous strategies' which leads to equilibrium concepts similar to Ellsberg equilibrium. However, the authors focus on the implications in mechanism design.

The paper is organized as follows. In Section 2 we explain how Ellsberg urns are understood as concealment device, in the line of von Neumann and Morgenstern's interpretation of mixed strategies. In Section ?? we develop the theoretical framework of Ellsberg games. The concept is applied to the three-player negotiation example in Section 3. Section 4 analyses the use of strategic ambiguity in two-person conflict and coordination games. A discussion of observational implications of Ellsberg games is pro-

vided in Section 5. We compare Ellsberg games to existing equilibrium concepts with ambiguity aversion in Section ??, and we conclude in Section 6.

2 Ellsberg Games

A game consists of a finite set N of players, a finite set of (pure) strategies $S_i, i \in N$ for each player, as well as a collection of payoff functions $u_i : S \rightarrow \mathbb{R}$ defined over strategy profiles $S = \times_{i \in N} S_i$. The normal form game is denoted $G = \langle N, (S_i), (u_i) \rangle$.

? introduced mixed strategies as probability vectors P_i over pure strategies S_i . The question then emerges how players evaluate profiles of such mixed strategies $P = (P_1, \dots, P_n)$; as the reader knows, von Neumann and Morgenstern and the complete subsequent literature adopt expected utility.

In this paper, we are going back to these foundations. Let us formalize the intuitive idea that players can create ambiguity and/or conceal his actions with the help of imprecise probabilistic devices, or Ellsberg urns. An Ellsberg urn is, for us, a triple $(\Omega, \mathcal{F}, \mathcal{P})$ of a nonempty set Ω of states of the world, a σ -field \mathcal{F} on Ω (where one can take the power set in case of a finite Ω), and a set of probability measures \mathcal{P} on the measurable space (Ω, \mathcal{F}) . This *set* of probability measures represents the Knightian uncertainty of the strategy.

Note that we allow the player to choose the degree of ambiguity of his urn. He tells the experimentalists of his laboratory to set up such and such an Ellsberg experiment that generates exactly the set of distributions \mathcal{P}_i . In this sense, the ambiguity in our formulation of the game is “objective”; it is not a matter of agents’ *beliefs* about the actions of other players, but rather a property of the device used to determine his action.

Technically, we model the Ellsberg urn of player i as a triple $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ as explained above. Player i acts in the game by choosing a measurable function (or Anscombe-Aumann act) $f_i : (\Omega_i, \mathcal{F}_i) \rightarrow \Delta S_i$ which specifies the classical mixed strategy played once the outcome of the Ellsberg urn is revealed. An *Ellsberg strategy* for player i is then a pair $((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i)$ of an Ellsberg urn and an act.

To finish the description of our Ellsberg game, we have to determine players’ payoffs. As we explained in the introduction, we follow the axiomatization of attitude towards objective but imprecise information in Gajdos, Hayashi, Tallon, and Vergnaud (2008)⁷

⁷The payoff from a profile of Ellsberg strategies is evaluated as a maxmin expected utility similar to the axiomatization of Gilboa and Schmeidler (1989), but with the difference of the decision maker facing objective instead of subjective ambiguity. Starting with Jaffray (1989), Giraud (2006) and Giraud and Tallon (2011) as well as Stinchcombe (2007) and Olszewski (2007) also make a case for

The payoff of player $i \in N$ at an Ellsberg strategy profile $((\Omega, \mathcal{F}, \mathcal{P}), f)$ is thus the minimal expected utility with respect to all different probability distributions in the closed and convex set \mathcal{P} ,

$$U_i(((\Omega, \mathcal{F}, \mathcal{P}), f)) := \min_{P_1 \in \mathcal{P}_1, \dots, P_n \in \mathcal{P}_n} \int_{\Omega_1} \dots \int_{\Omega_n} u_i(f(\omega)) dP_n \dots dP_1.$$

We call the described larger game an *Ellsberg game*. An *Ellsberg equilibrium* is, in the same spirit as Nash equilibrium, a profile of Ellsberg strategies $(((\Omega_i^*, \mathcal{F}_i^*, \mathcal{P}_i^*), f_i^*))_{i \in N}$ where no player has an incentive to deviate, i.e. for all players $i \in N$, all Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$, and all acts f_i for player i we have

$$U_i(((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)) \geq U_i(((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i), ((\Omega_{-i}^*, \mathcal{F}_{-i}^*, \mathcal{P}_{-i}^*), f_{-i}^*)).^8$$

2.1 Reduced Form Strategies

This definition of an Ellsberg game depends on the particular Ellsberg urn used by each player i . As there are arbitrarily many possible state spaces (in fact, the class of all state spaces is too large to be a well-defined set according to set theory), the definition of Ellsberg equilibrium might not seem very tractable. Fortunately, there is a more concise way to define Ellsberg equilibrium. The procedure is similar to the reduced form of a correlated equilibrium, see Aumann (1974) or Fudenberg and Tirole (1991). Instead of working with arbitrary Ellsberg urns, we note that the players' payoff depends, in the end, on the set of distributions that the Ellsberg urns and the associated acts induce on the set of strategies. One can then work with that set of distributions directly.

Definition 1. Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. A reduced form Ellsberg equilibrium of the game G is a profile of sets of probability measures $\mathcal{Q}_i^* \subseteq \Delta S_i$, such that for all players $i \in N$ and all sets of probability measures $\mathcal{Q}_i \subseteq \Delta S_i$ we have

$$\begin{aligned} & \min_{P_i \in \mathcal{Q}_i^*, P_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dP_{-i} dP_i \\ & \geq \min_{P_i \in \mathcal{Q}_i, P_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dP_{-i} dP_i. \end{aligned}$$

The two definitions of Ellsberg equilibrium are equivalent in the following sense.

objective ambiguity.

⁸Throughout the paper we follow the notational convention that $(f_i, f_{-i}^*) := (f_1^*, \dots, f_{i-1}^*, f_i, f_{i+1}^*, \dots, f_n^*)$. The same convention is used for profiles of pure strategies (s_i, s_{-i}) and probability distributions (P_i, P_{-i}) .

Theorem 1. *Ellsberg equilibrium and reduced form Ellsberg equilibrium are equivalent in the sense that every Ellsberg equilibrium $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ induces a payoff-equivalent reduced form Ellsberg equilibrium on $\Omega^* = S$; and every reduced form Ellsberg equilibrium \mathcal{Q}^* is an Ellsberg equilibrium $((S, \mathcal{F}, \mathcal{Q}^*), f^*)$ with f^* the constant act.*

We henceforth call a set $\mathcal{Q}_i \subseteq \Delta S_i$ an Ellsberg strategy whenever it is clear that we are in the reduced form context. In the definition of an Ellsberg game we assume that the Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ of all players $i \in N$ are stochastically independent. This is done by using product spaces as first suggested by Gilboa and Schmeidler (1989), p. 150. Different notions of stochastic independence in the context of ambiguity aversion have been discussed in the literature, see for example Klibanoff (2001), Bade (2011) and ?. In the present context of objective ambiguity in the form of Ellsberg urns the above notion by Gilboa and Schmeidler (1989) seems the most natural.

2.2 Ellsberg Equilibria Generalize Nash Equilibria

Note that the classical game is contained in our formulation: players just choose a singleton $\mathcal{P}_i = \{\delta_{\pi_i}\}$ that puts all weight on a particular (classical) mixed strategy π_i .

		Player 2	
		L	R
Player 1	T	3, 3	0, 0
	B	0, 0	1, 1

Figure 1: Strategic Ambiguity does not unilaterally make a player better off.

Now let (π_1, \dots, π_n) be a Nash equilibrium of the game G . Can any player unilaterally gain by creating ambiguity in such a situation? The answer is no. Take the game in Figure 1 and look at the pure strategy Nash equilibrium (B, R) with equilibrium payoff 1 for both players. If player 1 introduces ambiguity, he will play T in some states of the world (without knowing the exact probability of those states). But this does not help here because player 2 sticks to his strategy R , so playing T just leads to a payoff of zero. Unilateral introduction of ambiguity does not increase one's own payoff. We think that this is an important property of our formulation.

Theorem 2. *Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. Then a mixed strategy profile (π_1, \dots, π_n) of G is a Nash equilibrium of G if and only if the corresponding profile of singletons $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ with $\mathcal{P}_i = \{\delta_{\pi_i}\}$ is an Ellsberg equilibrium. In particular, Ellsberg equilibria exist when the strategy sets S_i are finite.*

By including and generalizing Nash equilibria, our formulation avoids the existence pitfalls that one encounters when players are assumed to play pure strategies and beliefs are uncertain about those pure actions.

2.3 Principle of Indifference in Distributions

We want to emphasize an important property of Ellsberg games (or ambiguity aversion in general): the best reply functions are no longer linear in the probabilities. As a consequence, the principle of indifference of classical game theory – when two pure strategies yield the same payoff, then the player is indifferent about mixing in any arbitrary way between the two strategies – does not carry over to Ellsberg games. When a player is indifferent between two Anscombe-Aumann acts, this does not imply that he is indifferent between all mixtures over these two acts. Thus, in an Ellsberg equilibrium players are in general *not* indifferent between playing the Ellsberg strategy or playing the pure strategies in its support. However, in Ellsberg equilibria the players are indifferent between all distributions contained in their Ellsberg equilibrium strategy. We call this the *Principle of Indifference in Distributions*, it is shown in the following.

Recall that a profile $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$ is a reduced form Ellsberg equilibrium if for all $i \in N$ and all $\mathcal{P}_i \subseteq \Delta S_i$,

$$U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*) \geq U_i(\mathcal{P}_i, \mathcal{P}_{-i}^*). \quad (1)$$

This formulation states that in equilibrium it is not profitable for any player to unilaterally deviate. Alternatively and in analogy with Nash equilibrium, an Ellsberg equilibrium can be defined as a profile of strategies, where each player only chooses strategies which maximize his (in Ellsberg games) minimal expected utility. That is, $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$ is a reduced form Ellsberg equilibrium if for all $i \in N$ and all $P_i \in \mathcal{P}_i^*$

$$P_i \in \arg \max_{\mathcal{P}_i \subseteq \Delta S_i} U_i(\mathcal{P}_i, \mathcal{P}_{-i}^*). \quad (2)$$

Conditions (1) and (2) are obviously equivalent. Thus we have

Proposition 1. *Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. A profile of Ellsberg strategies $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$ satisfies (1) if and only if it satisfies (2).*

We derive from the preceding proposition, that in an Ellsberg equilibrium $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$, for all $i \in N$,

$$P_i \in \arg \max_{P'_i \in \Delta S_i} U_i(P'_i, \mathcal{P}_{-i}^*) \quad \text{for all } P_i \in \mathcal{P}_i^*,$$

or, put differently,

$$U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*) = \max_{P_i \in \Delta S_i} \min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i}).$$

Thence, although U_i is defined as the minimal expected utility over all $\mathcal{P}_1, \dots, \mathcal{P}_n$, de facto player i uses only maximizers in his own Ellsberg equilibrium strategy \mathcal{P}_i^* , any other distribution would reduce his utility. In other words, all distributions P_i which player i uses in his Ellsberg equilibrium strategy yield the same utility, that is, player i is indifferent between any $P_i \in \mathcal{P}_i^*$. Thus we have

Theorem 3 (The Principle of Indifference in Distributions). *Let $(\mathcal{P}_1^*, \dots, \mathcal{P}_n^*)$ be an Ellsberg equilibrium of a normal form game $G = \langle N, (S_i), (u_i) \rangle$. Then for all $P_i \in \mathcal{P}_i^*$,*

$$\min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i}) = c^* \quad \text{for some } c^* \in \mathbb{R}.$$

An important property of the payoff functions is that they are linear in probabilities. In particular, with the equality established in Proposition 1, $U_i(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$ is linear on ΔS_i for each P_i and linear on \mathcal{P}_j^* for each $P_j \in \mathcal{P}_j^*$, $j \neq i$. This observation suffices to fulfill the assumptions (multilinear functions are convex in each variable) to Fan's Minimax Theorem in ?. Thus, we have the following minimax theorem for Ellsberg games.

Theorem 4 (Minimax Theorem 1). *In an Ellsberg equilibrium $(\mathcal{P}_i^*, \mathcal{P}_{-i}^*)$, for all $i \in N$,*

$$\max_{P_i \in \Delta S_i} \min_{P_{-i} \in \mathcal{P}_{-i}^*} u_i(P_i, P_{-i}) = \min_{P_{-i} \in \mathcal{P}_{-i}^*} \max_{P_i \in \Delta S_i} u_i(P_i, P_{-i}).$$

3 Non-Nash Outcomes in Ellsberg Equilibrium

Strategic ambiguity can lead to new phenomena that lie outside the scope of classical game theory. As our first example, we consider the following peace negotiation game taken from Greenberg (2000). There are two small countries who can either opt for peace, or war. If both countries opt for peace, all three players obtain a payoff of 4. If one of the countries does not opt for peace, war breaks out, but the superpower cannot decide whose action started the war. The superpower can punish one country and support the other. The game tree is in Figure 2 below.⁹

⁹We take the payoffs as in Greenberg's paper. In case the reader is puzzled by the slight asymmetry between country A and B in payoffs: it does not play a role for our argument. One could replace the payoffs 3 and 6 for country A by 0 and 9.

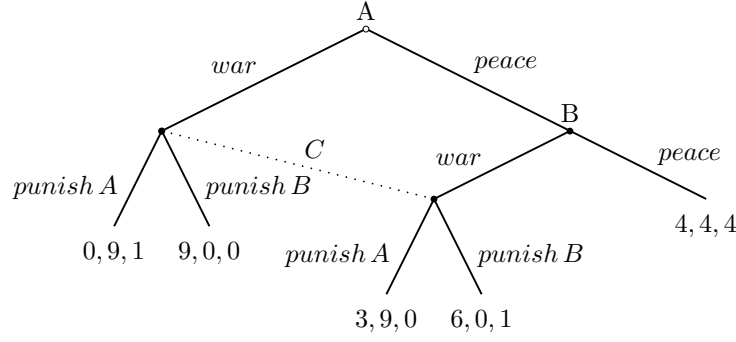


Figure 2: Peace Negotiation

As we deal here only with static equilibrium concepts, we also present the normal form, where country A chooses rows, country B columns, and the superpower chooses the matrix.

	<i>war</i>	<i>peace</i>		<i>war</i>	<i>peace</i>
<i>war</i>	0, 9, 1	0, 9, 1	<i>war</i>	9, 0, 0	9, 0, 0
<i>peace</i>	3, 9, 0	4, 4, 4	<i>peace</i>	6, 0, 1	4, 4, 4
	<i>punish A</i>			<i>punish B</i>	

Figure 3: Peace Negotiation in normal form

This game possesses a unique Nash equilibrium where country A mixes with equal probabilities, and country B opts for war; the superpower has no clue who started the war given these strategies. It is thus indifferent about whom to punish and mixes with equal probabilities as well. War occurs with probability 1. The resulting equilibrium payoff vector is (4.5, 4.5, 0.5).

If the superpower can create ambiguity (and if the countries A and B are ambiguity-averse), the picture changes. Suppose for simplicity, that the superpower creates maximal ambiguity by using a device that allows for any probability between 0 and 1 for its strategy *punish A*. The pessimistic players A and B are ambiguity-averse and thus maximize against the worst case. For both of them, the worst case is to be punished by the superpower, with a payoff of 0. Hence, both prefer to opt for peace given that the superpower creates ambiguity. As this leads to a very desirable outcome for the superpower, it has no incentive to deviate from this strategy. We have thus found an equilibrium where the strategic use of ambiguity leads to an equilibrium outcome outside the support of the Nash equilibrium outcome.

Let us formalize the above considerations. We claim that there is the following type

of Ellsberg equilibria. The superpower creates ambiguity about its decision; if this ambiguity is sufficiently large, both players fear to be punished by the superpower in case of war. As a consequence, they opt for peace.

In our game with just two actions for the superpower, we can identify an Ellsberg strategy with an interval $[P_0, P_1]$ where $P \in [P_0, P_1]$ is the probability that the superpower punishes country A. Suppose the superpower plays so with $P_0 < 4/9$ and $P_1 > 5/9$. Assume also that country B opts for *peace*. If A goes for war, it uses that prior in $[P_0, P_1]$ which minimizes its expected payoff, which is P_1 . This yields $U_A(\text{war}, \text{war}, [P_0, P_1]) = P_1 \cdot 0 + (1 - P_1) \cdot 9 < 4$. Hence, opting for peace is country A's best reply. The reasoning for country B is similar, but with the opposite probability P_0 . If both countries A and B go for peace, the superpower gets 4 regardless of what it does; in particular, the ambiguous strategy described above is optimal. We conclude that $(\text{peace}, \text{peace}, [P_0, P_1])$ is a (reduced form) Ellsberg equilibrium.

Proposition 2. *In Greenberg's game, the strategies $(\text{peace}, \text{peace}, [P_0, P_1])$ with $P_0 < 4/9$ and $P_1 > 5/9$ form an Ellsberg equilibrium.*

Note that this Ellsberg equilibrium is very different from the game's unique Nash equilibrium. In Nash equilibrium, war occurs in every play of the game; in our Ellsberg equilibrium, peace is the unique outcome. By using the strategy $[P_0, P_1]$ which is a set of probability distributions, the superpower *creates* ambiguity. This supports an Ellsberg equilibrium where players' strategies do not lie in the support of the unique Nash equilibrium. We also point out that the countries A and B use different *worst-case priors* in equilibrium; this is a typical phenomenon in Ellsberg equilibria that are supported by strategies which are not in the support of any Nash equilibrium of the game.

Greenberg refers to historic peace negotiations between Israel and Egypt (countries A and B in the negotiation example) mediated by the USA (superpower C) after the 1973 war. As explained by Kissinger (1982), p. 802, the fact that both Egypt and Israel were too afraid to be punished if negotiations broke down partly contributed to the success of the peace negotiations. This story is supported by our Ellsberg equilibrium, a first evidence that Ellsberg equilibria might capture some real world phenomena better than Nash equilibria.

Other equilibrium concepts for extensive form games (without Knightian uncertainty) such as conjectural equilibrium, Battigalli and Guaitoli (1988), self-confirming equilibrium, Fudenberg and Levine (1993), and subjective equilibrium, Kalai and Lehrer (1995), can also assure the peace equilibrium outcome in the example by Greenberg. A

discussion of Greenberg’s game close to ours is provided in Mukerji and Tallon (2004) and Lo (2009), however, the authors describe an equilibrium situation supported by subjective ambiguous beliefs, as opposed to our notion. Other equilibrium concepts for extensive form games with Knightian uncertainty are, e.g., Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2011) and Lo (1999). The relation of these equilibrium concepts to Ellsberg equilibrium are discussed in ?; we only want to stress here that in difference to the existing concepts the driving factor in Ellsberg equilibrium is that ambiguity is employed strategically and objectively.

4 Strategic Ambiguity in Two-Person Conflicts

Our approach to games has its most natural and fruitful applications to conflicts where players are at least to some degree in opposition to each other. We start this section by discussing a modified version of Matching Pennies to illustrate the phenomena of *immunization against ambiguity* and *nonlinearity of payoffs* that arise in Ellsberg games.

We then provide a general analysis of 2×2 -conflict games. While our predictions are broader than the classical Nash equilibrium, they remain restrictive, and, at least in principle, testable (see Section 5). We show that in zero-sum games, Ellsberg equilibria are value-preserving. Subsequently, we provide a characterization of 2×2 coordination games and explain how in such games the strategic use of ambiguity conceals maximin and Nash behavior. The section closes with the calculation of Ellsberg equilibria of circulant 3×3 games such as Rock Scissors Paper and modified versions of this game.

4.1 A Matching Pennies Example

To get an intuition for Ellsberg equilibria in games of conflict, we take a modified version of Matching Pennies as our example. The payoff matrix for this game is in Figure 4.

		Player 2	
		<i>HEAD</i>	<i>TAIL</i>
Player 1	<i>HEAD</i>	3, −1	−1, 1
	<i>TAIL</i>	−1, 1	1, −1

Figure 4: Modified Matching Pennies I

The Ellsberg equilibria for our first version of modified Matching Pennies are as follows. Player 1 plays *HEAD* with probability $P \in [1/2, P_1]$ for some $1/2 \leq P_1 \leq 1$ and player 2 plays *HEAD* with probability $Q \in [1/3, Q_1]$ for some $1/3 \leq Q_1 \leq 1/2$. This Ellsberg

equilibrium yields the same payoffs $1/3$ and 0 as in Nash equilibrium. We prove a more general theorem covering this case in Section 4.2.

Proposition 3. *In Modified Matching Pennies I, the Ellsberg equilibria are of the form $([1/2, P_1], [1/3, Q_1])$ for $1/2 \leq P_1 \leq 1$ and $1/3 \leq Q_1 \leq 1/2$.*

The typical Ellsberg equilibrium strategy thus takes the following form. Player 1 says: “I will play *HEAD* with a probability of at least 50%, but not less.” And Player 2 replies: “I will play *HEAD* with at least 33%, but not more than 50%.”

We point out two interesting effects that arise due to strategic ambiguity in this class of games. On the one hand, the Ellsberg equilibria are different from what one might expect first; in a game like the one above, one might intuitively guess that “full ambiguity” would be an Ellsberg equilibrium, as the natural generalization of “full randomness” (completely mixed Nash equilibrium). This is not the case.

On the other hand, due to the hedging or diversification effect provided by a (classical) mixed strategy when players are ambiguity-averse, players can immunize against the ambiguity used by their opponent. We call this effect *immunization against strategic ambiguity*.

Immunization against Strategic Ambiguity

In our modified version of Matching Pennies, the unique Nash equilibrium is that player 1 mixes uniformly over his strategies, and player 2 mixes with $(1/3, 2/3)$. This yields the equilibrium payoffs $1/3$ and 0 . One might guess that one can get an Ellsberg equilibrium where both players use a set of probability measures around the Nash equilibrium distribution as their strategy. This is not true.

The crucial point to understand here is the following. Players can immunize themselves against ambiguity; in the modified Matching Pennies example, player 1 can use the mixed strategy $(1/3, 2/3)$ to make himself independent of any ambiguity used by the opponent. Indeed, with this strategy, his expected payoff is $1/3$ against any mixed strategy of the opponent, and a fortiori against Ellsberg strategies as well. This strategy is also the unique best reply of player 1 to Ellsberg strategies with ambiguity around the Nash equilibrium; in particular, such strategic ambiguity is not part of an Ellsberg equilibrium.

Let us explain this somewhat more formally. An Ellsberg strategy for player 2 can be identified with an interval $[Q_0, Q_1] \subseteq [0, 1]$ where $Q \in [Q_0, Q_1]$ is the probability to play *HEAD*. Suppose player 2 uses many probabilities around $1/3$, so $Q_0 < 1/3 < Q_1$. The

(minimal) expected payoff for player 1 when he uses the mixed strategy with probability P for *HEAD* is then

$$\begin{aligned}
& \min_{Q_0 \leq Q \leq Q_1} 3PQ - P(1-Q) - (1-P)Q + (1-P)(1-Q) \\
&= \min \{Q_0(6P-2), Q_1(6P-2)\} + 1 - 2P \\
&= \begin{cases} Q_1(6P-2) + 1 - 2P & \text{if } P < 1/3 \\ 1/3 & \text{if } P = 1/3 \\ Q_0(6P-2) + 1 - 2P & \text{else.} \end{cases}
\end{aligned}$$

We plot the payoff function in Figure 5. By choosing the mixed strategy $P = 1/3$, player

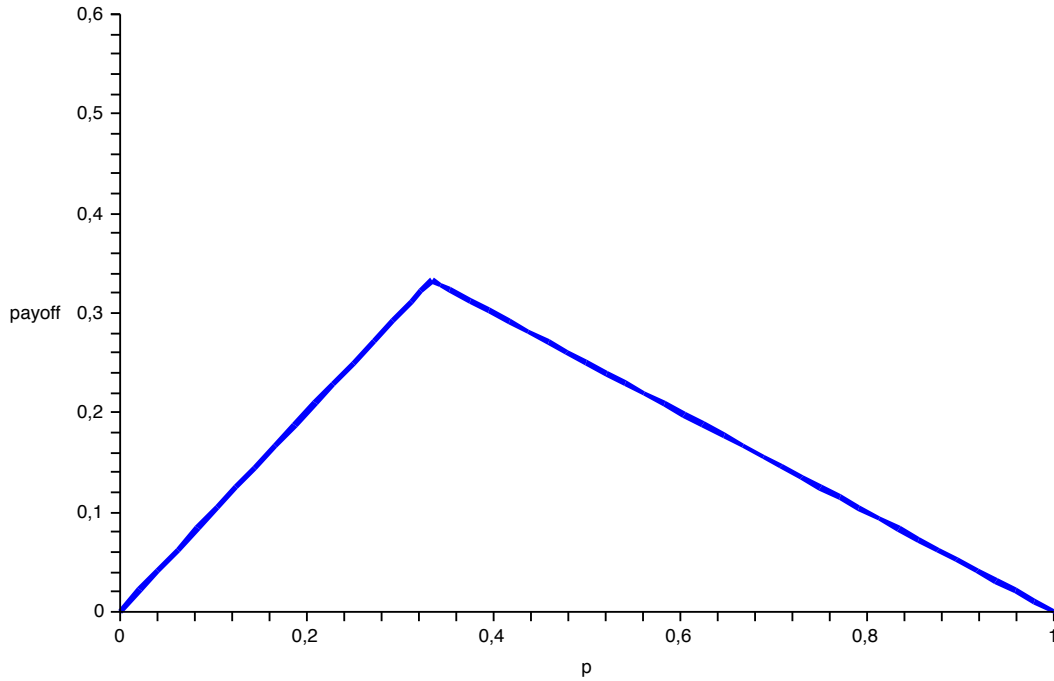


Figure 5: Player 1's (minimal expected) payoff as a function of the probability P of playing *HEAD* when player 2 uses the Ellsberg strategy $[1/4, 1/2]$.

1 becomes immune against any ambiguity and ensures the (Nash) equilibrium payoff of $1/3$. If there was an Ellsberg equilibrium with $P_0 < 1/2 < P_1$ and $Q_0 < 1/3 < Q_1$, then the minimal expected payoff would be below $1/3$. Hence, such Ellsberg equilibria do not exist. We call the strategy that renders the player immune against any ambiguity used

by the opponents *immunization strategy*.

Definition 2. Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. A mixed strategy $\bar{P}_i \in \Delta S_i$ for player i is called an immunization strategy, if there exists a $v_i \in \mathbb{R}$ such that $U_i(\bar{P}_i, \mathcal{P}_{-i}) = v_i$ for all sets $\mathcal{P}_{-i} \subseteq \Delta S_{-i}$.

Such immunization plays frequently a role in two-person games, and it need not always be the Nash equilibrium strategy that is used to render oneself immune. In fact, Nash equilibrium and immunization are in some sense opposite concepts: with a Nash equilibrium strategy the player wants to make his opponent indifferent between all his strategies, and with an immunization strategy the player wants to make *himself* indifferent. Consider, e.g., the slightly changed payoff matrix

		Player 2	
		HEAD	TAIL
Player 1	HEAD	1, −1	−1, 1
	TAIL	−2, 1	1, −1

Figure 6: Modified Matching Pennies II

In the unique Nash equilibrium, player 1 still plays both strategies with probability 1/2 (to render player 2 indifferent); however, in order to be immune against Ellsberg strategies, he has to play *HEAD* with probability 3/5. Then his payoff is $-1/5$ regardless of what player 2 does. This strategy does not play any role in Nash equilibrium, but note that the payoff to the strategy is the same as in the unique Nash equilibrium in which player 1 plays *HEAD* with probability 1/2 and player 2 with probability 2/5. In fact, every strategy of player 1 in the interval $[1/2, 3/5]$ yields the same maxmin payoff, which makes this strategy a candidate for an Ellsberg equilibrium strategy.

Immunization as Maximin Strategy

The modified Matching Pennies I game (Figure 4) has also been discussed in Aumann and Maschler (1972). The authors notice that in some (and not only in zero-sum!) games the maximin payoff is the same as the Nash equilibrium payoff, what makes in those games the maximin strategy more attractive than the Nash equilibrium strategy. In a very instructive paper, ? analyses this observation further and finds, that this is the case if and only if there exists a completely mixed Nash equilibrium and the maximin strategy is an *equalizer*. What the author calls an equalizer is the same as an immunization strategy, only for pure and mixed strategies and not for Ellsberg strategies, that is a

strategy \bar{P}_i for which the expected payoff is constant for all strategies $P_{-i} \in \Delta S_{-i}$. It is immediate that every equalizer strategy is an immunization strategy¹⁰.

We have the following theorem on the relation of immunization and maximin strategies in two-person normal form games.

Theorem 5. *Let G be a square two-person normal form game with a completely mixed Nash equilibrium (P^*, Q^*) . If player 1 (2) has an immunization strategy \bar{P} (\bar{Q}) in G , then \bar{P} (\bar{Q}) is a maximin strategy of player 1 (2).*

4.2 Ellsberg Equilibria in General 2×2 Conflict Games

We calculate the Ellsberg equilibria of general 2×2 conflict games. Consider the game with payoff matrix in Figure 7. We assume that $a, d > b, c$ and $e, h < f, g$.

		Player 2	
		L	R
Player 1	U	a, e	b, f
	D	c, g	d, h

Figure 7: General conflict game.

As before, P^* and Q^* are the probabilities with which U respectively L are played. Let

$$P^* = \frac{h - g}{e - f - g + h}, \quad \text{respectively} \quad Q^* = \frac{d - b}{a - b - c + d}$$

denote the Nash equilibrium strategies for player 1 and 2, respectively. The immunization strategies of each player are denoted by M_1 , respectively M_2 .

Proposition 4. *Let P^*, Q^* denote the mixed strategy Nash equilibria and M_1, M_2 the immunization strategies of player 1 and 2, respectively. Then the Ellsberg equilibria of the general conflict game are of the following form.*

	$M_1 \leq P^*$	$M_1 \geq P^*$
$M_2 \leq Q^*$	$([P_0, P^*], [Q^*, Q_1])$ $M_1 \leq P_0$	$([P_0, P^*], [Q_0, Q^*])$ $M_2 \leq Q_0$
$M_2 \geq Q^*$	$([P^*, P_1], [Q^*, Q_1])$ $Q_1 \leq M_2$	$([P^*, P_1], [Q_0, Q^*])$ $P_1 \leq M_1$

¹⁰ If the expected payoff $u_i(\bar{P}_i, P_{-i})$ is constant for all $P_{-i} \in \Delta S_{-i}$, then $\min_{P_{-i} \in \mathcal{P}_{-i}} u_i(\bar{P}_i, P_{-i})$ is also constant for all $\mathcal{P}_{-i} \subset \Delta S_{-i}$. The converse is also true, since if $\min_{P_{-i} \in \mathcal{P}_{-i}} u_i(\bar{P}_i, P_{-i})$ is constant for some fixed \bar{P}_i and all $\mathcal{P}_{-i} \subset \Delta S_{-i}$, then it is also constant for all $P_{-i} \in \Delta S_{-i}$.

If $M_1 = P^*$ or $M_2 = Q^*$, then an additional type of Ellsberg equilibria arises,

$$\begin{aligned} (P^*, [Q_0, Q_1]), \text{ where } Q_0 \leq Q^* \leq Q_1 & \quad \text{when } M_1 = P^*; \\ ([P_0, P_1], Q^*), \text{ where } P_0 \leq P^* \leq P_1 & \quad \text{when } M_2 = Q^*. \end{aligned}$$

In any case, we have the Ellsberg equilibrium which is identical to the Nash equilibrium in mixed strategies, (P^*, Q^*) .

We also see that our theory leads to a broader set of equilibria as the classical theory, but it is not arbitrary. The probabilities used in Ellsberg equilibrium do have to satisfy certain nontrivial bounds. In particular, one can test these bounds in the lab. We will come back to testability in Section 5.

4.3 Ellsberg Equilibria are Value-Preserving in Zero-Sum Games

We show that in two-person zero-sum games the Ellsberg equilibria all yield the same payoff, and this payoff is the value v of the game for player 1, and $-v$ for player 2.

Theorem 6 (Minimax Theorem 2). *Let G be a two-person zero-sum game. Then for all $i \in \{1, 2\}$ and every Ellsberg equilibrium $(\mathcal{P}^*, \mathcal{Q}^*)$ we have*

$$\max_{\mathcal{P}_i \subseteq \Delta S_i} \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} U_i(\mathcal{P}, \mathcal{Q}) = \min_{\mathcal{P}_{-i} \subseteq \Delta S_{-i}} \max_{\mathcal{P}_i \subseteq \Delta S_i} U_i(\mathcal{P}, \mathcal{Q}) = U_i(\mathcal{P}^*, \mathcal{Q}^*).$$

4.4 Ellsberg Equilibria in General 2×2 Coordination Games

Next we calculate the Ellsberg equilibria of general 2×2 coordination games. Consider the game with payoff matrix in Figure 8. We assume that $a, d \geq b, c$ and $e, h \geq f, g$ with $a - b - c + d \neq 0$ and $e - f - g + h \neq 0$.

		Player 2	
		L	R
Player 1	U	a, e	b, f
	D	c, g	d, h

Figure 8: General coordination game.

The conditions on the payoffs allow for different types of symmetric and non-symmetric coordination games. Let P^* and Q^* denote the Nash equilibrium strategies for player 1

and 2, respectively. The Nash equilibrium strategies can be computed using standard methods, they are

$$P^* = \frac{h - g}{e - f - g + h} \quad \text{and} \quad Q^* = \frac{d - b}{a - b - c + d}.$$

The immunization strategies of each player are denoted by M_1 and M_2 . Furthermore, player 1's Ellsberg strategy is denoted $[P_0, P_1]$, player 2's $[Q_0, Q_1]$. Then we have the following proposition.

Proposition 5. *Let P^*, Q^* denote the mixed Nash equilibrium strategies, and M_1, M_2 the immunization strategies of player 1 and 2, respectively. Then the Ellsberg equilibria of the general coordination game are of the following form.*

	$M_1 \leq P^*$	$M_1 \geq P^*$
$M_2 \leq Q^*$	$([P^*, P_1], [Q^*, Q_1])$	$([P^*, P_1], [Q_0, Q^*])$ $P_1 \leq M_1$ and $M_2 \leq Q_0$
$M_2 \geq Q^*$	$([P_0, P^*], [Q^*, Q_1])$ $M_1 \leq P_0$ and $Q_1 \leq M_2$	$([P_0, P^*], [Q_0, Q^*])$

If $M_1 = P^*$ or $M_2 = Q^*$, then an additional type of Ellsberg equilibria arises,

$$\begin{aligned} (P^*, [Q_0, Q_1]), \text{ where } Q_0 \leq Q^* \leq Q_1 & \quad \text{when } M_1 = P^*; \\ ([P_0, P_1], Q^*), \text{ where } P_0 \leq P^* \leq P_1 & \quad \text{when } M_2 = Q^*. \end{aligned}$$

In any case, the pure and mixed Nash equilibria, that is (U, L) , (D, R) and (P^*, Q^*) , are also Ellsberg equilibria.

Interestingly, the equilibria in the lower left and the upper right cell of the above table contain the equilibria $([M_1, P^*], [Q^*, M_2])$ and $([P^*, M_1], [M_2, Q^*])$, respectively. The associated coordination games are those of the type Battle of the Sexes. In these games, where despite the wish to coordinate the players are in some conflict of interest, we see that the strategic use of ambiguity conceals Nash behavior and maximin behavior.

4.5 Ellsberg Equilibria in Circulant 3×3 Games

We can apply the insights on the Ellsberg equilibria of 2×2 games in the preceding sections to investigate the Ellsberg equilibria of larger games. We discuss the Ellsberg equilibria of a special class of modified 3×3 zero-sum games, that is, modified *circulant* games. Circulant games have square payoff matrices in which the payoffs are circularly permuted in every row (column). For example, the classic game Rock Scissors Paper (RSP) is a circulant game, this is the game we analyze in this section. Our RSP has the payoff matrix given in Figure 9, the strategies are denoted R for Rock, S for Scissors, P for Paper.

		Player 2		
		R	S	P
Player 1	R	0, 0	1, -1	-1, 1
	S	-1, 1	0, 0	1, -1
	P	1, -1	-1, 1	0, 0

Figure 9: Rock Scissors Paper.

We fix some new notation for Ellsberg strategies when we deal with games with more than two pure strategies. An Ellsberg strategy for player 1 is, as before, denoted by \mathcal{P} , for player 2 by \mathcal{Q} . They are convex sets of probability distributions (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) in ΔS_1 and ΔS_2 , respectively, that is $\mathcal{P} \subseteq \Delta S_1$, $\mathcal{Q} \subseteq \Delta S_2$. An Ellsberg strategy \mathcal{P} is described as follows. Fix two vectors $x, y \in [0, 1]^3$ in the unit cube with components (x_1, x_2, x_3) and (y_1, y_2, y_3) . Then

$$\mathcal{P} = \left\{ (P_1, P_2, P_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 P_i = 1, P_i \geq 0, x_i \leq P_i \leq y_i, i = 1, 2, 3 \right\}. \quad (3)$$

Note that \mathcal{P} is completely described by giving the possible range of probabilities P_1 and P_2 . The range of P_3 (given by x_3 and y_3) then follows directly from the assumption that every triple $(P_1, P_2, P_3) \in \mathcal{P}$ is a probability distribution and some careful calculations. We use the following abbreviation for (3):

$$\mathcal{P} = (\{x_i \leq P_i \leq y_i\})_{i=1,2,3} = (\{x_1 \leq P_1 \leq y_1\}, \{x_2 \leq P_2 \leq y_2\}, \{x_3 \leq P_3 \leq y_3\}).$$

We write $\{x_i \leq P_i\}$ when $y_i = 1$, and $\{P_i \leq y_i\}$ when $x_i = 0$. We drop the subscript $i = 1, 2, 3$ when no confusion can arise. To describe an Ellsberg strategy \mathcal{Q} for player 2,

we use the vectors $w, z \in [0, 1]^3$. An Ellsberg strategy for player 1 is then for example

$$(\{1/3 \leq P_1 \leq 1/2\}, \{P_2 \leq 2/3\}, \{P_3 \leq 2/3\}).$$

When we calculate Ellsberg equilibria, we typically find a large number of equilibria which are very similar but differ in that one or more boundaries of the set of probability distributions are variable. Then we use the following notation. Fix $\hat{x}, \hat{y} \in [0, 1]^3$, then

$$\begin{aligned} & (\{\hat{x}_i \leq x_i \leq P_i \leq y_i \leq \hat{y}_i\}) \\ & := \{(\{x_i \leq P_i \leq y_i\}) \mid \text{for all } x, y \in [0, 1]^3 \text{ such that } \hat{x}_i \leq x_i \leq y_i \leq \hat{y}_i\}. \end{aligned}$$

As before, we suppress \hat{x}_i and \hat{y}_i when they are 0 or 1, respectively. Such a set of Ellsberg strategies is then for example

$$(\{1/3 \leq P_1 \leq y_1 \leq 1/2\}, \{x_2 \leq P_2 \leq 2/3\}, \{P_3 \leq y_3 \leq 2/3\}).$$

Note that this describes the set of Ellsberg strategies which necessarily include the probability distribution $(1/3, 2/3, 0)$ at the boundary of *each* Ellsberg strategy contained in the set. This is the type of set of Ellsberg strategies which we encounter frequently in Ellsberg equilibrium analysis.

Now we present the Ellsberg equilibria of RSP. The only Ellsberg equilibria these games have¹¹ are of the type (P^*, Q^*) $((P^*, Q^*))$, where P^* (Q^*) is the Nash equilibrium strategy of player 1 (2), and $Q^* \subseteq \Delta S_2$ ($P^* \subseteq \Delta S_1$) can be any convex set of probability distributions in which Q^* (P^*) is contained. This is the type of Ellsberg equilibrium that occurs in circulant games such as RSP.

We use the following notation. We denote by $P_1(Q_1)$ the probability with which player 1(2) plays R , $P_2(Q_2)$ the probability with which player 1(2) plays S , $P_3(Q_3)$ the probability with which player 1(2) plays P . The only Nash equilibrium of the game is $(P^*, Q^*) = ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$. Therefore, using the above considerations, we obtain the following proposition.

¹¹This can be deduced from Proposition 4, contact the authors for details.

Proposition 6. *The Ellsberg equilibria of Rock Scissors Paper are*

$$((\{P_i = 1/3\}), (\{w_i \leq Q_i \leq z_i\})),$$

where $0 \leq w_i \leq 1/3 \leq z_i \leq 1$ for all $i = 1, 2, 3$,

$$\text{and } ((\{x_i \leq P_i \leq y_i\}), (\{Q_i = 1/3\})),$$

where $0 \leq x_i \leq 1/3 \leq y_i \leq 1$ for all $i = 1, 2, 3$.

As in the matching pennies game, the situation changes when we slightly modify the original zero-sum game. We consider the following modification (Figure 10), where player 1 gets a payoff of 2, instead of 1, when (R, S) is played. Now the Nash equilibrium

		Player 2		
		R	S	P
Player 1	R	0, 0	2, -1	-1, 1
	S	-1, 1	0, 0	1, -1
	P	1, -1	-1, 1	0, 0

Figure 10: Modified Rock Scissors Paper.

is

$$(P^*, Q^*) = ((1/3, 1/3, 1/3), (1/3, 1/4, 5/12)).$$

As in 2×2 games, the immunization strategies play an important role in the Ellsberg equilibria of the modified RSP. In our modified game, they are

$$(M_1, M_2) = ((1/4, 1/3, 5/12), (1/3, 1/3, 1/3)).$$

We have the following proposition.

Proposition 7. *The Ellsberg equilibria of the modified Rock Scissors Paper are*

$$((\{1/3 \leq P_1 \leq y_1 \leq 2/3\}, \{P_2 = 1/3\}, \{x_3 \leq P_3 \leq 1/3\}),$$

$$(\{w_1 \leq Q_1 \leq z_1\}, \{1/4 \leq Q_2 \leq z_2 \leq 1/3\}, \{5/12 \leq Q_3 \leq z_3 \leq 3/4\})),$$

where $0 \leq w_1 \leq 1/3 = z_1$.

This can be read as player 1 using the strategy: “I will play Rock at least with probability $1/3$ but not with probability higher than $2/3$. I will play Paper with probability less than

1/3 and Scissors with probability 1/3.” Before we prove this, we discuss the intuition of the result. We know from the analysis of 2×2 games that ambiguity can only be a best response if the opponent uses his Nash equilibrium probability at the boundary of his Ellsberg strategy, and additionally this is the worst case measure which the player uses in his utility evaluation. In this case, the utility function has flat parts and the player best responds with ambiguity. When the opponent plays exactly his Nash equilibrium distribution (without using ambiguity himself), the whole utility function of the player is flat and he can play any Ellsberg strategy he likes, it will always be a best response. However, recall that this can only be an Ellsberg equilibrium if the opponent’s Nash equilibrium strategy is exactly his maximin strategy.

Consider the Ellsberg equilibrium in Proposition 7. We can draw the Ellsberg strategies as a projection into the 2-simplex (see, e.g., ? p. 36 for explanation) to understand how the characteristics reviewed above are apparent in 3×3 games. To this end we use the ‘largest’ Ellsberg equilibrium in the set, that is

$$((\{1/3 \leq P_1 \leq 2/3\}, \{P_2 = 1/3\}, \{P_3 \leq 1/3\}), \\ (\{Q_1 \leq 1/3\}, \{1/4 \leq Q_2 \leq 1/3\}, \{5/12 \leq Q_3 \leq 3/4\})).$$

In the equilateral triangle in Figures 11 and 12, for every point on the edge opposite the vertex R , the probability that R is played is zero. On the other hand, at the vertex R , R is played with probability one and S and P with probability zero. Sets of probability distributions, as we encounter in Ellsberg equilibria, are drawn as gray areas. The gray areas are the possible probabilities for each component of $(P_1, P_2, P_3) \in \mathcal{P}$ (in Figure 11) and $(Q_1, Q_2, Q_3) \in \mathcal{Q}$ (in Figure 12), the intersection is then the set of probability distributions which satisfy all three conditions of the Ellsberg strategy. The intersection is framed by a thick black line. As one can see in Figure 11, the Ellsberg equilibrium strategy of player 1 is a line, because $\{P_2 = 1/3\}$ contains only a single element. His Nash equilibrium strategy $P^* = (1/3, 1/3, 1/3)$ lies at the boundary of the Ellsberg equilibrium strategy. Player 2 plays a set of probability distributions depicted in Figure 12. Again, $Q^* = (1/3, 1/4, 5/12)$ lies at the boundary of the Ellsberg equilibrium strategy. As we have seen in 2×2 games, the ‘largest’ Ellsberg equilibrium is bounded by the immunization strategy M_2 .

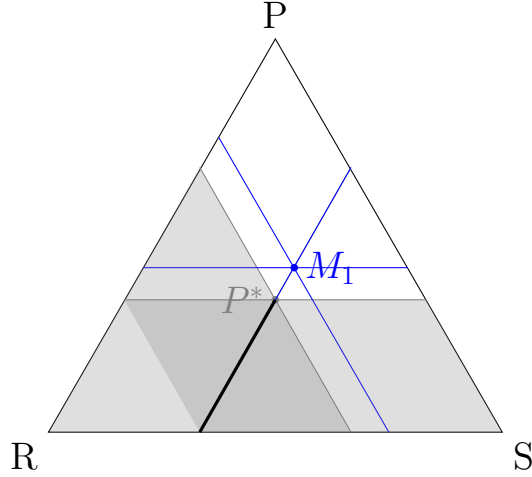


Figure 11: Ellsberg equilibrium strategy of player 1 in the modified RSP.

5 Observational Implications of Ellsberg Games, and Human Behavior

Human beings are not rational agents. This observation, trivial as it is for an educated person, has been amply documented in experiments. They are neither rational, nor are they all expected utility maximizers. Nevertheless, we can ask whether the aggregate behavior of human beings in the lab is or is not related to equilibrium concepts put forward by game theorists.

5.1 Observational Implications of Ellsberg Equilibria

Game Theory studies equilibrium outcomes of social conflicts when rational agents interact. Human beings are quite different from rational agents in general, so one can only expect to see a consistency with Nash equilibrium predictions and human behavior when the situation is controlled in such a way as to bring out the rational part of humans.

Nevertheless, it does make sense to ask what the observational implications of our theory are. For three player games, this is quite clear, as our theory predicts new equilibria outside the support of Nash equilibria; this is a testable implication, and we shall proceed one day to carry out such a test.

For two player games, the situation is more subtle. Both the Nash equilibrium and the Ellsberg equilibria have full support, so the only thing that we can learn from our

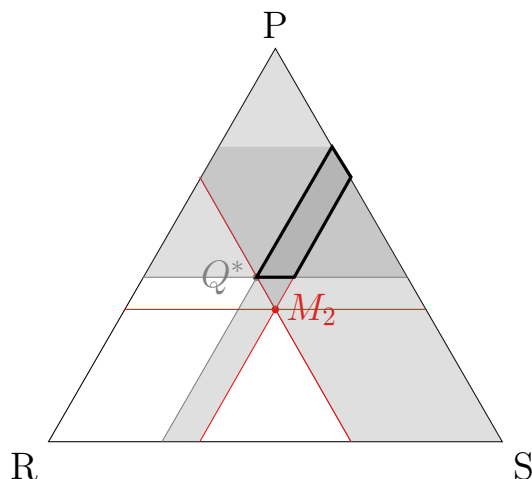


Figure 12: Ellsberg equilibrium strategy of player 2 in the modified RSP.

theory seems to be that either action is fine in a one shot game. This is indeed the stance of Bade (2011), in line with a number of predecessors.

There is, however, a way to distinguish the predictions of Ellsberg equilibria and Nash equilibria even in two player games. To understand this, we need first explain how the law of large numbers looks like under ambiguity. The classical law states that the frequency of *HEAD* in an infinite sequence of independent coin tosses will converge to the probability of *HEAD*. Now let us look at a typical Ellsberg urn that contains 100 balls, red and black, and we only know that the number of red balls is between 30 and 60. What can we say about the average frequency drawn from independent repetitions of the Ellsberg experiment? The natural guess would be that the average lies in the interval between 30% and 60% in the long run. This is indeed correct, and mathematical versions of that theorem have recently been proven, see Maccheroni and Marinacci (2005) and Epstein and Schneider (2003), e.g. Peng (2007) has obtained the result that the average frequency will indeed fluctuate between both bounds, and *every* point in the interval $[0.3, 0.6]$ is an accumulation point of the sequence.

What is then the empirical content of such laws of large numbers? If we adhere to the point of view that our observed humans play independently one shot games, and that they should play equilibrium strategies, then the average frequency will converge to the Nash equilibrium strategy according to the classical theory, and will fluctuate between two bounds according to the new Ellsberg theory.

We thus do get observational differences between the two theories; and we interpret

the Goeree-Holt results that we discuss below as a first evidence that our theory can accommodate deviations from Nash equilibrium observed in laboratories.

5.2 Human Behavior in Matching Pennies Games and Ellsberg Equilibria

Whereas the support of the Ellsberg and Nash equilibria is obviously the same in the two player games we studied, we do think that the Ellsberg equilibria reveal a new class of behavior not encountered in game theory before.

It might be very difficult for humans to play exactly a randomizing strategy with equal probabilities; indeed, the ability to do so has been a debate since the early days of game theory, and some claim that humans cannot randomize, see Dang (2009) for a recent account and references therein. Our result shows that it is not necessary to randomize exactly to support a similar equilibrium outcome (with the same expected payoff). It is just enough that your opponent knows that you are randomizing with some probability, and that it could be that this probability is one half, but not less. It is thus sufficient that the player is able to control the lower bound of his device. This might be easier to implement than the perfectly random behavior required in classical game theory.

In fact, there are experimental findings which suggest that the Ellsberg equilibrium strategy in the modified Matching Pennies game is closer to real behavior than the Nash equilibrium prediction. To illustrate this, let us consider the interesting results by Goeree and Holt (2001) who ran experiments on three different versions of Matching Pennies; the three payoff matrices can be seen in Table 1.

In the first game, we have a typical symmetric conflict game with a unique mixed Nash equilibrium in which both players randomize uniformly over both pure strategies. The aggregate play of humans in the experiment is closely consistent with the Nash equilibrium prediction, 48% of players choosing “Top” or “Left”, respectively.

Remember that the probabilities in a mixed strategy equilibrium are chosen in such a way as to render the opponent indifferent between her two pure strategies. As a consequence, if we change the payoffs of player 1 only (while keeping the ordering of payoffs), his Nash equilibrium strategy does not change because he has to make player 2 indifferent between her two pure actions, and her payoffs have not been modified.

In the second game, called the asymmetric Matching Pennies game, player 1 gets 320 instead of 80 in the upper left outcome. All other payoffs remain the same. Many humans now deviate from Nash, as is reported in brackets, 96% of the players taking the

		Left (48)	Right (52)
Symmetric	Top (48)	80,40	40,80
	Bottom (52)	40,80	80,40
		Left (16)	Right(84)
Asymmetric	Top (96)	320 , 40	40,80
	Bottom (4)	40,80	80,40
		Left (80)	Right (20)
Reversed	Top (8)	44 ,40	40,80
	Bottom (92)	40,80	80, 40

Table 1: The Goeree-Holt Results on three different versions of Matching Pennies.

action “Top”. Interestingly, also the humans playing the role of player 2 change their behavior, and most of them play “Right”, the best reply to “Top”.

In the third case, player 1’s payoff in the upper left outcome is decreased to a lowly 44. Then only 8% of players choose “Top”; 80% of humans in the role of player 2 choose “Left”.

While aggregate behavior by humans is certainly inconsistent with the predictions of Nash equilibrium, it is consistent with Ellsberg equilibria. We summarize the results in Table 2.

In the symmetric game, our Proposition 4 essentially predicts only Nash equilibrium behavior, and this is what we observe in the experiment as well.

In the asymmetric Matching Pennies game, the Nash equilibrium strategies are $P^* = 1/2$ for player 1 and $Q^* = 1/8$ for player 2. According to our proposition, the Ellsberg equilibria allow for probabilities in the interval $[1/2, 1]$ for player 1 choosing “Top”, and for the interval $[1/8, 1/2]$ for player 2 choosing “Left”. The observed percentages of 96% and 16% do lie in these intervals.

And in the “reversed” version of the game, the Nash equilibrium strategies are $P^* = 1/2$ and $Q^* = 10/11$. So we have the reversed relation $Q^* > P^*$. The Ellsberg equilibria allow for probabilities for “Top” in the interval $[0, 1/2]$ for player 1, and for probabilities in $[1/2, 10/11]$ for player 2. The aggregate observed quantities of 8% and 80% do lie in these intervals.

6 Conclusion

This article demonstrates that the strategic use of ambiguity is a relevant concept in game theory. Employing objective ambiguity as a strategic instrument leads to a new

Game	Nash Equilibrium		Ellsberg Equilibrium		Observations	
	Player 1	Player 2	Player 1	Player 2	Player 1	Player 2
symmetric	0.5	0.5	0.5	0.5	0.48	0.48
asymmetric	0.5	0.125	[0.5,1]	[0.125,0.5]	0.96	0.16
reversed	0.5	0.90	[0,0.5]	[0.5,0.90]	0.08	0.8

Table 2: Comparison of Nash and Ellsberg Predictions with the Experimental Observations. We record the probabilities (or intervals of probabilities) for each player to play the first pure strategy (“Top” resp. “Left”) and the observed aggregate frequency of these actions in the Goeree-Holt experiments.

class of equilibria not encountered in classic game theory. We point out that in many games players choose to be deliberately ambiguous to gain a strategic advantage.

In some games this results in equilibrium outcomes which cannot be obtained as Nash equilibria. The peace negotiation game provides an example of such Ellsberg equilibria. Games with more than two players offer a strategic possibility that is not available in two-person games, because a third player is able to induce the use of different probability distributions. Although countries A and B observe the same Ellsberg strategy played by the superpower C, due to their ambiguity aversion the countries use different probability distributions to assess their utility. We plan to say more on this power of the third player, as well as on immunization against strategic ambiguity in games with more than two players, in a companion paper.

However, also two-person 2×2 games with conflicting interests have Ellsberg equilibria which are different from classic mixed strategy Nash equilibria. There are equilibria in which both players create ambiguity. They use an Ellsberg strategy where they only need to control the lower (or upper) bound of their set of probability distributions. We argue that this device is easier to use for a player than playing one precise probability distribution like in mixed strategy Nash equilibrium. What makes this argument attractive is that the payoffs in these Ellsberg equilibria are the same as in the unique mixed Nash equilibrium and thus the use of ambiguous strategies in competitive games is indeed an option. Our argument is strengthened by experimental results. Without any further assumptions besides ambiguity aversion, Ellsberg equilibria can explain human non-Nash behavior in modified Matching Pennies games. In symmetric Matching Pennies, humans tend to play the Nash equilibrium which is also in line with our result that essentially no new equilibria emerge in such symmetric games.

Appendix

We provide here the proofs of Theorems 1, 2 and 5, of Propositions 4 and 5, Theorem 6 and Proposition 7 in this order (in which they appear in the text).

First we recap the definition of an Ellsberg equilibrium, which was stated in the text of Section 3.

Definition 3. Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. A profile

$$(((\Omega_1^*, \mathcal{F}_1^*, \mathcal{P}_1^*), f_1^*), \dots, ((\Omega_n^*, \mathcal{F}_n^*, \mathcal{P}_n^*), f_n^*))$$

of Ellsberg strategies is an Ellsberg equilibrium of G if no player has an incentive to deviate from $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$, i.e. for all players $i \in N$, all Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ and all acts f_i for player i we have

$$U_i(((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)) \geq U_i(((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i), ((\Omega_{-i}^*, \mathcal{F}_{-i}^*, \mathcal{P}_{-i}^*), f_{-i}^*)), \text{ that is}$$

$$\begin{aligned} & \min_{P_i \in \mathcal{P}_i^*, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} u_i(f_i^*(\omega_i), f_{-i}^*(\omega_{-i})) dP_{-i} dP_i \\ & \geq \min_{P_i \in \mathcal{P}_i, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i} \int_{\Omega_{-i}^*} u_i(f_i(\omega_i), f_{-i}^*(\omega_{-i})) dP_{-i} dP_i. \end{aligned}$$

Reduced form Ellsberg equilibrium is defined in Definition 1.

Proof of Theorem 1. “ \Leftarrow ” Let \mathcal{Q}^* be an Ellsberg equilibrium according to Definition 1. We choose the states of the world $\Omega = S$ to be the set of pure strategy profiles, thereby we see that player i uses the Ellsberg urn $(S_i, \mathcal{F}_i, \mathcal{Q}_i^*)$. We define the act $f_i^* : (S_i, \mathcal{F}_i) \rightarrow \Delta S_i$ to be the constant act that maps $f_i^*(s_i) = \{\delta_{s_i}\}$. $\{\delta_{s_i}\} \in \Delta S_i$ is the degenerate mixed strategy which puts all weight on the pure strategy s_i . Each measure $Q_i \in \mathcal{Q}_i^*$ has an image measure under f_i^* ,

$$Q_i \circ f_i^{*-1} : \{\delta_{s_i}\} \mapsto Q_i(f_i^{*-1}(\{\delta_{s_i}\})).$$

$Q_i \circ f_i^{*-1}$ can be identified with $Q_i \in \mathcal{Q}_i^*$. Thus the reduced form Ellsberg equilibrium strategy \mathcal{Q}_i^* can be written as the Ellsberg strategy $((S, \mathcal{F}, \mathcal{Q}^*), f^*)$. This strategy is an Ellsberg equilibrium according to Definition 3.

“ \Rightarrow ” Let now $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ be an Ellsberg equilibrium according to Definition 3. Every $P_i \in \mathcal{P}_i^*$ induces an image measure $P_i \circ f_i^{*-1}$ on ΔS_i that assigns a probability

to a distribution $f_i^*(\omega_i) \in \Delta S_i$ to occur.

To describe the probability that a pure strategy s_i is played, given a distribution P_i and an Ellsberg strategy $((\Omega_i^*, \mathcal{F}_i^*, \mathcal{P}_i^*), f_i^*)$, we integrate $f_i^*(\omega_i)(s_i)$ over all states $\omega_i \in \Omega_i$. Thus we can define Q_i to be:

$$Q_i(s_i) := \int_{\Omega_i^*} f_i^*(\omega_i)(s_i) dP_i. \quad (4)$$

Recall that \mathcal{P}_i is a closed and convex set of probability distributions. We get a measure Q_i on S_i for each $P_i \in \mathcal{P}_i \subseteq \Delta \Omega_i$. We call the resulting set of probability measures \mathcal{Q}_i^* .

$$\mathcal{Q}_i^*(s_i) := \left\{ Q_i(s_i) = \int_{\Omega_i^*} f_i^*(\omega_i)(s_i) dP_i \mid P_i \in \mathcal{P}_i^* \right\}.$$

\mathcal{Q}_i^* is closed and convex, since \mathcal{P}_i^* is.

Now suppose \mathcal{Q}^* was not a reduced form Ellsberg equilibrium. Then for some player $i \in N$ there existed a set \mathcal{Q}_i of probability measures on S_i that yields a higher minimal expected utility. This means we would have

$$\begin{aligned} & \min_{Q_i \in \mathcal{Q}_i, Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dQ_{-i} dQ_i \\ & > \min_{Q_i \in \mathcal{Q}_i^*, Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dQ_{-i} dQ_i \end{aligned} \quad (5)$$

for some $\mathcal{Q}_i \neq \mathcal{Q}_i^*$. Let Q'_i be the minimizer of the first expression, then it must be that $Q'_i \notin \mathcal{Q}_i^*$. We know that Q'_i is derived from some P'_i under the equilibrium act,

$$Q'_i(s_i) = \int_{\Omega_i^*} f_i^*(\omega_i)(s_i) dP'_i. \quad (6)$$

It follows that P'_i is not element of the equilibrium Ellsberg urn $(\Omega_i^*, \mathcal{F}_i^*, \mathcal{P}_i^*)$, that is $P'_i \notin \mathcal{P}_i^*$. Now it remains to show that in the original game P'_i yields a higher minimal expected utility than using \mathcal{P}_i^* . In that case $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ is not an Ellsberg equilibrium and the proof is complete.

Let player i use P'_i in his maxmin expected utility evaluation in the original game.

This yields

$$\begin{aligned} & \min_{P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} u_i(f_i^*(\omega_i), f_{-i}^*(\omega_{-i})) dP_{-i} dP_i' \\ &= \min_{P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) df_{-i}^*(\omega_{-i}) df_i^*(\omega_i) dP_{-i} dP_i'. \end{aligned} \quad (7)$$

Recall that we use u_i to be the utility function on S_i as well as on ΔS_i . We use equations (4) and (6) to rewrite the expression and get

$$\min_{Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dQ_{-i} dQ_i'. \quad (8)$$

We know by equation (5) that this is larger than the minimal expected utility over \mathcal{Q}_i^* and this gives

$$\begin{aligned} (8) &> \min_{Q_i \in \mathcal{Q}_i^*, Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dQ_{-i} dQ_i \\ &= \min_{P_i \in \mathcal{P}_i^*, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} u_i(f_i^*(\omega_i), f_{-i}^*(\omega_{-i})) dP_{-i} dP_i. \end{aligned}$$

Going back to equation (7) we see that this contradicts the assumption that the profile $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ was an Ellsberg equilibrium. Thus \mathcal{Q}^* is a reduced form Ellsberg equilibrium. \square

The main part of the proof of Theorem 2 is that every player can find for every profile of Ellsberg strategies a mixed strategy that gives him at least the same utility as his Ellsberg strategy. We first prove the latter result in the following lemma. We let π_i^* abbreviate the constant act that maps every state of the world to the mixed strategy $\pi_i^* \in \Delta S_i$.

Lemma 1. *Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. Then for any profile of Ellsberg strategies $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ and every $i \in N$ there exists a mixed strategy $\pi_i^* \in \Delta S_i$ such that $U_i(\{\delta_{\pi_i^*}\}, \mathcal{P}_{-i}) \geq U_i(\mathcal{P}_i, \mathcal{P}_{-i})$.*

Proof. If \mathcal{P}_i is a singleton $\{\delta_{\pi_i}\}$, we take $\pi_i^* := \pi_i$ and we are done. Let \mathcal{P}_i not be a singleton. Recall that $U_i(\mathcal{P}_i, \mathcal{P}_{-i})$ is defined to be the minimal expected payoff according to all distributions $P \in \mathcal{P}$. Thus, there must exist a $\pi_i^* \in \Delta S_i$ for which $\{\delta_{\pi_i^*}\} \in \mathcal{P}_i$ with $U_i(\{\delta_{\pi_i^*}\}, \mathcal{P}_{-i}) \geq U_i(\mathcal{P}_i, \mathcal{P}_{-i})$. \square

Proof of Theorem 2. Let \mathcal{P} with $\mathcal{P}_i = \{\delta_{\pi_i}\}$ be an Ellsberg equilibrium of G . So $U_i(\{\delta_{\pi_i}\}, \{\delta_{\pi_{-i}}\}) \geq U_i(\mathcal{P}'_i, \{\delta_{\pi_{-i}}\})$ holds for all $i \in N$ for all Ellsberg strategies $\mathcal{P}'_i \subseteq \Delta S_i$. In particular this holds for all singletons $\{\delta_{\pi'_i}\}$, so

$$U_i(\{\delta_{\pi_i}\}, \{\delta_{\pi_{-i}}\}) \geq U_i(\{\delta_{\pi'_i}\}, \{\delta_{\pi_{-i}}\})$$

and thus

$$u_i(\pi_i, \pi_{-i}) \geq u_i(\pi'_i, \pi_{-i}) \quad \text{for all } \pi'_i \in \Delta S_i.$$

Therefore the profile (π_1, \dots, π_n) is a Nash equilibrium of G .

Next, assume that (π_1, \dots, π_n) is a Nash equilibrium of G . Suppose it was not an Ellsberg equilibrium of G , that is there exists an Ellsberg strategy \mathcal{P}_i for some player i such that $U_i(\mathcal{P}_i, \{\delta_{\pi_{-i}}\}) > U_i(\{\delta_{\pi_i}\}, \{\delta_{\pi_{-i}}\})$. By Lemma 1 we have that there exists a π_i^* such that

$$U_i(\{\delta_{\pi_i^*}\}, \{\delta_{\pi_{-i}}\}) \geq U_i(\mathcal{P}_i, \{\delta_{\pi_{-i}}\}) > U_i(\{\delta_{\pi_i}\}, \{\delta_{\pi_{-i}}\}).$$

This is a contradiction with the assumption that (π_1, \dots, π_n) was a Nash equilibrium of the game G . \square

To prove Theorem 5 we use the following notation. A two-person normal form game G is described by the payoff matrices A and B of player 1 and player 2, we write $G = (A, B)$. We call G square when A and B are square matrices; in this section we only consider square games. The row vectors of A (B) are denoted by subscripts, a_i (b_i) $\in \mathbb{R}^n$, and the column vectors by superscripts, a^j (b^j) $\in \mathbb{R}^n$. In the following it will be convenient to write the expected utility of a mixed strategy P of player 1 with payoff matrix A as PAQ when player 2 plays mixed strategy Q . Transpose signs are suppressed.

A necessary condition for G to have a completely mixed Nash equilibrium is that no player has weakly or strictly dominated strategies. That is, no row (column) of A (B) is dominated by another row (column) or a convex combination of rows (columns). This condition can be expressed as follows. Let \tilde{A} be the $(n+1) \times n$ -matrix consisting of the matrix A with an additional last column $(1, \dots, 1)$, and \tilde{B} the $n \times (n+1)$ -matrix with an additional last row $(1, \dots, 1)$. Furthermore let $\tilde{k} = (k, \dots, k, 1)$ and $\tilde{l} = (l, \dots, l, 1)$ with $k, l \in \mathbb{R}$. Then G has a completely mixed Nash equilibrium $(P, Q) = (P^*, Q^*)$ when it is a nonnegative solution to the two systems of linear equations

$$\begin{aligned} P\tilde{B} &= \tilde{k}, \\ \tilde{A}Q &= \tilde{l}. \end{aligned}$$

The existence of immunization strategies can be expressed analogously: no column (row) of matrix A (B) is dominated by another column (row) or a convex combination of columns (rows). Let $\tilde{u} = (u, \dots, u, 1)$ and $\tilde{v} = (v, \dots, v, 1)$ with $u, v \in \mathbb{R}$. Player 1 has an immunization strategy $P = \bar{P}$ in G when the vector is a nonnegative solution to the system

$$P\tilde{A} = \tilde{u},$$

and player 2 has an immunization strategy $Q = \bar{Q}$ in G when the vector is a nonnegative solution to the system

$$\tilde{B}Q = \tilde{v}.$$

Note that for the existence of a completely mixed Nash equilibrium both solutions P^* and Q^* have to exist, whereas the immunization strategy is defined for a single player. To a square game G we define *associated zero-sum games* G^1 and G^2 . G^1 is the game with payoff matrices $(A, -A)$, G^2 the game with payoff matrices $(-B, B)$. We first prove the following lemma.

Lemma 2. *Let G be a square two-person normal form game, G^1 and G^2 its associated zero-sum games. If G has a completely mixed Nash equilibrium (P^*, Q^*) and player 1 (2) has an immunization strategy \bar{P} (\bar{Q}), then G^1 (G^2) has a Nash equilibrium (P_1^*, Q_1^*) ((P_2^*, Q_2^*)) where $P_1^* = \bar{P}$ and $Q_1^* = Q^*$ ($P_2^* = P^*$ and $Q_2^* = \bar{Q}$).*

Proof. We show this only for player 1, the case for player 2 follows analogously. If (P^*, Q^*) is a completely mixed Nash equilibrium of G , then Q^* solves the system $\tilde{A}Q = \tilde{l}$ for some $l \in \mathbb{R}$. Furthermore, if \bar{P} is an immunization strategy of player 1, then \bar{P} solves the system $P\tilde{A} = \tilde{u}$ for some $u \in \mathbb{R}$. Then \bar{P} also solves $P(-\tilde{A}) = -\tilde{u}$ and is therefore a Nash equilibrium strategy for player 1 of the game G^1 . \square

Now, we can prove Theorem 5.

Proof of Theorem 5. We prove the result only for player 1. \bar{P} is an immunization strategy of player 1 in the game G if and only if it is a nonnegative solution to the system $P\tilde{A} = \tilde{u}$. Now by assumption there exists a completely mixed Nash equilibrium (P^*, Q^*) of G , thus with Lemma 2 there exists a completely mixed Nash equilibrium (P_1^*, Q_1^*) in G^1 . P_1^* is therefore a nonnegative solution to the system $P(-\tilde{A}) = \tilde{k}$ for some $k \in \mathbb{R}$. Then P_1^* also solves the system $P\tilde{A} = (-\tilde{k})$ and we see that P_1^* must be the immunization strategy \bar{P} of player 1.

Now, since G^1 is a zero-sum game, \bar{P} is by the classic Minimax Theorem for zero-sum games also the maximin strategy in G^1 , that means

$$\bar{P} \in \arg \max_P \min_Q PAQ.$$

This condition only depends on the payoff matrix A of player 1, thus $\bar{P}(= P_1^*)$ is the maximin strategy in any game where player 1 has the payoff matrix A , in particular in the game G .

To show the analog statement for player 2 we use the associated zero-sum game G^2 . \square

Remark 1. *Lemma 1 in ? proves a result very close to our Theorem 5, but with different means by making use of geometric properties of the various types of strategies. The interpretation of the result gets an interesting twist when we use the associated zero-sum games as in our proof. A player who is immunizing himself against strategic ambiguity blindfolds himself and drops the strategic element of the game. Instead of playing the original game, he plays the associated zero-sum game which results from his own payoff matrix. Thus, he plays his optimal strategy \bar{P} , the immunization strategy, against an imaginary self that receives the negative of his own payoff.*

Proof of Proposition 4. The Nash equilibrium strategies follow from the usual analysis. To calculate the Ellsberg equilibria of the general conflict game (Figure 7), we first derive the utility functions of player 1 and player 2. Due to the assumption that $a, d > b, c$ and $e, h < f, g$, the denominator $a - b - c + d$ is positive, and the denominator $e - f - g + h$ is negative. This reflects the competitiveness of the game in the payoff functions; player 1 uses Q_0 as a minimizer when $P > \frac{d-c}{a-b-c+d}$, and on the contrary, player 2 uses P_0 as a minimizer when $Q < \frac{h-f}{e-f-g+h}$.

$$\begin{aligned} U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \leq Q \leq Q_1} aPQ + bP(1-Q) + c(1-P)Q + d(1-P)(1-Q) \\ &= \min_{Q_0 \leq Q \leq Q_1} Q((a-b-c+d)P + c-d) + (b-d)P + d \\ &= \begin{cases} Q_0((a-b-c+d)P + c-d) + (b-d)P + d & \text{if } P > \frac{d-c}{a-b-c+d}, \\ \frac{(b-d)(c-d)}{a-b-c+d} + d & \text{if } P = \frac{d-c}{a-b-c+d}, \\ Q_1((a-b-c+d)P + c-d) + (b-d)P + d & \text{if } P < \frac{d-c}{a-b-c+d}, \end{cases} \end{aligned}$$

$$\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{P_0 \leq P \leq P_1} ePQ + g(1-P)Q + fP(1-Q) + h(1-P)(1-Q) \\
&= \min_{P_0 \leq P \leq P_1} P((e-f-g+h)Q + f-h) + (g-h)Q + h \\
&= \begin{cases} P_0((e-f-g+h)Q + f-h) + (g-h)Q + h & \text{if } Q < \frac{h-f}{e-f-g+h}, \\ \frac{(g-h)(h-f)}{e-f-g+h} + h & \text{if } Q = \frac{h-f}{e-f-g+h}, \\ P_1((e-f-g+h)Q + f-h) + (g-h)Q + h & \text{if } Q > \frac{h-f}{e-f-g+h}. \end{cases}
\end{aligned}$$

We see that

$$M_1 = \frac{d-c}{a-b-c+d}, \quad \text{respectively} \quad M_2 = \frac{h-f}{e-f-g+h}$$

are indeed the immunization strategies of player 1 and 2 respectively. At the boundaries the utility functions become

$$\begin{aligned}
U_1(0, [Q_0, Q_1]) &= (c-d)Q_1 + d, & U_2([P_0, P_1], 0) &= (f-h)P_0 + h, \\
U_1(1, [Q_0, Q_1]) &= (a-b)Q_0 + b, & U_2([P_0, P_1], 1) &= (e-g)P_1 + g.
\end{aligned}$$

The payoff function of player 1 is constant when $Q_0 = Q_1 = Q^*$, and the payoff function of player 2 is constant when $P_0 = P_1 = P^*$. The best response correspondences for both players are listed below. We see immediately that, of course, there cannot be an Ellsberg equilibrium in pure strategies: when player 2 plays L , player 1 best responds U , and when player 1 plays U , player 2 best responds R . The intersections of the best response correspondences are discussed in detail below.

$$Q_0 > Q^* : \quad B_1([Q_0, Q_1]) = 1 \quad (1)$$

$$Q_0 = Q^* < Q_1 : \quad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [M_1, 1]\} \quad (2)$$

$$Q_0 < Q^* < Q_1 : \quad B_1([Q_0, Q_1]) = M_1 \quad (3)$$

$$Q_0 < Q^* = Q_1 : \quad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, M_1]\} \quad (4)$$

$$Q_1 < Q^* : \quad B_1([Q_0, Q_1]) = 0 \quad (5)$$

$$Q_0 = Q^* = Q_1 : \quad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1]\} \quad (6)$$

$P_0 > P^* :$	$B_2([P_0, P_1]) = 0$
$P_0 = P^* < P_1 :$	$B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, M_2]\}$
$P_0 < P^* < P_1 :$	$B_2([P_0, P_1]) = M_2$
$P_0 < P^* = P_1 :$	$B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [M_2, 1]\}$
$P_1 < P^* :$	$B_2([P_0, P_1]) = 1$
$P_0 = P^* = P_1 :$	$B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\}$

To find the Ellsberg equilibria, we look at the different cases in turn.

(1) $Q_0 > Q^*$: player 1 responds $P_0 = P_1 = 1$ and player 2 chooses $Q_0 = Q_1 = 0$, thus this is not an Ellsberg equilibrium.

(2) $\left[\frac{d-b}{a-b-c+d}, Q_1\right] \Rightarrow [P_0, P_1] \subseteq \left[\frac{d-c}{a-b-c+d}, 1\right]$, then, depending on the size of M_1 in relation to P^* , player 1 has the following choices:

- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: player 1 can play either
 - $\left[\frac{h-g}{e-f-g+h}, P_1\right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:
 - * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$ with $Q_1 \leq \frac{h-f}{e-f-g+h}$, we thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1\right], \left[\frac{d-b}{a-b-c+d}, Q_1\right]\right),$$

$$\text{where } Q_1 \leq \frac{h-f}{e-f-g+h},$$

that is $([P^*, P_1], [Q^*, Q_1])$, where $Q_1 \leq M_2$.

- * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.
- or $\left[P_0, \frac{h-g}{e-f-g+h}\right]$ with $\frac{d-c}{a-b-c+d} \leq P_0 \leq \frac{h-g}{e-f-g+h}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1\right]$. Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:
 - * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$, then the best response of player 1

is $[P_0, P_1] \subseteq [\frac{d-c}{a-b-c+d}, 1]$. We thus have here the Ellsberg equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h} \right], \left[\frac{d-b}{a-b-c+d}, Q_1 \right] \right),$$

$$\text{where } \frac{d-c}{a-b-c+d} \leq P_0,$$

that is $([P_0, P^*], [Q^*, Q_1])$, where $M_1 \leq P_0$.

* $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.

• $\frac{d-c}{a-b-c+d} \geq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 0$, thus no Ellsberg equilibrium arises.

(3) $Q_0 < Q^* < Q_1$: player 1 responds with $P_0 = P_1 = M_1$. Only when $M_1 = P^*$, player 1 sticks to his strategy and we get the equilibrium $(P^*, [Q_0, Q_1])$, where $Q_0 < Q^* < Q_1$.

(4) $[Q_0, \frac{d-b}{a-b-c+d}] \Rightarrow [P_0, P_1] \subseteq [0, \frac{d-c}{a-b-c+d}]$, then, depending on the size of M_1 in relation to P^* , player 1 has the following choices:

• $\frac{d-c}{a-b-c+d} \geq \frac{h-g}{e-f-g+h}$: player 1 can play either
 – $\left[P_0, \frac{h-g}{e-f-g+h} \right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1 \right]$.
 Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:

* $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: $[Q_0, \frac{d-b}{a-b-c+d}]$ with $\frac{h-f}{e-f-g+h} \leq Q_0$, we thus have here the Ellsberg equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h} \right], \left[Q_0, \frac{d-b}{a-b-c+d} \right] \right),$$

$$\text{where } \frac{h-f}{e-f-g+h} \leq Q_0,$$

that is $([P_0, P^*], [Q_0, Q^*])$, where $M_2 \leq Q_0$.

* $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.

– or $\left[\frac{h-g}{e-f-g+h}, P_1\right]$ with $\frac{h-g}{e-f-g+h} \leq P_1 \leq \frac{d-c}{a-b-c+d}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:

* $\frac{d-b}{a-b-c+d} \leq \frac{h-f}{e-f-g+h}$: $[Q_0, \frac{d-b}{a-b-c+d}]$, then the best response of player 1 is $[P_0, P_1] \subseteq \left[0, \frac{d-c}{a-b-c+d}\right]$. We thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1\right], \left[Q_0, \frac{d-b}{a-b-c+d}\right]\right),$$

$$\text{where } P_1 \leq \frac{d-c}{a-b-c+d},$$

that is $([P^*, P_1], [Q_0, Q^*])$, where $P_1 \leq M_1$.

* $\frac{d-b}{a-b-c+d} \geq \frac{h-f}{e-f-g+h}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.

• $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 0$, thus no Ellsberg equilibrium arises.

(5) $Q_1 < Q^*$: player 1 responds $P_0 = P_1 = 0$ and player 2 chooses $Q_0 = Q_1 = 1$ thereafter, thus this is not an Ellsberg equilibrium.

(6) $Q_0 = Q^* = Q_1$: player 1 responds with $[P_0, P_1] \subseteq [0, 1]$. Only when $M_2 = Q^*$, player 2 sticks to his strategy and we get the equilibrium $([P_0, P_1], Q^*)$, where $P_0 < P^* < P_1$.

□

Theorem 6 follows from the following three lemmas. The first lemma concerns the maximization and minimization of a linear function u on some convex set ΔS (for the moment, S does not denote a set of strategy profiles, but simply a set with m elements).

Lemma 3. *P is a probability distribution on the set $S = \{s_1, \dots, s_m\}$. Let $u(P)$ be a linear function that maps the m -dimensional vector P into the real numbers. We denote by ΔS the set of probability distributions P , and by $\mathcal{P} \subset \Delta S$ a closed and convex subset of ΔS . Then*

$$\max_{P \in \Delta S} \min_{P \in \mathcal{P}} u(P) = \max_{P \in \Delta S} u(P), \quad (9)$$

$$\text{and } \min_{P \in \Delta S} \min_{P \in \mathcal{P}} u(P) = \min_{P \in \Delta S} u(P). \quad (10)$$

Proof. We start by showing equation (9). It is evident that

$$\max_{P \in \Delta S} u(P) \geq \max_{\mathcal{P} \subseteq \Delta S} \min_{P \in \mathcal{P}} u(P),$$

so it remains to show that the left hand side is always greater than or equal to the right hand side of (9). Since the function $u(P)$ is linear, there exists a $P' \in \Delta S$ with

$$\max_{P \in \Delta S} u(P) = u(P').$$

Now we have

$$\max_{\mathcal{P} \subseteq \Delta S} \min_{P \in \mathcal{P}} u(P) \geq \max_{\mathcal{P} \subseteq \Delta S, P' \in \mathcal{P}} \min_{P \in \mathcal{P}} u(P),$$

since maximizing over a smaller number of subsets of ΔS is necessarily less than or equal to the original maximization. Making the set over which we maximize even smaller, we obtain

$$\max_{\mathcal{P} \subseteq \Delta S, P' \in \mathcal{P}} \min_{P \in \mathcal{P}} u(P) \geq \max_{P'} \min_{P=P'} u(P) = u(P')$$

and the equality (9) is shown. The argument for equation (10) is analog:

$$\min_{\mathcal{P} \subseteq \Delta S} \min_{P \in \mathcal{P}} u(P) \geq \min_{P \in \Delta S} u(P)$$

is evident. Furthermore, there exists $P' \in \Delta S$ such that $\min_{P \in \Delta S} u(P) = u(P')$, and

$$\begin{aligned} \min_{\mathcal{P} \subseteq \Delta S} \min_{P \in \mathcal{P}} u(P) &\leq \min_{\mathcal{P} \subseteq \Delta S, P' \in \mathcal{P}} \min_{P \in \mathcal{P}} u(P) \\ &\leq \min_{P'} \min_{P=P'} u(P) = u(P') \end{aligned}$$

and the equality holds. □

Lemma 4. *Let G be a zero-sum game with two players. Then the following holds.*

$$\min_{\mathcal{Q} \subseteq \Delta S_2} \max_{\mathcal{P} \subseteq \Delta S_1} U_1(\mathcal{P}, \mathcal{Q}) \stackrel{(1)}{=} \max_{\mathcal{P} \subseteq \Delta S_1} \min_{\mathcal{Q} \subseteq \Delta S_2} U_1(\mathcal{P}, \mathcal{Q}) \stackrel{(2)}{=} u_1(P^*, Q^*) = v, \quad (11)$$

$$\text{and } \min_{\mathcal{P} \subseteq \Delta S_1} \max_{\mathcal{Q} \subseteq \Delta S_2} U_2(\mathcal{P}, \mathcal{Q}) \stackrel{(1)}{=} \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{\mathcal{P} \subseteq \Delta S_1} U_2(\mathcal{P}, \mathcal{Q}) \stackrel{(2)}{=} -u_1(P^*, Q^*) = -v. \quad (12)$$

Proof. We start by showing equality (2) of equation (11), followed by equality (1). (12) is shown below. For all equalities we use the fact that for all linear functions $f(x, y)$,

$\min_x \min_y f(x, y) = \min_y \min_x f(x, y)$, and Lemma 3.

$$\begin{aligned}
\max_{\mathcal{P} \subseteq \Delta S_1} \min_{\mathcal{Q} \subseteq \Delta S_2} U_1(\mathcal{P}, \mathcal{Q}) &= \max_{\mathcal{P} \subseteq \Delta S_1} \min_{\mathcal{Q} \subseteq \Delta S_2} \min_{P \in \mathcal{P}} \min_{Q \in \mathcal{Q}} u_1(P, Q) \\
&= \max_{\mathcal{P} \subseteq \Delta S_1} \min_{\mathcal{Q} \subseteq \Delta S_2} \min_{Q \in \mathcal{Q}} \min_{P \in \mathcal{P}} u_1(P, Q) \stackrel{\text{Lemma 3}}{=} \max_{\mathcal{P} \subseteq \Delta S_1} \min_{Q \in \Delta S_2} \min_{P \in \mathcal{P}} u_1(P, Q) \\
&= \max_{\mathcal{P} \subseteq \Delta S_1} \min_{P \in \mathcal{P}} \min_{Q \in \Delta S_2} u_1(P, Q) \stackrel{\text{Lemma 3}}{=} \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_1(P, Q) = u_1(P^*, Q^*) = v.
\end{aligned}$$

To proof equality (1) of equation (11) we need the Minimax Theorem 1 which we presented in Theorem 4. Then we have

$$\begin{aligned}
\min_{\mathcal{Q} \subseteq \Delta S_2} \max_{\mathcal{P} \subseteq \Delta S_1} U_1(\mathcal{P}, \mathcal{Q}) &= \min_{\mathcal{Q} \subseteq \Delta S_2} \max_{\mathcal{P} \subseteq \Delta S_1} \min_{P \in \mathcal{P}} \min_{Q \in \mathcal{Q}} u_1(P, Q) \\
&= \min_{\mathcal{Q} \subseteq \Delta S_2} \max_{P \in \Delta S_1} \min_{Q \in \mathcal{Q}} u_1(P, Q) \stackrel{\text{Thm. 4}}{=} \min_{\mathcal{Q} \subseteq \Delta S_2} \min_{Q \in \mathcal{Q}} \max_{P \in \Delta S_1} u_1(P, Q) \\
&= \min_{Q \in \Delta S_2} \max_{P \in \Delta S_1} u_1(P, Q) = u_1(P^*, Q^*) = \max_{\mathcal{P} \subseteq \Delta S_1} \min_{\mathcal{Q} \subseteq \Delta S_2} U_1(\mathcal{P}, \mathcal{Q}).
\end{aligned}$$

We now come to equality (2) of equation (12).

$$\begin{aligned}
\max_{\mathcal{Q} \subseteq \Delta S_2} \min_{\mathcal{P} \subseteq \Delta S_1} U_2(\mathcal{P}, \mathcal{Q}) &= \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{\mathcal{P} \subseteq \Delta S_1} \min_{Q \in \mathcal{Q}} \min_{P \in \mathcal{P}} u_2(P, Q) \\
&= \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{\mathcal{P} \subseteq \Delta S_1} \min_{P \in \mathcal{P}} \min_{Q \in \mathcal{Q}} u_2(P, Q) \stackrel{\text{Lemma 3}}{=} \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{P \in \Delta S_1} \min_{Q \in \mathcal{Q}} u_2(P, Q) \\
&= \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{Q \in \mathcal{Q}} \min_{P \in \Delta S_1} u_2(P, Q) \stackrel{\text{Lemma 3}}{=} \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_2(P, Q) \\
&= \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} -u_1(P, Q) = -u_1(P^*, Q^*) = v.
\end{aligned}$$

Lastly, we show equality (1) of equation (12).

$$\begin{aligned}
\min_{\mathcal{P} \subseteq \Delta S_1} \max_{\mathcal{Q} \subseteq \Delta S_2} U_2(\mathcal{P}, \mathcal{Q}) &= \min_{\mathcal{P} \subseteq \Delta S_1} \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{Q \in \mathcal{Q}} \min_{P \in \mathcal{P}} u_2(P, Q) \\
&= \min_{\mathcal{P} \subseteq \Delta S_1} \max_{Q \in \Delta S_2} \min_{P \in \mathcal{P}} u_2(P, Q) \stackrel{\text{Thm. 4}}{=} \min_{\mathcal{P} \subseteq \Delta S_1} \min_{P \in \mathcal{P}} \max_{Q \in \Delta S_2} -u_1(P, Q) \\
&= \min_{P \in \Delta S_1} \max_{Q \in \Delta S_2} -u_1(P, Q) = -u_1(P^*, Q^*) = \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{\mathcal{P} \subseteq \Delta S_1} U_2(\mathcal{P}, \mathcal{Q}).
\end{aligned}$$

□

Lemma 5. *Let $(\mathcal{P}^*, \mathcal{Q}^*)$ be an Ellsberg equilibrium of the two-person zero-sum game G .*

Then

$$U_1(\mathcal{P}^*, \mathcal{Q}^*) = u_1(P^*, Q^*) = v, \quad (13)$$

$$\text{and } U_2(\mathcal{P}^*, \mathcal{Q}^*) = -u_1(P^*, Q^*) = -v. \quad (14)$$

Proof. We start by showing equation (13). Equation (14) follows analogously.

$$\begin{aligned} U_2(\mathcal{P}^*, \mathcal{Q}^*) &\geq U_2(\mathcal{P}^*, \mathcal{Q}) \text{ for all } \mathcal{Q} \subseteq \Delta S_2 \\ \Rightarrow U_2(\mathcal{P}^*, \mathcal{Q}^*) &= \max_{\mathcal{Q} \subseteq \Delta S_2} U_2(\mathcal{P}^*, \mathcal{Q}) \\ \Rightarrow \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_2(P, Q) &= \max_{\mathcal{Q} \subseteq \Delta S_2} \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_2(P, Q) \\ \Rightarrow \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_2(P, Q) &= \max_{Q \in \Delta S_2} \min_{P \in \mathcal{P}^*} u_2(P, Q) \\ \Rightarrow \max_{P \in \mathcal{P}^*} \max_{Q \in \mathcal{Q}^*} u_1(P, Q) &= \min_{Q \in \Delta S_2} \max_{P \in \mathcal{P}^*} u_1(P, Q) \\ \Rightarrow \max_{P \in \mathcal{P}^*} \max_{Q \in \mathcal{Q}^*} u_1(P, Q) &\leq \min_{Q \in \Delta S_2} \max_{P \in \Delta S_1} u_1(P, Q). \end{aligned}$$

Furthermore,

$$\begin{aligned} U_1(\mathcal{P}^*, \mathcal{Q}^*) &\geq U_1(\mathcal{P}, \mathcal{Q}^*) \text{ for all } \mathcal{P} \subseteq \Delta S_1 \\ \Rightarrow U_1(\mathcal{P}^*, \mathcal{Q}^*) &= \max_{\mathcal{P} \subseteq \Delta S_1} U_1(\mathcal{P}, \mathcal{Q}^*) \\ \Rightarrow \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) &= \max_{\mathcal{P} \subseteq \Delta S_1} \min_{P \in \mathcal{P}} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) \\ \Rightarrow \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) &= \max_{P \in \Delta S_1} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) \\ \Rightarrow \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q) &\geq \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_1(P, Q). \end{aligned}$$

From the above relations and the classical Minimax Theorem follows that

$$\max_{P \in \mathcal{P}^*} \max_{Q \in \mathcal{Q}^*} u_1(P, Q) \leq \min_{Q \in \Delta S_2} \max_{P \in \Delta S_1} u_1(P, Q) = \max_{P \in \Delta S_1} \min_{Q \in \Delta S_2} u_1(P, Q) \leq \min_{P \in \mathcal{P}^*} \min_{Q \in \mathcal{Q}^*} u_1(P, Q)$$

and we finally have $U_1(\mathcal{P}^*, \mathcal{Q}^*) = u_1(P^*, Q^*) = v$. \square

From Lemma 3, 4 and 5 follows Theorem 6.

Proof of Proposition 5. The Nash equilibrium strategies follow from the usual analysis. To calculate the Ellsberg equilibria of the general coordination game (Figure 8), we

first derive the utility functions of player 1 and player 2. Due to the assumption that $a, d \geq b, c$ and $e, h \geq f, g$, with $a - b - c + d \neq 0$ and $e - f - g + h \neq 0$, the denominators $a - b - c + d$ and $e - f - g + h$ are positive. This reflects in the payoff functions that the game is a coordination game; player 1 uses Q_0 as a minimizer when $P > \frac{d-c}{a-b-c+d}$, and, the same way, player 2 uses P_0 as a minimizer when $Q > \frac{h-f}{e-f-g+h}$.

$$\begin{aligned}
U_1(P, [Q_0, Q_1]) &= \min_{Q_0 \leq Q \leq Q_1} aPQ + bP(1-Q) + c(1-P)Q + d(1-P)(1-Q) \\
&= \min_{Q_0 \leq Q \leq Q_1} Q((a-b-c+d)P + c-d) + (b-d)P + d \\
&= \begin{cases} Q_0((a-b-c+d)P + c-d) + (b-d)P + d & \text{if } P > \frac{d-c}{a-b-c+d}, \\ \frac{(b-d)(c-d)}{a-b-c+d} + d & \text{if } P = \frac{d-c}{a-b-c+d}, \\ Q_1((a-b-c+d)P + c-d) + (b-d)P + d & \text{if } P < \frac{d-c}{a-b-c+d}, \end{cases}
\end{aligned}$$

$$\begin{aligned}
U_2([P_0, P_1], Q) &= \min_{P_0 \leq P \leq P_1} ePQ + g(1-P)Q + fP(1-Q) + h(1-P)(1-Q) \\
&= \min_{P_0 \leq P \leq P_1} P((e-f-g+h)Q + f-h) + (g-h)Q + h \\
&= \begin{cases} P_0((e-f-g+h)Q + f-h) + (g-h)Q + h & \text{if } Q > \frac{h-f}{e-f-g+h}, \\ \frac{(g-h)(h-f)}{e-f-g+h} + h & \text{if } Q = \frac{h-f}{e-f-g+h}, \\ P_1((e-f-g+h)Q + f-h) + (g-h)Q + h & \text{if } Q < \frac{h-f}{e-f-g+h}. \end{cases}
\end{aligned}$$

We see that

$$M_1 = \frac{d-c}{a-b-c+d}, \quad \text{respectively} \quad M_2 = \frac{h-f}{e-f-g+h}$$

are the immunization strategies of player 1 and 2, respectively. At the boundaries the utility functions become

$$\begin{aligned}
U_1(0, [Q_0, Q_1]) &= (c-d)Q_1 + d, & U_2([P_0, P_1], 0) &= (f-h)P_1 + h, \\
U_1(1, [Q_0, Q_1]) &= (a-b)Q_0 + b, & U_2([P_0, P_1], 1) &= (e-g)P_0 + g.
\end{aligned}$$

The payoff function of player 1 is constant for all P , when $Q_1 = Q_0 = Q^*$, and the payoff function of player 2 is constant for all Q , when $P_1 = P_0 = P^*$. We calculate the best

response correspondences:

$$Q_0 > Q^* : \quad B_1([Q_0, Q_1]) = 1 \quad (1)$$

$$Q_0 = Q^* < Q_1 : \quad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [M_1, 1]\} \quad (2)$$

$$Q_0 < Q^* < Q_1 : \quad B_1([Q_0, Q_1]) = M_1 \quad (3)$$

$$Q_0 < Q^* = Q_1 : \quad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, M_1]\} \quad (4)$$

$$Q_1 < Q^* : \quad B_1([Q_0, Q_1]) = 0 \quad (5)$$

$$Q_0 = Q^* = Q_1 : \quad B_1([Q_0, Q_1]) = \{[P_0, P_1] \subseteq [0, 1]\} \quad (6)$$

$$P_0 > P^* : \quad B_2([P_0, P_1]) = 1$$

$$P_0 = P^* < P_1 : \quad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [M_2, 1]\}$$

$$P_0 < P^* < P_1 : \quad B_2([P_0, P_1]) = M_2$$

$$P_0 < P^* = P_1 : \quad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, M_2]\}$$

$$P_1 < P^* : \quad B_2([P_0, P_1]) = 0$$

$$P_0 = P^* = P_1 : \quad B_2([P_0, P_1]) = \{[Q_0, Q_1] \subseteq [0, 1]\}$$

To find the Ellsberg equilibria, we look at the different cases in turn.

(1) $Q_0 > Q^*$: player 1 responds $P_0 = P_1 = 1$. If also player 2 chooses $Q_0 = Q_1 = 1$, this is the Ellsberg equilibrium that is identical to the pure Nash equilibrium (U, L) .

(2) $\left[\frac{d-b}{a-b-c+d}, Q_1\right] \Rightarrow [P_0, P_1] \subseteq \left[\frac{d-c}{a-b-c+d}, 1\right]$, then, depending on the size of M_1 in relation to P^* , player 1 has the following choices:

- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: player 1 can play either
 - $\left[\frac{h-g}{e-f-g+h}, P_1\right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1\right]$. Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:

* $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$, we thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1\right], \left[\frac{d-b}{a-b-c+d}, Q_1\right]\right),$$

that is $([P^*, P_1], [Q^*, Q_1])$.

- * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.
- or $\left[P_0, \frac{h-g}{e-f-g+h}\right]$ with $\frac{d-c}{a-b-c+d} \leq P_0 \leq \frac{h-g}{e-f-g+h}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:
 - * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: $\left[\frac{d-b}{a-b-c+d}, Q_1\right]$ with $Q_1 \leq \frac{h-f}{e-f-g+h}$, then the best response of player 1 is $[P_0, P_1] \subseteq \left[\frac{d-c}{a-b-c+d}, 1\right]$. We thus have here the Ellsberg equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h}\right], \left[\frac{d-b}{a-b-c+d}, Q_1\right]\right),$$

where $\frac{d-c}{a-b-c+d} \leq P_0$ and $Q_1 \leq \frac{h-f}{e-f-g+h}$,

that is $([P_0, P^*], [Q^*, Q_1])$, where $M_1 \leq P_0$ and $Q_1 \leq M_2$.

- * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.
 - $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 1$, thus no Ellsberg equilibrium arises.
- (3) $Q_0 < Q^* < Q_1$: player 1 responds with $P_0 = P_1 = M_1$. Only when $M_1 = P^*$, player 1 sticks to his strategy and we get the equilibrium $(P^*, [Q_0, Q_1])$, where $Q_0 < Q^* < Q_1$.
- (4) $\left[Q_0, \frac{d-b}{a-b-c+d}\right] \Rightarrow [P_0, P_1] \subseteq \left[0, \frac{d-c}{a-b-c+d}\right]$, then, depending on the size of M_1 in relation to P^* , player 1 has the following choices:
- $\frac{d-c}{a-b-c+d} \geq \frac{h-g}{e-f-g+h}$: player 1 can play either
 - $\left[P_0, \frac{h-g}{e-f-g+h}\right]$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[0, \frac{h-f}{e-f-g+h}\right]$. Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:
 - * $\frac{h-f}{e-f-g+h} \geq \frac{d-b}{a-b-c+d}$: $\left[Q_0, \frac{d-b}{a-b-c+d}\right]$, we thus have here the Ellsberg

equilibrium

$$\left(\left[P_0, \frac{h-g}{e-f-g+h} \right], \left[Q_0, \frac{d-b}{a-b-c+d} \right] \right),$$

that is $([P_0, P^*], [Q_0, Q^*])$.

- * $\frac{h-f}{e-f-g+h} \leq \frac{d-b}{a-b-c+d}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 0$, thus no Ellsberg equilibrium arises.
- or $\left[\frac{h-g}{e-f-g+h}, P_1 \right]$ with $\frac{h-g}{e-f-g+h} \leq P_1 \leq \frac{d-c}{a-b-c+d}$, then the best response of player 2 is $[Q_0, Q_1] \subseteq \left[\frac{h-f}{e-f-g+h}, 1 \right]$. Now, depending on the size of M_2 in relation to Q^* , player 2 has the following choices:
 - * $\frac{d-b}{a-b-c+d} \geq \frac{h-f}{e-f-g+h}$: $\left[Q_0, \frac{d-b}{a-b-c+d} \right]$ with $\frac{h-f}{e-f-g+h} \leq Q_0$, then the best response of player 1 is $[P_0, P_1] \subseteq \left[0, \frac{d-c}{a-b-c+d} \right]$. We thus have here the Ellsberg equilibrium

$$\left(\left[\frac{h-g}{e-f-g+h}, P_1 \right], \left[Q_0, \frac{d-b}{a-b-c+d} \right] \right),$$

where $P_1 \leq \frac{d-c}{a-b-c+d}$ and $\frac{h-f}{e-f-g+h} \leq Q_0$,

that is $([P^*, P_1], [Q_0, Q^*])$, where $P_1 \leq M_1$ and $M_2 \leq Q_0$.

- * $\frac{d-b}{a-b-c+d} \leq \frac{h-f}{e-f-g+h}$: for all $[Q_0, Q_1]$ the best response is $P_0 = P_1 = 1$, thus no Ellsberg equilibrium arises.
- $\frac{d-c}{a-b-c+d} \leq \frac{h-g}{e-f-g+h}$: to any $[P_0, P_1]$ the best response is $Q_0 = Q_1 = 1$, thus no Ellsberg equilibrium arises.

- (5) $Q_1 < Q^*$: player 1 responds $P_0 = P_1 = 0$, if player 2 chooses $Q_0 = Q_1 = 0$ this is the Ellsberg equilibrium that is identical to the pure Nash equilibrium (D, R) .
- (6) $Q_0 = Q^* = Q_1$: player 1 responds with $[P_0, P_1] \subseteq [0, 1]$. Only when $Q^* = M_2$, player 2 sticks to his strategy and we get the equilibrium $([P_0, P_1], Q^*)$, where $P_0 < P^* < P_1$.

□

Proof of Proposition 7. We start by calculating the minimal expected utility functions of player 1 and 2. Player 1 minimizes over Q_1 and Q_2 . Q_3 is expressed as $1 - Q_1 - Q_2$.

The lower and upper bounds of Q_1 and Q_2 are w_1, z_1 and w_2, z_2 , respectively.

$$\begin{aligned}
& U_1(P_1, P_2, Q_1, Q_2) \\
&= \min_{Q_1, Q_2} 2P_1Q_2 - P_1(1 - Q_1 - Q_2) - P_2Q_1 \\
&\quad + P_2(1 - Q_1 - Q_2) + (1 - P_1 - P_2)Q_1 - (1 - P_1 - P_2)Q_2 \\
&= \min_{Q_2} \begin{cases} w_1(1 - 3P_2) + 4P_1Q_2 - P_1 + P_2 - Q_2 & \text{if } P_2 < 1/3, \\ 4P_1Q_2 - P_1 - Q_2 + 1/3 & \text{if } P_2 = 1/3, \\ z_1(1 - 3P_2) + 4P_1Q_2 - P_1 + P_2 - Q_2 & \text{if } P_2 > 1/3 \end{cases} \\
&= \begin{cases} \begin{cases} w_2(4P_1 - 1) - 3P_2w_1 + w_1 - P_1 + P_2 & \text{if } P_1 > 1/4, \\ w_1 - 3P_2w_1 - 1/4 + P_2 & \text{if } P_1 = 1/4, \\ z_2(4P_1 - 1) - 3P_2w_1 + w_1 - P_1 + P_2 & \text{if } P_1 < 1/4, \end{cases} & P_2 < 1/3, \\ \\ \begin{cases} w_2(4P_1 - 1) + 1/3 - P_1 & \text{if } P_1 > 1/4, \\ 1/12 & \text{if } P_1 = 1/4, \\ z_2(4P_1 - 1) + 1/3 - P_1 & \text{if } P_1 < 1/4, \end{cases} & P_2 = 1/3, \\ \\ \begin{cases} w_2(4P_1 - 1) - 3P_2z_1 + z_1 - P_1 + P_2 & \text{if } P_1 > 1/4, \\ z_1 - 3P_2z_1 - 1/4 + P_2 & \text{if } P_1 = 1/4, \\ z_2(4P_1 - 1) - 3P_2z_1 + z_1 - P_1 + P_2 & \text{if } P_1 < 1/4, \end{cases} & P_2 > 1/3. \end{cases}
\end{aligned}$$

Player 2 minimizes over P_1 and P_2 , P_3 is expressed as $1 - P_1 - P_2$. The lower and upper

bounds of P_1 and P_2 are x_1, y_1 and x_2, y_2 , respectively.

$$\begin{aligned}
& U_2(P_1, P_2, Q_1, Q_2) \\
&= \min_{P_1, P_2} -P_1 Q_2 + P_1(1 - Q_1 - Q_2) + P_2 Q_1 \\
&\quad - P_2(1 - Q_1 - Q_2) - (1 - P_1 - P_2)Q_1 + (1 - P_1 - P_2)Q_2 \\
&= \min_{P_2} \begin{cases} x_1(1 - 3Q_2) + 3P_2 Q_1 - P_2 - Q_1 + Q_2 & \text{if } Q_2 < 1/3, \\ 3P_2 Q_1 - P_2 - Q_1 + 1/3 & \text{if } Q_2 = 1/3, \\ y_1(1 - 3Q_2) + 3P_2 Q_1 - P_2 - Q_1 + Q_2 & \text{if } Q_2 > 1/3 \end{cases} \\
&= \begin{cases} \begin{cases} x_2(3Q_1 - 1) - 3Q_2 x_1 + x_1 - Q_1 + Q_2 & \text{if } Q_1 > 1/3, \\ x_1 - 3Q_2 x_1 - 1/3 + Q_2 & \text{if } Q_1 = 1/3, \\ y_2(3Q_1 - 1) - 3Q_2 x_1 + x_1 - Q_1 + Q_2 & \text{if } Q_1 < 1/3, \end{cases} & Q_2 < 1/3, \\ \begin{cases} x_2(3Q_1 - 1) + 1/3 - Q_1 & \text{if } Q_1 > 1/3, \\ 0 & \text{if } Q_1 = 1/3, \\ y_2(3Q_1 - 1) + 1/3 - Q_1 & \text{if } Q_1 < 1/3, \end{cases} & Q_2 = 1/3, \\ \begin{cases} x_2(3Q_1 - 1) - 3Q_2 y_1 + y_1 - Q_1 + Q_2 & \text{if } Q_1 > 1/3, \\ y_1 - 3Q_2 y_1 - 1/3 + Q_2 & \text{if } Q_1 = 1/3, \\ y_2(3Q_1 - 1) - 3Q_2 y_1 + y_1 - Q_1 + Q_2 & \text{if } Q_1 < 1/3, \end{cases} & Q_2 > 1/3. \end{cases}
\end{aligned}$$

Now we proceed as follows to derive the Ellsberg equilibria. Recall the Nash equilibria and immunization strategies of the game:

$$(P^*, Q^*) = ((1/3, 1/3, 1/3), (1/3, 1/4, 5/12)),$$

$$\text{and } (M_1, M_2) = ((1/4, 1/3, 5/12), (1/3, 1/3, 1/3)).$$

Player 1 has to use his Nash equilibrium strategies at the boundary of his Ellsberg equilibrium strategy, except for the component P_2 , where the Nash equilibrium probability is the same as the immunization strategy. Furthermore, the set of probability distributions in the Ellsberg equilibrium may not extend across the immunization strategy.

Hence, for the first component for player 1, we consider either $\{1/4 \leq x_1 \leq P_1 \leq 1/3\}$, or $\{1/3 \leq P_1\}$. In Ellsberg equilibrium, player 2 uses $1/3$ as his worst case measure, thus in the first case we only consider those elements of U_2 which use y_1 as worst case measure, in the second case only those which use x_1 . y_1 is only used when $Q_2 \geq 1/3$.

Since the range of Q_2 must contain $1/4$ at the boundary, which is impossible when $Q_2 \geq 1/3$, $\{1/4 \leq x_1 \leq P_1 \leq 1/3\}$ cannot be an Ellsberg equilibrium strategy. We proceed with the second case: x_1 is only used for $Q_2 \leq 1/3$, this is compatible with the Nash equilibrium strategy and yields $\{1/4 \leq Q_2 \leq z_2 \leq 1/3\}$. Finally we check if $\{1/3 \leq P_1\}$ is a best response to $\{1/4 \leq Q_2 \leq z_2 \leq 1/3\}$. Player 1 must use w_2 in his utility evaluation U_1 , this is true whenever $P_1 \geq 1/4$ which is compatible with the strategy $\{1/3 \leq P_1\}$. Thence we have found a best response pair.

In the second component, player 1 can play $\{P_2 = 1/3\}$, which is his Nash equilibrium probability and therefore makes player 2 indifferent between all $\{w_1 \leq Q_1 \leq z_1\}$, and at the same time his best response to player 2 playing any probability $\{w_1 \leq Q_1 \leq z_1\}$ with $0 \leq w_1 \leq 1/3 \leq z_1 \leq 1$. Or, conversely, player 2 can play $\{Q_1 = 1/3\}$ and player 1 $\{x_2 \leq P_2 \leq y_2\}$ with $0 \leq x_2 \leq 1/3 \leq y_2 \leq 1$. The latter case collapses to Nash equilibrium. The restrictions on w_1, z_1 , which are $0 \leq w_1 \leq 1/3 = z_1$, follow from the third component and the fact that each element of the Ellsberg equilibrium strategy must be a probability distribution.

We cannot determine the range for P_3 and Q_3 from the two utility functions, since P_3 and Q_3 are only implicitly given. Thus, we derive the utility functions again, now using $(P_1, 1 - P_1 - P_3, P_3)$ and $(1 - Q_2 - Q_3, Q_2, Q_3)$ as probabilities. Since we already know that only the boundaries x_1 and w_2 are relevant for the equilibrium, we get

$$\begin{aligned}
& U_1(P_1, P_3, Q_2, Q_3) \\
&= \min_{Q_2, Q_3} 2P_1Q_2 - P_1Q_3 - (1 - P_1 - P_3)(1 - Q_2 - Q_3) \\
&\quad + (1 - P_1 - P_3)Q_3 + P_3(1 - Q_2 - Q_3) - P_3Q_2 \\
&= \begin{cases} w_3(2 - 3P_1 - 3P_3) + w_2(P_1 - 3P_3 + 1) + 2P_3 + P_1 - 1 & \text{if } P_3 < 2/3 - P_1, \\ w_2(5/3 - 4P_3) + P_3 - 1/3 & \text{if } P_3 = 2/3 - P_1, \\ z_3(2 - 3P_1 - 3P_3) + w_2(P_1 - 3P_3 + 1) + 2P_3 + P_1 - 1 & \text{if } P_3 > 2/3 - P_1, \end{cases}
\end{aligned}$$

when $P_1 > 3P_3 - 1$.

On the other hand,

$$\begin{aligned}
& U_2(P_1, P_3, Q_2, Q_3) \\
&= \min_{P_1, P_3} -P_1 Q_2 + P_1 Q_3 + (1 - P_1 - P_3)(1 - Q_2 - Q_3) \\
&\quad - (1 - P_1 - P_3)Q_3 - P_3(1 - Q_2 - Q_3) + P_3 Q_2 \\
&= \begin{cases} x_3(3Q_2 + 3Q_3 - 2) + x_1(3Q_3 - 1) - Q_2 - 2Q_3 + 1 & \text{if } Q_2 > 2/3 - Q_3, \\ x_1(3Q_3 - 1) - Q_3 + 1/3 & \text{if } Q_2 = 2/3 - Q_3, \\ y_3(3Q_2 + 3Q_3 - 2) + x_1(3Q_3 - 1) - Q_2 - 2Q_3 + 1 & \text{if } Q_2 < 2/3 - Q_3, \end{cases}
\end{aligned}$$

when $Q_3 > 1/3$.

Now we use the following reasoning: we see from U_2 that Q_3 must be greater than (or in fact equal to, this is suppressed in the shortened statement of U_2) $1/3$. Since $5/12$ is the Nash equilibrium probability of Q_3 , we get the two following possible probability sets for Q_3 :

$$\text{either } \{1/3 \leq w_3 \leq Q_3 \leq 5/12\}, \text{ or } \{5/12 \leq Q_3 \leq z_3\}.$$

In the first case, player 1 must use z_3 in his utility evaluation, according to U_1 this is the case only when $P_3 \geq 2/3 - P_1$. We know that $P_1 \geq 1/3$, therefore $P_3 \geq 1/3$. This leads to the set of probabilities $\{1/3 \leq P_3 \leq y_3 \leq 5/12\}$ for P_3 , since $1/3$ as the Nash equilibrium probability must be part of the set. Here x_3 is used by player 2, hence from U_2 we see this is only the case if $Q_2 \geq 2/3 - Q_3$. $Q_3 \geq 1/3$ and therefore Q_2 must be greater than $1/3$. This is not possible, since we have already found Q_2 to be played with the set of probabilities $\{1/4 \leq Q_2 \leq z_2 \leq 1/3\}$.

Consider the second possible set for Q_2 . Here w_3 is used by player 1, and hence $P_2 \leq 1/3$. This leads to $\{x_3 \leq P_3 \leq 1/3\}$ for P_3 . Player 2 uses y_3 and we conclude that Q_2 must be less than or equal to $1/3$. This is exactly what we found to be true for Q_2 before. Therefore we have found the Ellsberg equilibrium of RSP to be

$$\begin{aligned}
& ((\{1/3 \leq P_1 \leq y_1 \leq 2/3\}, \{P_2 = 1/3\}, \{x_3 \leq P_3 \leq 1/3\}), \\
& (\{w_1 \leq Q_1 \leq z_1\}, \{1/4 \leq Q_2 \leq z_2 \leq 1/3\}, \{5/12 \leq Q_3 \leq z_3 \leq 3/4\})), \\
& \text{where } 0 \leq w_1 \leq 1/3 = z_1.
\end{aligned}$$

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